Part V

Rankin-Selberg method and periods of modular forms

Hidenori Katsurada

0 Introduction

Let $f(z) = \sum_{m=1}^{\infty} a(m)\exp(2\pi imz)$ and $g(z) = \sum_{m=1}^{\infty} b(m)\exp(2\pi imz)$ be cusp forms for $SL_2(\mathbf{Z})$. Then the Rankin-Selberg method gives an integral representation of the Dirichlet series, called the Rankin-Selberg convolution product, defined by

$$L(s,f,g) = \sum_{m=1}^{\infty} a(m)\overline{b(m)}m^{-s}.$$

This method was first introduced by Rankin [27] and Selberg [29] independently. Since then, it has fully developed for several types of modular forms, and has become one of the most useful tools for studying modular forms and their L-functions. In particular, it plays a very important role in proving analytic properties (meromorphy, functional equation etc.) of several automorphic L-functions. As for this, the reader is referred to excellent surveys by Bump [2] and [3].

In this paper, we give another application of the Rankin-Selberg method, which expresses the period of a cuspidal Hecke eigenform in terms of the special values of automorphic L-functions related to it. Here we mean by the period of a cusp form f the Petersson product $\langle f, f\rangle$ of f in almost all cases. The main purposes of this paper are as follows:

(1) to survey Petersson's formula for the period of an elliptic cusp form and its application;
(2) to survey Kohnen-Zagier's formula for the period of a Hecke eigenform of half integral weight;
(3) to give an outline of the proof of Ikeda's conjecture on the period of the Ikeda lift.

To explain them more precisely, first let f be a cusp form of integral weight k for $\Gamma_0(N)$. Then, in Section 2, we give Petersson's formula, which expresses the period $\langle f, f\rangle$ in terms of the residue of the Rankin-Selberg convolution product $L(s,f,f)$ at $s = k$. (cf. Proposition 2.3.) This is due to Petersson [26]. As an application, we express $\langle f, f\rangle$ in terms of the adjoint L-function of f evaluated at $s = 1$ in case f is a normalized Hecke eigenform (cf. Theorem 2.4.) This topic is rather elementary and well-known but instructive for our later investigation. So I will explain it precisely. Furthermore, we consider the algebraicity of the special values of several L-functions.

Next let f be a Hecke eigenform in the Kohnen plus subspace of cusp forms of weight $k+1/2$ for $\Gamma_0(4)$, and $S(f)$ the normalized Hecke eigenform of weight $2k$ for $SL_2(\mathbf{Z})$ corresponding to f under the Shimura correspondence. Then, in Section 3, we explain Kohnen-Zagier's formula, which expresses the ratio $\dfrac{\langle S(f), S(f)\rangle}{\langle f, f\rangle}$ of the periods in terms of the Fourier coefficient of f and the central critical value of the twisted Hecke L-function of $S(f)$ (cf. Theorem 3.4.) This type of result was first given by Waldspurger [39] in more general setting from the automorphic representation theoretic view point, and later was refined for the above special case by Kohnen and Zagier [23]. We here remark that a certain Rankin-Selberg convolution product without Euler product plays an important role in proving Theorem 3.4.

Finally for a normalized Hecke eigenform f of weight $2k-n$ for $SL_2(\mathbf{Z})$ with k,n even, let $\hat{f}$ be the Ikeda lift of f (cf. [9]). Furthermore let $\tilde{f}$ be a Hecke eigenform in the Kohnen plus subspace of cusp forms of weight $k-n/2+1/2$ for $\Gamma_0(4)$ such that $S(\tilde{f})=f$. Then Ikeda [10] among others conjectured that the ratio $\dfrac{\langle \hat{f},\hat{f}\rangle}{\langle \tilde{f},\tilde{f}\rangle}$ would be expressed in terms of the special values of the Hecke L-function and the adjoint L-function of f. This has been proved by the author and Kawamura (cf. [18]). In Sections 5 and 6, we give an outline of the proof. Here we also would like to emphasize that several Rankin-convolution products with or without Euler product play important roles in the proof. As for an application of this period relation to congruence between Ikeda lifts and non-Ikeda lifts, the reader is referred to [17] and [15].

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Notation. For a complex number x we put $\mathbf{e}(x)=\exp(2\pi i x)$. For a commutative ring R, we denote by $M_{mn}(R)$ the set of (m,n)-matrices with entries in R. In particular put $M_n(R)=M_{nn}(R)$. Put $GL_m(R)=\{A\in M_m(R)\mid \det A\in R^*\}$, where $\det A$ denotes the determinant of a square matrix A, and R^* denotes the unit group of R. For an (m,n)-matrix X and an (m,m)-matrix A, we write $A[X]={}^tXAX$, where ${}^tX$ denotes the transpose of X. Let $Sym_n(R)$ denote the set of symmetric matrices of degree n with entries in R. Furthermore, let $\mathcal{L}_n$ denote the set of half-integral matrices of degree n over $\mathbf{Z}$, that is, $\mathcal{L}_n$ is the set of symmetric matrices of degree n whose (i,j)-component belongs to $\mathbf{Z}$ or $\frac{1}{2}\mathbf{Z}$ according as $i=j$ or not. If S is a subset of $Sym_n(\mathbf{R})$ with $\mathbf{R}$ the field of real numbers, we denote by $S_{>0}$ (resp. $S_{\geq 0}$) the subset of S consisting of positive definite (resp. semi-positive definite) matrices. Let R be a commutative ring. Then a subgroup G of $GL_n(R)$ acts on the set $Sym_n(R)$ in the following way:

$$G\times Sym_n(R)\ni (g,A)\longrightarrow A[g]\in Sym_n(R).$$

For a subset S of $Sym_n(R)$, we denote by S/G the set of equivalence classes of S with respect to G.

1 Siegel modular forms

In this section we review modular forms of integral or half-integral weight. Put $J_n=\begin{pmatrix} O_n & -1_n \\ 1_n & O_n\end{pmatrix}$, where 1_n denotes the unit matrix of degree n. For a subring K of $\mathbf{R}$ put

$$GSp_n^+(K)=\{\gamma\in GL_{2n}(K)\mid J_n[\gamma]=\nu(\gamma)J_n \text{ with some } \nu(\gamma)>0\},$$

and

$$Sp_n(K)=\{\gamma\in GSp_n^+(K)\mid J_n[\gamma]=J_n\}.$$

We call $GSp_n^+(\mathbf{R})$ the group of proper symplectic similitudes of degree n, and $\Gamma^{(n)}=Sp_n(\mathbf{Z})$. For a positive integer N we define the principal congruence subgroup $\Gamma^{(n)}(N)$ of $\Gamma^{(n)}$ of level N by

$$\Gamma^{(n)}(N)=\{\gamma\in\Gamma^{(n)}\mid \gamma\equiv 1_{2n} \bmod N\}.$$

A subgroup Γ of $\Gamma^{(n)}$ is called a congruence subgroup of $\Gamma^{(n)}$ if Γ contains some principal congruence subgroup. For a positive integer N, we denote by $\Gamma_0^{(n)}(N)$ the subgroup of $\Gamma^{(n)}$ consisting of matrices whose lower left $n\times n$ block is congruent to O_n modulo N. Clearly $\Gamma_0^{(n)}(N)$ is a congruence subgroup of $\Gamma^{(n)}$. Let

$$\mathbf{H}_n = \{Z \in Sym_n(\mathbf{C}) \mid \mathrm{Im}(Z) > 0\}$$

be Siegel's upper half-space of degree n. Write $\gamma \in GSp_n^+(\mathbf{R})$ as $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ with $A, B, C, D \in M_n(\mathbf{R})$ and for $Z \in \mathbf{H}_n$ put $\gamma(Z) = (AZ+B)(CZ+D)^{-1}$. Then $\gamma(Z)$ also belongs to $\mathbf{H}_n$ and we can define an action of the group $GSp_n^+(\mathbf{R})$ on $\mathbf{H}_n$ in this way. Furthermore put $j(\gamma, Z) = \det(CZ+D)$.

First we define a modular form of integral weight. Let k be a positive integer. For a function f on $\mathbf{H}_n$ we define $f|_k\gamma$ as

$$(f|_k\gamma)(Z) = \det(\gamma)^{k/2} j(\gamma, Z)^{-k} f(\gamma(Z)).$$

We simply write $f|\gamma$ for $f|_k\gamma$ if there is no confusion. Then $f|_k$ defines an action of $GSp_n^+(\mathbf{R})$ on f, that is, we have $f|_k(\gamma_1\gamma_2) = (f|_k\gamma_1)|_k\gamma_2$ for any $\gamma_1, \gamma_2 \in GSp_n^+(\mathbf{R})$. Let Γ be a congruence subgroup of $\Gamma^{(n)}$ which contains some $\Gamma^{(n)}(N)$, and χ a character of Γ trivial on $\Gamma^{(n)}(N)$. A function f on $\mathbf{H}_n$ is called a C^∞-modular form of weight k and character χ for Γ if it satisfies the following conditions:

(i) f is a C^∞-function on $\mathbf{H}_n$;

(ii) $(f|_k\gamma)(Z) = \chi(\gamma)f(Z)$ for any $\gamma \in \Gamma$.

We call a C^∞-modular form f a holomorphic modular form if
(i) f is holomorphic on $\mathbf{H}_n$;
(ii) for any $\gamma \in \Gamma^{(n)}$, $f|_k\gamma$ has the following Fourier expansion

$$f|_k\gamma(Z) = \sum_{A \in \mathcal{L}_{n\geq 0}} c_{f|_k\gamma}(A)\mathbf{e}(\mathrm{tr}(AZ/N)),$$

where tr denotes the trace of a matrix.
We call $f(Z)$ a cusp form if
(iii) $c_{f|_k\gamma}(A) = 0$ unless A is positive-definite.
We note that we have the following Fourier expansion

$$f(Z) = \sum_{A \in \mathcal{L}_{n\geq 0}} c(A)\mathbf{e}(\mathrm{tr}(AZ))$$

if f is a modular form for $\Gamma_0(N)$.

Next we define a modular form of half-integral weight. Let k be a half-integer. In this case, for a function f on $\mathbf{H}_n$ and $\gamma \in GSp_n^+(\mathbf{R})$ we can also define $f|_k\gamma$ as

$$(f|_k\gamma)(Z) = \det(\gamma)^{k/2} j(\gamma, Z)^{-k} f(\gamma(Z)),$$

where $j(\gamma, Z)^{-k}$ is an appropriately defined single valued holomorphic function. However, this does not define an action of $GSp_n^+(\mathbf{R})$. To overcome this obstacle, we define a group $\mathfrak{G}\mathfrak{S}\mathfrak{p}_n^+(\mathbf{R})$ as follows. Namely let $\mathfrak{G}\mathfrak{S}\mathfrak{p}_n^+(\mathbf{R})$ denote the set of all couples $(\gamma, \phi(Z))$ formed by an element $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in GSp_n^+(\mathbf{R})$ and a holomorphic function on $\mathbf{H}_n$ such that

$$\phi(Z)^2 = t \det\gamma^{-1/2} \det(CZ + D)$$

with $t \in \mathbf{C}^1$, where $\mathbf{C}^1 = \{t \in \mathbf{C} \mid |t| = 1\}$. We define a law of multiplication in $\mathfrak{GSp}_n^+(\mathbf{R})$ by

$$(\gamma_1, \phi_1)(\gamma_2, \phi_2) = (\gamma_1\gamma_2, \phi_1(\gamma_2(Z))\phi_2(Z)).$$

By this, we make $\mathfrak{GSp}_n^+(\mathbf{R})$ a group. Let $P : \mathfrak{GSp}^+(\mathbf{R}) \longrightarrow GSp_n^+(\mathbf{R})$ be the natural projection map, and put $\mathfrak{GSp}_n^+(K) = P^{-1}(GSp_n^+(K))$ and $\mathfrak{Sp}_n(K) = P^{-1}(Sp_n(K))$ for a subring K of $\mathbf{R}$. In particular we put $\mathfrak{GL}_2^+(K) = \mathfrak{GSp}_1^+(K)$. We define the action of $\xi \in \mathfrak{GSp}_n^+(\mathbf{R})$ on $\mathbf{H}_n$ as that of $P(\xi)$ on $\mathbf{H}_n$. For a function f on $\mathbf{H}_n$ and $\xi = (\gamma, \phi) \in \mathfrak{GSp}_n^+(\mathbf{R})$ we define $f|_{2k}\xi$ as

$$(f|_{2k}\xi)(Z) = \phi(Z)^{-2k} f(\gamma(Z)).$$

This defines an action of $\mathfrak{GSp}_n^+(\mathbf{R})$ on f. Now we define a function $\theta(Z)$ on $\mathbf{H}_n$ as

$$\theta(Z) = \sum_{\mathbf{m} \in M_{n1}(\mathbf{Z})} \mathbf{e}(Z[\mathbf{m}]).$$

Put $\tilde{j}(\gamma, Z) = \dfrac{\theta(\gamma(Z))}{\theta(Z)}$ for $\gamma \in \Gamma_0^{(n)}(4)$. Then we remark that

$$\tilde{j}(\gamma, Z)^2 = (-1)^{(\det D - 1)/2} j(\gamma, Z)$$

for $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$. Then $(\gamma, \tilde{j}(\gamma, Z))$ belongs to $\mathfrak{GSp}_n^+(\mathbf{R})$ and the map $I : \Gamma_0^{(n)}(4) \ni \gamma \mapsto (\gamma, \tilde{j}(\gamma, Z)) \in \mathfrak{GSp}_n^+(\mathbf{R})$ is an injective homomorphism of groups. For a congruence subgroup Γ contained in $\Gamma_0^{(n)}(4)$ we use the same symbol Γ to denote the image $I(\Gamma)$. Now let k be a half-integer, assume that Γ is contained in $\Gamma_0^{(n)}(4)$ and that it contains some $\Gamma^{(n)}(N)$. Furthermore let χ be a character of Γ trivial on $\Gamma^{(n)}(N)$. A function f on $\mathbf{H}_n$ is called a C^∞-modular form of weight k for Γ if it satisfies the following conditions:

(i) f is a C^∞-function on $\mathbf{H}_n$;

(ii) $(f|_{2k}\gamma)(Z) = \chi(\gamma)f(Z)$ for any $\gamma \in \Gamma$;

We call a C^∞-modular form f a holomorphic modular form if
(i) f is holomorphic on $\mathbf{H}_n$;
(ii) for any $\gamma \in \mathfrak{Sp}_n(\mathbf{Z})$, $f|_{2k}\gamma$ has the following Fourier expansion

$$f|_{2k}\gamma(Z) = \sum_{A \in \mathcal{L}_{n \geq 0}} c_{f|_{2k}\gamma}(A)\mathbf{e}(\mathrm{tr}(AZ/N)).$$

In particular we call $f(Z)$ a cusp form if
(iii) $c_{f|_{2k}\gamma}(A) = 0$ unless A is positive-definite.
We note that we have the following Fourier expansion

$$f(Z) = \sum_{A \in \mathcal{L}_{n \geq 0}} c(A)\mathbf{e}(\mathrm{tr}(AZ))$$

if f is a modular form for $\Gamma_0(N)$.

For an integer or half-integer k, we denote by $\mathfrak{M}_k(\Gamma, \chi)$ (resp. $\mathfrak{M}_k^\infty(\Gamma, \chi)$) the space of holomorphic (resp. C^∞-) modular forms of weight k and character χ for Γ. We denote by $\mathfrak{S}_k(\Gamma, \chi)$ the subspace of $\mathfrak{M}_k(\Gamma, \chi)$ consisting of cusp forms. If χ is the trivial character, we simply write $\mathfrak{M}_k(\Gamma, \chi)$ as

$\mathfrak{M}_k(\mathit{\Gamma})$, and the others. Let χ be a Dirichlet character mod N. Then the map $\mathit{\Gamma}_0(N) \ni \gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \mapsto \chi(\det D)$ is a character of $\mathit{\Gamma}_0(N)$, which we denote by the same symbol χ. In this case we understand that $\mathfrak{M}_k(\mathit{\Gamma}_0(N))$ is $\mathfrak{M}_k(\mathit{\Gamma}_0(N), \psi)$ with ψ the trivial character mod N. Let $f \in \mathfrak{M}_k(\mathit{\Gamma}_0(N), \psi)$ with ψ a Dirichlet character mod N. Let dv denote the invariant volume element on $\mathbf{H}_n$ defined by $dv = \det(\mathrm{Im}(Z))^{-n-1} \wedge_{1 \le j \le l \le n} (dx_{jl} \wedge dy_{jl})$. Here for $Z \in \mathbf{H}_n$ we write $Z = (x_{jl}) + i(y_{jl})$ with real matrices (x_{jl}) and (y_{jl}). For two C^∞-modular forms f and g of weight k for $\mathit{\Gamma}^{(n)}$ we define the Petersson scalar product $\langle f, g \rangle$ by

$$\langle f, g \rangle = [\mathit{\Gamma}^{(n)} : \mathit{\Gamma}\{\pm 1\}]^{-1} \int_{\Phi_{\mathit{\Gamma}}} f(Z)\overline{g(Z)} \det(\mathrm{Im}(Z))^k dv,$$

provided the integral converges. Here $\Phi_{\mathit{\Gamma}}$ is a fundamental domain for $\mathbf{H}_n$ modulo $\mathit{\Gamma}$.

Now we review a general Hecke theory for modular forms. A more precise Hecke theory will be explained in Sections 2,3 and 4. For a while put $G_n = GSp_n^+(\mathbf{Q})$ or $\mathfrak{G}\mathfrak{S}\mathfrak{p}_n^+(\mathbf{Q})$, and $\mathit{\Gamma}_n = Sp_n(\mathbf{Z})$ or $\mathit{\Gamma}_0^{(n)}(4)$ according as $G_n = GSp_n^+(\mathbf{Q})$ or $\mathfrak{G}\mathfrak{S}\mathfrak{p}_n^+(\mathbf{Q})$. Let K be a commutative ring with unity. Let Δ be a sub-semigroup of G_n and $\mathit{\Gamma}$ a congruence subgroup of $\mathit{\Gamma}_n$. We denote by $\mathcal{R}_K(\mathit{\Gamma}, \Delta)$ the module of all the K-finite formal sum of the double coset $\mathit{\Gamma}\gamma\mathit{\Gamma}$ with $\gamma \in \Delta$. We define the following multiplication law: for two double cosets $\mathit{\Gamma}\gamma\mathit{\Gamma}$ and $\mathit{\Gamma}\gamma'\mathit{\Gamma}$ write

$$\mathit{\Gamma}\gamma\mathit{\Gamma} = \bigcup_i \mathit{\Gamma}\gamma_i$$

and

$$\mathit{\Gamma}\gamma'\mathit{\Gamma} = \bigcup_j \mathit{\Gamma}\gamma'_j,$$

and we define $\mathit{\Gamma}\gamma\mathit{\Gamma}\mathit{\Gamma}\gamma'\mathit{\Gamma}$ as

$$\mathit{\Gamma}\gamma\mathit{\Gamma}\mathit{\Gamma}\gamma'\mathit{\Gamma} = \sum_{\gamma''} c(\gamma''; \gamma, \gamma')\mathit{\Gamma}\gamma''\mathit{\Gamma},$$

where $c(\gamma''; \gamma, \gamma') = \#\{(i,j) \mid \mathit{\Gamma}\gamma'' = \mathit{\Gamma}\gamma_i\gamma'_j\}$. Under this multiplication, $\mathcal{R}_K(\mathit{\Gamma}, \Delta)$ becomes an associative algebra over K, which we call the Hecke algebra over K associated with the Hecke pair $(\mathit{\Gamma}, \Delta)$. Now we consider the action of $\mathcal{R}_K(\mathit{\Gamma}, \Delta)$ on $\mathfrak{M}_k(\mathit{\Gamma}, \chi)$. First let k be an integer. Let $\mathit{\Gamma}$ be a congruence subgroup of $Sp_n(\mathbf{Z})$ and Δ a sub-semigroup of $Sp_n^+(\mathbf{Q})$. We assume that χ can be extended to Δ, which we denote by the same symbol χ. Furthermore assume that if $\alpha\gamma\alpha^{-1} \in \mathit{\Gamma}$ for $\gamma \in \mathit{\Gamma}, \alpha \in \Delta$, then $\chi(\alpha\gamma\alpha^{-1}) = \chi(\gamma)$. Let $T = \mathit{\Gamma}\gamma\mathit{\Gamma}$ be an element of $\mathcal{R}_K(\mathit{\Gamma}, \Delta)$. Write T as $T = \cup_\gamma \mathit{\Gamma}\gamma$ and for $f \in \mathfrak{M}_k(\mathit{\Gamma}, \chi)$ define the Hecke operator $|_k T$ associated to T as

$$f|_k T = \det(\gamma)^{k/2-(n+1)/2} \sum_\gamma \overline{\chi(\gamma)} f|_k \gamma.$$

This expression does not depend on the choice of γ, and $f|_k T$ belongs to $\mathfrak{M}_k(\mathit{\Gamma}, \chi)$. We call this action the Hecke operator as usual (cf. [1].) Next let k be a half-integer. Then, for a congruence subgroup of $\mathit{\Gamma}_0^{(n)}(4)$ and a sub-semigroup Δ of $\mathfrak{G}\mathfrak{S}\mathfrak{p}_n^+(\mathbf{Q})$, we can define the action of $\mathcal{R}_K(\mathit{\Gamma}, \Delta)$ on $\mathfrak{M}_k(\mathit{\Gamma}, \psi)$ in a similar way. If f is an eigenfunction of a Hecke operator $T \in \mathcal{R}_K(\mathit{\Gamma}, \Delta)$ we denote by $\lambda_f(T)$ its eigenvalue. We call $f \in \mathfrak{M}_k(\mathit{\Gamma}, \chi)$ a Hecke eigenform for $\mathcal{R}_K(\mathit{\Gamma}, \Delta)$ if it is a common eigenfunction of all Hecke operators in $\mathcal{R}_K(\mathit{\Gamma}, \Delta)$.

2 Elliptic modular forms

Throughout this section and the next, we simply write $\Gamma_0^{(1)}(N)$ as $\Gamma_0(N)$. Let $f \in \mathfrak{M}_k(\Gamma_0(N), \phi)$. Then we have the following Fourier expansion of f:

$$f(z) = \sum_{m=0}^{\infty} a(m)\mathbf{e}(mz).$$

Then for a Dirichlet character χ we define the Dirichlet series

$$L(s, f, \chi) = \sum_{m=1}^{\infty} a(m)\chi(m)m^{-s}.$$

We briefly review the properties of $L(s, f, \chi)$ following [24]. The Dirichlet series $L(s, f, \chi)$ can be continued to a meromorphic function on the whole s-plane. Furthermore $L(s, f, \chi)$ is entire if f is a cusp form. Let

$$\Delta_0(N) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbf{Z}) \mid c \equiv 0 \bmod N, (a, N) = 1, ad - bc > 0\}.$$

Put $\mathcal{R}_0(N) = \mathcal{R}_{\mathbf{Z}}(\Gamma_0(N), \Delta_0(N))$. For integers l, m such that $l|m$ and $(l, N) = 1$ define an element $T(l, m)$ of $\mathcal{R}_0(N)$ by

$$T(l, m) = \Gamma_0(N) \begin{pmatrix} l & 0 \\ 0 & m \end{pmatrix} \Gamma_0(N).$$

In particular put $T(p) = T(1, p)$ for a prime number p. Then $\mathcal{R}_0(N)$ is the polynomial ring over $\mathbf{Z}$ generated by $T(p), T(p, p)$ with all prime numbers p prime to N, and $T(q)$ with prime numbers q dividing N. We simply call f a Hecke eigenform if it is a Hecke eigenform for $\mathcal{R}_0(N)$. Now assume that f is a normalized Hecke eigenform, that is, f is a Hecke eigenform with the first Fourier coefficient 1. Put $\lambda(m) = \lambda_f(m)$. Then we have

$$a(m) = \lambda(m).$$

We also have

$$\lambda(mn) = \lambda(m)\lambda(n)$$

if $(m, n) = 1$, and

$$\lambda(p^2) = \lambda(p)^2 - \psi(p)p^{k-1}$$

for any prime number p. Thus we have

$$L(s, f, \chi) = \prod_p (1 - \lambda(p)p^{-s}\chi(p) + p^{k-1-2s}\psi(p)\chi(p)^2)^{-1}.$$

We call $L(s, f, \chi)$ Hecke's L-function of f twisted by χ. We write $\lambda(p)$ as

$$\lambda(p) = p^{k/2-1/2}(\alpha_p + \beta_p) \text{ and } \alpha_p\beta_p = \psi(p)$$

with α_p, β_p complex numbers. Then $L(s, f, \chi)$ can also be expressed as

$$L(s, f, \chi) = \prod_p \{(1 - \alpha_p p^{k/2-1/2-s}\chi(p))(1 - \beta_p p^{k/2-1/2-s}\chi(p))\}^{-1}.$$

If χ is the principal character, we simply write $L(s,f,\chi)$ as $L(s,f)$. In Section 3, we will express $L(s,f)$ for a Hecke eigenform f in $\mathfrak{M}_k(SL_2(\mathbf{Z}))$ in another way.

Now for $f(z)=\sum_{m=0}^{\infty}a(m)\mathbf{e}(mz)\in\mathfrak{M}_k(\Gamma_0(N),\phi)$ and $g(z)=\sum_{m=0}^{\infty}b(m)\mathbf{e}(mz)\in\mathfrak{M}_l(\Gamma_0(N),\psi)$ we then define the Rankin-Selberg convolution product as

$$D(s,f,g)=\sum_{m=1}^{\infty}a(m)\overline{b(m)}m^{-s}.$$

We consider an integral expression of $D(s,f,g)$ in case f or g is a cusp form. Let λ be a non-negative integer and χ a Dirichlet character mod N such that $\chi(-1)=(-1)^{\lambda}$. We define the Eisenstein series $E_{\lambda,N}(z,s;\chi)$ by

$$E_{\lambda,N}(z,s;\chi)=y^s\sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0(N)}\chi(\gamma)j(\gamma,z)^{-\lambda}|j(\gamma,z)|^{-2s}.$$

Proposition 2.1. *Let λ,N and χ be as above.*
(1) *Put*

$$\mathfrak{E}(s)=\Gamma(s+\lambda)E_{\lambda,N}(z,s,\chi),$$

where $\Gamma(s)$ is Gamma function. Then as a function of s, $\mathfrak{E}(s)$ can be continued to a meromorphic function on the whole s-plane. Furthermore $\mathfrak{E}(s)$ is entire if $\lambda\neq 0$ or χ is non-trivial. If $\lambda=0$ and χ is trivial, $\mathfrak{E}(s)$ has a simple pole at $s=1$ with the residue

$$\frac{3}{N\pi}\prod_{p|N}(1+p^{-1})^{-1}.$$

(2) *If $\lambda\geq 3$, or $\lambda=2$ and $\chi\neq 1$. Then $E_{\lambda,N}(z,0,\chi)$ belongs to $\mathfrak{M}_\lambda(\Gamma_0(N),\overline{\chi})$, and in particular if χ is a primitive character, we have the following Fourier expansion*

$$E_{\lambda,N}(z,0,\chi)=1+\frac{2}{L(1-\lambda,\chi)}\sum_{n=1}^{\infty}(\sum_{d|n}\overline{\chi}(d)d^{\lambda-1})\mathbf{e}(nz),$$

where $L(s,\chi)$ is the Dirichlet L-function associated with χ. Furthermore, if χ is trivial, then we have

$$E_{2,N}(z,0,\chi)=\frac{c}{4\pi y}+\sum_{n=0}^{\infty}c_n\mathbf{e}(nz)$$

with rational numbers c and c_n.

Now by using so called the Rankin-Selberg method we have the following:

Theorem 2.2. *Let $f(z)=\sum_{m=0}^{\infty}a(m)\mathbf{e}(mz)\in\mathfrak{S}_k(\Gamma_0(N),\phi)$ and $g(z)=\sum_{m=0}^{\infty}b(m)\mathbf{e}(mz)\in\mathfrak{M}_l(\Gamma_0(N),\psi)$. Assume that $k\geq l$. Then we have*

$$(4\pi)^{-s}\Gamma(s)D(s,f,g)=\int_{\Phi_{\Gamma_0(N)}}f(z)\overline{g(z)E_{k-l,N}(z,\bar{s}+1-k,\overline{\phi}\psi)}y^{k-2}dxdy$$

Proof. Put

$$I=\int_{\Phi_{\Gamma_0(N)}}f(z)\overline{g(z)E_{k-l,N}(z,\bar{s}+1-k,\overline{\phi}\psi)}y^{k-2}dxdy.$$

Then we have

$$I=\int_{\Phi_{\Gamma_0(N)}} f(z)\overline{g(z)} \sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0(N)} \mathrm{Im}(z)^{s+1}\phi(\gamma)\overline{\psi(\gamma)j(\gamma,z)}^{l-k}|j(\gamma,z)|^{-2s-2+2k}y^{-2}dxdy.$$

For any $\gamma\in\Gamma_0(N)$ we have

$$f(z)\overline{g(z)}\mathrm{Im}(z)^{s+1}\phi(\gamma)\overline{\psi(\gamma)j(\gamma,z)}^{l-k}|j(\gamma,z)|^{-2s-2+2k}=f(\gamma(z))\overline{g(\gamma(z))}\mathrm{Im}(\gamma(z))^{s+1}.$$

Thus we have

$$\begin{aligned}I&=\int_{\cup_{\gamma\in\Gamma_\infty\backslash\Gamma_0(N)}\gamma(\Phi_{\Gamma_0(N)})} f(z)\overline{g(z)}\mathrm{Im}(z)^{s+1}y^{-2}dxdy\\&=\int_{\mathcal{R}} f(z)\overline{g(z)}\mathrm{Im}(z)^{s+1}y^{-2}dxdy,\end{aligned}$$

where $\mathcal{R}=\{z\in\mathbf{H}\mid |\mathrm{Re}(z)|\le 1/2\}$. Thus we have

$$\begin{aligned}I&=\int_0^\infty\int_{-1/2}^{1/2}\sum_{m,n=0}^\infty a(m)\overline{b(n)}\exp(2\pi i(m-n)x-2\pi(m+n)y)y^{s+1}y^{-2}dxdy\\&=\int_0^\infty\sum_{m,n=0}^\infty a(m)\overline{b(n)}\int_{-1/2}^{1/2}\exp(2\pi i(m-n)x-2\pi(m+n)y)y^{s+1}y^{-2}dxdy\\&=\int_0^\infty\sum_{m=1}^\infty a(m)\overline{b(m)}\exp(-4\pi my)y^{s+1}y^{-2}dy\\&=\sum_{m=1}^\infty a(m)\overline{b(m)}(4\pi m)^{-s}\Gamma(s).\end{aligned}$$

This proves the assertion.

By (1) of Propositions 2.1 and Theorem 2.2 we have

Proposition 2.3. *Let f and g be as in Theorem 2.2. Then $D(s,f,g)$ can be continued to a meromorphic function on the whole s-plane. Furthermore we have*

$$\mathrm{Res}_{s=k}D(s,f,f)=\frac{(4\pi)^k}{\Gamma(k)}\langle f,f\rangle\frac{3}{\pi}.$$

Let $f(z)=\sum_{m=1}^\infty a(m)\mathbf{e}(mz)$ be a normalized Hecke eigenform in $\mathfrak{S}_k(\Gamma_0(N),\phi)$ and α_p and β_p the complex numbers defined as above. We then define the adjoint L-function $L(s,f,\chi,\mathrm{Ad})$ of f twisted by χ as

$$L(s,f,\chi,\mathrm{Ad})=\prod_p\{(1-\alpha_p^2p^{-s}\chi(p))(1-\beta_p^2p^{-s}\chi(p))(1-p^{-s}\alpha_p\beta_p\chi(p))\}^{-1}.$$

If χ is the principal character, we simply write $L(s,f,\chi,\mathrm{Ad})$ as $L(s,f,\mathrm{Ad})$. In Section 4, we will express $L(s,f,\mathrm{Ad})$ for a Hecke eigenform f in $\mathfrak{M}_k(SL_2(\mathbf{Z}))$ in another way. Put $f_\rho(z)=\sum_{m=1}^\infty\overline{a(m)}\mathbf{e}(mz)$. Then we note that we can write $D(s,f,f_\rho)$ as

$$D(s,f,f_\rho)=\prod_p(1-\alpha_p^2\beta_p^2p^{2k-2-2s})\prod_p\{(1-\alpha_p^2p^{k-1-s})(1-\beta_p^2p^{k-1-s})(1-p^{k-1-s}\alpha_p\beta_p)^2\}^{-1}.$$

We note that $\alpha_p\beta_p = \phi(p)$. Thus by comparing the Euler products of $L(s, f, \mathrm{Ad})$ and $D(s+k-1, f, f_\rho)$ we have

$$(*) \qquad L(s,\phi)L(s,f,\mathrm{Ad}) = L(2s,\phi^2)D(s+k-1,f,f_\rho).$$

In particular, if $f \in \mathfrak{S}_k(\Gamma_0(N))$, we have $f(z) = f_\rho(z)$, and $\alpha_p\beta_p = 0$ or 1 according as p divides N or not. Thus we have

$$\zeta_N(s)L(s,f,\mathrm{Ad}) = \zeta_N(2s)D(s+k-1,f,f),$$

where $\zeta_N(s) = \zeta(s)\prod_{p|N}(1-p^{-s})$ with $\zeta(s)$ Riemann's zeta function. Thus by Proposition 2.3 we have

Theorem 2.4. *Under the above notation and the assumption, $L(s,f,\mathrm{Ad})$ can be continued to a meromorphic function on the whole s-plane. In particular, if $f \in \mathfrak{S}_k(\Gamma_0(N))$, then we have*

$$\frac{L(1,f,\mathrm{Ad})}{\pi^{k+1}\langle f,f\rangle} = \frac{2^{2k-1}}{\Gamma(k)}\prod_{p|N}(1+p^{-1}).$$

Remark. (1) Similarly for any character χ, $L(s,f,\chi,\mathrm{Ad})$ can be continued to a meromorphic function on the whole s-plane. Furthermore, if $f \in \mathfrak{S}_k(\Gamma_0(N))$, then $L(s,f,\chi,\mathrm{Ad})$ is entire for any character χ. As for this, see [31].
(2) By Theorem 2.4, the period $\langle f,f\rangle$ of a normalized Hecke eigenform $f \in \mathfrak{S}_k(\Gamma_0(N))$ can be expressed as the adjoint L-function of f evaluated at $s=1$ and some elementary quantities. The same formula holds for a normalized Hecke eigenform in $\mathfrak{S}_k(\Gamma_0(N),\phi)$ with non-trivial character ϕ.

In the rest of this section, we consider the special values of several L-functions of a modular form. For a Hecke eigenform f in $\mathfrak{S}_k(\Gamma_0(N),\phi)$, we denote by $\mathbf{Q}(f)$ the field generated over $\mathbf{Q}$ by all the eigenvalues of the Hecke operators, and call it the Hecke field of f. First, by using (2) of Proposition 2.1 and Theorem 2.2, we have

Proposition 2.5 ([32]) *Let $f \in \mathfrak{S}_k(\Gamma_0(N),\phi)$ and $g \in \mathfrak{M}_l(\Gamma_0(N),\psi)$ be normalized Hecke eigenforms. Assume that $k > l$, and let m be an integer such that $\frac{1}{2}(k+l-2) < m < k$. Then $\dfrac{D(m,f,g)}{\pi^k\langle f,f\rangle}$ belongs to $\mathbf{Q}(f)\mathbf{Q}(g)$.*

Put $\Gamma_{\mathbf{C}}(s) = \dfrac{2\Gamma(s)}{(2\pi)^s}$. Then by applying Proposition 2.5, we get the following (cf. [32],[33].)

Theorem 2.6. (1) *Let ψ be a Dirichlet character mod N. Let f be a normalized Hecke eigenform in $\mathfrak{S}_k(\Gamma_0(N),\psi)$. Then there exist complex numbers $u_\pm(f)$ uniquely determined up to $\mathbf{Q}(f)^\times$ multiple such that $\dfrac{\Gamma_{\mathbf{C}}(m)L(m,f,\chi)}{\tau(\chi)u_j(f)} \in \mathbf{Q}(f)(\chi)$ for any integer $0 < m \le k-1$ and a Dirichlet character χ such that $j = (-1)^m\chi(-1)$, where $\tau(\chi)$ is the Gauss sum.*

Finally we consider the algebraicity of the adjoint L-function. Proposition 2.5 holds for two modular forms of different weights. Thus we cannot derive the above type of result for the adjoint L-function from Proposition 2.5. However, by using a variant of the Rankin-Selberg method we can get the following.

Theorem 2.7. ([35],[40]) *Let ψ be a Dirichlet character mod N. Let f be a Hecke eigenform in $\mathfrak{S}_k(\Gamma_0(N),\psi)$. Let χ be a Dirichlet character. Let m be a positive integer not greater than $k-1$ and $\chi(-1)=(-1)^{m-1}$. Put*

$$\mathbf{L}(m,f,\chi,\mathrm{Ad})=\frac{\Gamma_{\mathbf{C}}(m)\Gamma_{\mathbf{C}}(m+k-1)L(m,f,\chi,\mathrm{Ad})}{\langle f,f\rangle}.$$

Then $\mathbf{L}(m,f,\chi,\mathrm{Ad})$ belongs to $\mathbf{Q}(f)(\chi)$, and in particular it is algebraic.

3 Half-integral weight modular forms

Let N be a positive integer. Let $h(z)=\sum_m c_h(m)\mathbf{e}(mz)\in\mathfrak{S}_{k+1/2}(\Gamma_0(4N),\chi)$ and $g(z)=\sum_m c_g(m)\mathbf{e}(mz)\in\mathfrak{S}_{l+1/2}(\Gamma_0(4N),\psi)$. We then define

$$D(s,h,g)=\sum_{m=1}^{\infty}c_h(m)\overline{c_g(m)}m^{-s}.$$

Then similarly to the integral weight case, we have the following.

Proposition 3.1. *Assume that $k\geq l$. Then we have*

$$\Gamma(s)(4\pi)^{-s}D(s,h,g)=\int_{\Phi_{\Gamma_0(4N)}}h(z)\overline{g(z)E_{k-l,4N}(z,\bar{s}+1/2-k,\omega)}y^{k-3/2}dxdy,$$

where $\omega(d)=(\frac{(-1)^{k-l}}{d})\overline{\chi(d)}\psi(d)$. In particular, we have

$$\Gamma(s)(4\pi)^{-s}D(s,h,h)=\int_{\Phi_{\Gamma_0(4N)}}|h(z)|^2\overline{E_{0,4N}(z,\bar{s}+1/2-k,\omega)}y^{k-3/2}dxdy.$$

In particular, the period $\langle h,h\rangle$ can be expressed in terms of the residue of the convolution product

$$D(s,h,h)=\sum_{m=1}^{\infty}\frac{|c_h(m)|^2}{m^s}.$$

However there is no formula like Theorem 2.4. Instead we express the ratio of the period of h to that of its Shimura correspondence $S(h)$ in terms of $L(k/2,S(h))$.

Let N be an odd positive integer. We define the Kohnen plus subspace $\mathfrak{M}^+_{k+1/2}(\Gamma_0(4N))$ of $\mathfrak{M}_{k+1/2}(\Gamma_0(4N))$ by

$$\mathfrak{M}^+_{k+1/2}(\Gamma_0(4N))$$
$$=\{f(z)=\sum_{m=0}^{\infty}c_f(m)\mathbf{e}(mz)\mid c_f(m)=0\text{ unless }m\equiv(-1)^k\bmod 4\text{ or }m\equiv 0\bmod 4\}.$$

We also put $\mathfrak{S}^+_{k+1/2}(\Gamma_0(4N))=\mathfrak{S}_{k+1/2}(\Gamma_0(4N))\cap\mathfrak{M}^+_{k+1/2}(\Gamma_0(4N))$. We note that $\mathfrak{M}^+_{k+1/2}(\Gamma_0(4N))$ can also be defined as the eigenspace of a certain linear operator acting on $\mathfrak{M}_{k+1/2}(\Gamma_0(4N))$, and the canonical projection pr from $\mathfrak{M}_{k+1/2}(\Gamma_0(4N))$ to $\mathfrak{M}^+_{k+1/2}(\Gamma_0(4N))$ is given by

$$\mathrm{pr}(g)=(-1)^{[(k+1)/2]}\frac{1}{3\sqrt{2}}(\sum_{\nu\bmod 4}g|_{2k+1}\xi\alpha^*_\nu)+\frac{1}{3}g,$$

where

$$\xi = (\begin{pmatrix} 4 & 1 \\ 0 & 4 \end{pmatrix}, \exp(\pi i/4)), \alpha_\nu^* = (\alpha_\nu, \frac{\theta(\alpha_\nu(z))}{\theta(z)})$$

with $\alpha_\nu = \begin{pmatrix} 1 & 0 \\ 4N\nu & 1 \end{pmatrix}$ (cf. [21].) Now to explain the Shimura correspondence, let $\theta(z) = \sum_{m=-\infty}^{\infty} \mathbf{e}(-m^2 z)$ be the theta series defined in Section 1, and put $\theta_t(z) = \theta(tz)$ for each positive integer t. We note that $\theta_t(z)$ belongs to $\mathfrak{M}_{1/2}(\Gamma_0(4t), (\frac{D_t}{*}))$. where D_t is the discriminant of $\mathbf{Q}(\sqrt{t})$ and $(\frac{D_t}{*})$ denotes the Kronecker symbol. We denote by $\tilde{\Delta}_0(4N)$ the subalgebra of $\mathcal{R}_{\mathbf{Z}}(\Gamma_0(4N), \mathfrak{GL}_2^+(\mathbf{Q}))$ generated by all the $\Gamma_0(4N)(\begin{pmatrix} 1 & 0 \\ 0 & p^2 \end{pmatrix}, p^{1/2})\Gamma_0(4N)'$ s with p prime numbers not dividing N. Then $\tilde{\Delta}_0(4N)$ acts on $\mathfrak{M}_{k+1/2}^+(\Gamma_0(4N))$ as in Kohnen [21]. We simply call an element $h \in \mathfrak{M}_{k+1/2}^+(\Gamma_0(4N))$ a Hecke eigenform if h is a Hecke eigenform for $\tilde{\Delta}_0(4N)$. Then we have the following (cf. Kohnen [21], Shimura [30].)

Theorem 3.2. (1) *Let N be an odd positive integer. Let k be a positive integer and D a fundamental discriminant such that $(-1)^k D > 0$. For $h(z) = \sum_{m=1}^{\infty} c(m)\mathbf{e}(mz) \in \mathfrak{S}_{k+1/2}^+(\Gamma_0(4N))$, we define $\mathcal{S}_{k,4N,D}(h)(z)$ by*

$$\mathcal{S}_{k,4N,D}(h)(z) = \sum_{m=1}^{\infty} \sum_{\substack{d|m \\ (d,N)=1}} (\frac{D}{d}) d^{k-1} c(|D|(m/d)^2)\mathbf{e}(mz).$$

Then $\mathcal{S}_{k,4N,D}(h)(z)$ belongs to $\mathfrak{S}_{2k}(\Gamma_0(N))$. The map $\mathcal{S}_{k,4N,D}$ is an isomorphism from $\mathfrak{S}_{k+1/2}(\Gamma_0(4N))$ to $\mathfrak{S}_{2k}(\Gamma_0(N))$. In particular if $N = 1$, then the map

$$\sum_{m=0}^{\infty} c(m)\mathbf{e}(mz) \mapsto \frac{c(0)L(1-k, (\frac{D}{d}))}{2} + \sum_{m=1}^{\infty} \sum_{d|m} (\frac{D}{d}) d^{k-1} c(|D|(m/d)^2)\mathbf{e}(mz)$$

induces an isomorphism from $\mathfrak{M}_{k+1/2}^+(\Gamma_0(4))$ to $\mathfrak{M}_{2k}(SL_2(\mathbf{Z}))$ whose restriction to $\mathfrak{S}_{k+1/2}^+(\Gamma_0(4))$ is equal to $\mathcal{S}_{k,4,D}$. This map will be denoted by the same symbol $\mathcal{S}_{k,4,D}$.
(2) *Let D be as above. Then, for any Hecke eigenform $h \in \mathfrak{S}_{k+1/2}^+(\Gamma_0(4))$, $\mathcal{S}_{k,4,D}(h)$ is a Hecke eigenform in $\mathfrak{S}_{2k}(SL_2(\mathbf{Z}))$, and we have*

$$D(s, h, \theta_{|D|})L(s-k+1, (\frac{D}{*})) = c(|D|)L(s, \mathcal{S}_{k,4,D}(h)).$$

We call $\mathcal{S}_{k,4N,D}$ the Shimura correspondence associated to D. For a Hecke eigenform h in $\mathfrak{S}_{k+1/2}^+(\Gamma_0(4))$, let $S(h)$ be the normalized Hecke eigenform in $\mathfrak{S}_{2k}(SL_2(\mathbf{Z}))$ such that $\mathbf{Q}(\mathcal{S}_{k,4,D}(h)) = \mathbf{Q}(S(h))$. This $S(h)$ is uniquely determined by h and does not depend on the choice of D. We also call $S(h)$ the normalized Hecke eigenform corresponding to h under the Shimura correspondence. As for the recent development related to Theorem 3.2, see Ueda [36]. Now as for the period of a half-integral weight cusp form, we have the following (cf. Shimura [33].)

Theorem 3.3. *Let h be a Hecke eigenform in $\mathfrak{S}_{k+1/2}^+(\Gamma_0(4))$. Assume that all the Fourier coefficients of h belong to a field K. Then we have*

$$\frac{\pi i \langle h, h \rangle}{u_-(S(h))} \in K.$$

Roughly speaking, the period $\langle h,h\rangle$ is nothing but $u_-(S(h))$. Now our main result of this section is to give the ratio $\dfrac{\langle h,h\rangle}{\langle S(h),S(h)\rangle}$ in terms of $L(k/2,S(h),\chi)$ with some quadratic character χ. This type of result was first given by Waldspurger in automorphic representation theoretic view point. Here we give a refinement of it in a special case where h is in the Kohnen plus subspace due to Kohnen and Zagier. As for the recent progress of this theme, see Sakata [28].

Theorem 3.4. (Kohnen-Zagier [23]) *Let h be a Hecke eigenform in $\mathfrak{S}^+_{k+1/2}(\Gamma_0(4))$. Then for any fundamental discriminant D such that $(-1)^{n/2}D>0$ we have*

$$\frac{|c_h(|D|)|^2}{\langle h,h\rangle}=\frac{\Gamma(k)}{\pi^k}\frac{|D|^{k-1/2}L(k,S(h),(\frac{D}{*}))}{\langle S(h),S(h)\rangle}.$$

To prove the above theorem, we consider the following Eisenstein series: Let D be a fundamental discriminant such that $(-1)^kD>0$. We then put

$$G_{k,D}(z,s)=E_{k,|D|}(z,s,(\frac{D}{*})),$$

$$\tilde{G}_{k,D}(z,s)=G_{k,D}(4z,s)-2^{-k-2s}(\frac{D}{2})G_{k,D}(2z,s).$$

Furthermore put

$$G_{k,D}(z)=\frac{L(1-k,(\frac{D}{*}))}{2}G_{k,D}(z,0),$$

and

$$\tilde{G}_{k,D}(z)=\frac{L(1-k,(\frac{D}{*}))}{2}\tilde{G}_{k,D}(z,0).$$

We note that $G_{k,D}(z,s)$ and $\tilde{G}_{k,D}(z,s)$ belong to $\mathfrak{M}_k^\infty(\Gamma_0(|D|),(\frac{D}{*}))$ and $\mathfrak{M}_k^\infty(\Gamma_0(4|D|),(\frac{D}{*}))$, respectively, and, in particular if $k\ge 3$, then $G_{k,D}(z)$ and $\tilde{G}_{k,D}(z)$ belong to $\mathfrak{M}_k(\Gamma_0(|D|),(\frac{D}{*}))$ and $\mathfrak{M}_k(\Gamma_0(4|D|),(\frac{D}{*}))$, respectively. We define $\mathcal{F}_D(z,s)$ and $\mathcal{G}_D(z,s)$ as

$$\mathcal{F}_D(z,s)=\mathrm{Tr}_1^{|D|}(G_{k,D}(z)G_{k,D}(z,s)),$$

and

$$\mathcal{G}_D(z,s)=\frac{3}{2}(1-(\frac{D}{2})2^{-k-2s})^{-1}\mathrm{pr}(\mathrm{Tr}_4^{4|D|}(\tilde{G}_{k,D}(z,s)\theta(|D|z))).$$

Here for positive integers N,M such that $N|M$ we denote by Tr_N^M the trace map from $\mathfrak{M}_l^\infty(\Gamma_0(M))$ to $\mathfrak{M}_l^\infty(\Gamma_0(N))$ given by

$$\mathrm{Tr}_N^M(f)=\sum_{\gamma\in\Gamma_0(M)\backslash\Gamma_0(N)}f(\gamma(z))$$

for $f\in\mathfrak{M}_l^\infty(\Gamma_0(M))$. Furthermore put

$$\mathcal{F}_D(z)=\frac{L(1-k,(\frac{D}{*}))}{2}\mathcal{F}_D(z,0),$$

and

$$\mathcal{G}_D(z)=\frac{L(1-k,(\frac{D}{*}))}{2}\mathcal{G}_D(z,0).$$

We note that $\mathcal{F}_D(z)$ and $\mathcal{G}_D(z)$ belong to $\mathfrak{M}_{2k}(SL_2(\mathbf{Z}))$, and $\mathfrak{M}^+_{k+1/2}(\Gamma_0(4))$, respectively, if $k \geq 3$.

Proposition 3.5. *Let D be a fundamental discriminant such that $(-1)^k D > 0$. Let f be a normalized Hecke eigenform in $\mathfrak{S}_{2k}(SL_2(\mathbf{Z}))$. Then we have*

$$\langle f, \mathcal{F}_D \rangle = \frac{\Gamma(2k-1)L(1-k, (\frac{D}{*}))}{2(4\pi)^{2k-1}} \frac{L(2k-1, f)L(k, f, (\frac{D}{*}))}{L(k, (\frac{D}{*}))}.$$

Proof. Put $I(s) = \langle f, \mathcal{F}_D(*, \bar{s}) \rangle$. Then we have

$$I(s) = \int_{\Phi_{SL_2(\mathbf{Z})}} \sum_{\gamma \in \Gamma_0(|D|) \backslash SL_2(\mathbf{Z})} f(\gamma(z)) \overline{G_{k,D}(\gamma(z)) G_{k,D}(\gamma(z), \bar{s})} y^{2k-2} dxdy$$

$$= \int_{\Phi_{\Gamma_0(|D|)}} f(z) \overline{G_{k,D}(z) G_{k,D}(z, \bar{s})} y^{2k-2} dxdy.$$

Thus by Theorem 2.2, for $s \in \mathbf{C}$ with sufficiently large real part, we have

$$I(s) = \frac{\Gamma(2k-1+s)}{(4\pi)^{2k-1+s}} L(2k-1+s, f, G_{k,D})$$

and therefore by (2) of Proposition 2.1 we have

$$I(s) = \frac{\Gamma(2k-1+s)}{(4\pi)^{2k-1+s}} \frac{L(2k-1+s, f)L(k+s, f, (\frac{D}{*}))}{L(k+2s, (\frac{D}{*}))}.$$

The both hand-sides of the above are holomorphic at $s = 0$. thus the assertion holds.

Proposition 3.6. *Let D and f be as in Proposition 3.5, and $g(z) = \sum_{m=1}^{\infty} c_g(m) \mathbf{e}(mz)$ be a Hecke eigenform in $\mathfrak{S}^+_{k+1/2}(\Gamma_0(4))$ such that $\mathcal{S}_{k,4,D}(g) = c_g(|D|) f$. Then we have*

$$\langle g, \mathcal{G}_D \rangle = \frac{\Gamma(k-1/2)L(1-k, (\frac{D}{*}))}{4(4\pi)^{k-1/2} L(k, (\frac{D}{*}))|D|^{k-1/2}} L(2k-1, f) c_g(|D|).$$

Proof. The assertion can be proved in a way similar to Proposition 3.5 by using Propositions 3.1 and 3.2.

Proof of Theorem 3.4. Let $\{g_\nu\}$ be an orthogonal basis of $\mathfrak{S}^+_{k+1/2}(\Gamma_0(4))$ consisting of Hecke eigenforms. Furthermore, we define the Eisenstein series $E_{k+1/2}(z)$ and $G^+_{k+1/2}(z)$ as

$$E_{k+1/2}(z) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(4)} (\frac{\theta(\gamma(z))}{\theta(z)})^{-2k-1},$$

and

$$G^+_{k+1/2}(z) = \zeta(1-2k)(E_{k+1/2}(z) + 2^{-2k-1}(1+(-1)^k i) z^{-k-1/2} E_{k+1/2}(\frac{-1}{4z}).$$

Then $E_{k+1/2}(z)$ and $G^+_{k+1/2}(z)$ belong to $\mathfrak{M}_{k+1/2}(\Gamma_0(4))$ and $\mathfrak{M}^+_{k+1/2}(\Gamma_0(4))$, respectively. Then $\mathcal{G}_D$ can be written as

$$\mathcal{G}_D(z) = \lambda G^+_{k+1/2}(z) + \sum_\nu \lambda_\nu g_\nu(z)$$

with $\lambda, \lambda_\nu \in \mathbf{C}$. Then we have

$$\lambda_\nu = \frac{\langle \mathcal{G}_D, g_\nu \rangle}{\langle g_\nu, g_\nu \rangle}.$$

Let $\{f_\nu\}$ be the basis of $\mathfrak{S}_{2k}(SL_2(\mathbf{Z}))$ consisting of normalized Hecke eigenforms such that $S(g_\nu) = f_\nu$.

We note that

$$\mathcal{S}_{k,4,D}(G^+_{k+1/2}) = L(1-k, (\frac{D}{*}))G_{2k},$$

$$\mathcal{S}_{k,4,D}(g_\nu) = c_{g_\nu}(|D|)f_\nu,$$

and

$$\mathcal{S}_{k,4,D}(\mathcal{G}_D) = \mathcal{F}_D.$$

Thus we have

$$\mathcal{F}_D(z) = \lambda G_{2k}(z) + \sum_\nu \lambda_\nu c_{g_\nu}(|D|)f_\nu(z),$$

and therefore we have

$$\langle f_\nu, \mathcal{F}_D \rangle = \lambda_\nu c_{g_\nu}(|D|)\langle f_\nu, f_\nu \rangle = c_{g_\nu}(|D|)\langle f_\nu, f_\nu \rangle \frac{\langle \mathcal{G}_D, g_\nu \rangle}{\langle g_\nu, g_\nu \rangle}.$$

Thus the assertion follows from Propositions 3.5 and 3.6 by remarking $L(2k-1, f)L(1-k, (\frac{D}{*})) \neq 0$.

4 Ikeda's conjecture on the period of the Ikeda lift

The Rankin-Selberg method plays an important role also in investigating the period relation of Siegel modular forms. Here we apply it to Ikeda's conjecture. We restrict ourselves to the full modular case, and we define standard L-function. Let $\mathbf{L}_n = \mathcal{R}_{\mathbf{Q}}(\Gamma^{(n)}, GSp_n^+(\mathbf{Q}))$, and for each prime number p let $\mathbf{L}_{np} = \mathcal{R}_{\mathbf{Q}}(\Gamma^{(n)}, GSp_n^+(\mathbf{Q}) \cap GL_{2n}(\mathbf{Z}[p^{-1}]))$. Then $\mathbf{L}_{np}$ is a subalgebra of $\mathbf{L}_n$, and $\mathbf{L}_n$ is generated by all the $\mathbf{L}_{np}$'s. Now let

$$T(p) = \Gamma^{(n)}(1_n \perp p1_n)\Gamma^{(n)},$$

and

$$(p^\pm) = \Gamma^{(n)}(p^\pm 1_{2n})\Gamma^{(n)}.$$

Furthermore, for $i = 1, ..., n-1$ put

$$T_i(p^2) = \Gamma^{(n)}(1_{n-i} \perp p1_i \perp p^2 1_{n-i} \perp p1_i)\Gamma^{(n)}.$$

Then $\mathbf{L}_{np}$ is generated over $\mathbf{Q}$ by $(p^\pm), T(p)$ and $T_i(p^2)$ $(i = 1, ..., n-1)$. We now review the Satake p-parameters of $\mathbf{L}_{np}$. Let $\mathbf{P}_n = \mathbf{Q}[X_0^\pm, X_1^\pm, ..., X_n^\pm]$ be the ring of Laurent polynomials in $X_0, X_1, ..., X_n$ over $\mathbf{Q}$. Let $\mathbf{W}_n$ be the group of $\mathbf{Q}$-automorphisms of $\mathbf{P}_n$ generated by all permutations in variables $X_1, ..., X_n$ and by the automorphisms $\tau_1, ..., \tau_n$ defined by

$$\tau_i(X_0) = X_0X_i, \tau_i(X_i) = X_i^{-1}, \tau_i(X_j) = X_j \ (j \neq i).$$

Furthermore a group $\tilde{\mathbf{W}}_n$ isomorphic to $\mathbf{W}_n$ acts on the set $\mathbf{T}_n = (\mathbf{C}^\times)^{n+1}$ in a way similar to the above. Then there exists a $\mathbf{Q}$-algebra isomorphism Ω_{np}, called the Satake isomorphism, from $\mathbf{L}_{np}$ to the $\mathbf{W}_n$-invariant subring $\mathbf{P}_n^{\mathbf{W}_n}$ of $\mathbf{P}_n$. Then for a non-zero $\mathbf{C}$-algebra homomorphism λ from $\mathbf{L}_{np}$ to $\mathbf{C}$, there exists an element $(\alpha_0(p, \lambda), \alpha_1(p, \lambda), ..., \alpha_n(p, \lambda))$ of $\mathbf{T}_n$ such that

$$\lambda(\Omega_{np}^{-1}(F(X_0, X_1, ..., X_n))) = F(\alpha_0(p, \lambda), \alpha_1(p, \lambda), ..., \alpha_n(p, \lambda))$$

for $F \in \mathbf{P}_n^{\mathbf{W}_n}$. The equivalence class of $(\alpha_0(p,\lambda),\alpha_1(p,\lambda),...,\alpha_n(p,\lambda))$ under the action of $\tilde{\mathbf{W}}_n$ is uniquely determined by λ. Now assume that f is a Hecke eigenform in $\mathfrak{M}_k(\Gamma^{(n)})$. Then for each prime number p, the map $\lambda_{f,p} : \mathbf{L}_{np} \ni T \mapsto \lambda_f(T)$ defines a non-zero $\mathbf{C}$-homomorphism from $\mathbf{L}_{np}$ to $\mathbf{C}$. We denote by $(\alpha_0(p),\alpha_1(p),....,\alpha_n(p))$ the Satake p-parameters of $\mathbf{L}_{np}$ determined by $\lambda_{f,p}$. Let χ be a Dirichlet character. We then define the spinor L-function $L(s,f,\chi,\mathrm{Sp})$ of f twisted by χ as

$$L(s,f,\chi,\mathrm{Sp}) = \prod_p \{(1-\alpha_0(p)p^{-s}\chi(p)) \prod_{r=1}^{n} \prod_{1\le i_1<...<i_r\le n} (1-\alpha_{i_1}(p)...\alpha_{i_r}(p)p^{-s}\chi(p))\}^{-1}.$$

We also define the standard L-function $L(s,f,\chi,\mathrm{St})$ of f twisted by χ as

$$L(s,f,\chi,\mathrm{St}) = \prod_p \prod_{i=1}^{n} \{(1-p^{-s}\chi(p))(1-\alpha_i(p)p^{-s}\chi(p))(1-\alpha_i(p)^{-1}p^{-s}\chi(p))\}^{-1}.$$

Let f be a Hecke eigenform in $\mathfrak{S}_k(SL_2(\mathbf{Z}))$, and α_p and β_p be the complex numbers in Section 2. Then we have $\beta_p = \alpha_p^{-1}$. We note that we can take $\alpha_0(p) = p^{k/2-1/2}\alpha_p$ and $\alpha_1(p) = \alpha_p^{-2}$ as the Satake p-parameters determined by f. Thus we have

$$L(s,f,\chi,\mathrm{Sp}) = L(s,f,\chi),$$

and

$$L(s,f,\chi,\mathrm{St}) = L(s,f,\chi,\mathrm{Ad}).$$

For an integer $D \in \mathbf{Z}$ such that $D \equiv 0$ or $1 \bmod 4$, put $\mathfrak{d}_D$ be the discriminant of $\mathbf{Q}(\sqrt{D})$, and put $\mathfrak{f}_D = \sqrt{\frac{D}{\mathfrak{d}_D}}$. Let n be a positive even integer. For an element $T \in \mathcal{L}_{n>0}$, put $\mathfrak{d}_T = \mathfrak{d}_{(-1)^{n/2}\det(2T)}, \mathfrak{f}_T = \mathfrak{f}_{(-1)^{n/2}\det(2T)}$, and $\chi_T = (\frac{\mathfrak{d}_T}{*})$. Now we define the local Siegel series $b_p(T,s)$ by

$$b_p(T,s) = \sum_{R \in Sym_n(\mathbf{Z}[1/p])/Sym_n(\mathbf{Z})} \mathbf{e}(\mathrm{tr}(TR))p^{-ord_p(\mu_p(R))s},$$

where $\mu_p(R) = [R\mathbf{Z}_p^n + \mathbf{Z}_p^n : \mathbf{Z}_p^n]$. We remark that there exists a unique polynomial $F_p(T,X)$ in X such that

$$b_p(T,s) = F_p(T,p^{-s})\frac{(1-p^{-s})\prod_{i=1}^{n/2}(1-p^{2i-2s})}{1-\chi_T(p)p^{n/2-s}}$$

(cf. Kitaoka [19]). Now let k be a positive even integer. Let

$$f(z) = \sum_{m=1}^{\infty} a(m)\mathbf{e}(mz)$$

be a normalized Hecke eigenform in $\mathfrak{S}_{2k-n}(\Gamma^{(1)})$. Furthermore let

$$\tilde{f}(z) = \sum_e c(e)\mathbf{e}(ez)$$

be a cuspidal Hecke eigenform in the Kohnen plus subspace $\mathfrak{S}^+_{k-n/2+1/2}(\Gamma_0(4))$ such that $S(\tilde{f}) = f$. We define a Fourier series $I_n(f)(Z)$ in $Z \in \mathbf{H}_n$ by

$$I_n(f)(Z) = \sum_{T \in \mathcal{L}_{n>0}} a_{I_n(f)}(T)\mathbf{e}(\mathrm{tr}(TZ)),$$

where

$$a_{I_n(f)}(T) = c(|\mathfrak{d}_T|) \prod_p (p^{k-n/2-1/2}\alpha_p)^{ord_p(\mathfrak{f}_T)} \prod_p F_p(T, p^{-(n+1)/2}\alpha_p^{-1}).$$

Then Ikeda [9] showed the following:

Theorem 4.1. *$I_n(f)(Z)$ is a Hecke eigenform in $\mathfrak{S}_k(\Gamma^{(n)})$ whose standard L-function is*

$$\zeta(s) \prod_{i=1}^{n} L(s+k-i, f).$$

We call $I_n(f)$ the Ikeda lift of f. We note that $I_n(f)$ is uniquely determined by $\tilde{f}$. We also note that $I_2(f)$ is the Saito-Kurokawa lift of f.

To formulate Ikeda's conjecture, put

$$\Gamma_{\mathbf{R}}(s) = \pi^{-s/2}\Gamma(s/2).$$

Furthermore put

$$\xi(s) = \Gamma_{\mathbf{R}}(s)\zeta(s)$$

and

$$\widetilde{\xi}(s) = \Gamma_{\mathbf{C}}(s)\zeta(s).$$

Put

$$\Lambda(s,\, f, \chi) = \Gamma_{\mathbf{C}}(s)L(s,\, f, \chi)\tau(\chi)^{-1}.$$

Furthermore put

$$\widetilde{\Lambda}(s,\, f,\, \mathrm{Ad}) = \Gamma_{\mathbf{C}}(s)\Gamma_{\mathbf{C}}(s+2k-n-1)L(s,\, f,\, \mathrm{Ad}).$$

Now we have the following diagram of liftings:

$$\begin{array}{ccccc} \mathfrak{S}^{+}_{k-n/2+1/2}(\Gamma_0(4)) & \leftrightarrow & \mathfrak{S}_{2k-n}(\Gamma^{(1)}) & \to & \mathfrak{S}_k(\Gamma^{(n)}) \\ \tilde{f} & \leftrightarrow & f & \mapsto & I_n(f) \end{array}$$

Then Ikeda [10] proposed the following conjecture:

Conjecture A. *We have*

$$\frac{\langle I_n(f),\, I_n(f)\rangle}{\langle \tilde{f},\, \tilde{f}\rangle} = 2^{\alpha(n,\,k)}\Lambda(k,\, f)\widetilde{\xi}(n) \prod_{i=1}^{n/2-1} \tilde{\Lambda}(2i+1,\, f,\, \mathrm{Ad})\, \widetilde{\xi}(2i),$$

where $\alpha(n,\, k)$ is a certain integer depending only on n and k.

Remark. When $n=2$, Conjecture A holds true. Namely, Kohnen and Skoruppa [22] showed that

$$\frac{\langle I_2(f),\, I_2(f)\rangle}{\langle \tilde{f},\, \tilde{f}\rangle} = 2^{k-2}\Lambda(k,\, f)\widetilde{\xi}(2).$$

(See also Oda [25].)

Now we have

Theorem 4.2 (Katsurada-Kawamura [18]) *Conjecture A holds true for any even positive integer n.*

By this result combined with Theorem 3.4, we get the following result.

Theorem 4.3. *For any fundamental discriminant D such that $(-1)^{n/2}D > 0$ and $L(k-n/2, f, (\frac{D}{*})) \neq 0$ we have*

$$\frac{\langle I_n(f),\, I_n(f)\rangle}{\langle f,\, f\rangle^{n/2}} = \frac{a_{n,k}|c(|D|)|^2\Lambda(k,\, f)}{|D|^{k-n/2}\Lambda(k-n/2, f, (\frac{D}{*}))}\widetilde{\xi}(n)\prod_{i=1}^{n/2-1}\frac{\widetilde{\Lambda}(2i+1,\, f,\, \mathrm{Ad})}{\langle f,\, f\rangle}\widetilde{\xi}(2i)$$

with some algebraic number $a_{n,k}$ depending only on n, k.

By Theorems 2.6 and 2.7, we see that $\dfrac{\Lambda(k,\, f)}{\Lambda(k-n/2, f, (\frac{D}{*}))}$ and $\dfrac{\widetilde{\Lambda}(2i+1,\, f,\, \mathrm{Ad})}{\langle f,\, f\rangle}$ for $i = 1, ..., n/2-1$ are algebraic numbers and belong to the Hecke field $\mathbf{Q}(f)$. Thus we get the following corollary.

Corollary. *In addition to the above assumption, assume that all the Fourier coefficients of $\tilde{f}$ are algebraic. Then $\dfrac{\langle I_n(f),\, I_n(f)\rangle}{\langle f,\, f\rangle^{n/2}}$ is algebraic.*

This has been already proved by Furusawa [6] in case $n = 2$, and by Choie and Kohnen [4] in general case. Thus our result can be regarded as a refinement of theirs. We also remark that we can apply Theorem 4.3 to solve a problem concerning the congruence between Ikeda lifts and non-Ikeda lifts. This was announced in [17], and the details will be discussed in [15] (see also [14].)

5 Rankin-Selberg Dirichlet series associated with the Fourier-Jacobi expansion of the Ikeda lift

In this section and the next, we give an outline of the proof of Theorem 4.2. First we give an explicit formula for a certain Rankin-convolution product associated with the Fourier-Jacobi expansion of the Ikeda lift, and express its residue in terms of its period. First we review Jacobi forms of integral index. Let $\mathrm{H}_{1,n}(\mathbf{R})$ be the real Heisenberg group of characteristic $(1, n)$, that is, the set

$$\mathrm{H}_{1,n}(\mathbf{R}) = \mathbf{R}^{2n} \times \mathbf{R} = \{[X,\, \kappa] \,|\, X \in \mathbf{R}^{2n},\, \kappa \in \mathbf{R}\}$$

with the following group-structure: for $[X_i,\, \kappa_i] \in \mathrm{H}_{1,n}(\mathbf{R})$ $(i = 1, 2)$,

$$[X_1,\, \kappa_1] * [X_2,\, \kappa_2] = [X_1 + X_2,\, \kappa_1 + \kappa_2 + X_1 J_n\, {}^tX_2].$$

Since the group $GSp_n^+(\mathbf{R})$ acts on $\mathrm{H}_{1,n}(\mathbf{R})$ by

$$[X,\, \kappa]\cdot\gamma = [\nu(\gamma)^{-1}X\gamma,\, \nu(\gamma)^{-1}\kappa] \quad ([X,\, \kappa] \in \mathrm{H}_{1,n}(\mathbf{R}),\, \gamma \in GSp_n^+(\mathbf{R})),$$

we can define the semi-direct product $GSp_n^+(\mathbf{R})^J = GSp_n^+(\mathbf{R}) \ltimes \mathrm{H}_{1,n}(\mathbf{R})$, that is, the set

$$GSp_n^+(\mathbf{R}) \ltimes \mathrm{H}_{1,n}(\mathbf{R}) = GSp_n^+(\mathbf{R}) \times \mathrm{H}_{1,n}(\mathbf{R})$$

with the following group-structure: for $g_i = (\gamma_i,\, [X_i,\, \kappa_i]) \in GSp_n^+(\mathbf{R}) \ltimes \mathrm{H}_{1,n}(\mathbf{R})$ $(i = 1, 2)$,

$$g_1g_2 \;=\; (\gamma_1\gamma_2,\, ([X_1,\, \kappa_1]\cdot\gamma_2) * [X_2,\, \kappa_2])$$

$$= \quad (\gamma_1\gamma_2,\, [\nu(\gamma_2)^{-1}X_1\gamma_2 + X_2,\, \nu(\gamma_2)^{-1}\kappa_1 + \kappa_2 + \nu(\gamma_2)^{-1}X_1\gamma_2 J_n\, {}^tX_2]).$$

For simplicity, we denote any element of $GSp_n^+(\mathbf{R})^J$ by $[\gamma,\, X,\, \kappa] = (\gamma,\, [X,\, \kappa])$ with $\gamma \in GSp_n^+(\mathbf{R})$, $X \in \mathbf{R}^{2n}$ and $\kappa \in \mathbf{R}$.

Remark. For any $g = [\gamma,\, X,\, \kappa] \in GSp_n^+(\mathbf{R})^J$, we write $\gamma = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ and $X = (\lambda,\, \mu)$, in which A, B, C, D are $n \times n$ matrices and λ, μ are n-vectors. Then we define g' by

$$g' = \begin{pmatrix} \nu & 0 & 0 & 0 \\ 0 & A & 0 & B \\ 0 & 0 & 1 & 0 \\ 0 & C & 0 & D \end{pmatrix} \begin{pmatrix} 1 & \lambda & \kappa & \mu \\ 0 & 1_n & {}^t\mu & 0_n \\ 0 & 0 & 1 & 0 \\ 0 & 0_n & -\,{}^t\lambda & 1_n \end{pmatrix},$$

where $\nu = \nu(\gamma)$. Then we easily see that $g' \in GSp_{n+1}^+(\mathbf{R})$ and the correspondence $g \mapsto g'$ defines an injective group-homomorphism.

We also define a subgroup $\Gamma^{(n),J}$ of $GSp_n^+(\mathbf{R})^J$ by $\Gamma^{(n),J} = \Gamma^{(n)} \ltimes \mathrm{H}_{1,n}(\mathbf{Z})$, where $\mathrm{H}_{1,n}(\mathbf{Z}) = \mathrm{H}_{1,n}(\mathbf{R}) \cap (\mathbf{Z}^{2n} \times \mathbf{Z})$. Let k and m be non-negative integers. For any $[\gamma,\, X,\, \kappa] \in GSp_n^+(\mathbf{R})^J$, we decompose γ and X into $n \times n$ blocks $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ and n-vectors $(\lambda,\, \mu)$, respectively. For any function $\phi(\tau,\, z)$ on $\mathbf{H}_n \times \mathbf{C}^n$, we define

$$\begin{aligned} &(\phi|_{k,\,m}[\gamma,\, X,\, \kappa])(\tau,\, z) \\ &\quad = e^{m\nu}(\kappa + \tau[\,{}^t\lambda] + 2\lambda\, {}^tz + \lambda\, {}^t\mu - (C\tau + D)^{-1}C\,[\,{}^t(z + \lambda\tau + \mu)]) \\ &\qquad \times \det(C\tau + D)^{-k}\phi(\gamma\langle\tau\rangle,\, \nu(z + \lambda\tau + \mu)(C\tau + D)^{-1}), \end{aligned}$$

where we write $\nu = \nu(\gamma)$. Then for any $g_i = [\gamma_i,\, X_i,\, \kappa_i] \in GSp_n^+(\mathbf{R})^J$ $(i = 1,\, 2)$, we have

$$(\phi|_{k,\,m}\, g_1)|_{k,\,m\nu}\, g_2 = \phi|_{k,\,m}\, (g_1 g_2),$$

where we write $\nu = \nu(\gamma_1)$. Moreover, we denote the actions of $\gamma \in GSp_n^+(\mathbf{R})$ and $X \in \mathbf{Z}^{2n}$ by

$$\phi|_{k,\,m}\, \gamma = \phi|_{k,\,m}\, [\gamma,\, 0,\, 0],$$

and

$$\phi|_m\, X = \phi|_{k,\,m}\, [1_{2n},\, X,\, 0],$$

respectively. Then for any $\gamma,\, \gamma' \in GSp_n^+(\mathbf{R})$ and $X,\, X' \in \mathbf{Z}^{2n}$, we have

$$\begin{cases} (\phi|_{k,\,m}\, \gamma)|_{k,\,m\nu}\, \gamma' = \phi|_{k,\,m}\, (\gamma\gamma'), \\ (\phi|_m\, X)|_m\, X' = \phi|_m\, (X + X'), \\ (\phi|_{k,\,m}\, \gamma)|_{m\nu}\, (\nu^{-1}X\gamma) = (\phi|_m\, X)|_{k,\,m}\, \gamma, \end{cases}$$

where we write $\nu = \nu(\gamma)$. Let k and m be positive integers. A holomorphic function ϕ on $\mathbf{H}_n \times \mathbf{C}^n$ is called a *(holomorphic) Jacobi form* of degree n, weight k and index m if it satisfies the following two conditions:

(i) $\phi|_{k,m}\gamma = \phi$ for any $\gamma \in \Gamma^{(n),J}$,

(ii) ϕ has a Fourier expansion of the form

$$\phi(\tau,\, z) = \sum_{T \in \mathcal{L}_n,\, r \in \mathbf{Z}^n} c_\phi(T,\, r) e(\mathrm{tr}(T\tau) + r\, {}^tz)$$

with $c_\phi(T,\, r) = 0$ unless $4mT - {}^trr \geq 0$. If ϕ satisfies the stronger condition $c_\phi(T,\, r) = 0$ unless $4mT - {}^trr > 0$, it is called a *Jacobi cusp form*. We denote by $J_{k,\,m}(\Gamma^{(n),J})$ and $J_{k,\,m}^{\mathrm{cusp}}(\Gamma^{(n),J})$ the

C-vector spaces of the (holomorphic) Jacobi forms and Jacobi cusp forms of degree n, weight k and index m, respectively. If $\phi,\, \psi \in J_{k,\,m}(\Gamma^{(n),J})$ and $\phi\psi \in J_{2k,\,2m}^{\rm cusp}(\Gamma^{(n),J})$, then we can define the Petersson inner product of ϕ and ψ by

$$\langle \phi,\, \psi\rangle = \int_{\Phi_{\Gamma^{(n)},J}} \phi(\tau,\, z)\overline{\psi(\tau,\, z)}\det(v)^{k-n-2}\exp(-4\pi m v^{-1}[{}^t y])\, dudvdxdy,$$

where $\Phi_{\Gamma^{(n)},J}$ is a fundamental domain for $\mathbf{H}_n \times \mathbf{C}^n$ modulo $\Gamma^{(n),J}$, and $\tau = u + iv \in \mathbf{H}_n$, $z = x + iy \in \mathbf{C}^n$. As is well-known, the Petersson inner product defines a Hermitian inner product on $J_{k,\,m}^{\rm cusp}(\Gamma^{(n),J})$. We also have a Hecke theory for Jacobi forms, but we omit the details of it.

Now we consider a certain Rankin-Selberg Dirichlet series associated with the Fourier-Jacobi expansion of a Siegel cusp form. Let $F \in \mathfrak{S}_k(\Gamma^{(n)})$. Then we have the following Fourier expansion:

$$F(Z) = \sum_{B \in \mathcal{L}_{n>0}} A(B)\mathbf{e}(\mathrm{tr}(BZ)) \quad (Z \in \mathbf{H}_n).$$

Writing $Z = \begin{pmatrix} \tau' & z \\ {}^t z & \tau \end{pmatrix}$ with $\tau \in \mathbf{H}_{n-1}$, $z \in \mathbf{C}^{n-1}$ and $\tau' \in \mathbf{H}_1$, we have the Fourier-Jacobi expansion of F of type $(1,\, n-1)$ as follows:

$$F\Bigl(\begin{pmatrix} \tau' & z \\ {}^t z & \tau \end{pmatrix}\Bigr) = \sum_{N=1}^{\infty} \phi_N(\tau,\, z)\mathbf{e}(N\tau').$$

Here $\phi_N(\tau,\, z)$ is the N-th Fourier-Jacobi coefficient of F and defined as follows:

$$\phi_N(\tau,\, z) = \sum_{\substack{T \in \mathcal{L}_{n-1},\, r \in \mathbf{Z}^{n-1}, \\ 4NT - {}^t rr > 0}} A\Bigl(\begin{pmatrix} N & r/2 \\ {}^t r/2 & T \end{pmatrix}\Bigr)\mathbf{e}(\mathrm{tr}(T\tau) + r\,{}^t z).$$

Then it is easily shown that $\phi_N \in J_{k,\,N}^{\rm cusp}(\Gamma^{(n-1),J})$ for each $N \in \mathbf{Z}_{>0}$.

Now we define a Dirichlet series $D_1(s, F)$ as

$$D_1(s; F) = \zeta(2s - 2k + 2n)\sum_{N=1}^{\infty} \langle \phi_N, \phi_N\rangle N^{-s},$$

where $\langle \phi_N, \phi_N\rangle$ is the Petersson product defined on the space $J_{k,\,N}^{\rm cusp}(\Gamma^{(n-1),J})$. Then, as for the analytic properties of $D_1(s; F)$ the reader is referred to [38], where they are proved by using the Rankin-Selberg method:

Proposition 5.1. *Let* $\Gamma_{n,k}(s) = \pi^{k-n}(2\pi)^{-2s}\Gamma(s)\Gamma(s-k+n)$*, Then the function*

$$\mathcal{D}_1(s; F) = \Gamma_{n,k}(s)\, D_1(s; F)$$

has meromorphic continuation to the whole s-plane, and has simple poles at $s = k$ and $s = k - n$ with the residue $\frac{1}{2}\langle F, F\rangle$. Furthermore, it satisfies the following functional equation:

$$\mathcal{D}_1(s; F) = \mathcal{D}_1(2k - n - s; F).$$

Now we give an explicit formula for $D_1(s, I_n(f))$:

Theorem. 5.2. ([16], Main Theorem) *Let n and k be positive even integers s.t. $k > n+1$. Let f be a normalized Hecke eigenform in $\mathfrak{S}_{2k-n}(\Gamma^{(1)})$, and $\phi_1 = \phi_{I_n(f),1}$ the first Fourier-Jacobi coefficient of $I_n(f)$. Then we have*

$$D_1(s; I_n(f)) = \langle \phi_1,\, \phi_1 \rangle\, \zeta(s-k+1)\zeta(s-k+n)L(s,\, f).$$

By taking the residues of the both sides of Theorem 5.2, we have

Corollary. *Under the same assumption as above, we have*

$$\frac{\langle I_n(f),\, I_n(f)\rangle}{\langle \phi_1,\, \phi_1\rangle} = 2^{-k+n-1}\Lambda(k,\, f)\widetilde{\xi}(n). \tag{5.1}$$

Remark. In [16], we incorrectly quoted Yamazaki's result [38]. Namely, "$\langle F, G\rangle$" on the page 2026, line 14 of [16] should read "$\frac{1}{2}\langle F, G\rangle$" and therefore "$2^{2k-n+1}$" on the page 2027, line 7 of [16] should read "2^{2k-n}."

We give an outline of the proof of Theorem 5.2. Let

$$I_n(f)\left(\begin{pmatrix} \tau' & z \\ {}^t z & \tau \end{pmatrix}\right) = \sum_{N=1}^{\infty} \phi_N(\tau,\, z)\mathbf{e}(N\tau').$$

First we use the following fact due to Hayashida [7]:

Proposition 5.3. ([16], Theorem 4) *For each N and m, there is a homomorphism*

$$D_f(N) : J_{k,\,m}^{\mathrm{cusp}}(\Gamma^{(n-1),J}) \longrightarrow J_{k,\,mN}^{\mathrm{cusp}}(\Gamma^{(n-1),J})$$

such that $D_f(N)(\phi_m) = \phi_{mN}$.

We note that $D_f(N)$ coincides with the usual shift operator V_N in Eichler-Zagier [5] and with $D_{n-1}(N)$ in Yamazaki [37] in case $n = 2$. However it does not so in general, and depends on f. Next we use the fact concerning the adjoint operator of $D_f(N)$. To explain it more precisely, for a positive integer N put

$$\Psi_p(N;\, \alpha_p) = \frac{\alpha_p^{\delta+1} - \alpha_p^{-(\delta+1)}}{\alpha_p - \alpha_p^{-1}} + p^{-(n-1)/2} \cdot \frac{\alpha_p^{\delta} - \alpha_p^{-\delta}}{\alpha_p - \alpha_p^{-1}}$$

where $\delta = \mathrm{ord}_p(N)$, and α_p is the complex number defined in Section 2. Then we have the following:

Proposition 5.4. ([16], Lemma 2) *Let $D_f^*(N) : J_{k,\,mN}^{\mathrm{cusp}}(\Gamma^{(n-1),J}) \longrightarrow J_{k,\,N}^{\mathrm{cusp}}(\Gamma^{(n-1),J})$ be the adjoint operator of $D_f(N)$, that is*

$$\langle D_f(N)(\phi),\, \psi\rangle = \langle \phi,\, D_f^*(N)(\psi)\rangle$$

for any $\phi \in J_{k,\,m}^{\mathrm{cusp}}(\Gamma^{(n-1),J})$ and $\psi \in J_{k,\,mN}^{\mathrm{cusp}}(\Gamma^{(n-1),J})$. Then

$$D_f^*(N)D_f(N)(\phi_1) = \sum_{d|N} d^{k-1}(Nd^{-1})^{k-(n+1)/2} \prod_{p|Nd^{-1}} \Psi_p(Nd^{-1}; \alpha_p)\phi_1.$$

Proof of Theorem 5.2. By using the above two propositions we have

$$\sum_{N=1}^{\infty}\langle \phi_N,\phi_N\rangle N^{-s}=\sum_{N=1}^{\infty}\langle D_f(N)(\phi_1),D_f(N)(\phi_1)\rangle N^{-s}$$

$$=\sum_{N=1}^{\infty}\langle \phi_1,D_f^*(N)D_f(N)(\phi_1)\rangle N^{-s}$$

$$=\sum_{N=1}^{\infty}\langle \phi_1,\phi_1\rangle N^{-s}\sum_{d|N}d^{k-1}(Nd^{-1})^{k-(n+1)/2}\prod_{p|Nd^{-1}}\Psi_p(Nd^{-1};\alpha_p)$$

$$=\langle \phi_1,\,\phi_1\rangle\,\frac{\zeta(s-k+1)\zeta(s-k+n)L(s,\,f)}{\zeta(2s-2k+2n)}.$$

Thus the assertion holds.

6 Rankin-Selberg Dirichlet series associated with the Ibukiyama correspondence of the Fourier-Jacobi coefficient of the Ikeda lift

To prove Conjecture A, we rewrite it in terms of the residue of the Rankin-Selberg convolution product of a certain half-integral weight modular form. Let l be a positive integer. Let $F(Z)\in\mathfrak{S}_{l-1/2}(\Gamma_0^{(m)}(4))$. Then $F(Z)$ has the following Fourier expansion:

$$F(Z)=\sum_{A\in\mathcal{L}_{m>0}}a_F(A)\mathbf{e}(\mathrm{tr}(AZ)).$$

We define the Rankin-Selberg convolution product $R(s,F)$ of F as

$$R(s,F)=\sum_{A\in\mathcal{L}_{m>0}/SL_m(\mathbf{Z})}\frac{|a_F(A)|^2}{e(A)(\det A)^s},$$

where $e(A)=\#\{X\in SL_m(\mathbf{Z})\mid A[X]=A\}$. Let

$$\mathcal{L}'_{m>0}=\{A\in\mathcal{L}_{m>0}\mid A\equiv -\,{}^t rr \bmod 4\mathcal{L}_m \text{ for some } r\in\mathbf{Z}^m\}.$$

We note the r in the above definition is uniquely determined modulo $2\mathbf{Z}^m$ by A, which will be denoted by r_A. Now we define the generalized Kohnen plus subspace of weight $l-1/2$ with respective to $\Gamma_0^{(m)}(4)$ as

$$\mathfrak{S}^+_{l-1/2}(\Gamma_0^{(m)}(4))=\{F(Z)=\sum_{A\in\mathcal{L}_{m>0}}c(A)\mathbf{e}(\mathrm{tr}(AZ))\in\mathfrak{S}_{l-1/2}(\Gamma_0^{(m)}(4))|$$

$$c(A)=0 \text{ unless } A\in\mathcal{L}'_{m>0}\}.$$

Then there exists a correspondence between the space of Jacobi-forms of index 1 and the generalized Kohnen plus space due to Ibukiyama. To explain this, let $\phi(Z,z) \in J_{l,1}^{\mathrm{cusp}}(\Gamma^{(m),J})$. Then we have the following Fourier-Jacobi expansion:

$$\phi(Z,\, z) = \sum_{\substack{T \in \mathcal{L}_m,\, r \in \mathbf{Z}^m, \\ 4T - {}^t rr > 0}} c(T,r)\mathbf{e}(\mathrm{tr}(TZ) + r^t z).$$

We call two elements (T,r) and (T',r') of $\mathcal{L}_m \times \mathbf{Z}^m$ are $SL_m(\mathbf{Z})$-equivalent with each other and wright $(T,r) \sim (T',r')$ if there exists an element $g \in SL_m(\mathbf{Z})$ such that $T' - {}^t r'r'/4 = (T - {}^t rr/4)[g]$. We then define a Dirichlet series $R(s,\phi)$ as

$$R(s,\phi) = \sum_{(T,r)} \frac{c(T,r)}{(\det(T - {}^t rr/4))^s e(T - {}^t rr/4)},$$

where (T,r) runs over a complete set of representatives of $SL_m(\mathbf{Z})$-equivalence classes of $\mathcal{L}_m \times \mathbf{Z}^m$ such that $T - {}^t rr/4 \in \mathcal{L}_{m>0}$. Now $\phi(Z,z)$ can also be expressed as follows:

$$\phi(Z,z) = \sum_{r \in \mathbf{Z}^m/2\mathbf{Z}^m} h_r(Z)\theta_r(Z,z),$$

where $h_r(Z)$ is a holomorphic function on $\mathbf{H}_m$, and

$$\theta_r(Z,z) = \sum_{\lambda \in M_{1,m}(\mathbf{Z})} \mathbf{e}(\mathrm{tr}(Z[{}^t(\lambda + 2^{-1}r)]) + 2(\lambda + 2^{-1}r)^t z).$$

We note that $h_r(Z)$ have the following Fourier expansion:

$$h_r(Z) = \sum_T c(T,r)\mathbf{e}(\mathrm{tr}((T - {}^t rr/4)Z)),$$

where T runs over all elements of $\mathcal{L}_m$ such that $T - {}^t rr/4$ is positive definite. Put $\mathbf{h}(Z) = (h_r(Z))_{r \in \mathbf{Z}^m/2\mathbf{Z}^m}$. Then $\mathbf{h}$ is a vector valued modular form of weight $l - 1/2$ for $\Gamma^{(m)}$. Namely, for each $\gamma \in \Gamma^{(m)}$, we have

$$\mathbf{h}(\gamma(Z)) = J(\gamma, Z)\mathbf{h}(\gamma(Z)),$$

where $J(\gamma,Z)$ is an $m \times m$ matrix with entries in holomorphic functions on $\mathbf{H}_m$ such that $\overline{{}^t J(\gamma,Z)} J(\gamma,Z) = |j(\gamma,Z)|^{2l-1} 1_m$. In particular, we have

$$\sum_{r \in \mathbf{Z}^m/2\mathbf{Z}^m} h_r(\gamma(Z))\overline{h_r(\gamma(Z))} = |j(\gamma,Z)|^{2l-1} \sum_{r \in \mathbf{Z}^m/2\mathbf{Z}^m} h_r(Z)\overline{h_r(Z)}.$$

We then put

$$\sigma_m(\phi)(Z) = \sum_{r \in \mathbf{Z}^m/2\mathbf{Z}^m} h_r(4Z).$$

Then Ibukiyama [8] showed the following:

Proposition 6.1. *Let l be an even positive integer. Then σ_m gives a $\mathbf{C}$-linear isomorphism*

$$\sigma_m : J_{l,1}^{\mathrm{cusp}}(\Gamma^{(m),J}) \cong \mathfrak{S}_{l-1/2}^{+}(\Gamma_0^{(m)}(4)).$$

We call σ_m the Ibukiyama correspondence. We note that we have

$$\sigma_m(\phi) = \sum_{A \in Sym_m(\mathbf{Z})_{>0}} c((A + {}^t r_A r_A)/4, r_A)\mathbf{e}(\mathrm{tr}(AZ)),$$

where $r = r_A$ denote an element of $\mathbf{Z}^m$ such that $A + {}^t r_A r_A \in 4\mathcal{L}_m$. This r_A is uniquely determined up to modulo $2\mathbf{Z}^m$, and $c((A + {}^t r_A r_A)/4, r_A)$ does not depend on the choice of the representative of r_A mod $2\mathbf{Z}^m$. Furthermore, we have

$$R(s, \sigma_m(\phi)) = \sum_{A \in \mathcal{L}'_{m>0}/SL_m(\mathbf{Z})} \frac{|c((A + {}^t rr)/4, r)|^2}{e(A) \det A^s},$$

and therefore we have

$$R(s, \phi) = 2^{2sm} R(s, \sigma_m(\phi)).$$

By using the Rankin-Selberg method, we can prove the following analytic properties of $R(s, \phi)$:

Proposition 6.2. *Let l be a positive integer. Let $\phi(Z, z) \in J_{l,1}^{\mathrm{cusp}}(\Gamma^{(m),J})$. Put*

$$\mathcal{R}(s, \phi) = \gamma_m(s)\xi(2s + m + 2 - 2l) \prod_{i=1}^{[m/2]} \xi(4s + 2m + 4 - 4l - 2i)R(s, \phi),$$

where

$$\gamma_m(s) = 2^{1-2sm} \prod_{i=1}^{m} \Gamma_{\mathbf{R}}(2s - i + 1).$$

Then $\mathcal{R}(s, \phi)$ has a meromorphic continuation to the whole s-plane, and has the following functional equation:

$$\mathcal{R}(2l - 3/2 - m/2 - s, \phi) = \mathcal{R}(s, \phi).$$

Furthermore it has a simple pole at $s = l - 1/2$ with the residue

$$2^{m+1} \prod_{i=1}^{[m/2]} \xi(2i + 1)\langle \phi, \phi \rangle.$$

Now let l be a positive even integer. For $F \in \mathfrak{S}^+_{l-1/2}(\Gamma_0^{(m)}(4))$ put

$$\mathcal{R}(s, F) = \prod_{i=1}^{m} \Gamma_{\mathbf{R}}(2s - i + 1)$$

$$\times \xi(2s + m - 2l + 2) \prod_{i=1}^{[m/2]} \xi(4s + 2m - 4l + 4 - 2i)R(s, F).$$

We note that

$$\mathcal{R}(s, \sigma_m(\phi)) = 2^{-1}\mathcal{R}(s, \phi)$$

for $\phi \in J_{l,1}^{\mathrm{cusp}}(\Gamma^{(m),J})$. Thus we have

Corollary. *In addition to the notation and the assumption as Proposition 6.2, assume that l is even. $\mathcal{R}(s, \sigma_m(\phi))$ has a meromorphic continuation to the whole s-plane, and has the following functional equation:*

$$\mathcal{R}(2l - 3/2 - m/2 - s, \sigma_m(\phi)) = \mathcal{R}(s, \sigma_m(\phi)).$$

Furthermore it has a simple pole at $s = l - 1/2$ with the residue

$$2^m \prod_{i=1}^{[m/2]} \xi(2i+1)\langle \phi, \phi \rangle.$$

Now we recall the following diagram of liftings:

$$\begin{array}{ccc} \mathfrak{S}^+_{k-(n-1)/2}(\Gamma_0^{(1)}(4)) \ni \tilde{f} & \longrightarrow & f \in \mathfrak{S}_{2k-n}(\Gamma^{(1)}) \\ & & \downarrow \\ & & I_n(f) \in \mathfrak{S}_k(\Gamma^{(n)}) \\ & & \downarrow \\ \mathfrak{S}^+_{k-1/2}(\Gamma_0^{(n-1)}(4)) \ni \sigma_{n-1}(\phi_1) & \longleftarrow & \phi_1 \in J^{\mathrm{cusp}}_{k,1}(\Gamma^{(n-1),J}) \end{array}$$

Under the above notation, we propose a conjecture:

Conjecture B.

$$\mathrm{Res}_{s=k-1/2}\mathcal{R}(s, \sigma_{n-1}(\phi_1)) = 2^{\beta(n,k)}\langle \tilde{f}, \tilde{f}\rangle \prod_{i=1}^{n/2-1} \widetilde{\xi}(2i)\xi(2i+1)\widetilde{\Lambda}(2i+1, f, \mathrm{Ad}),$$

where $\beta(n,k)$ is a certain integer depending only on n and k.

Then, by Corollary to theorem 5.2, we can rewrite Conjecture A as follows:

Theorem 6.3. *Under the above notation and the assumption Conjecture A is equivalent to Conjecture B.*

To prove Conjecture B, we give an explicit formula for $R(s, \sigma_{n-1}(\phi_1))$ for the first Fourier-Jacobi coefficient of $I_n(f)$. To do this, we reduce the problem to local computations. Let

$$\mathcal{L}'_{m,p} = \{A \in \mathcal{L}_{m,p} \mid A \equiv -\,{}^t rr \bmod 4\mathcal{L}_{m,p} \text{ for some } r \in \mathbf{Z}_p^m\}.$$

Furthermore we put $Sym_m(\mathbf{Z}_p)_e = 2\mathcal{L}_{m,p}$. We note that we have $\mathcal{L}'_{m,p} = Sym_m(\mathbf{Z}_p)_e = \mathcal{L}_{m,p} = Sym_m(\mathbf{Z}_p)$ if $p \neq 2$. Let m be a positive even integer. Let $T \in \mathcal{L}'_{m-1,p}$. Then there exists an element $r_T \in \mathbf{Z}_p^{m-1}$ such that $T^{(1)} = \begin{pmatrix} 1 & r_T/2 \\ {}^t r_T/2 & (T + {}^t r_T r_T)/4 \end{pmatrix}$ belongs to $\mathcal{L}_{m,p}$. Thus we can define $\mathfrak{d}_T$ and $\mathfrak{f}_T$ as $\mathfrak{d}_{T^{(1)}}$ and $\mathfrak{f}_{T^{(1)}}$, respectively. These do not depend on the choice of r_T. We define a polynomial $F_p^{(1)}(T, X)$ and a Laurent polynomial $\tilde{F}_p^{(1)}(T, X)$ by

$$F_p^{(1)}(T, X) = F_p(\begin{pmatrix} 1 & r_T/2 \\ {}^t r_T/2 & (T + {}^t r_T r_T)/4 \end{pmatrix}, X),$$

and

$$\tilde{F}_p^{(1)}(T, X) = X^{-\nu_p(\mathfrak{f}_T)} F_p^{(1)}(T, p^{-(n+1)/2}X).$$

where $r = r_T$ is an element of $\mathbf{Z}_p^{n-1}$ such that $T + {}^t rr \in 4\mathcal{L}_{n-1}$. Let B be a half-integral matrix over $\mathbf{Z}_p$ of degree n. Now let m and l be positive integers such that $m \geq l$. Then for non-degenerate symmetric matrices A and B of degree m and l respectively with entries in $\mathbf{Z}_p$ we define the local density $\alpha_p(A,B)$ and the primitive local density $\beta_p(A,B)$ representing B by A as

$$\alpha_p(A,B) = 2^{-\delta_{m,l}} \lim_{e\to\infty} p^{e(-ml+l(l+1)/2)} \#\mathcal{A}_e(A,B),$$

and

$$\beta_p(A,B) = 2^{-\delta_{ml}} \lim_{e\to\infty} p^{e(-ml+l(l+1)/2)} \#\mathcal{B}_e(A,B),$$

where

$$\mathcal{A}_e(A,B) = \{X \in M_{ml}(\mathbf{Z}_p)/p^e M_{ml}(\mathbf{Z}_p) \mid A[X] - B \in p^e Sym_l(\mathbf{Z}_p)_e\},$$

and

$$\mathcal{B}_e(A,B) = \{X \in \mathcal{A}_e(A,B) \mid \mathrm{rank}_{\mathbf{Z}_p/p\mathbf{Z}_p} X = l\}.$$

In particular we write $\alpha_p(A) = \alpha_p(A,A)$. Furthermore put

$$M(A) = \sum_{A' \in \mathcal{G}(A)} \frac{1}{e(A')}$$

for a positive definite symmetric matrix A of degree $n-1$ with entries in $\mathbf{Z}$, where $\mathcal{G}(A)$ denotes the set of $SL_{n-1}(\mathbf{Z})$-equivalence classes belonging to the genus of A. Then by Siegel's main theorem on the quadratic forms, we have

$$M(A) = e_{n-1}\kappa_{n-1} \det A^{n/2} \prod_p \alpha_p(A)^{-1}$$

where $e_{n-1} = 1$ or 2 according as $n = 2$ or not, and

$$\kappa_{n-1} = 2^{2-n}\pi^{-n(n-1)/4} \prod_{i=1}^{n-1} \Gamma(i/2)$$

(cf. [20], Theorem 6.8.1.) Let $l = \pm 1$. Then put

$$\mathcal{F}_p = \{d_0 \in \mathbf{Z}_p \mid \nu_p(d_0) \le 1\}$$

if p is an odd prime, and

$$\mathcal{F}_2 = \{d_0 \in \mathbf{Z}_2 \mid d_0 \equiv 1 \bmod 4 \text{ or } d_0/4 \equiv -1 \bmod 4 \text{ or } \nu_2(d_0) = 3\}.$$

For $d_0 \in \mathcal{F}_p$ and a $GL_{n-1}(\mathbf{Z})_p$-invariant function ω_p on $\mathcal{L}_{n-1,p}^{\times}$ put

$$H_{n-1,p}(d_0;\omega_p,X,Y,t) = \sum_{A \in \mathcal{L}'_{n-1,p}(d_0)/GL_{n-1}(\mathbf{Z}_p)} \omega_p(A) t^{\nu_p(\det A)} \frac{\tilde{F}_p^{(1)}(A,X)\tilde{F}_p^{(1)}(A,Y)}{\alpha_p(A)},$$

where

$$\mathcal{L}'_{n-1,p}(d_0) = \{A \in \mathcal{L}'_{n-1,p} \mid \mathfrak{d}_{(-1)^{n/2}\det A} = d_0\}.$$

Let ι_p be the constant function of $\mathcal{L}_{n-1,p}^{\times}$ taking the value 1, and ε_p the function of $\mathcal{L}_{n-1,p}^{\times}$ assigning the Hasse invariant of A for $A \in \mathcal{L}_{n-1,p}^{\times}$. Let $\mathcal{F}$ denote the set of fundamental discriminants, and for $l = \pm 1$, put $\mathcal{F}^{(l)} = \{d_0 \in \mathcal{F} \mid ld_0 > 0\}$. It is easily shown that the Fourier coefficient $c_{\sigma_{n-1}(\phi_1)}(T)$ of $\sigma_{n-1}(\phi_1)$ is uniquely determined by the genus to which T belongs. Thus, by using the same method as in Theorem 2.2 of [13], similarly to [11], Theorem 3.3, (1), and [12], Theorem 3.2, we have

Theorem 6.4. *Under the same notation and the assumption as Conjecture A, we have*

$$R(s,\sigma_{n-1}(\phi_1)) = \frac{e_{n-1}}{2}\kappa_{n-1}2^{-(k-n/2-1/2)(n-2)}$$
$$\times \sum_{d_0\in\mathcal{F}^{((-1)^{n/2})}} |c_{\tilde{f}}(|d_0|)|^2|d_0|^{n/2-k+1/2}$$
$$\times(\prod_p H_{n-1,p}(d_0;\iota_p,\alpha_p,\alpha_p,p^{-s+k-1/2}) + \prod_p H_{n-1,p}(d_0;\varepsilon_p,\alpha_p,\alpha_p,p^{-s+k-1/2})).$$

It is rather elaborate to compute $H_{n-1,p}(d_0;\omega_p,\alpha_p,\alpha_p,p^{-s+k-1/2})$ for $\omega_p=\iota_p,\varepsilon_p$. But anyway we get the following explicit formula for them. For details, see [18].

Theorem 6.5. *Let $d_0\in\mathcal{F}_p$ and put $\xi_0=(\frac{d_0}{p})$.*
(1). *We have*

$$H_{n-1,p}(d_0;\iota_p,X,Y,t)$$
$$=(2^{-(n-1)(n-2)/2}t^{n-2})^{\delta_{2,p}}\phi_{(n-2)/2}(p^{-2})^{-1}(p^{-1}t)^{\nu(d_0)}(1-p^{-n}t^2)\prod_{i=1}^{n/2-1}(1-p^{-2n+2i}t^4)$$
$$\times\frac{(1+p^{-2}t^2)(1+p^{-3}\xi_0^2t^2)-p^{-5/2}t^2\xi_0(X+X^{-1}+Y+Y^{-1})}{(1-p^{-2}XYt^2)(1-p^{-2}XY^{-1}t^2)(1-p^{-2}X^{-1}Yt^2)(1-p^{-2}X^{-1}Y^{-1}t^2)}$$
$$\times\frac{1}{\prod_{i=1}^{n/2-1}(1-p^{-2i-1}XYt^2)(1-p^{-2i-1}XY^{-1}t^2)(1-p^{-2i-1}X^{-1}Yt^2)(1-p^{-2i-1}X^{-1}Y^{-1}t^2)}.$$

(2). *We have*

$$H_{n-1,p}(d_0;\varepsilon_p,X,Y,t)=((-1)^{n(n-2)/8}2^{-(n-1)(n-2)/2}t^{n-2})^{\delta_{2,p}}$$
$$\times((-1)^{n/2},(-1)^{n/2}d_0)_p\phi_{(n-2)/2}(p^{-2})^{-1}(1-p^{-n}t^2)\prod_{i=1}^{n/2-1}(1-p^{-2n+2i}t^4)(tp^{-n/2})^{\nu(d_0)}$$
$$\times\frac{(1+p^{-n}t^2)(1+p^{-n-1}\xi_0^2t^2)-p^{-1/2-n}t^2\xi_0(X+X^{-1}+Y+Y^{-1})}{\{(1-p^{-n}XYt^2)(1-p^{-n}XY^{-1}t^2)(1-p^{-n}X^{-1}Yt^2)(1-p^{-n}X^{-1}Y^{-1}t^2)}$$
$$\times\frac{1}{\prod_{i=1}^{n/2-1}(1-p^{-2i}XYt^2)(1-p^{-2i}XY^{-1}t^2)(1-p^{-2i}X^{-1}Yt^2)(1-p^{-2i}X^{-1}Y^{-1}t^2)},$$

where $(a,b)_p$ is the Hilbert symbol of $a,b\in\mathbf{Q}_p$.

The following result can be easily proved.

Proposition 6.6. *Let f be a normalized Hecke eigenform in $\mathfrak{S}_{2k-n}(\Gamma^{(1)})$. Then we have*

$$R(s,\tilde{f})=L(2s-2k+n+1,f,\mathrm{Ad})\sum_{d_0\in\mathcal{F}^{((-1)^{n/2})}}|c_{\tilde{f}}(|d_0|)|^2|d_0|^{-s}$$
$$\times\prod_p((1+p^{-2s+2k-n-1})(1+p^{-2s+2k-n-2}(\frac{d_0}{p})^2)-2p^{-2s+2k-n-3/2}(\frac{d_0}{p})a(p)).$$

Now by Theorems 6.4 and 6.5, and Proposition 6.6, we get the following result.

Theorem 6.7. *For a normalized Hecke eigenform* $f \in \mathfrak{S}_{2k-n}(\Gamma^{(1)})$, *let* $\widetilde{f} \in \mathfrak{S}^{+}_{k-n/2+1/2}(\Gamma_0(4))$ *and* $\phi_1 = \phi_{I_n(f),1} \in J^{\mathrm{cusp}}_{k,1}(\Gamma^{(n-1),J})$ *be as above. Put* $\lambda_n = \frac{e_{n-1}}{2}\prod_{i=1}^{n/2-1}\widetilde{\xi}(2i)$. *Then, we have*

$$R(s,\sigma_{n-1}(\phi_1)) = \lambda_n 2^{(-s-1/2)(n-2)}\zeta(2s+n-2k+1)^{-1}\prod_{i=1}^{\frac{n-2}{2}}\zeta(4s+2n-4k+2-2j)^{-1}$$

$$\times\{R(s-n/2+1,\tilde{f})\zeta(2s-2k+3)\prod_{i=1}^{\frac{n-2}{2}}L(2s-2k+2i+2,f,\mathrm{Ad})\zeta(2s-2k+2i+2)$$

$$+(-1)^{n(n-2)/8}R(s,\tilde{f})\zeta(2s-2k+n+1)\prod_{i=1}^{\frac{n-2}{2}}L(2s-2k+2i+1,f,\mathrm{Ad})\zeta(2s-2k+2i+1)\}.$$

Now by taking the residues of the both sides of Theorem 6.7, we have

Theorem 6.8. *Conjecture B holds true for any even positive integer* n.

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Muroran Institute of Technology
27-1 Mizumoto, Muroran, 050-8585, Japan
E-mail: hidenori@mmm.muroran-it.ac.jp