CHAPTER VII

Around log-terminal singularities

In this chapter, we discuss singularities arising from the consideration on the minimal model theory of higher-dimensional algebraic varieties. The notion of terminal singularities and that of canonical singularities are introduced by Reid in the study of singularities on minimal models ([**113**], [**114**]). In the minimal model program, we consider not only normal varieties themselves but also the pairs consisting of normal varieties and effective $\mathbb{Q}$-divisors. Notions of singularities can be defined similarly for such pairs. In the middle of 1980's, there appeared a summary [**61**] of minimal model program for higher dimensional varieties, where the notions of log-terminal, log-canonical, and weakly log-terminal are explained. The definition of log-terminal in [**61**] is different from the one used in the classification theory of open surfaces, in the sense that the latter allows a $\mathbb{Q}$-divisor with multiplicity one. Shokurov [**132**] introduced his original definition of log-terminal (it was written *log terminal*) in order to prove the log-flip conjectures, which coincides in dimension two with the one used in the classification theory of open surfaces. The notion of log-terminal in [**61**] is given a different name and called *Kawamata log terminal* or klt in [**132**] and [**74**]. However, Shokurov's notion of log terminal seems to have no good meaning for application. The notion of divisorial log terminal (dlt) in [**132**] and [**74**] is useful for the log minimal model program. In [**134**], the notion of dlt is shown to be equivalent to the notion of weakly log-terminal if we consider only simple normal crossing divisors in the definition given in [**61**]. Unfortunately, however, the notion of dlt is not a property well-defined for analytic germs. Fujita's definition of log terminal in [**27**] dealt with the analytic local situation. In the early 1990's, the author introduced another notion of log-terminal, named strongly log-canonical, which is closer to the notion of log-canonical. It is a property well-defined for analytic germs and has many useful properties for the minimal model program.

In this chapter, we introduce the notions of *admissible*, *quasi log-terminal*, and *strongly log-canonical*, for pairs (X, Δ) consisting of normal varieties and effective $\mathbb{R}$-divisors. These notions are analytically local in nature. These are defined and discussed in §**1**. In the definition of admissible pairs, the $\mathbb{R}$-divisor $K_X + \Delta$ need not to be $\mathbb{R}$-Cartier. A new proof of rationality of canonical singularities is also given in §**1**. The minimal model program for strongly log-canonical pairs is mentioned in §**2** and a relation between admissible singularities and ω-sheaves is explained in §**3**.

§1. Admissible and strongly log-canonical singularities

§1.a. Admissible singularities. We now prepare a sufficient condition for a singularity to be rational, by using which we can prove the rationality of canonical singularities.

1.1. Theorem *Let $f\colon Y \to X$ be a locally projective surjective morphism from a non-singular variety onto a normal variety. Suppose that there is an effective divisor R such that $\mathrm{R}^i f_*\mathcal{O}_Y(R) = 0$ for $i > 0$ and that the natural homomorphism $f_*\mathcal{O}_Y \to f_*\mathcal{O}_Y(R)$ is an isomorphism. Then X has only rational singularities.*

Proof. Let $Y \to V \to X$ be the Stein factorization. If V has only rational singularities, then so does X. Therefore we may assume that $V \simeq X$ or equivalently, $\mathcal{O}_X \simeq f_*\mathcal{O}_Y$. In the derived category $D^+(\mathcal{O}_X)$, the composite $\mathcal{O}_X \to \mathrm{R}\, f_*\mathcal{O}_Y \to \mathrm{R}\, f_*\mathcal{O}_Y(R)$ is a quasi-isomorphism. Thus

$$\mathrm{R}\, f_*\mathcal{O}_Y \sim_{\mathrm{qis}} \mathcal{O}_X \oplus L^\bullet$$

for a bounded complex $L^\bullet$. By duality (cf. [**37**], [**117**]), we have

$$\mathrm{R}\, f_*\omega_Y[\dim Y] \sim_{\mathrm{qis}} \mathrm{R}\,\mathcal{H}om(\mathrm{R}\, f_*\mathcal{O}_Y, \omega_X^\bullet).$$

Thus $\mathrm{R}\, f_*\omega_Y[\dim Y] \sim_{\mathrm{qis}} \omega_X^\bullet \oplus G^\bullet$ for a complex $G^\bullet$. By **V.3.7**-(1), $\mathcal{H}^{-i}(\omega_X^\bullet)$ is torsion-free. Thus it is zero except for $i = \dim X$. Hence X is Cohen-Macaulay. Let $Y' \to Y$ and $\mu\colon X' \to X$ be bimeromorphic morphisms from non-singular varieties such that

(1) the morphism $g\colon Y' \to X'$ is induced,
(2) g is a smooth morphism outside a normal crossing divisor of X'.

Then $\mathrm{R}^i g_*\omega_{Y'}$ is a locally free sheaf and $\mathrm{R}^p \mu_*(\mathrm{R}^i g_*\omega_{Y'}) = 0$ for $i \geq 0$ and $p > 0$ by **V.3.7**. In particular, $\mathrm{R}^d g_*\omega_{Y'} \simeq \omega_{X'}$, where $d := \dim Y - \dim X$. Thus

$$\mathrm{R}^d f_*\omega_Y \simeq \mu_*\omega_{X'} \simeq \mathcal{H}^{-\dim X}(\omega_X^\bullet \oplus G^\bullet).$$

Therefore $\mu_*\omega_{X'} \simeq \omega_X$. Hence X has only rational singularities. □

1.2. Definition Let (X, Δ) be a pair of a normal variety X and an effective $\mathbb{R}$-divisor Δ with $\llcorner \Delta \lrcorner = 0$. It is called *strictly admissible* if there exist a bimeromorphic morphism $f\colon Y \to X$ from a non-singular variety and a $\mathbb{Q}$-divisor E on Y satisfying the following conditions:

(1) $\mathrm{Supp}\langle E\rangle$ is a normal crossing divisor;
(2) $\ulcorner E \urcorner$ is an f-exceptional effective divisor;
(3) $-f_*E \geq \Delta$;
(4) $E - K_Y$ is f-ample.

If there is an open covering $\{U_\lambda\}$ of X such that $(U_\lambda, \Delta|_{U_\lambda})$ is strictly admissible for any λ, then (X, Δ) is called *admissible* or having only *admissible singularities.* A normal variety X is said to have only *admissible singularities* if $(X, 0)$ is admissible.

If (X, Δ) is admissible, then X has only admissible singularities. The admissible singularity is rational by **1.1**. Moreover, we have:

1.3. Lemma *Let (X, Δ) be a pair of normal variety and effective $\mathbb{R}$-divisor. Then (X, Δ) is admissible if and only if, for any point $x \in X$, there exist an open neighborhood U of x and an effective $\mathbb{Q}$-divisor Δ' of U such that $\Delta' \geq \Delta|_U$ and (U, Δ') is log-terminal.*

PROOF. Let $f\colon Y \to X$ and E be the bimeromorphic morphism and the $\mathbb{Q}$-divisor, respectively, in **1.2**. Then there are an open neighborhood U of x, an integer $m > 1$, and a non-singular effective divisor A of $f^{-1}(U)$ such that $m(E - K_Y)|_{f^{-1}(U)} \sim A$ and $\operatorname{Supp}(\langle E\rangle|_{f^{-1}(U)} + A)$ is a normal crossing divisor. If we set

$$\Delta' := f_*((1/m)A - E|_{f^{-1}(U)}),$$

then (U, Δ') is log-terminal, since

$$f^*(K_U + \Delta') = K_{f^{-1}(U)} + (1/m)A - E|_{f^{-1}(U)}.$$

Conversely suppose that (X, Δ') is log-terminal for a $\mathbb{Q}$-divisor Δ' with $\Delta' \geq \Delta$. Let $f\colon Y \to X$ be a bimeromorphic morphism from a non-singular variety and set $R' := K_Y - f^*(K_X + \Delta')$. We may assume that there is an effective divisor B such that $-B$ is f-ample and $\operatorname{Supp} B \cup \operatorname{Supp}\langle R'\rangle$ is a normal crossing divisor. Note that $\ulcorner R' \urcorner$ is f-exceptional and effective. Then $R' - \delta B - K_Y$ is f-ample and $\ulcorner R' - \delta B \urcorner = \ulcorner R' \urcorner$ for $0 < \delta \ll 1$ over an open neighborhood of any point in X. Thus the $\mathbb{Q}$-divisor $E := R' - \delta B$ satisfies the required condition for (X, Δ) to be admissible. □

1.4. Lemma *Let (X, Δ) be a strictly admissible pair and let $f\colon Y \to X$ be a bimeromorphic morphism from a non-singular variety with a $\mathbb{Q}$-divisor E satisfying the condition of* **1.2**. *Let $\mu\colon Z \to Y$ be a projective bimeromorphic morphism from a non-singular variety and let $g := f \circ \mu$. Suppose that the union of μ-exceptional locus and $\mu^{-1}(\operatorname{Supp}\langle E\rangle)$ is a normal crossing divisor. Then, for any relatively compact open subset $U \subset X$, there is a $\mathbb{Q}$-divisor E' of $g^{-1}(U)$ such that*

(1) $\operatorname{Supp}\langle E'\rangle$ *is a normal crossing divisor,*
(2) $\ulcorner E' \urcorner$ *is a g-exceptional effective divisor,*
(3) $-g_*E' = (-f_*E)|_U$,
(4) $E' - K_{g^{-1}(U)}$ *is a g-ample $\mathbb{Q}$-divisor.*

PROOF. There is a μ-exceptional effective divisor B such that $-B$ is μ-ample. Hence $\mu^*(E - K_Y) - \delta B$ is g-ample over U for $0 < \delta \ll 1$. Since $K_Z - \mu^* K_Y$ is an effective μ-exceptional divisor, the $\mathbb{Q}$-divisor

$$E' := K_Z - \mu^*(K_Y - E) - \delta B$$

satisfies the conditions by **II.4.3**-(2). □

1.5. Lemma *Let (X, Δ) be a pair of normal variety and effective $\mathbb{Q}$-divisor. Then (X, Δ) is admissible if and only if, for any relatively compact open subset $U \subset X$, there exist a positive integer m, a bimeromorphic morphism $g\colon Z \to U$ from a non-singular variety, and a divisor F of Z such that*

(1) $m\Delta|_U$ *is a $\mathbb{Z}$-divisor,*

(2) $\operatorname{Supp}\langle(1/m)F\rangle$ *is a normal crossing divisor,*
(3) $\ulcorner(1/m)F\urcorner$ *is a g-exceptional effective divisor,*
(4) $g^*\mathcal{O}_U(-mK_X - m\Delta)/(\mathrm{tor}) \simeq \mathcal{O}_Z(F - mK_Z)$.

Proof. First suppose that (X, Δ) is admissible. Let $U \subset X$ be a relatively compact open subset and let $\mathcal{U}_i \subset X$ $(1 \le i \le l)$ be a finite number of open subsets such that $(\mathcal{U}_i, \Delta|_{\mathcal{U}_i})$ is strictly admissible and $U \subset \bigcup_{i=1}^{l} \mathcal{U}_i$. Then, for every i, there exist bimeromorphic morphisms $f_i \colon Y_i \to \mathcal{U}_i$ and $\mathbb{Q}$-divisor E_i of Y_i satisfying the same condition as **1.2** for $(\mathcal{U}_i, \Delta|_{\mathcal{U}_i})$. By replacing $\mathcal{U}_i$ with a relatively compact open subset of $\mathcal{U}_i$, we may assume that there is a positive integer m such that $m\Delta$ is a $\mathbb{Z}$-divisor, $m(E_i - K_{Y_i})$ are Cartier, and the evaluation homomorphism

$$f_i^* f_{i*}\mathcal{O}_{Y_i}(m(E_i - K_{Y_i})) \to \mathcal{O}_{Y_i}(m(E_i - K_{Y_i}))$$

is surjective for any i. Let $g \colon Z \to U$ be a bimeromorphic morphism from a non-singular variety such that the union of the g-exceptional locus and $g^{-1}(\operatorname{Supp}\Delta)$ is a normal crossing divisor and that $g^*\mathcal{O}_X(-mK_X - m\Delta))/(\mathrm{tor})$ is an invertible sheaf. Then there is a $\mathbb{Z}$-divisor F of Z such that $\operatorname{Supp} F$ is a normal crossing divisor and the invertible sheaf above is isomorphic to $\mathcal{O}_Z(F - mK_Z)$. For each i, let $\varphi_i \colon M_i \to f_i^{-1}(U \cap \mathcal{U}_i)$ be a bimeromorphic morphism from a non-singular variety such that $\psi_i \colon M_i \dashrightarrow g^{-1}(\mathcal{U}_i)$ is holomorphic. Since $f_{i*}\mathcal{O}_{Y_i}(mE_i - mK_{Y_i}) \subset \mathcal{O}_{\mathcal{U}_i}(-m(\Delta + K_X))$, we have $\psi_i^*(F - mK_Z) \ge \varphi_i^*(mE_i - mK_{Y_i})$. By the logarithmic ramification formula **II.4.3**, we have:

$$K_{M_i} + \Delta_i - \psi_i^*(\ulcorner(1/m)F\urcorner) = \psi_i^*(K_Z - (1/m)F) + R_i,$$
$$K_{M_i} + \Delta_i' - \varphi_i^*(\ulcorner E_i \urcorner) = \varphi_i^*(K_{Y_i} - E_i) + R_i',$$

for effective $\mathbb{Q}$-divisors Δ_i, Δ_i' with $\llcorner \Delta_i \lrcorner = \llcorner \Delta_i' \lrcorner = 0$, for ψ_i-exceptional effective divisors R_i, and for φ_i-exceptional effective divisors R_i'. Hence

$$\psi_i^*(\ulcorner(1/m)F\urcorner) + \Delta_i' + R_i \ge \varphi_i^*(\ulcorner E_i \urcorner) + \Delta_i + R_i'.$$

We have $\ulcorner(1/m)F\urcorner \ge 0$, since $\ulcorner E_i \urcorner \ge 0$, $\llcorner \Delta_i' \lrcorner = 0$, and R_i is ψ_i-exceptional. Thus g and F satisfy the required conditions. Next suppose the existence of such g and F. By **II.4.3**, we may replace Z by a blowing-up, and hence we may assume that there is an effective $\mathbb{Z}$-divisor B such that $-B$ is g-ample and $\operatorname{Supp}(B + F)$ is normal crossing. Thus, over any relatively compact open subset of X, $(1/m)F - \delta B - K_Z$ is g-ample and $\ulcorner(1/m)F - \delta B\urcorner = \ulcorner(1/m)F\urcorner$ for a rational number $0 < \delta \ll 1$. Therefore, (X, Δ) is admissible by $g_*((1/m)F - \delta B) \le -\Delta$. □

1.6. Proposition *Let (X, Δ) be an admissible pair. Then $(U, \Delta|_U)$ is strictly admissible for any relatively compact open subset $U \subset X$.*

Proof. For a relative compact open subset $U' \supset \overline{U}$, there is a positive integer m such that $(U', (1/m)\ulcorner m\Delta\urcorner|_{U'})$ is admissible. Thus, by the proof of **1.5**, $(U, (1/m)\ulcorner m\Delta\urcorner|_U)$ is strictly admissible. Therefore $(U, \Delta|_U)$ is strictly admissible. □

1.7. Lemma *Let (X, Δ) be a pair of normal variety and effective $\mathbb{R}$-divisor such that $K_X + \Delta$ is $\mathbb{R}$-Cartier. Then it is log-terminal if and only if it is admissible.*

PROOF. We may replace X by an open subset freely. Suppose first that (X, Δ) is log-terminal. Let $f : Y \to X$ be a bimeromorphic morphism from a non-singular variety. We may assume there is an f-exceptional divisor B such that $-B$ is f-ample and that the union of $\operatorname{Supp} B$, $f^{-1}(\operatorname{Supp} \Delta)$, and the f-exceptional locus is a normal crossing divisor. We set $R := K_Y - f^*(K_X + \Delta)$. Then $R - \delta B - K_Y$ is f-ample for $\delta > 0$. We can choose δ so that $\ulcorner R - \delta B \urcorner = \ulcorner R \urcorner$. Since $f_*(R - \delta B) \leq -\Delta$, (X, Δ) is admissible.

Next, suppose that (X, Δ) is admissible. Then $(X, (1/m_1)\ulcorner m_1 \Delta \urcorner)$ is admissible for some positive integer m_1. By **1.5**, there exist a bimeromorphic morphism $g : Z \to X$ from a non-singular variety, a divisor F of Z, and a positive integer m which satisfy the condition of **1.5** for $(X, (1/m_1)\ulcorner m_1 \Delta \urcorner)$. Then we have $R \geq (1/m)F$ for the $\mathbb{R}$-divisor $R = K_Z - g^*(K_X + \Delta)$, by **III.5.1**. Thus $\ulcorner R \urcorner$ is a g-exceptional effective divisor. Hence (X, Δ) is log-terminal. □

§1.b. Quasi log-terminal and strongly log-canonical singularities. Fujita introduced the following 'log terminal' in [**27**]:

1.8. Definition Let (X, Δ) be a log-canonical pair. It is called *log terminal* in Fujita's sense if, for any bimeromorphic morphism $f : Y \to X$ from a non-singular variety, for the $\mathbb{R}$-divisor $R := K_Y - f^*(K_X + \Delta)$, and for any prime f-exceptional divisor Γ with $\operatorname{mult}_\Gamma R = -1$, X is non-singular and Δ is a reduced normal crossing divisor at a general point of $f(\Gamma)$.

Remark If (X, Δ) is weakly log-terminal, then it is log terminal in Fujita's sense. Let $D \subset \mathbb{C}^3$ be a hypersurface defined by the equation: $z^2 = xy^2$, which is called a Whitney umbrella. Then $(\mathbb{C}^3, D)$ is not weakly log-terminal but log terminal in Fujita's sense.

1.9. Definition Let (X, Δ) be a pair of normal complex analytic variety and effective $\mathbb{R}$-divisor. The pair (X, Δ) is said to be *strongly log-canonical* if, locally on X, there exist a bimeromorphic morphism $f : Y \to X$ from a non-singular variety and $\mathbb{R}$-divisors R and G on Y satisfying the following conditions:

(1) $\operatorname{Supp} R \cup \operatorname{Supp} G$ is a normal crossing divisor;
(2) $R - K_Y$ is f-numerically trivial;
(3) $f_* R = -\Delta$;
(4) G is f-ample;
(5) $\operatorname{mult}_\Gamma R \geq -1$ for a prime component Γ of R;
(6) If a prime component Γ of R satisfies $\operatorname{mult}_\Gamma R = -1$, then $\operatorname{mult}_\Gamma G > 0$;
(7) A prime component Γ of G with $\operatorname{mult}_\Gamma G > 0$ is either a component of R or an f-exceptional divisor.

1.10. Lemma *Let (X, Δ) be a strongly log-canonical pair and let Δ' be an effective $\mathbb{R}$-divisor with $\Delta' \leq \Delta$ and $\operatorname{mult}_\Gamma \Delta' < \operatorname{mult}_\Gamma \Delta$ for any prime component Γ of Δ. Then (X, Δ') is admissible.*

Proof. Let $f\colon Y \to X$, R, and G be as in **1.9**. We can take a small positive number α such that $\ulcorner R + \alpha G \urcorner$ is an f-exceptional effective divisor. Since $R + \alpha G - K_Y$ is f-ample, (X, Δ_α) is admissible for $\Delta_\alpha := -f_*(R + \alpha G)$. If α is sufficiently small, then $\Delta_\alpha \geq \Delta'$. Hence (X, Δ') is admissible. □

1.11. Lemma *The pair (X, Δ) is strongly log-canonical if and only if (X, Δ) is log-canonical and X is admissible.*

Proof. Suppose that (X, Δ) is strongly log-canonical. By **1.10**, X has only rational singularities. Therefore, $K_X + \Delta$ is $\mathbb{R}$-Cartier and we can write $K_Y = f^*(K_X + \Delta) + R$. Hence (X, Δ) is log-canonical. Next suppose that (X, Δ) is log-canonical and X is admissible. There exist a bimeromorphic morphism $f\colon Y \to X$ from a non-singular variety and a $\mathbb{Q}$-divisor E' of Y such that

(1) the union of the f-exceptional locus, $f^{-1}(\operatorname{Supp} \Delta)$, and $\operatorname{Supp} E'$ is a normal crossing divisor,
(2) $E' - K_Y$ is f-ample,
(3) $\ulcorner E' \urcorner$ is an f-exceptional effective divisor.

For the $\mathbb{R}$-divisor $R = K_Y - f^*(K_X + \Delta)$, we have $\operatorname{mult}_\Gamma R \geq -1$ for any prime component Γ. Let G be the f-ample $\mathbb{R}$-divisor $E' - R$. Then $\operatorname{mult}_\Gamma G > 0$, if $\operatorname{mult}_\Gamma R = -1$. Therefore (X, Δ) is strongly log-canonical. □

1.12. Definition A pair (X, Δ) of normal variety and effective $\mathbb{R}$-divisor is called *quasi log-terminal* if (X, Δ) is log-canonical and (X, Δ') is admissible for any effective $\mathbb{R}$-divisor $\Delta' \leq \Delta$ with $\llcorner \Delta' \lrcorner = 0$.

If (X, Δ) is log terminal in Fujita's sense, then it is quasi log-terminal by [**27**, (1.8)]. If (X, Δ) is quasi log-terminal, then $(X, \langle \Delta \rangle)$ is admissible. In particular, $(U, \Delta|_U)$ is log-terminal for $U := X \setminus \operatorname{Supp} \llcorner \Delta \lrcorner$.

1.13. Lemma *Let (X, Δ) be a log-canonical pair such that $(U, \Delta|_U)$ is log-terminal for $U := X \setminus \operatorname{Supp}(\llcorner \Delta \lrcorner)$. Suppose that there is an effective $\mathbb{R}$-Cartier divisor D such that $\operatorname{Supp}(\llcorner \Delta \lrcorner) \subset \operatorname{Supp} D \subset \operatorname{Supp} \Delta$. Then (X, Δ) is quasi log-terminal.*

Proof. We have a bimeromorphic morphism $f\colon Y \to X$ from a non-singular variety such that the union of f-exceptional locus and $f^{-1}(\operatorname{Supp} \Delta)$ is a normal crossing divisor. Let R be the $\mathbb{R}$-divisor $K_Y - f^*(K_X + \Delta)$. If Γ is a prime divisor with $\operatorname{mult}_\Gamma R = -1$, then $f(\Gamma) \subset \operatorname{Supp}(\llcorner \Delta \lrcorner)$. Let $\Delta' \leq \Delta$ be an effective $\mathbb{R}$-divisor with $\llcorner \Delta' \lrcorner = 0$ and $\langle \Delta' \rangle \geq \langle \Delta \rangle$. Then, locally on X, there is a positive number α such that, for the $\mathbb{R}$-divisor $G := R + \alpha f^* D$, $\ulcorner G \urcorner$ is an effective f-exceptional divisor and $-f_* G \geq \Delta'$. We may assume that there is an f-exceptional effective divisor B such that $-B$ is f-ample. Then $G - \delta B - K_Y$ is f-ample and $\ulcorner G - \delta B \urcorner = \ulcorner G \urcorner$ for $0 < \delta \ll 1$. Thus (X, Δ') is admissible. □

1.14. Lemma *Let (X, Δ) be a log-canonical pair. Suppose that there is an effective $\mathbb{R}$-Cartier divisor D such that $\operatorname{Supp} D = \operatorname{Supp} \Delta$. Then the following two conditions are mutually equivalent:*

(1) $(X \setminus \mathrm{Supp}(\Delta), 0)$ *is log-terminal*;
(2) (X, Δ) *is strongly log-canonical.*

PROOF. (1) ⇒ (2): Let $f\colon Y \to X$ be a bimeromorphic morphism from a non-singular variety and let $R = K_Y - f^*(K_X + \Delta)$. Then, locally over X, $\ulcorner R + \delta f^* D \urcorner$ is an f-exceptional effective divisor and $R + \delta f^* D - K_Y$ is f-numerically trivial for a sufficiently small positive number δ. Therefore X is admissible.

(2) ⇒ (1) follows from **1.7**. □

1.15. Corollary *Let (X, Δ) be a log-canonical pair such that every prime component of Δ is $\mathbb{Q}$-Cartier.*

(1) *(X, Δ) is quasi log-terminal if and only if $(U, \Delta|_U)$ is log-terminal for $U = X \setminus \mathrm{Supp}(\llcorner \Delta \lrcorner)$.*
(2) *(X, Δ) is strongly log-canonical if and only if $(X \setminus \mathrm{Supp}\,\Delta, 0)$ is log-terminal.*

In particular, if X is $\mathbb{Q}$-factorial and if $(X \setminus \mathrm{Supp}\,\Delta, 0)$ is log-terminal for a log-canonical pair (X, Δ), then X has only admissible singularities.

1.16. Example We shall give three examples of pairs related to the properties: log terminal in Fujita's sense, quasi log-terminal, and strongly log-canonical. (1) is an example of strongly log-canonical singularities which is not quasi log-terminal. (2) and (3) are examples of quasi log-terminal singularities which are not log terminal in Fujita's sense.

(1) Let X be a non-singular surface and let L_i $(i = 1, 2, 3)$ be smooth prime divisors intersecting transversely each other only at a point x. Then $(X, (2/3)(L_1 + L_2 + L_3))$ is strongly log-canonical.
(2) Let X be a non-singular surface and let L_1 and L_2 be smooth prime divisors intersecting only at a point x. Suppose that the local intersection number is 2. Then $(X, L_1 + (1/2)L_2)$ is quasi log-terminal.
(3) Let Y be a non-singular threefold and let $S = \sum_{i=1}^{4} S_i$ be a simple normal crossing divisor satisfying the following conditions:
    (a) $C := S_1 \cap S_2$ is a non-singular rational curve;
    (b) $S_3 \cap S_4 = \emptyset$;
    (c) $S_1 \cdot C = S_2 \cdot C = -1$ and $S_3 \cdot C = S_4 \cdot C = 1$.

    Let $f\colon Y \to X$ be the contraction of the curve C. Then $(X, f_* S)$ is quasi log-terminal.

§2. Minimal model program

We shall consider a kind of minimal model program for (X, Δ), where X is a projective variety. But, by using the same technique as in [**98**] (cf. Chapter **II**, §**5.d**), we can generalize to the relative case of complex analytic varieties.

2.1. Lemma *Let (X, Δ) be a pair of a normal projective variety and an effective $\mathbb{R}$-divisor. It is admissible if and only if there is an effective $\mathbb{Q}$-divisor $\Delta' \geq \Delta$ such that (X, Δ') is log-terminal.*

Proof. By the argument of **1.3**, we have only to show the existence of Δ' assuming that (X,Δ) is admissible. Since X is compact, (X,Δ) is strictly admissible by **1.6**. Thus there are a bimeromorphic morphism $f\colon Y \to X$ and a $\mathbb{Q}$-divisor E satisfying the conditions of **1.2**. Let H be an ample divisor of X. Then $mE - mK_Y + mlf^*H$ is very ample for some positive integers m, l. Let D be a general non-singular member of $|mE - mK_Y + mlf^*H|$ such that $\operatorname{Supp}\langle E\rangle \cup \operatorname{Supp} D$ is a normal crossing divisor. Then $E - (1/m)D - K_Y$ is f-numerically trivial and $\ulcorner E - (1/m)D \urcorner = \ulcorner E \urcorner$. Therefore (X,Δ') is log-terminal for $\Delta' = f_*((1/m)D - E)$. □

Let us fix a normal projective variety X and an effective $\mathbb{R}$-divisor Δ such that (X,Δ) has only strongly log-canonical singularities.

2.2. Lemma *Let D be a $\mathbb{Q}$-Cartier divisor such that $D - (K_X + \Delta)$ is ample. Then there is an effective $\mathbb{Q}$-divisor Δ_0 such that (X,Δ_0) is log-terminal and $D \sim_{\mathbb{Q}} K_X + \Delta_0$.*

Proof. Since X has only admissible singularities, there is an effective $\mathbb{Q}$-divisor Δ_1 such that (X,Δ_1) is log-terminal by **2.1**. Let $f\colon Y \to X$ be a birational morphism from a non-singular projective variety such that there is an effective $\mathbb{Q}$-divisor B with $-B$ being f-ample and that the union of the f-exceptional locus, $f^{-1}(\operatorname{Supp}\Delta)$, $f^{-1}(\operatorname{Supp}\Delta_1)$, and $\operatorname{Supp} B$ is a normal crossing divisor. Then

$$K_Y = f^*(K_X + \Delta) + R = f^*(K_X + \Delta_1) + R_1$$

for an $\mathbb{R}$-divisor R and a $\mathbb{Q}$-divisor R_1. Let $\Delta_\alpha := (1-\alpha)\Delta + \alpha\Delta_1$ for $0 < \alpha < 1$. Then

$$K_Y = f^*(K_X + \Delta_\alpha) + (1-\alpha)R + \alpha R_1.$$

Hence (X,Δ_α) is log-terminal for $0 < \alpha \ll 1$. Thus there are rational numbers $0 < \alpha \ll 1$ and $0 < \delta \ll 1$ such that $\ulcorner (1-\alpha)R + \alpha R_1 - \delta B \urcorner \geq 0$, $D - (K_X + \Delta_\alpha)$ is ample, and

$$f^*(D - (K_X + \Delta_\alpha)) - \delta B = f^*D + (1-\alpha)R + \alpha R_1 - \delta B - K_Y$$

is ample. We can take a sufficiently large positive integer m such that

$$f^*D + (1/m)\llcorner m(1-\alpha)R \lrcorner + \alpha R_1 - \delta B - K_Y \sim_{\mathbb{Q}} (1/m)C$$

for a non-singular divisor $C \subset Y$. Let us define a $\mathbb{Q}$-divisor

$$\Delta_0 := f_*(\delta B + (1/m)C - (1/m)\llcorner m(1-\alpha)R \lrcorner - \alpha R_1).$$

Then Δ_0 is effective and (X,Δ_0) is log-terminal for suitable choices of m and C. Here $D \sim_{\mathbb{Q}} K_X + \Delta_0$. □

2.3. Lemma *There is a sequence of effective $\mathbb{Q}$-divisors $\{\Delta_n\}_{n=1}^{\infty}$ such that every (X,Δ_n) is log-terminal and $\lim_{n\to\infty} c_1(K_X + \Delta_n) = c_1(K_X + \Delta)$.*

PROOF. Since $K_X + \Delta$ is $\mathbb{R}$-Cartier, there is a sequence of $\mathbb{Q}$-Cartier divisors $\{L_m\}_{m=1}^{\infty}$ such that $\lim_{m\to\infty} c_1(L_m) = c_1(K_X + \Delta)$. Let A be an ample divisor. Then, for any positive integer n, there is a positive integer m_n such that $L_m + (1/n)A - (K_X + \Delta)$ is ample for $m \geq m_n$. By **2.2**, there is an effective $\mathbb{Q}$-divisor Δ_n such that (X, Δ_n) is log-terminal and that $L_{m_n} + (1/n)A \sim_{\mathbb{Q}} K_X + \Delta_n$. Thus $\lim_{n\to\infty} c_1(K_X + \Delta_n) = c_1(K_X + \Delta)$. □

2.4. Corollary *Let D be an $\mathbb{R}$-Cartier divisor such that $D - (K_X + \Delta)$ is ample. Then there is an effective $\mathbb{Q}$-divisor Δ_0 such that (X, Δ_0) is log-terminal and $D - (K_X + \Delta_0)$ is ample.*

The following is the base-point free theorem in the strongly log-canonical case:

2.5. Proposition *If D is a nef Cartier divisor of X such that $aD - (K_X + \Delta)$ is ample for a positive integer a, then* $\mathrm{Bs}\,|mD| = \emptyset$ *for $m \gg 0$.*

PROOF. By **2.4**, we may assume that Δ is a $\mathbb{Q}$-divisor and (X, Δ) is log-terminal. The result is known in this case (cf. [**61**]). □

The following theorem is considered to be a generalization of usual base-point free theorem in the minimal model theory (cf. [**25**, (A5)], [**57**, Theorem 1]):

2.6. Theorem *Let D be a $\mathbb{Q}$-Cartier divisor of X. Suppose that $D-(K_X+\Delta)$ is ample and D admits a Zariski-decomposition $\mu^*D = P_\sigma(\mu^*D) + N_\sigma(\mu^*D)$ for a birational morphism $\mu\colon Y \to X$ from a non-singular projective variety, where $P := P_\sigma(\mu^*D)$ is nef. Then P is a semi-ample $\mathbb{Q}$-divisor. Moreover, if P' is a $\mathbb{Z}$-divisor numerically equivalent to qP for some $q > 0$, then* $\mathrm{Bs}\,|mP'| = \emptyset$ *for $m \gg 0$.*

PROOF. By **2.4**, we may assume that Δ is a $\mathbb{Q}$-divisor and (X, Δ) is log-terminal. By replacing Y by X, we may assume the following conditions are also satisfied for $P := P_\sigma(D)$ and $A := N_\sigma(D) - \Delta$:

(1) P is nef;
(2) $P + A - K_X$ is ample;
(3) $\mathrm{Supp}\langle A\rangle$ is a normal crossing divisor;
(4) $\ulcorner A \urcorner$ is an effective divisor;
(5) $P_\sigma(tP + \ulcorner A \urcorner) = tP$ for any $t \geq 1$.

Then, by [**57**, Theorem 3], we infer that $\mathrm{h}^0(X, \llcorner mP \lrcorner) = \mathrm{h}^0(X, \llcorner mD \lrcorner) \neq 0$ for some positive integer $m > 0$. Furthermore, $\mathrm{Bs}\,|\llcorner mP \lrcorner| \subset \mathrm{Bs}\,|mD|$ for $m > 0$ with mD being Cartier. Thus, by the argument in the proof of [**57**, Theorem 1], we infer that P is a semi-ample $\mathbb{Q}$-divisor. The remaining things are derived from [**25**, (A5)]. □

We have the following rationality theorem also by **2.4**:

2.7. Theorem *Let F be a face of the cone $\overline{\mathrm{NE}}(X)$ such that $(K_X + \Delta)\cdot z < 0$ for any $z \in F \setminus \{0\}$. Then there is a nef Cartier divisor D such that*

$$F = D^\perp \cap \overline{\mathrm{NE}}(X) = \{z \in \overline{\mathrm{NE}}(X) \mid D \cdot z = 0\}.$$

Therefore we also have the following cone theorem:

2.8. Theorem

$$\overline{\mathrm{NE}}(X) = \overline{\mathrm{NE}}(X)_{(K_X+\Delta)} + \sum \mathbf{R}_j,$$

where $\overline{\mathrm{NE}}(X)_{(K_X+\Delta)} = \{z \in \overline{\mathrm{NE}}(X) \mid (K_X + \Delta) \cdot z \geq 0\}$*,* $\mathbf{R}_j$ *is an extremal ray, and* $\sum \mathbf{R}_j$ *is locally polyhedral.*

Each extremal ray $\mathbf{R} \subset \overline{\mathrm{NE}}(X)$ defines a fiber space $\varphi_{\mathbf{R}} \colon X \to Z$ into a normal projective variety such that

(1) $\rho(X) = \rho(Z) + 1$,
(2) $-(K_X + \Delta)$ is $\varphi_{\mathbf{R}}$-ample,
(3) for an irreducible curve C of X, its numerical class $\mathrm{cl}(C)$ is contained in $\mathbf{R}$ if and only if $\varphi_{\mathbf{R}}(C)$ is a point.

The morphism $\varphi_{\mathbf{R}}$ is called the *contraction* morphism of $\mathbf{R}$.

Suppose that $\varphi_{\mathbf{R}} \colon X \to Z$ is not a birational morphism. Then $\dim Z < \dim X$ and Z has only rational singularities by **2.4** and **1.1**. Furthermore, by **3.3** below, Z has only admissible singularities.

2.9. Lemma *Let* $\varphi\colon X \to Z$ *be a birational morphism of normal projective varieties and let* Δ *be an effective* $\mathbb{R}$*-divisor of* X*.*

(1) *Suppose that* (X, Δ') *is admissible for an* $\mathbb{R}$*-divisor* $\Delta' \leq \Delta$*,* (X, Δ) *is log-canonical, and that* $-(K_X+\Delta)$ *is* φ*-ample. Then* $(Z, \varphi_*\Delta')$ *is admissible.*
(2) *Suppose that* φ *is an isomorphism in codimension one and* $(Z, \varphi_*\Delta)$ *is admissible. Then* (X, Δ) *is admissible.*

Proof. (1) Let $f\colon Y \to X$ be a birational morphism from a non-singular projective variety such that a $\mathbb{Q}$-divisor E of Y satisfies the condition of **1.2** for (X, Δ'). Let R be the $\mathbb{R}$-divisor $K_Y - f^*(K_X + \Delta)$. We may assume that $\operatorname{Supp} R \cup \operatorname{Supp} E$ is a normal crossing divisor. Then $(1-\varepsilon)R + \varepsilon E - K_Y$ is relatively ample over Z for $0 < \varepsilon \ll 1$. Thus $(Z, \varphi_*\Delta')$ is admissible, since $(1-\varepsilon)\Delta + \varepsilon\Delta' \geq \Delta'$.

(2) is trivial. □

Suppose that the contraction morphism $\varphi_{\mathbf{R}}\colon X \to Z$ of the extremal ray $\mathbf{R}$ is birational and there is an exceptional divisor. If X is $\mathbb{Q}$-factorial, then the exceptional locus is a prime divisor and $(Z, \varphi_{\mathbf{R}*}\Delta)$ has only strongly log-canonical singularities by **2.9**-(1). Similarly, if (X, Δ) is quasi log-terminal and if X is $\mathbb{Q}$-factorial, then so is $(Z, \varphi_{\mathbf{R}*}\Delta)$.

Next suppose that $\varphi_{\mathbf{R}}\colon X \to Z$ is isomorphic in codimension one. Then $(Z, \varphi_{\mathbf{R}*}\Delta')$ is admissible for any $0 \leq \Delta' \leq \Delta$ with (X, Δ') being admissible, by **2.9**-(1). The existence of the flip for $\varphi_{\mathbf{R}}$ is unknown. However, the existence for any log-terminal pair (X, Δ) with Δ being $\mathbb{Q}$-divisor implies that for any strongly log-canonical pair. Suppose that $X^+ \to Z$ is the flip and Δ^+ is the proper transform of Δ. Then, by **2.9**-(2), (X^+, Δ^+) has only strongly log-canonical singularities. Similarly, if (X, Δ) is quasi log-terminal, then so is (X^+, Δ^+).

Thus we expect to consider the minimal model program/problem starting from (X, Δ) with only strongly log-canonical singularities where X is $\mathbb{Q}$-factorial.

§3. ω-sheaves and log-terminal singularities

Here, we shall treat general normal complex analytic varieties. The following lemma is proved by the same argument as in **1.1**. But this result is weaker than **3.2** below.

3.1. Lemma *If there is a non-zero locally free ω-sheaf on a normal variety Y, then Y has only rational singularities.*

Proof. Let $f\colon X \to Y$ be a proper surjective morphism from a Kähler manifold such that a direct summand $\mathcal{F}$ of $\mathrm{R}^j f_*\omega_X$ is locally free for some j. We may assume that there is a factorization $f\colon X \to Z \to Y$ such that

(1) Z is a non-singular variety,
(2) $\pi\colon X \to Z$ is smooth outside a normal crossing divisor of Z,
(3) $\mu\colon Z \to Y$ is a bimeromorphic morphism.

Then we have an injection $\mu^*\mathcal{F} \hookrightarrow \mathrm{R}^j \pi_*\omega_X$. By taking the direct images by μ_*, we have the following morphism in the derived category $D_c^+(\mathcal{O}_Y)$ by **V.3.7**:

$$\mathrm{R}\,\mu_*(\mu^*\mathcal{F}) \to \mathrm{R}\,\mu_*(R^j\pi_*\omega_X) \sim_{\mathrm{qis}} \mathrm{R}^j f_*\omega_X \to \mathcal{F}.$$

Hence there is a complex $\mathcal{G}^\bullet$ such that

$$\mathrm{R}\,\mu_*(\mu^*\mathcal{F}) \sim_{\mathrm{qis}} \mathcal{F} \oplus \mathcal{G}^\bullet.$$

By duality (cf. [**37**], [**117**]), we have

$$\mathrm{R}\,\mathcal{H}om(\mathrm{R}\,\mu_*(\mu^*\mathcal{F}), \omega_Y^\bullet) \sim_{\mathrm{qis}} \mathrm{R}\,\mu_*\,\mathrm{R}\,\mathcal{H}om(\mu^*\mathcal{F}, \omega_Z^\bullet) \sim_{\mathrm{qis}} \mathcal{F}^\vee \otimes \mu_*\omega_Z[\dim Y],$$

where $\omega_Y^\bullet$ is the dualizing complex. Hence

$$\mathrm{R}\,\mathcal{H}om(\mathcal{F}, \omega_Y^\bullet) \sim_{\mathrm{qis}} \mathcal{F}^\vee \otimes \omega_Y[\dim Y]$$

and there is a surjective homomorphism

$$\mathcal{F}^\vee \otimes \mu_*\omega_Z \twoheadrightarrow \mathcal{F}^\vee \otimes \omega_Y.$$

Therefore Y has only rational singularities. □

Let X be a normal variety with only admissible singularities. Then, for any relatively compact open subset $X' \subset X$, there are a bimeromorphic morphism $f\colon Y \to X'$ from a non-singular variety and a $\mathbb{Q}$-divisor E of Y such that

(1) $\mathrm{Supp}\langle E\rangle$ is a normal crossing divisor,
(2) $\ulcorner E \urcorner$ is an f-exceptional effective divisor, and
(3) $E - K_Y$ is f-ample.

Then $\mathcal{O}_Y(\ulcorner E \urcorner)$ is an ω-sheaf by **V.3.10**. Thus $\mathcal{O}_{X'}$ is an ω-sheaf. Conversely, the same argument as **V.3.32** proves following:

3.2. Proposition *Let Z be a normal variety such that $\mathcal{O}_Z$ is an ω-sheaf. Then there exist a bimeromorphic morphism $\varphi\colon M \to Z$ from a non-singular variety M and a φ-nef $\mathbb{Q}$-divisor D of M such that* $\mathrm{Supp}\langle D\rangle$ *is a normal crossing divisor and*

$$\mathcal{O}_Z \simeq \varphi_*\omega_M(\ulcorner D \urcorner).$$

In particular, Z has only admissible singularities.

Therefore, a normal variety X has only admissible singularities if and only if $\mathcal{O}_X$ is an ω-sheaf locally on X.

3.3. Corollary *Let $f\colon X \to Y$ be a projective surjective morphism of normal varieties. Suppose that (X, Δ) is log-terminal and there is an effective $\mathbb{Q}$-Cartier $\mathbb{Z}$-divisor E satisfying the following conditions*:

(1) *$E - (K_X + \Delta)$ is f-nef and f-abundant.*
(2) *the canonical homomorphism $f_*\mathcal{O}_X \to f_*\mathcal{O}_X(E)$ is an isomorphism.*

Then Y has only admissible singularities.

Proof. We may assume that Y is Stein and we may replace Y by a relatively compact open subset. By **V.3.12**, we infer that $\mathcal{O}_X(E)$ is an ω-sheaf. Since $\mathcal{O}_Y$ is a direct summand of $f_*\mathcal{O}_X$, the conclusion is derived from **3.2**. $\square$