Introduction.

Purpose of this book.

This book stems from lectures given at the Universities of Tōhoku and Hokkaidō.

Its main purpose is to introduce the reader to p-adic analytic geometry and to the theory of p-adic analytic functions and differential equations, by focusing on the theme of *period mappings.* Of course, this approach is not meant to replace more systematic expositions of p-adic analysis or geometry found in a number of good treatises, but to complement them.

It is our general rule, in this book, to follow as closely as possible the complex theory, and to go back and forth between the complex and the p-adic worlds. We hope that this will make the text of interest both to some complex geometers and to some arithmetic geometers.

In the course of Chapter I, this approach will eventually become a kaleidoscope of half-correspondences and broken echoes. We hope that the reader will then have gained enough hindsight and wariness about these analogies, and will enjoy seeing unity being restored at a deeper level in Chapters II and III.

We have chosen the theme of period mappings because of its central role in the nineteen-century mathematics as a fertile place of interaction between differential equations, group theory, algebraic and differential geometry, topology (and even number theory). In fact, it was a guiding thread in the early harmonious development of these branches of mathematics, from Gauss and Riemann to Klein and Poincaré (*cf.* [**Gray86**]).

The origins lie in Gauss' largely unpublished work on elliptic functions on one hand (rediscovered and extended by Abel and by Jacobi), and on the hypergeometric differential equation in the complex domain on the other hand. Gauss knew the connection between the two topics: the inverse of the indefinite integral $\int \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}$ is a single-valued function with two independent periods ω_1 and ω_2 which are solutions of the hypergeometric differential equation with parameters $(\frac{1}{2}, \frac{1}{2}; 1)$ in the variable k^2.

Riemann's point of view of the 'Riemann surface' of a multivalued complex function has given a geometric framework for all of complex analysis. He applied this idea with equal success to Jacobi's inversion problem for more general indefinite algebraic integrals on one hand; and to the elucidation of the paradoxical polymorphism of hypergeometric functions which had puzzled Gauss and Kummer by introducing the concept of monodromy, on the other hand. He studied in detail the monodromy of the multivalued map $k^2 \mapsto \tau = \omega_2/\omega_1$, the first and basic example of a 'period mapping' , whose inverse is single-valued. He also showed (and Schwarz rediscovered) that under certain conditions, the quotient of two solutions of more general hypergeometric equations maps the upper half-plane onto a curvilinear triangle.

Group-theoretic aspects of monodromy were studied by Jordan, and the geometrization of complex analysis around the concept of automorphic function was carried on by Klein, who once described his work as 'blending Galois with Riemann'(1). The achievement fell to Poincaré. Starting from Fuchs' problem of finding all second-order differential equations for which a quotient of solutions τ admits a single-valued inverse, Poincaré founded the theory of 'uniformizing differential equations' (in modern terminology), and eventually recognized that every Riemann surface of genus > 1 should arise from the action of a discrete group of non-euclidean moves on the upper half-plane.

What about non-archimedean analogues of this saga?

It is clear that the development of p-adic analysis and geometry took a strikingly different route.

The very beginnings looked similar, indeed: J. Tate introduced rigid geometry as a proper framework for the uniformization $\mathbb{C}_p^\times/q^{\mathbb{Z}}$ of elliptic curves with multiplicative reduction; B. Dwork developed p-adic analysis starting from the hypergeometric differential equation with parameters $(\frac{1}{2}, \frac{1}{2}; 1)$, where he discovered the essential notion of 'Frobenius structure'. But it was immediately clear that since solutions of p-adic differential equations do not converge up to the next singularity, no faithful counterpart of complex monodromy could take place.

From there on, different p-adic theories grew apart, with their own languages, most of them relying on sophisticated parts of contemporary algebraic geometry, and all claiming some analogy with "the complex case":

- the theory of differential equations matured slowly (with a strong orientation toward applications to exponential sums), struggling with the problems of singularities, without being able to tackle global problems,
- crystalline theory offered a global viewpoint on differential equations (oriented toward the cohomology of varieties in positive characteristic), but did not help to understand singularities,
- the theory of p-adic representations and p-adic Hodge theory developed independently of differential equations, as did the several avatars of rigid geometry, motivated by idiosyncratic problems.

However, it is the author's opinion that the situation has somewhat changed over the last years, that isolated branches are merging by fits and

(1)quoted from [**Gray86**, p. 179].

starts[2], and that a synthesis is gradually emerging. This encourages to hope that, after many twists and turns, period mappings can indeed become a unifying theme in the p-adic context. This book is intended to be a contribution in this direction, by bringing closer the p-adic theory to its complex precursor — from periods and monodromy up to triangle groups. Much remains to be done in order to achieve comparable harmony and clarity.

Contents of this book.

Chapter I is preparatory. It deals with the problems of analytic continuation — with emphasis on the case of solutions of differential equations — and periods of abelian integrals, in the p-adic context.

Multivalued complex-analytic functions can be handled in two different, but essentially equivalent, ways:

1) in a geometric way using Riemann surfaces, coverings and paths;

2) as limits of algebraic functions; this less orthodox way leads to a more Galois-theoretic viewpoint on analytic continuation.

Both ways are practicable in the p-adic context, but eventually lead to completely different theories of analytic continuation.

Following the first way demands to have at disposal p-adic spaces which are locally arcwise connected. Surprisingly enough in view of the fractal nature of p-adic numbers, such a nice p-adic geometry does exist: it has been built by V. Berkovich (from the categorical viewpoint, it is essentially equivalent to rigid geometry, in the sense of Tate and Raynaud). In this framework, the monodromy of differential equations can be analyzed in the usual way; but a new phenomenon occurs: it is no longer true that there exists a basis of solutions around every ordinary point. This approach is therefore limited to a rather special class of connections. We shall see in the sequel how the theory of p-adic period mappings provides interesting examples in this class.

Following way 2), one encounters Dwork's notion of Frobenius structure, which has often been considered as a plausible substitute for monodromy in the p-adic context. We discuss so-called unit-root crystals and overconvergence, and illustrate these notions by Dwork's treatment of the p-adic Gamma function Γ_p, the Gross-Koblitz formula, and by a detailed study of the p-adic hypergeometric function $F(\frac{1}{2}, \frac{1}{2}, 1; z)$.

Abelian periods show themselves in two different ways:

1) as integrals of algebraic differentials over loops,

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[2] let us mention notably the incursion of rigid geometry into the crystalline viewpoint and into the geometric theory of finite coverings, the maturity of index theory and its applications to algebraic geometry in positive characteristic, the new connections between p-adic representations and differential equations on annuli...

2) when the integrand depends on a parameter z, as solutions of certain linear differential equations in z (Picard-Fuchs, or Gauss-Manin).

Both ways are passable in the p-adic context, and again lead to completely different theories.

Following up the first way — integrals being interpreted as limits of Riemann sums — leads to P. Colmez' construction of abelian p-adic periods (in the sense of Fontaine-Messing). These periods relate the Tate module (p-primary torsion points) to the first De Rham cohomology module of a given abelian variety over a p-adic field. They do not live in $\mathbb{C}_p$, but in a certain p-adic ring $\mathrm{B}_{\mathrm{dR}}^+$ (of which $\mathbb{C}_p$ is a quotient), and it is not possible in general to express them function-theoretically, when the abelian variety moves in a family. Nevertheless, we stress some particular cases where this can actually be done, throwing a bridge between ways 1) and 2).

The second way — Dwork's viewpoint of periods as solutions of Gauss-Manin connections — will prevail at the modular level, when p-divisible groups and filtered De Rham modules attached to families of abelian varieties will be compared not directly, but via their moduli spaces by the period mapping.

This way of looking at periods lacks arithmetic structure: namely, a rational, or at least a $\overline{\mathbb{Q}}$-structure, on the $\mathbb{C}_p$-space of solutions, to convey some arithmetical meaning to the periods. We discuss some cases where such a canonical structure exists ('p-adic Betti lattices'), notably the case of abelian varieties with supersingular reduction. We get here well-defined p-adic periods (completely different from the Fontaine-Messing periods), and compute them in terms of Γ_p-values in the case of elliptic curves with complex multiplication (p-adic analogue of the Lerch-Chowla-Selberg formula).

Chapter II is an introduction to the geometric theory of p-adic period mappings, in the sense of Drinfeld-Rapoport-Zink.

We have tried to keep prerequisites at a minimum, and to emphasize as much as possible the analogies between the complex and p-adic contexts. Basic definitions about p-divisible groups and crystals are recalled, and the proof of some basic results is sketched.

The theory of period mappings attaches to a family of algebraic varieties its periods, viewed abstractly as a moving point in a suitable grassmannian. Due to constraints of Riemann-type, the period mapping actually takes its values in an open subset of the grassmannian, the period domain, which is a symmetric domain. It is multivalued, the ambiguity being described by the projective monodromy of the Gauss-Manin connection.

The younger p-adic theory is far less advanced: at present, there is a theory of p-adic period mappings only for p-divisible groups or closely related geometric objects. For want of wide range, it has nevertheless gained richness and depth.

In the presentation of Drinfeld-Rapoport-Zink, one starts by constructing a moduli space $\mathcal{M}$ for p-divisible groups which are (quasi-)isogenous to a given one in characteristic p. The *p-adic period mapping* $\mathcal{P}$ then relates this moduli space to a suitable grassmannian which parametrizes the Hodge filtration in the Dieudonné module. There are again constraints of Riemann-type, which force the period mapping to take its values in an open subset of the grassmannian, the period domain, which is a 'symmetric domain'. The best known example of such a period domain is the Drinfeld space $\mathbb{C}_p \setminus \mathbb{Q}_p$, a p-adic analogue of the double-half-plane $\mathbb{C} \setminus \mathbb{R}$.

In the situation where the p-divisible groups are algebraizable, *i.e.* come from p-primary torsion of abelian varieties parametrized by a certain Shimura variety $\mathcal{S}h$, the moduli space $\mathcal{M}$ provides a uniformization of some tubular region in the Shimura variety $\mathcal{S}h$.

We show that, just as in the complex case, $\mathcal{P}$ can be described in terms of quotients of solutions of the associated Gauss-Manin connection (this feature does not seem to appear in the literature, except in the old special case of Dwork's period mapping for elliptic curves with ordinary reduction). This allows to give explicit formulas 'à la Dwork' for $\mathcal{P}$ in many cases.

The most interesting cases, investigated by Drinfeld-Rapoport-Zink, arise when $\mathcal{S}h$ itself (more accurately: a whole connected component) is uniformized by $\mathcal{M}$: $\mathcal{S}h$ is then a quotient of $\mathcal{M}$ by an arithmetic discrete group Γ. We show that, up to replacing $\mathcal{S}h$ by a finite covering (to kill torsion in Γ), the solutions of the Gauss-Manin connection extend to global multivalued functions: in other words, the Gauss-Manin connection has global monodromy in the sense of chapter I, and the projective monodromy group coincides with Γ.

We review the case of Shimura curves attached to quaternion algebras over totally real fields (Čerednik-Drinfeld-Boutot-Zink): there is a global p-adic uniformization when p divides the discriminant of the quaternion algebra. We then construct, using the p-adic Betti lattices of chapter I, a canonical $\overline{\mathbb{Q}}$-space of solutions of the Gauss-Manin connection which is stable under global monodromy.

Chapter III explores the group-theoretic aspects of the theory of period mappings, in the p-adic context.

The modern presentation of ramified coverings, uniformization, and of the Gauss-Riemann-Fuchs-Schwarz theory uses the notions of orbifold and uniformizing differential equation. We develop p-adic counterparts of these notions.

The right notion of ramified covering to adopt is not obvious, because étale coverings are only exceptionally topological coverings, in the p-adic context. Berkovich's geometry provides a very convenient framework for this kind of problems, since his spaces are locally arcwise connected. In the first section, we discuss the formalism of fundamental groups attached to categories of (possibly infinite) étale coverings satisfying simple axioms. Although the definition of these topological groups is simple and natural, their topology itself may be very complicated (not locally compact in general).

It turns out that there are too few topological coverings and too many étale coverings, in general, to provide a workable theory of monodromy. To remedy this, we introduce in the second section the notion of *tempered étale covering*. Such coverings are essentially built from (possibly infinite) topological coverings of finite étale coverings. They are classified by tempered fundamental groups, which seem to be the right p-adic equivalents, in many ways, of fundamental groups of complex manifolds. These groups are not discrete in general, but nevertheless often possess many infinite discrete quotients. We present a large selection of examples.

Tempered étale coverings are well-suited to the definition of p-adic orbifold charts and of the category of *p-adic orbifolds* (they replace the unramified coverings, over $\mathbb{C}$). As in the complex case, orbifolds and orbifold fundamental groups are the tools for a theory of ramified coverings, developed in section 4.

Before turning to the p-adic analogue of the *uniformizing differential equation* attached to an orbifold of dimension one (via Schwarzian derivatives), we discuss local and global monodromy of p-adic differential equations in section 3.

We outline the Christol-Mebkhout theory of p-adic slopes of differential equations over annuli, and the relation with Galois representations of local fields of characteristic p.

We then define and study the *p-adic étale Riemann-Hilbert functor*, which attaches a vector bundle with integrable connection to any discrete representation of the étale fundamental group of a p-adic manifold; connections in the image are characterized by the fact that the étale sheaf of germs of solutions is locally constant. This is a vast generalization of the phenomena of global monodromy studied in Chapter I; for instance, the differential equation $y' = y$ over the affine line belong to this class.

Uniformizing differential equations of orbifolds of dimension one also belong to this class, and the representation actually factors through the tempered fundamental group. The case of a Shimura orbifold is of special interest: the period mapping (complex or p-adic) is given by a quotient of two solutions of a fuchsian differential equation defined over a number field, which can be interpreted as uniformizing differential equation either over $\mathbb{C}$ or over $\mathbb{C}_p$.

In Section 5, we examine the case of Schwarz orbifolds: the projective line with $0, 1, \infty$ as branched points (endowed with suitable finite multiplicities). Over $\mathbb{C}$, the uniformizing differential equations are of hypergeometric type, with projective monodromy group identified with the orbifold fundamental group, namely with a (cocompact) triangle group.

In the p-adic case, we define *p-adic triangle groups* to be projective monodromy groups of those hypergeometric differential equations which are in the image of the p-adic étale Riemann-Hilbert functor: thus by definition, there exists a finite étale covering of $\mathbb{C}_p \setminus \{0,1\}$ over which the hypergeometric function extends to a global multivalued analytic function.

We give a purely geometric description of these discrete subgroups of $PSL_2(\mathbb{C}_p)$ (without reference to differential equations; it is perhaps here that we are closest to Fuchs and Schwarz). From this, and recent combinatorial work by F. Kato, it follows that infinite p-adic triangle groups exist only for $p \leq 5$.

We then construct the p-adic analogues of Takeuchi's list of arithmetic triangle groups, *i.e.* the list of all "arithmetic" p-adic triangle groups for every p, using the Čerednik-Drinfeld-Boutot-Zink uniformization of Shimura curves and p-adic period mapping. Special values of the corresponding hypergeometric functions at CM points are expressed in terms of Γ_p-values.

On the style.

The first chapter is rather down-to-earth. It has kept something of the informal style of lecture notes, and this also holds to a lesser extent for the second chapter; the level is inhomogeneous, the pace may sometimes be brisk, proofs are often omitted and replaced by references. In contrast, the last and longest chapter is devoted to a systematic exposition of new material.

We hope that our constant function-theoretic viewpoint brings some unity to the exposition, throughout the chapters. We have tried to keep them (and even the sections) as logically independent as possible. Thus, for example:

only subsections 1.5 and 5.3 of chapter I are needed in the sequel;

until section 7, chapter II is almost self-contained;

until subsection 4.7, chapter III is almost self-contained;

reading fragments of III should be enough to grasp the ins and outs of p-adic triangle groups.

Sections I.3 and III.6 are small pieces of 'computational mathematics' (without computer) intended as testing ground for notions developed in the preceding sections.

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first of all, V. Berkovich's views on p-adic geometry: without this extremely convenient way of seeing p-adic spaces, our systematic to-ings and fro-ings between complex and p-adic worlds would have been impossible;

Rapoport-Zink's book on period spaces for p-divisible groups has been a constant reference for Chapter II;

in Chapter III, our treatment has been much influenced by the work of M. van der Put on p-adic discontinuous groups, by J. De Jong's study

of étale coverings, and no less by M. Yoshida's orbifold viewpoint on the (complex) Gauss-Schwarz theory.

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