CHAPTER 11

Appendix

11.1. Existence of complements

PROPOSITION 11.1.1 ([**Sh3**]). *Let $f\colon X \to Z \ni o$ be a contraction from a surface and D a boundary on X such that $K_X + D$ is lc and $-(K_X + D)$ is f-nef and f-big. Then*

(i) *the linear system $|-m(K_X + D)|$ is base point free for some $m \in \mathbb{N}$;*
(ii) *$K_X + D$ is n-complementary near $f^{-1}(o)$ for some $n \in \mathbb{N}$;*
(iii) *the Mori cone $\overline{NE}(X/Z)$ is polyhedral and generated by irreducible curves.*

We hope that this fact has higher dimensional generalizations (cf. [**K3**], see also M. Reid's Appendix to [**Sh2**]).

PROOF. First we prove (i). We consider only the case of compact X. In the case $\dim Z \geq 1$ there are stronger results (see Theorem 6.0.6). Applying a log terminal modification 3.1.1, we may assume that $K_X + D$ is dlt (and X is smooth). Set $C := \lfloor D \rfloor$, $B := \{D\}$. Note that C is connected by Connectedness Lemma. Take sufficiently large and divisible $n \in \mathbb{N}$ and consider the exact sequence

$$0 \longrightarrow \mathcal{O}_X(-n(K_X + D) - C) \longrightarrow \mathcal{O}_X(-n(K_X + D)) \longrightarrow \mathcal{O}_C(-n(K_X + D)) \longrightarrow 0.$$

By Kawamata-Viehweg Vanishing [**KMM**, 1-2-6],

$$H^1(X, \mathcal{O}_X(-n(K_X + D) - C)) = H^1(X, \mathcal{O}_X(K_X + B - (n+1)(K_X + D))) = 0.$$

Therefore $C \cap \operatorname{Bs}|-n(K_X + D)| = \operatorname{Bs}\bigl|-n(K_X + D)|_C\bigr|$.

We claim that $\operatorname{Bs}\bigl|-n(K_X + D)|_C\bigr| = \varnothing$. Indeed, if C is not a tree of rational curves, then $p_a(C) = 1$ and C is either a smooth elliptic curve or a wheel of smooth rational curves (see Lemma 6.1.7). Moreover, $\operatorname{Supp}B \cap C = \varnothing$. But then $(K_X + D)|_C = (K_X + C)|_C = K_C = 0$ and $\operatorname{Bs}\bigl|-n(K_X + D)|_C\bigr| = \varnothing$ in this case. Note also that here we have an 1-complement by Lemma 8.3.8. Assume now that C is a tree of smooth rational curves. Then $\bigl|-n(K_X + D)|_{C_i}\bigr|$ is base point free on each component $C_i \subset C$ whenever $-n(K_X + D)$ is Cartier. Hence so is $\bigl|-n(K_X + D)|_C\bigr|$. This proves our claim.

Thus we have shown that $C \cap \operatorname{Bs}|-n(K_X + D)| = \varnothing$. Let $L \in |-n(K_X + D)|$ be a general member. Then $K_X + D + \frac{1}{n}L$ is dlt near C (see 1.3.2). By Connectedness

Lemma, $K_X + D + \frac{1}{n}L$ is lc everywhere. Hence $K_X + D + \frac{1}{n}L$ is a $\mathbb{Q}$-complement of $K_X + D$. The fact that $|-n(K_X + D)|$ is free outside of C can be proved in a usual way (see e.g., [**R**], [**K3**]). We omit it.

(ii) is obvious. Let us prove (iii). Clearly, we may assume that $\rho(X) \geq 2$. It follows by 11.2.2 that any $(K_X + D)$-negative extremal ray R is generated by an irreducible curve C. By Proposition 11.2.5, $-(K_X + D) \cdot C \leq 2$. Let $\varphi\colon X \to Y \subset \mathbb{P}^N$ be the contraction given by the linear system $-m(K_X + D)$ for sufficiently big and divisible $m \in \mathbb{N}$. Then $\deg \varphi(C) \leq 2$. This implies that C belongs to a finite number of algebraic families. Thus the cone $\overline{NE}(X)$ is polyhedral outside of $\overline{NE}(X) \cap \{z \mid (K_X + D) \cdot z = 0\}$. Now consider the extremal ray R such that $(K_X + D) \cdot R = 0$. By the Hodge Index Theorem, $R^2 < 0$. Thus, by Proposition 11.2.1 R is generated by an irreducible curve, say C. Since $(K_X + D) \cdot C = 0$, we have that φ contracts this curve to a point. Therefore there is a finite number of such curves, so $\overline{NE}(X)$ is polyhedral everywhere. □

11.2. Minimal Model Program in dimension two

The log Minimal Model Program in dimension two is much easier than in higher dimensions. Following [**A**] and [**KK**] (see also [**Sh4**]) we present two main theorems 11.2.2 and 11.2.3 of MMP in the surface case. First we note that in the surface case it is possible to define the *numerical pull back* of any $\mathbb{Q}$-Weil divisor under birational contractions (see e.g., [**S1**]). Therefore all definitions of 1.1 can be given for arbitrary normal surface (we need not the $\mathbb{Q}$-Cartier assumption). It turns out a posteriory that any numerically lc pair (X, B) satisfies the property that $K_X + B$ is $\mathbb{Q}$-Cartier [**KM**, Sect. 4.1], [**Ma**]. Similarly, the dlt property of (X, D) implies that the surface X is $\mathbb{Q}$-factorial [**KM**, Sect. 4.1]. For surfaces there is an alternative way to define the numerical equivalence: two 1-cycles $\Upsilon_1, \Upsilon_2 \in Z_1(X/Z)$ are said to be numerically equivalent if $L \cdot \Upsilon_1 = L \cdot \Upsilon_2$ for *all* Weil divisors L (not only for those, that are $\mathbb{Q}$-Cartier). This gives also an alternative way to define $N_1(X/Z)$, $\rho(X/Z)$, and $\overline{NE}(X/Z)$ and leads to a possibly larger dimensional space $N_1(X/Z)$. We use the standard definition of the numerical equivalence and $N_1(X/Z)$ [**KMM**].

The following properties are well known (see e.g., [**KM**, 1.21–1.22] or [**Ko3**, Ch. II, Lemma 4.12]).

PROPOSITION 11.2.1 (Properties of the Mori cone). *Let X be a normal projective surface.*

(i) *Let z be an element of $N_1(X)$ such that $z^2 > 0$ and $z \cdot H > 0$ for some ample divisor H. Then z is contained in the interior of $\overline{NE}(X)$.*
(ii) *Let $C \subset X$ be an irreducible curve. If $C^2 \leq 0$, then the class $[C]$ is in the boundary of $\overline{NE}(X)$. If $C^2 < 0$, then the ray $\mathbb{R}_+[C]$ is extremal.*
(iii) *Let $R \subset \overline{NE}(X)$ be an extremal ray such that $R^2 < 0$. Then R is generated by an irreducible curve.*

PROOF. We prove only (iii). Take a 1-cycle Z so that $[Z] \in R$, $[Z] \neq 0$ and Z_i a sequence of effective 1-cycles whose limit is Z. Write $Z_i = \sum_j a_{i,j} C_j$, where

C_j are distinct irreducible curves. Since $0 > Z^2 = \lim Z \cdot Z_i$, there is at least one curve $C = C_k$ such that $Z \cdot C < 0$. Write $Z_i = c_i C + \sum_{j \neq k} a_{i,j} C_j$, $c_i \geq 0$. Then

$$0 > C \cdot Z = \lim C \cdot Z_i \geq (\lim c_i) C^2.$$

Thus $C^2 < 0$ and $\lim c_i > 0$. Pick $0 < c < \lim c_i$. Then $Z_i - cC$ is effective for $i \gg 0$ and $Z = cC + \lim(Z_i - cC)$ Since R is an extremal ray, this implies that $[C] \in R$. □

THEOREM 11.2.2 (The Cone Theorem). *Let X be a normal projective surface and $K_X + B$ be an effective $\mathbb{R}$-Cartier divisor. Let A be an ample divisor on X. Then for any $\varepsilon > 0$ the Mori-Kleiman cone of effective curves $\overline{NE}(X)$ in $N_1(X)$ can be written as*

$$\overline{NE}(X) = \overline{NE}_{K+B+\varepsilon A}(X) + \sum R_k$$

where, as usual, the first part consists of cycles that have positive intersection with $K + B + \varepsilon A$ and R_k are finitely many extremal rays. Each of the extremal rays is generated by an effective curve.

THEOREM 11.2.3 (Contraction Theorem). *Let X be a projective surface with log canonical $K_X + B$. Let R be a $(K_X + B)$-negative extremal ray. Then there exists a nontrivial projective morphism $\phi\colon X \to Z$ such that $\phi_*(\mathcal{O}_X) = \mathcal{O}_Z$ and $\phi(C) = pt$ if and only if the class of C belongs to R. Moreover, if ϕ is birational and $K_X + B$ is lc (resp. klt) then $K_Z + \phi_* B$ is lc (resp. klt).*

REMARK 11.2.4. Notation as above.

(i) If $\dim Z = 1$, then all fibers of ϕ are irreducible smooth rational curves and X has only rational singularities [**KK**].
(ii) If $\dim Z = 2$, then $C \simeq \mathbb{P}^1$ and $K_Z + \phi_* B$ is plt at $\phi(C)$.

PROPOSITION 11.2.5 (Properties of extremal curves). *Let (X, B) be a normal projective log surface and R a $(K_X + B)$-negative extremal ray on X. Assume that $K_X + B$ is lc. If $R^2 \leq 0$, then for any irreducible curve C such that $[C] \in R$ we have $-(K_X + D) \cdot C \leq 2$. If $R^2 > 0$, then X is covered by a family of rational curves C_λ such that $-(K_X + D) \cdot C_\lambda \leq 3$.*

PROOF. Let $\mu\colon \tilde{X} \to X$ be the minimal resolution and $K_{\tilde{X}} + \tilde{B} = \mu^*(K_X + B)$ the crepant pull back.

Consider the case $R^2 \leq 0$. Let $\tilde{C}$ be the proper transform of C. Then

$$-(K_X + B) \cdot C = -(K_{\tilde{X}} + \tilde{B}) \cdot \tilde{C} \leq -(K_{\tilde{X}} + \tilde{C}) \cdot \tilde{C} \leq 2$$

because $\tilde{C}^2 \leq C^2 \leq 0$ and $\tilde{B}$ is a boundary.

Now we assume that $R^2 > 0$. Then $-(K_X + B)$ is ample (see 11.2.1). Thus (X, B) is a log del Pezzo surface. By Corollary 5.4.3, $\tilde{X}$ is birationally ruled. It is well known, that in this situation $\tilde{X}$ is covered by a family of rational curves $\tilde{C}_\lambda$ such that $-K_{\tilde{X}} \cdot \tilde{C}_\lambda \leq 3$. Take $C_\lambda = \mu(\tilde{C}_\lambda)$. Then

$$-(K_X + B) \cdot C_\lambda = -(K_{\tilde{X}} + \tilde{B}) \cdot \tilde{C}_\lambda \leq -K_{\tilde{X}} \cdot \tilde{C}_\lambda \leq 3.$$

□