CHAPTER 2

Inversion of adjunction

2.1. Two-dimensional toric singularities and log canonical singularities with a reduced boundary

2.1.1. If the cyclic group $\mathbb{Z}_m$ acts linearly on $\mathbb{C}^n$ by

$$x_1 \to \varepsilon^{a_1} x_1, \qquad x_2 \to \varepsilon^{a_2} x_2, \dots, x_n \to \varepsilon^{a_n} x_n,$$

where ε is a chosen primitive root of degree m of unity, we call the integers $a_1, \dots, a_n$ the *weights* of the action. In this case, the quotient is denoted by $\mathbb{C}^n/\mathbb{Z}_m(a_1, \dots, a_n)$. It is clear that the weights are defined modulo m and also depend on the choice of the primitive root ε.

Let (Z, Q) be a two-dimensional quotient singularity $\mathbb{C}^2/\mathbb{Z}_m(1, q)$, where $\gcd(q, m) = 1$ (in particular, this means that $\mathbb{Z}_m$ acts on $\mathbb{C}^2$ freely in codimension one). Then this singularity is toric, hence it is klt. The minimal resolution is obtained as a sequence of *weighted blowups* (see 3.2). The dual graph is a chain

$$\overset{-a_1}{\bigcirc} \text{ --- } \overset{-a_2}{\bigcirc} \text{ --- } \cdots \text{ --- } \overset{-a_{r-1}}{\bigcirc} \text{ --- } \overset{-a_r}{\bigcirc}, \qquad \text{with} \quad a_i \geq 2,$$

where the sequence $a_1, a_2, \dots, a_r$ is obtained from the continued fraction decomposition of m/q (see [**Hi**] or [**Br**]):

$$\frac{m}{q} = a_1 - \cfrac{1}{a_2 - \cfrac{1}{\cdots \cfrac{1}{a_r}}}. \tag{2.1}$$

Now we give the classification of two-dimensional log canonical singularities with nonempty reduced boundary, following Kawamata [**K**]. Note that this is much easier than the classification of all two-dimensional log canonical singularities.

THEOREM 2.1.2 ([**K**, 9.6], [**Ut**, ch. 3]). *Let $X \ni P$ be an analytic germ of a two-dimensional normal singularity and $X \supset C$ a (possibly reducible) reduced curve. Assume that $K_X + C$ is plt. Then*

$$(X, C) \simeq (\mathbb{C}^2, \{x = 0\})/\mathbb{Z}_m(1, a), \quad \textit{with} \quad \gcd(a, m) = 1.$$

In particular, C is irreducible and smooth. In this case, $K_X + C$ has index m and the graph of the minimal resolution of $(X \supset C \ni P)$ is of type

where the black vertex ● corresponds to the proper transform of C, and the white ones ○ correspond to exceptional divisors. The numbers attached to white vertices are self-intersection numbers.

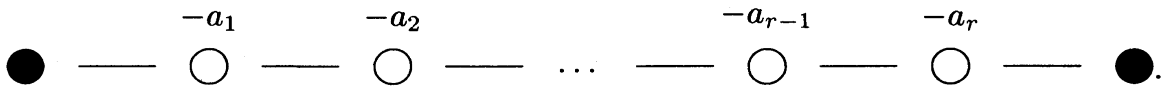
SKETCH OF PROOF. Let m be the index of C (i.e., $mC \sim 0$) and $\psi \colon X' \to X$ the corresponding cyclic m-cover. Then $C' := \psi^{-1}(C)_{\mathrm{red}}$ is a Cartier divisor and $K_{X'} + C'$ is plt. This gives that X' and C' are smooth and $X' \simeq \mathbb{C}^2$ up to analytic isomorphism. □

THEOREM 2.1.3 ([**K**, 9.6], [**Ut**, ch. 3]). *Let $X \ni P$ be an analytic germ of a two-dimensional normal singularity and $X \supset C$ a (possibly reducible) reduced curve. Assume that $K_X + C$ is lc but not plt. Then just one of the following two possibilities holds:*

(i) *C has two smooth components,*

$$(X, C) \simeq (\mathbb{C}^2, \{xy = 0\})/\mathbb{Z}_m(1, a), \quad \textit{with} \quad \gcd(a, m) = 1.$$

The index of $K_X + C$ is equal to 1 (i.e., $K_X + C \sim 0$). The graph of the minimal resolution of $(X \supset C \ni P)$ is of the form

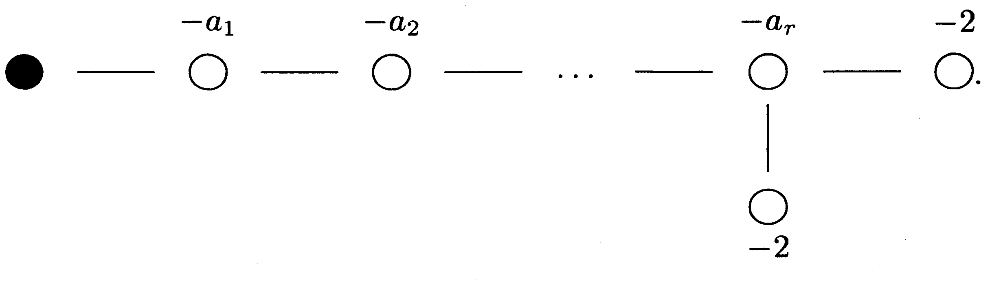


In this case, $K_X + C$ is not dlt for $m > 1$ and dlt for $m = 1$.

(ii) *The curve C is smooth and irreducible,*

$$(X, C) \simeq (\mathbb{C}^2, \{xy = 0\})/\mathbb{D}_m,$$

where $\mathbb{D}_m \subset \mathrm{GL}_2(\mathbb{C})$ is a subgroup of dihedral type without reflections (see [**Br**] *for precise description of $\mathbb{D}_m$). In this case, $K_X + C$ is not dlt and is of index two (i.e., $2(K_X + C) \sim 0$), the log canonical cover is the singularity from (i) and the graph of the minimal resolution of $(X \supset C \ni P)$ is of type*

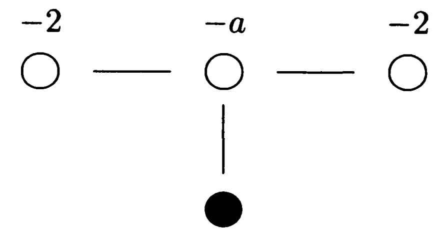


The degenerate case $r = 1$ is included here (then $\mathbb{D}_m$ is a cyclic group).

COROLLARY 2.1.4. *Let (X, D) be a log variety. Assume that $K_X + D$ is lc and $W \subset X$ an irreducible subvariety of codimension two. Assume that $W \subset \lfloor D \rfloor$. Then near a general point $w \in W$ there is an analytic isomorphism between $(X, \lfloor D \rfloor, W)$ and the product of a surface singularity from 2.1.2 or (i)-(ii) of 2.1.3 by $\mathbb{C}^{\dim X - 2}$.*

EXERCISE 2.1.5 (cf. 2.1.7). Assume that in the conditions of the theorem above C is a Cartier divisor. Show without using the theorem that then $(X, C) \simeq (\mathbb{C}^2, \mathbb{C}^1)$ or X is a Du Val point of type A_n, and C is its general hyperplane section.

EXERCISE 2.1.6. Express in the form $\mathbb{C}^2/\mathbb{Z}_m(1, q)$ the singularity with the minimal resolution



EXAMPLE 2.1.7. Let (Z, Q) be a Du Val singularity of type A_n given by the equation $x^2 + y^2 + z^{n+1}$ and C the hyperplane section given by $z = 0$. Then (Z, C) is a lc pair as in (i) of Theorem 2.1.3. Similarly, for the case (ii) of Theorem 2.1.3 we can take (Z, Q) of type D_n given by the equation $x^2 + y^2z + z^{n-1}$, $n \geq 4$ and C as $\{z = 0\}_{\text{red}}$.

2.2. Adjunction

EXAMPLE 2.2.1. Let $X = X_n \subset \mathbb{P}^{n+1}$ be a two-dimensional projective cone over a rational normal curve $C_n \subset \mathbb{P}^n$ and $L \subset X$ its generator. The group of classes of Weil divisors modulo linear equivalence is generated by the class of L: $\mathrm{Weil}_{\mathrm{lin}}(X) \simeq \mathbb{Z} \cdot L$ and nL is the class of the hyperplane section of X. Thus we have $L|_L = \frac{1}{n}P$, where P is the class of a point on $L \simeq \mathbb{P}^1$. It is also easy to compute that $K_X \sim -(n+2)L$. This yields

$$(K_X + L)|_L - K_L = -(n+1)L|_L + 2P = (1 - 1/n)P.$$

This is one instance where the *standard coefficients* (see 2.2.5) arise naturally.

This example shows that adjunction formula in its usual form fails for the case of Weil divisors. This phenomenon was first observed by M. Reid and is called also *subadjunction*. Shokurov [**Sh2**, §3] introduced the notion of different for the difference $(K_X + L)|_L - K_L$ (see also [**KMM**, 5-1-9], [**Ut**, ch. 16]). The corresponding ideal sheaf sometimes is called the *conductor* ideal.

The following construction is a codimension two construction, i.e. the variety X may always be replaced with any open subset $X \setminus Z$, where $\mathrm{codim}_X Z \geq 3$.

PROPOSITION-DEFINITION 2.2.2. Let X be a normal variety and $S \subset X$ a reduced subscheme of pure codimension one. For simplicity we assume that $K_X + S$

is lc in codimension two. Then by Theorem 2.1.2 and Theorem 2.1.3, S has only normal crossings in codimension one. In particular, the scheme S is Gorenstein in codimension one. Then there exists naturally defined an effective $\mathbb{Q}$-Weil divisor $\mathrm{Diff}_S(0)$, called the *different*, such that

$$(K_X + S)|_S = K_S + \mathrm{Diff}_S(0).$$

Now let B be a $\mathbb{Q}$-divisor, which is $\mathbb{Q}$-Cartier in codimension two. Then the different for $K_X + S + B$ is defined by the formula

$$(K_X + S + B)|_S = K_S + \mathrm{Diff}_S(B).$$

In particular, if B is a boundary and $K_X + S + B$ is lc in codimension two, then by 2.1.2 and 2.1.3, B is $\mathbb{Q}$-Cartier in codimension two. Moreover, none of the components of $\mathrm{Diff}_S(B)$ are contained in the singular locus of S.

EXAMPLE 2.2.3. Let $Q \subset \mathbb{P}^4$ be a quadratic cone over $xy = zt$ and $S \subset Q$ a plane. Then $S|_S = 0$ modulo codimension two subsets and $(K_Q + S)|_S = K_S$. Therefore $\mathrm{Diff}_S(0) = 0$. This shows that codimension three singularities are not essential for 2.2.2.

The following theorem allows us to compute coefficients of the different and shows that computations in Example 2.2.1 are very general.

THEOREM 2.2.4 ([**Sh2**, 3.9], [**Ut**, 16.6]). *In the conditions of Theorem 2.1.2 and Theorem 2.1.3 for the different* $\mathrm{Diff}_C(0)$ *at* P *we have*

(i) *If* $K_X + C$ *is plt, then* $\mathrm{Diff}_C(0) = (1 - 1/m)P$, *where* m *is the index of* $K_X + C$ *(see Theorem 2.1.2).*
(ii) *If* $(X \supset C \ni P)$ *is as in (i) of Theorem 2.1.3, then* $\mathrm{Diff}_C(0) = 0$.
(iii) *If* $(X \supset C \ni P)$ *is as in (ii) of Theorem 2.1.3, then* $\mathrm{Diff}_C(0) = P$.

2.2.5. Notation. Put

$$\Phi_{\mathbf{sm}} := \{1 - 1/m \mid m \in \mathbb{N} \cup \{\infty\}\}.$$

We distinguish this set because it naturally appears in the Adjunction Formula. Latter we will see that the class of boundaries with coefficients $\in \Phi_{\mathbf{sm}}$ is closed under finite Galois morphisms (see 1.2) and Adjunction Formula (Corollary 2.2.9, cf. [**Ko1**]). We say that the boundary $D = \sum d_i D_i$ has *standard coefficients*, if $d_i \in \Phi_{\mathbf{sm}}$ for all i. Unfortunately the property $\in \Phi_{\mathbf{sm}}$ is not closed under crepant birational transformations (see 1.1.6) to avoid this difficulty Shokurov considered the class of boundaries with coefficients from the set

$$\Phi_{\mathbf{m}} := \Phi_{\mathbf{sm}} \cup [6/7, 1].$$

The following result is very important for applications and is called *Inversion of Adjunction.*

THEOREM 2.2.6 ([**Sh2**, 3.3, 3.12, 5.13], [**Ut**, 17.6]). *Notation as in 2.2.2. Let* B *be an effective* $\mathbb{Q}$*-divisor on* X *such that* S *and* B *have no common components and* $\lfloor B \rfloor = 0$. *Assume that* $K_X + S + B$ *is* $\mathbb{Q}$*-Cartier. Then* $K_X + S + B$ *is plt near* S *if and only if* S *is normal and* $K_S + \mathrm{Diff}_S(B)$ *is klt.*

COROLLARY 2.2.7 ([**Ut**, 17.7]). *Let X be a normal variety, S an irreducible divisor and let B, B' be effective $\mathbb{Q}$-divisors such that S and $B + B'$ have no common components and $\lfloor B \rfloor = 0$. Assume that $K_X + S + B$ and B' are $\mathbb{Q}$-Cartier and $K_X + S + B$ is plt. Then $K_X + S + B + B'$ is lc near S if and only if so is $K_S + \operatorname{Diff}_S(B + B')$.*

COROLLARY 2.2.8 ([**Sh2**, 3.10]). *Let X be a normal variety, S a reduced Weil divisor on X and $B = \sum b_i B_i$ a boundary on X such that S and B have no common components. Assume that $K_X + S + B$ is plt. Then $\operatorname{Diff}_S(B)$ has the form*

$$\operatorname{Diff}_S(B) = \sum_{P_i} \left(\frac{m_i - 1}{m_i} + \sum_j \frac{b_j n_{i,j}}{m_i} \right) P_i,$$

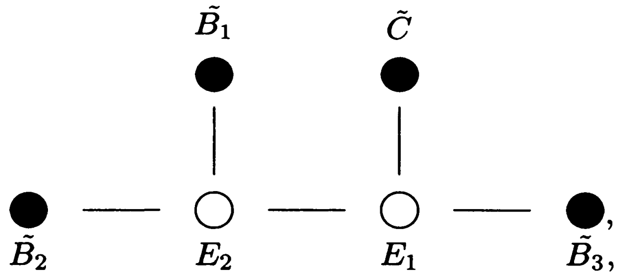
where each P_i is a prime divisor on S, m_i is the index of S at a general point of P_i, and $n_{i,j} \in \mathbb{N}$. Moreover, assume that B has only standard coefficients. Then so has $\operatorname{Diff}_S(B)$. More precisely, if $B = \sum(1 - 1/r_i)B_i$ and P is a prime divisor on S, then P is contained in at most one component, say B_i, of B and the coefficient of $\operatorname{Diff}_S(B)$ along P is equal to $1 - \frac{1}{m_i r_i}$.

COROLLARY 2.2.9. *Notation as in 2.2.8. Then*

$$B \in \Phi_{\mathbf{sm}} \implies \operatorname{Diff}_S(B) \in \Phi_{\mathbf{sm}},$$
$$B \in \Phi_{\mathbf{m}} \implies \operatorname{Diff}_S(B) \in \Phi_{\mathbf{m}}.$$

The example below shows that Inversion of Adjunction fails in the case of noneffective divisors.

EXAMPLE 2.2.10. Consider the following smooth curves on $\mathbb{C}^2$: $C := \{x = 0\}$, $B_1 := \{y = x^2\}$, $B_2 := \{y = 2x^2\}$, $B_3 := \{y = x\}$ and consider the subboundary $B := bB_1 + bB_2 + (\frac{3}{2} - 3b)B_3$, where $\frac{1}{2} < b < 1$. Then $\operatorname{Diff}_C(B) = (\frac{3}{2} - b)(\mathrm{pt})$ because C intersects transversally B_1, B_2, B_3. Hence $K_C + \operatorname{Diff}_C(B)$ is klt. On the other hand $K_{\mathbb{C}^2} + C + B$ is not lc. Indeed, a log resolution of $(\mathbb{C}^2, C + B)$ can be obtained by two blowing ups:



where E_1 is a -2-curve and E_2 is a -1-curve. It is easy to compute

$$a(E_1, C + B) = -\frac{3}{2} + b > -1, \qquad a(E_2, C + B) = -\frac{1}{2} - b < -1.$$

Therefore $K_{\mathbb{C}^2} + C + B$ is not lc at the origin.

For dlt singularities there are only weaker results, which use generalizations of the definition of dlt singularities to the case of nonnormal varieties (cf. [**Sh2**, 3.2.3, 3.6, 3.8], [**Ut**, 17.5, 16.9]):

PROPOSITION 2.2.11 ([**Sz**]). *Let $(X, S+B)$ be a log variety, where S is reduced, $\lfloor B \rfloor = 0$ and S, B have no common components. Assume that $K_X + S + B$ is dlt. Then $K_S + \operatorname{Diff}_S(B)$ is generalized divisorial log terminal.*

EXAMPLE 2.2.12. Let $(X \ni P)$ be a germ of three-dimensional terminal singularity. Then by [**RY**] a general divisor $F \in |-K_X|$ has only Du Val singularity at P. Hence by Theorem 2.2.6, $K_X + F$ is plt (and even canonical, because $K_X + F$ is Cartier).

EXERCISE 2.2.13. Let H be a general hyperplane section of the canonical quotient singularity $X := \mathbb{C}^3/\mathbb{Z}_3(1,1,1)$. Show that $K_X + H$ is not plt.

EXAMPLE 2.2.14 (cf. [**R1**, Sect. 1]). Let $(X \ni P)$ be a normal three-dimensional $\mathbb{Q}$-Gorenstein singularity and $H \ni P$ a hyperplane section. Assume that the singularity $(H \ni P)$ is Du Val. Then by Inversion of Adjunction, $(X \ni P)$ is canonical. Moreover, if $(X \ni P)$ is isolated, then it is terminal.

EXERCISE 2.2.15. Let $(X \ni o, D)$ be a germ of a normal singularity. Assume that $K_X + D$ is lc and each component of $\lfloor D \rfloor$ is $\mathbb{Q}$-Cartier. Prove that $\lfloor D \rfloor$ has at most $\dim X$ components.

We also have a more general results:

EXAMPLE 2.2.16. Let $(X \ni o, D = \sum d_i D_i)$ be a germ of a normal singularity of dimension ≤ 3. Assume that $K_X + D$ is lc and each component of D is $\mathbb{Q}$-Cartier. Then $\sum d_i \leq \dim X$. Indeed, by taking cyclic covers étale in codimension one we obtain $(X' \ni o', D' = \sum d'_i D'_i)$ such that $K_{X'} + D'$ is lc and each component of D' is Cartier. Obviously, $\sum d'_i \geq \sum d_i$. It is known that in dimension ≤ 3 there exists a divisor E over $X' \ni o'$ such that $a(E, 0) \leq \dim X - 1$ (see Kawamata's appendix to [**Sh2**] and [**M**]). Then $-1 \leq a(E, D') \leq a(E, 0) - \sum d'_i$. This yields $\sum d_i \leq \dim X$. Moreover, if the equality holds, then X' is smooth. In this case, $X' \to X$ gives the universal cover of the smooth locus of X. Therefore $X' \to X$ is a quotient by a finite group, say G, which acts freely in codimension one. Then we have also $\sum d'_i = \sum d_i$. Hence G does not permute components of D'. So G must be abelian. This shows that the equality $\sum d_i = \dim X$ implies that X is analytically isomorphic to a toric singularity and $\lfloor D \rfloor$ is contained in the toric boundary.

Actually, the above result can be proved in any dimension without using [**M**]:

THEOREM 2.2.17 ([**Ut**, 18.22], [**A1**]). *Let $(X \ni o, D = \sum d_i D_i)$ be a germ of a log variety such that $K_X + D$ is lc. Assume that all the D_i are $\mathbb{Q}$-Cartier at o. Then $\sum d_i \leq \dim X$. Moreover, the equality holds only if $X \ni o$ is a cyclic quotient singularity.*

Let $(X/Z \ni o, D)$ be a log variety. Then $(X/Z \ni o, \lfloor D \rfloor)$ is said to be a *toric pair* if there are analytic isomorphisms $\pi\colon X \to X^T$, $Z \to Z^T$ and the commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\ \pi\ } & X^T \\ \downarrow & & \downarrow \\ Z & \longrightarrow & Z^T \end{array}$$

where $X^T \to Z^T$ is an algebraic toric contraction and $\lfloor \pi(D) \rfloor$ is the toric boundary (i.e., $\lfloor \pi(D) \rfloor$ is contained in the set $X^T \setminus \{\text{open orbit}\}$).

Shokurov proposed the following generalization of 2.2.17.

CONJECTURE 2.2.18 ([**Sh3**]). *Let $(X/Z \ni o, D = \sum d_i D_i)$ be a log variety such that $K_X + D$ is lc and $-(K_X + D)$ is nef over Z. Then**

$$\operatorname{rkWeil_{alg}}(X) \geq \sum d_i - \dim X. \tag{2.2}$$

If X is $\mathbb{Q}$-factorial, then

$$\rho(X/Z) \geq \sum d_i - \dim X. \tag{2.3}$$

Moreover, equalities hold only if $(X/Z \ni o, \lfloor D \rfloor)$ is a toric pair.

In the case when Z is a point and $\rho(X) = 1$ the inequality (2.3) was proved in [**Ut**, 18.24], see also [**A1**]. Shokurov [**Sh3**] proved this conjecture in dimension two; see theorems 8.5.1 and 8.5.2.

Note that inequality (2.3) is stronger than (2.2):

PROPOSITION 2.2.19. *Notation as in 2.2.18. Assume that the pair (X, D) has at least one minimal log terminal ($\mathbb{Q}$-factorial) modification $f\colon (\tilde{X}, \tilde{D} = \sum \tilde{d}_i \tilde{D}_i) \to (X, D)$ (see 3.1.3). Then*

$$\operatorname{rkWeil_{alg}}(X) - \sum d_i + \dim X \geq$$
$$\operatorname{rkWeil_{alg}}(\tilde{X}) - \sum \tilde{d}_i + \dim \tilde{X} \geq \rho(\tilde{X}/Z) - \sum \tilde{d}_i + \dim \tilde{X}.$$

PROOF. Let us prove, for example, the first inequality. Write $\tilde{D} = \sum d_i B_i + \sum_{j=1}^r E_j$, where each B_i is the proper transform of D_i and $\sum_{j=1}^r E_j$ is the (reduced) exceptional divisor. Then $\sum \tilde{d}_i = r + \sum d_i$. From the exact sequence

$$\bigoplus_{i=1}^r \mathbb{Z} \cdot E_i \longrightarrow \operatorname{Weil_{alg}}(\tilde{X}) \longrightarrow \operatorname{Weil_{alg}}\left(\tilde{X} \setminus \sum E_i\right) \longrightarrow 0$$

(cf. [**Ha**, Ch. II, 6.5]) we have

*Shokurov pointed out that the stronger version of inequality (2.2) should be $\operatorname{Weil}(X)/\operatorname{Weil_0} \geq \sum d_i - \dim X$, where $\operatorname{Weil_0} \subset \operatorname{Weil}(X)$ is the subgroup of all numerically trivial over Z (and $\mathbb{Q}$-Cartier) divisors.

$$\mathrm{rkWeil}_{\mathrm{alg}}(\tilde{X}) \leq \mathrm{Weil}_{\mathrm{alg}}\left(\tilde{X} \setminus \sum E_i\right) + r = \mathrm{rkWeil}_{\mathrm{alg}}(X) + r.$$

□

EXAMPLE 2.2.20. Let $X = Z$ be the hypersurface singularity given in $\mathbb{C}^4$ by the equation $xy = zt$. Consider four planes $D_1 := \{x = z = 0\}$, $D_2 := \{x = z = 0\}$, $D_3 := \{y = z = 0\}$, $D_4 := \{y = t = 0\}$. Let $D := \sum D_i$. It is easy to check that $K_X + D$ is lc. The group $\mathrm{Weil}_{\mathrm{alg}}(X)$ is generated by D_1 and D_2. However, $D_1 + D_2 \sim 0$ (because it is Cartier). Thus $\mathrm{rkWeil}_{\mathrm{alg}}(X) = 1$ and $\sum d_i = 4$. We have equality in (2.2) and the pair (X, D) is toric.

2.3. Connectedness Lemma

The most essential part of the proof of Theorem 2.2.6 is the following result which was proved firstly by Shokurov [**Sh2**, 5.7] in dimension two and latter by Kollár [**Ut**, 17.4], [**Ko2**, 7.4] in arbitrary dimension.

THEOREM 2.3.1 (Connectedness Lemma). *Let $f\colon X \longrightarrow Z$ be a contraction and $D = \sum d_i D_i$ an effective $\mathbb{Q}$-divisor on X such that $K_X + D$ is $\mathbb{Q}$-Cartier. Assume that $-(K_X + D)$ is f-nef and f-big. Let*

$$h\colon Y \xrightarrow{g} X \xrightarrow{f} Z$$

be a log resolution. Write

$$K_Y = g^*(K_X + D) + E^{(+)} - E^{(-)},$$

where $E^{(-)} \geq 0$ and the coefficients of $E^{(+)}$ are > -1, and the coefficients of $E^{(-)}$ are ≥ 1. Then $\mathrm{Supp}E^{(-)}$ is connected in a neighborhood of any fiber of h.

Note that in the case when f is birational, the big condition holds automatically.

PROOF. We have

$$\lceil E^{(+)} \rceil - \lfloor E^{(-)} \rfloor = K_X - g^*(K_X + D) + \left\{-E^{(+)}\right\} + \left\{E^{(-)}\right\}.$$

From this by Kawamata-Viehweg Vanishing Theorem [**KMM**, 1-2-3],

$$R^1 f_* \mathcal{O}_Y\left(\lceil E^{(+)} \rceil - \lfloor E^{(-)} \rfloor\right) = 0.$$

Applying f_* to an exact sequence

$$0 \longrightarrow \mathcal{O}_Y\left(\lceil E^{(+)} \rceil - \lfloor E^{(-)} \rfloor\right) \longrightarrow \mathcal{O}_Y\left(\lceil E^{(+)} \rceil\right) \longrightarrow \mathcal{O}_{\lfloor E^{(-)} \rfloor}\left(\lceil E^{(+)} \rceil\right) \longrightarrow 0$$

we get the surjectivity of the map

$$h_* \mathcal{O}_Y\left(\lceil E^{(+)} \rceil\right) \longrightarrow h_* \mathcal{O}_{\lfloor E^{(-)} \rfloor}\left(\lceil E^{(+)} \rceil\right).$$

Let E_i be a component $\lceil E^{(+)} \rceil$. Then either E_i is g-exceptional or E_i is the proper transform of some D_i whose coefficient $d_i < 1$. Thus $\lceil E^{(+)} \rceil$ is g-exceptional and

$$h_* \mathcal{O}_Y \left(\left\lceil E^{(+)} \right\rceil \right) = f_* \left(\mathcal{O}_X \left(g_* \left(\left\lceil E^{(+)} \right\rceil \right) \right) \right) = \mathcal{O}_Z.$$

Assume that in a neighborhood of some fiber $h^{-1}(z)$, $z \in Z$ the set $\lfloor E^{(-)} \rfloor$ has two connected components F_1 and F_2. Then

$$h_* \mathcal{O}_{\lfloor E^{(-)} \rfloor} \left(\left\lceil E^{(+)} \right\rceil \right)_{(z)} = h_* \mathcal{O}_{F_1} \left(\left\lceil E^{(+)} \right\rceil \right)_{(z)} + h_* \mathcal{O}_{F_2} \left(\left\lceil E^{(+)} \right\rceil \right)_{(z)},$$

and both terms do not vanish. Hence $h_* \mathcal{O}_{\lfloor E^{(-)} \rfloor} \left(\lceil E^{(+)} \rceil \right)_{(z)}$ cannot be a quotient of the cyclic module $\mathcal{O}_{z,Z} \simeq h_* \mathcal{O}_Y \left(\lceil E^{(+)} \rceil \right)_{(z)}$. □

DEFINITION 2.3.2 ([**Sh2**, 3.14]). Let X be a normal variety and $D = \sum d_i D_i$ a $\mathbb{Q}$-divisor on X such that $K_X + D$ is $\mathbb{Q}$-Cartier. We say that a subvariety $W \subset X$ is a *center of log canonical singularities*, if there exists a birational contraction $f \colon Y \to X$ and a divisor E (not necessarily f-exceptional) with discrepancy $a(E, D, X) \leq -1$ such that $f(E) = W$. The union of all centers of lc singularities is called *the locus of log canonical singularities* of (X, D) and is denoted by $\mathrm{LCS}(X, D)$.

COROLLARY 2.3.3. *Notation as in Theorem 2.3.1. Then the set* $\mathrm{LCS}(X, D)$ *is connected in a neighborhood of any fiber of* f.