Chapter 3

Arrangements

3.1 Basic Constructions

Let $\mathcal{A}$ be an arrangement in V and let $L = L(\mathcal{A})$ be the set of nonempty intersections of elements of $\mathcal{A}$. An element $X \in L$ is called an **edge** of $\mathcal{A}$. Define a **partial order** on L by $X \leq Y \iff Y \subseteq X$. Note that this is reverse inclusion. Thus V is the unique minimal element of L. (Ordinary inclusion also gives a partial order preferred by many authors.) Define a **rank** function on L by $r(X) = \text{codim} X$. Thus $r(V) = 0$, $r(H) = 1$ for $H \in \mathcal{A}$. Recall that the rank of $\mathcal{A}$, $r(\mathcal{A})$, is the maximal number of linearly independent hyperplanes in $\mathcal{A}$. It is also the maximal rank of any element in $L(\mathcal{A})$. We call $\mathcal{A}$ **central** if $\cap_{H \in \mathcal{A}} H \neq \emptyset$, where $T = \cap_{H \in \mathcal{A}} H$ is called the center. The ℓ–arrangement $\mathcal{A}$ is called **essential** if it has an element of rank ℓ. Equivalently, $\mathcal{A}$ contains ℓ linearly independent hyperplanes.

Let $N = N(\mathcal{A}) = \cup_{H \in \mathcal{A}} H$ be the divisor of $\mathcal{A}$ and let $M = M(\mathcal{A}) = V - N(\mathcal{A})$ be the complement of $\mathcal{A}$. Recall that V has coordinates $u_1, \ldots, u_\ell$ and we defined a linear polynomial α_H with $\ker \alpha_H = H$ for each hyperplane $H \in \mathcal{A}$. The product $Q(\mathcal{A}) = \prod_{H \in \mathcal{A}} \alpha_H$ is a **defining polynomial** for $\mathcal{A}$. It is unique up to a constant. The next four constructions will be used later.

Coning [OT1, 1.15]: The affine ℓ-arrangement $\mathcal{A}$ gives rise to a central $(\ell+1)$-arrangement $\mathbf{c}\mathcal{A}$, called the **cone** over $\mathcal{A}$. Let $\tilde{Q}$ be the homogenized $Q(\mathcal{A})$ with respect to the new variable u_0. Then $Q(\mathbf{c}\mathcal{A}) = u_0 \tilde{Q}$ and $|\mathbf{c}\mathcal{A}| = |\mathcal{A}| + 1$. There is a natural embedding of $\mathcal{A}$ in $\mathbf{c}\mathcal{A}$ in the subspace $u_0 = 1$. Note that this embedding does not intersect $\ker u_0 = H_\infty$, the "infinite" hyperplane. Here $M(\mathbf{c}\mathcal{A}) \simeq M(\mathcal{A}) \times \mathbb{C}^*$.

Projective closure: Embed $V = \mathbb{C}^\ell$ in complex projective space $\mathbb{CP}^\ell$ and call the complement of V the infinite hyperplane, $\bar{H}_\infty$. Let $\bar{H}$ be the projective closure of H and write $\bar{\mathcal{A}} = \cup_{H \in \mathcal{A}} \bar{H}$. We call $\mathcal{A}_\infty = \bar{\mathcal{A}} \cup \{\bar{H}_\infty\}$ the **projective closure** of $\mathcal{A}$. It is an arrangement in $\mathbb{CP}^\ell$. Let $u_0, u_1, \ldots, u_\ell$ be projective coordinates in $\mathbb{CP}^\ell$ so that $\bar{H}_\infty = \ker u_0$. Then $\bar{H} = \ker \tilde{\alpha}_H$ where tilde denotes the homogenized polynomial, $Q(\mathcal{A}_\infty) = u_0\tilde{Q}(\mathcal{A})$, $|\mathcal{A}_\infty| = |\mathcal{A}| + 1$, and $M(\mathcal{A}_\infty) \simeq M(\mathcal{A})$.

Projective quotient: Given a nonempty central $(\ell+1)$-arrangement $\mathcal{C}$, we obtain a projective ℓ-arrangement $\mathbb{P}\mathcal{C}$ by viewing the defining homogeneous polynomial $Q(\mathcal{C})$ as a polynomial in projective coordinates. There is a natural bijection between coning and projective closure, provided the infinite hyperplanes agree. Here $|\mathcal{C}| = |\mathbb{P}\mathcal{C}|$ and $M(\mathbb{P}\mathcal{C}) \simeq M(\mathcal{C})/\mathbb{C}^*$.

Deconing [OT1, p.15]: Given a nonempty central $(\ell+1)$-arrangement $\mathcal{C}$ and a hyperplane $H \in \mathcal{C}$, we define an affine ℓ-arrangement $\mathbf{d}_H\mathcal{C}$, called the decone of $\mathcal{C}$ with respect to H. We construct the projective quotient $\mathbb{P}\mathcal{C}$ and choose coordinates so that $\mathbb{P}H = \ker u_0$ is the hyperplane at infinity. By removing it, we obtain the affine arrangement $\mathbf{d}_H\mathcal{C} = \mathbb{P}\mathcal{C} - \mathbb{P}H$. Note that $Q(\mathbf{d}_H\mathcal{C}) = Q(\mathcal{C})|_{u_0=1}$ and $|\mathbf{d}_H\mathcal{C}| = |\mathcal{C}| - 1$. Here $M(\mathcal{C}) \simeq M(\mathbf{d}_H\mathcal{C}) \times \mathbb{C}^*$.

These constructions are interrelated in the diagram below.

$$\begin{array}{ccccccc} & & \mathbf{c}\mathcal{A} & & & & \\ & & \uparrow & \searrow & & & \\ \mathbf{d}_{H_\infty}\mathbf{c}\mathcal{A} & = & \mathcal{A} & \longrightarrow & \mathcal{A}_\infty & = & \mathbb{P}\mathbf{c}\mathcal{A} \end{array}$$

Example 3.1.1. *Let $\mathcal{A}$ be the Selberg 2-arrangement defined by*

$$Q(\mathcal{A}) = u_1(u_1-1)u_2(u_2-1)(u_1-u_2).$$

We label the hyperplanes in the order given by the the factors in Q and write j in place of H_j in Figure 3.1, where we also display $L(\mathcal{A})$. Here

$$Q(\mathcal{A}_\infty) = u_0u_1(u_1-u_0)u_2(u_2-u_0)(u_1-u_2)$$

and $L(\mathcal{A}_\infty)$ contains the additional edges $\{\infty, 12\infty, 34\infty, 5\infty\}$.

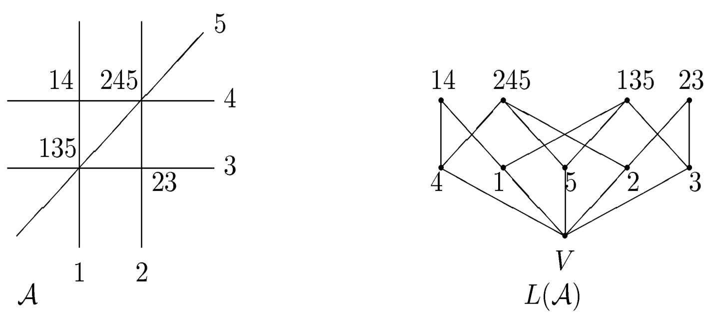

Figure 3.1: The Selberg Arrangement, I

Let $\mu : L \to \mathbb{Z}$ be the **Möbius function** of L defined by $\mu(V) = 1$, and for $X > V$ by the recursion $\sum_{Y \le X} \mu(Y) = 0$. The **characteristic polynomial** of $\mathcal{A}$ is $\chi(\mathcal{A}, t) = \sum_{X \in L} \mu(X) t^{\dim X}$. We get from [OT1, 2.51]:

Proposition 3.1.2. $\chi(\mathbf{c}\mathcal{A}, t) = (t-1)\chi(\mathcal{A}, t)$. ☐

This implies that if $\mathcal{C}$ is a central arrangement, then $\chi(\mathbf{d}_H\mathcal{C}, t)$ is independent of $H \in \mathcal{C}$. Thus we may write $\chi(\mathbf{d}\mathcal{C}, t)$.

Definition 3.1.3. *Given an edge $X \in L$ define a subarrangement $\mathcal{A}_X$ of $\mathcal{A}$ by $\mathcal{A}_X = \{H \in \mathcal{A} \mid X \subseteq H\}$. Here $\mathcal{A}_V$ is the empty ℓ-arrangement Φ_ℓ and if $X \neq V$, then $\mathcal{A}_X$ has center X in any arrangement. Define an arrangement $\mathcal{A}^X$ in X by $\mathcal{A}^X = \{X \cap H \mid H \in \mathcal{A} \setminus \mathcal{A}_X \text{ and } X \cap H \neq \emptyset\}$. We call $\mathcal{A}^X$ the* **restriction** *of $\mathcal{A}$ to X. The deletion-restriction triple is a nonempty arrangement $\mathcal{A}$ and $H \in \mathcal{A}$ together with $\mathcal{A}' = \mathcal{A}\backslash\{H\}$ and $\mathcal{A}'' = \mathcal{A}^H$. We call H a* **separator** *if $r(\mathcal{A}') < r(\mathcal{A})$.*

We get from [OT1, 2.57]:

Proposition 3.1.4. $\chi(\mathcal{A}, t) = \chi(\mathcal{A}', t) - \chi(\mathcal{A}'', t)$. ☐

3.2 Dense Edges

Let $\mathcal{C}$ be a central arrangement in V with center $T(\mathcal{C}) = \bigcap_{H \in \mathcal{C}} H \neq \emptyset$. We call $\mathcal{C}$ **decomposable** if there exist nonempty subarrangements $\mathcal{C}_1$ and $\mathcal{C}_2$ so that $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$ is a disjoint union and after a linear coordinate change the defining polynomials for $\mathcal{C}_1$ and $\mathcal{C}_2$ have no common variables. This is equivalent to the existence of two nonempty central arrangements so that $\mathcal{C}$ is their product in the sense of [OT1, 2.13]. If $\mathcal{C}$ is decomposed into $\mathcal{C}_1$ and $\mathcal{C}_2$, we write

$$\mathcal{C} = \mathcal{C}_1 \uplus \mathcal{C}_2.$$

It is easy to see that

$$\mathcal{C} = \mathcal{C}_1 \uplus \mathcal{C}_2 \Leftrightarrow r(\mathcal{C}) = r(\mathcal{C}_1) + r(\mathcal{C}_2) \Leftrightarrow T(\mathcal{C}_1) + T(\mathcal{C}_2) = V.$$

Definition 3.2.1. *Let $\mathcal{A}$ be an arrangement. An edge $X \in L$ is called* **dense** *in $\mathcal{A}$ if and only if the central arrangement $\mathcal{A}_X$ is not decomposable.*

The terminology is due to Varchenko [V2, 10.6.7]. A similar concept appeared in the work of Esnault-Schechtman-Viehweg [ESV].

Example 3.2.2. *The dense edges are $\{1, 2, 3, 4, 5, 135, 245\}$ in the Selberg arrangement 3.1.1. The additional dense edges in its projective closure are $\{\infty, 12\infty, 34\infty\}$.*

Lemma 3.2.3. *Let V be any vector space with subspaces A, B, C, D. Then*

$$(A \cap B) + (C \cap D) = (A + C) \cap (B + D) \Longleftrightarrow (A + B) \cap (C + D) = (A \cap C) + (B \cap D).$$

Proof. We show $\Rightarrow$. The inclusion $(A+B)\cap(C+D)\supseteq(A\cap C)+(B\cap D)$ is clear. For the reverse inclusion, let $a\in A$, $b\in B$, $c\in C$, $d\in D$ and assume $a+b=c+d$. Then $a-c=d-b\in(A+C)\cap(B+D)$. By assumption, we may find $x\in A\cap B$ and $y\in C\cap D$ with $a-c=d-b=x+y$. Then $a-x=y+c\in A\cap C$ and $b+x=d-y\in B\cap D$. Thus $a+b=(a-x)+(b+x)\in(A\cap C)+(B\cap D)$. □

Lemma 3.2.4. *Let $\mathcal{C}$ be a nonempty central arrangement with $H\in\mathcal{C}$. If $\mathcal{C}'$ and $\mathcal{C}''$ are decomposable, then $\mathcal{C}$ is decomposable.*

Proof. If H is a separator, then $\mathcal{C}=\mathcal{C}'\uplus\{H\}$ is a decomposition and we are done. Thus we may assume that H is not a separator. Suppose that $\mathcal{C}'=\mathcal{C}_1\uplus\mathcal{C}_2$ and $\mathcal{C}''=\mathcal{B}_1\uplus\mathcal{B}_2$. Let $\pi:\mathcal{C}'\to\mathcal{C}''$ be the natural surjection defined by $\pi(K)=K\cap H$. Let $\mathcal{C}_3=\pi^{-1}(\mathcal{B}_1)$ and $\mathcal{C}_4=\pi^{-1}(\mathcal{B}_2)$. Then we have $\mathcal{C}=\mathcal{C}_3\cup\mathcal{C}_4\cup\{H\}$ (disjoint). Also

$$\begin{aligned} r(\mathcal{C})-1 &= r(\mathcal{C}'')=r(\mathcal{B}_1)+r(\mathcal{B}_2)\\ &= r(\mathcal{C}_3\cup\{H\})-1+r(\mathcal{C}_4\cup\{H\})-1. \end{aligned}$$

Thus we obtain $r(\mathcal{C})=r(\mathcal{C}_3\cup\{H\})+r(\mathcal{C}_4\cup\{H\})-1$. Since H is not a separator, $r(\mathcal{C})=r(\mathcal{C}')$ so we get $r(\mathcal{C})=r(\mathcal{C}_1)+r(\mathcal{C}_2)$. If $T(\mathcal{C}_1)\subseteq H$, then $r(\mathcal{C})=r(\mathcal{C}_1\cup\{H\})+r(\mathcal{C}_2)$ and we are done. Thus we may assume that $T(\mathcal{C}_1)\nsubseteq H$. Similarly we may assume $T(\mathcal{C}_2)\nsubseteq H$. If $T(\mathcal{C}_3)\nsubseteq H$, then $r(\mathcal{C}_3)=r(\mathcal{C}_3\cup\{H\})-1$. Thus $r(\mathcal{C})=r(\mathcal{C}_3)+r(\mathcal{C}_4\cup\{H\})$ and we are done. So we may assume $T(\mathcal{C}_3)\subseteq H$. Similarly we may assume $T(\mathcal{C}_4)\subseteq H$. Define

$$A=T(\mathcal{C}_1\cap\mathcal{C}_3),\quad B=T(\mathcal{C}_1\cap\mathcal{C}_4),\quad C=T(\mathcal{C}_2\cap\mathcal{C}_3),\quad D=T(\mathcal{C}_2\cap\mathcal{C}_4).$$

Note that $A\cap B=T(\mathcal{C}_1)$ and $C\cap D=T(\mathcal{C}_2)$. Since $\mathcal{C}'=\mathcal{C}_1\uplus\mathcal{C}_2$, we have $(A\cap B)+(C\cap D)=V$. Note that

$$(A\cap B)+(C\cap D)\subseteq(A+C)\cap(B+D)$$

in general. Thus

$$(A\cap B)+(C\cap D)=(A+C)\cap(B+D).$$

By Lemma 3.2.3 we have

$$(A+B)\cap(C+D)=(A\cap C)+(B\cap D).$$

Note that $A\cap C=T(\mathcal{C}_3)$ and $B\cap D=T(\mathcal{C}_4)$. Thus

$$(A+B)\cap(C+D)=T(\mathcal{C}_3)+T(\mathcal{C}_4)\subseteq H.$$

Since $T(\mathcal{C}_1)\nsubseteq H$ and $T(\mathcal{C}_1)\subseteq A+B$, we have $A+B\nsubseteq H$. Similarly $C+D\nsubseteq H$. Therefore $A+B\neq V$ and $C+D\neq V$. Thus $\mathcal{C}_1$ is not decomposed into $\mathcal{C}_1\cap\mathcal{C}_3$

and $\mathcal{C}_1 \cap \mathcal{C}_4$. Similarly $\mathcal{C}_2$ is not decomposed into $\mathcal{C}_2 \cap \mathcal{C}_3$ and $\mathcal{C}_2 \cap \mathcal{C}_4$. Therefore we conclude

$$r(\mathcal{C}_1) < r(\mathcal{C}_1 \cap \mathcal{C}_3) + r(\mathcal{C}_1 \cap \mathcal{C}_4), \quad r(\mathcal{C}_2) < r(\mathcal{C}_2 \cap \mathcal{C}_3) + r(\mathcal{C}_2 \cap \mathcal{C}_4).$$

Finally we have

$$\begin{aligned} r(\mathcal{C}_3) + r(\mathcal{C}_4) - 1 &= r(\mathcal{C}) = r(\mathcal{C}_1) + r(\mathcal{C}_2) \\ &\leq r(\mathcal{C}_1 \cap \mathcal{C}_3) + r(\mathcal{C}_1 \cap \mathcal{C}_4) - 1 + r(\mathcal{C}_2 \cap \mathcal{C}_3) + r(\mathcal{C}_2 \cap \mathcal{C}_4) - 1 \\ &= r(\mathcal{C}_3) + r(\mathcal{C}_4) - 2. \end{aligned}$$

This is a contradiction. □

Definition 3.2.5. *Let* $\mathsf{D}_j(\mathcal{A})$ *denote the set of dense edges of dimension* j *in* $L(\mathcal{A})$ *and let* $\mathsf{D}(\mathcal{A}) = \cup_{j \geq 0} \mathsf{D}_j$.

We prove two properties of dense edges needed in Chapter 4.

Lemma 3.2.6. *Let* $\mathcal{A} \subset \mathcal{B}$. *If* $X \in \mathsf{D}(\mathcal{A})$, *then* $X \in \mathsf{D}(\mathcal{B})$.

Proof. Suppose not. Then $\mathcal{B}_X$ is decomposable, so we have $\mathcal{A}_X \subseteq \mathcal{B}_X = \mathcal{B}_1 \uplus \mathcal{B}_2$ with nonempty subarrangements. Since $\mathcal{A}_X$ is indecomposable, we may assume that $\mathcal{A}_X \subseteq \mathcal{B}_1$. Then $X = T(\mathcal{A}_X) \supseteq T(\mathcal{B}_1) \supseteq T(\mathcal{B}_X) = X$, so $X = T(\mathcal{B}_1)$. Since $r(X) = r(\mathcal{B}_1) + r(\mathcal{B}_2) = r(X) + r(\mathcal{B}_2)$, we conclude that $r(\mathcal{B}_2) = 0$ and $\mathcal{B}_2$ is empty. □

Lemma 3.2.7. *Let* $\mathcal{C} = \mathcal{C}_1 \uplus \cdots \uplus \mathcal{C}_m$ *be a central arrangement with an irreducible decomposition.*
(1) If $T_i = T(\mathcal{C}_i)$, *then* $\mathcal{C}_i = \mathcal{C}_{T_i}$.
(2) For $0 \leq j \leq \ell - 2$ *we have a disjoint union* $\mathsf{D}_j(\mathcal{C}) = \bigcup_{i=1}^m \mathsf{D}_j(\mathcal{C}_i)$.

Proof. (1) The inclusion $\mathcal{C}_i \subseteq \mathcal{C}_{T_i}$ is clear. Recall that the hyperplanes of $\mathcal{C}_i$ may be written in disjoint sets of variables. Let $\langle \mathcal{C}_i \rangle$ denote the vector space spanned by the corresponding variables. If $H \in \mathcal{C}_{T_i}$, then $\alpha_H \in \langle \mathcal{C}_i \rangle$. If $H \in \mathcal{C}_j$, then $\alpha_H \in \langle \mathcal{C}_j \rangle$ so $i = j$.

(2) Let $X \in \mathsf{D}_j(\mathcal{C})$. Since $\mathcal{C}_X$ is indecomposable, there is a unique i with $\mathcal{C}_X \subseteq \mathcal{C}_i$. Then $(\mathcal{C}_i)_X = \mathcal{C}_X$ is indecomposable and hence $X \in \mathsf{D}_j(\mathcal{C}_i)$. Conversely, if $X \in \mathsf{D}_j(\mathcal{C}_i)$, then $(\mathcal{C}_i)_X$ is indecomposable. It follows from Lemma 3.2.6 that $\mathcal{C}_X$ is indecomposable, so $X \in \mathsf{D}_j(\mathcal{C})$. □

3.3 The β Invariant

In Chapter 4 we will show that certain conditions on the dense edges are sufficient to compute local system cohomology groups explicitly. To determine these conditions we need to know which edges of a given arrangement are dense. In higher dimensions it is difficult to use the definition directly. H. Crapo [Cr] introduced the beta invariant (in a different form) and proved the results in this section, although the original proof of [Cr, Lemma to Theorem 2] is incomplete. The argument given in Lemma 3.2.4 is a completed version of the original proof. Corollary 3.3.5 provides a numerical criterion to decide which edges are dense.

Definition 3.3.1. *Let $\mathcal{A}$ be an arrangement of rank r. Define its* **beta invariant** *by*

$$\beta(\mathcal{A}) = (-1)^r \chi(\mathcal{A}, 1).$$

Crapo defined his invariant only for nonempty central arrangements. For a central $(r+1)$-arrangement $\mathcal{C}$ it is

$$(-1)^r \frac{d}{dt}\chi(\mathcal{C}, 1).$$

We note the connection with our invariant.

Proposition 3.3.2. *If $\mathcal{C}$ is a central $(r+1)$-arrangement, then*

$$\beta(\mathbf{d}\mathcal{C}) = (-1)^r \frac{d}{dt}\chi(\mathcal{C}, 1).$$

Proof. Differentiate both sides of Proposition 3.1.2 with respect to t, set $t = 1$, and multiply by $(-1)^r$. □

Proposition 3.3.3. *If H is not a separator, then $\beta(\mathcal{A}) = \beta(\mathcal{A}') + \beta(\mathcal{A}'')$.*

Proof. This follows from Proposition 3.1.4 and the fact that $r(\mathcal{A}'') = r(\mathcal{A}) - 1 = r(\mathcal{A}') - 1$. □

Theorem 3.3.4. *Let $\mathcal{C}$ be a nonempty central arrangement. Then*
(1) if $\mathcal{C}$ is decomposable, then $\beta(\mathbf{d}\mathcal{C}) = 0$,
(2) $\beta(\mathbf{d}\mathcal{C}) \geq 0$,
(3) if $\beta(\mathbf{d}\mathcal{C}) = 0$, then $\mathcal{C}$ is decomposable.

Proof. (1) Suppose $\mathcal{C}$ is decomposable. Then $\mathcal{C}$ is a product of two nonempty central arrangements. It follows from Proposition 3.1.2 that their characteristic polynomials are divisible by $(t-1)$. Thus $(t-1)^2$ divides $\chi(\mathcal{C}, t)$ by [OT1, Lemma 2.50]. We see from Proposition 3.1.2 that $\beta(\mathbf{d}\mathcal{C}) = 0$.

(2) We argue by induction on $|\mathcal{C}|$. If $|\mathcal{C}| = 1$, then $\beta(\mathbf{d}\mathcal{C}) = 1$. Suppose $|\mathcal{C}| > 1$. Let $H \in \mathcal{C}$ and $\mathcal{C}' = \mathcal{C} - \{H\}$. If H is a separator, then $\mathcal{C} = \{H\} \uplus \mathcal{C}'$, so $\beta(\mathbf{d}\mathcal{C}) = 0$ by (1). If H is not a separator, then $\beta(\mathbf{d}\mathcal{C}) = \beta(\mathbf{d}\mathcal{C}') + \beta(\mathbf{d}\mathcal{C}'')$ by Proposition 3.3.3. The conclusion follows because $\beta(\mathbf{d}\mathcal{C}') \geq 0$, $\beta(\mathbf{d}\mathcal{C}'') \geq 0$ by the induction assumption. Thus $\beta(\mathbf{d}\mathcal{C}) \geq 0$.

(3) We argue by induction on $|\mathcal{C}|$. If $|\mathcal{C}| = 1$, then $\beta(\mathbf{d}\mathcal{C}) \neq 0$. Thus $|\mathcal{C}| > 1$. Let $H \in \mathcal{C}$. If H is a separator, we are done. If H is not a separator, then $0 = \beta(\mathbf{d}\mathcal{C}) = \beta(\mathbf{d}\mathcal{C}') + \beta(\mathbf{d}\mathcal{C}'')$. Since $\beta(\mathbf{d}\mathcal{C}') \geq 0$ and $\beta(\mathbf{d}\mathcal{C}'') \geq 0$ by (2), we have $\beta(\mathbf{d}\mathcal{C}') = \beta(\mathbf{d}\mathcal{C}'') = 0$. By the induction assumption, both $\mathcal{C}'$ and $\mathcal{C}''$ are decomposable. It follows from Lemma 3.2.4 that $\mathcal{C}$ is decomposable. □

Corollary 3.3.5. *Let $\mathcal{A}$ be an arrangement and let $X \in L(\mathcal{A})$. The following conditions are equivalent:*
(1) X is dense,
(2) $\mathcal{A}_X$ is not decomposable,
(3) $\beta(\mathbf{d}\mathcal{A}_X) \neq 0$,
(4) $\beta(\mathbf{d}\mathcal{A}_X) > 0$. □

Example 3.3.6. *In every arrangement the hyperplanes are dense and the whole space is not. In the examples below we determine the dense edges of codimension ≥ 2. If a projective arrangement has normal crossings, then it has no dense edges. In the projective 2-arrangement called Ceva(3) defined by*

$$Q = (u_0^3 - u_1^3)(u_0^3 - u_2^3)(u_1^3 - u_2^3)$$

all twelve points are dense. In the projective 3-arrangement defined by

$$Q = u_0 u_1 u_2 u_3 (u_1 - u_2)(u_1 - u_3)(u_2 - u_3)$$

the four lines which are contained in three planes each are dense, but there are no dense points.

We conclude with two topological interpretations of the β-invariant.

Theorem 3.3.7. *Let $\mathrm{Poin}(M,t) = \sum \dim H^p(M,\mathbb{C})t^p$ be the Poincaré polynomial of M. By [OT1, 5.93], $\mathrm{Poin}(M,t) = (-t)^\ell \chi(\mathcal{A}, -t^{-1})$. In particular, $\beta(\mathcal{A}) = |e(M)|$ is the absolute value of the euler characteristic of the complement.* □

Definition 3.3.8. *We say that $\mathcal{A}$ is a* **complexified real** *arrangement if the polynomials α_H have real coefficients. In this case let $V_{\mathbb{R}} = \mathbb{R}^\ell$ be the real part of V and let $M_{\mathbb{R}} = M \cap V_{\mathbb{R}}$ be the real complement. It is a disjoint union of open convex subsets called chambers. Let $\mathsf{ch}(\mathcal{A})$ denote the set of chambers in $M_{\mathbb{R}}$. If $\mathcal{A}$ is essential, then some chambers may be bounded. Let $\mathsf{bch}(\mathcal{A})$ denote the set of bounded chambers in $M_{\mathbb{R}}$.*

Zaslavsky [Za] proved:

Theorem 3.3.9. *Let $\mathcal{A}$ be a real ℓ-arrangement. Then $|\mathsf{ch}(\mathcal{A})| = (-1)^\ell \chi(\mathcal{A}, -1)$. If $\mathcal{A}$ is essential, then $|\mathsf{bch}(\mathcal{A})| = \beta(\mathcal{A})$.* □