Advanced Studies in Pure Mathematics 72, 2017
Geometry, Dynamics, and Foliations 2013
pp. 301–348

Cones of foliations almost without holonomy

John Cantwell and Lawrence Conlon

Abstract.

On sutured 3-manifolds M, we classify taut foliations almost without holonomy up to isotopy. We assume that the compact leaves lie in ∂M. The classification is given by finitely many convex, polyhedral cones in $H^1(M;\mathbb{R})$ which have disjoint interiors. The classes in the interiors of these cones determine the isotopy classes. This work relies heavily on the Handel–Miller classification of the isotopy classes of end-periodic surface automorphisms. While the Handel–Miller theory was not published by the originators, the authors have given a complete account elsewhere.

§1. Introduction

Throughout this paper, $(M, \mathcal{F})$ is a smoothly foliated 3-manifold, M being compact and oriented and $\mathcal{F}$ being transversely oriented, codimension one, and taut. Recall that a foliation is taut if each leaf meets either a closed transversal or a transverse arc from one leaf in ∂M to another. Such foliations are interesting because they are Reebless and hence, by the well known theorems of S. P. Novikov [28], reflect topological features of the manifold. For a detailed discussion of Reebless foliations, see [4, Chapter 9].

In this note, we classify such foliations that are almost without holonomy.

Definition 1.1. *A taut, transversely oriented, codimension one foliation $\mathcal{F}$ of a compact 3-manifold M will be said to be* almost without holonomy *if only the compact leaves can have nontrivial holonomy. The foliation is of depth 1 if it fibers the complement of the compact leaves over S^1.*

Received June 21, 2014.
Revised August 7, 2015.
2010 *Mathematics Subject Classification.* 57R30.
Key words and phrases. foliated 3-manifold, cone, asymptotic cycles, almost without holonomy, isotopy.

We will suppose that all compact leaves have holonomy, lie in ∂M and have negative Euler characteristic. It may be that the boundary decomposes as $\partial M = \partial_\tau M \cup \partial_\pitchfork M$, separated by convex corners, where $\partial_\tau M$ is the union of the compact leaves and $\partial_\pitchfork M$ is met transversely by the foliation. The components of $\partial_\pitchfork M$ are annuli and/or tori. We will always assume tautness not only for $\mathcal{F}$ but for $\mathcal{F}|\partial_\pitchfork M$. The decomposition $\partial M = \partial_\tau M \cup \partial_\pitchfork M$ makes M a *sutured manifold* in the sense of Gabai [19]. In standard sutured manifold notation, $\partial_\pitchfork M = \gamma$ and $\partial_\tau M = R(\gamma)$. It is important to emphasize that the sutured structure includes a choice of transverse orientations of the components of $R(\gamma)$. All foliations of sutured manifolds are required to be transversely oriented in a way consistent with the transverse orientation of $R(\gamma)$.

Smooth foliations almost without holonomy fall into two classes. Those of depth 1 fiber $M^\circ = M \smallsetminus \partial_\tau M$ over S^1, and the remaining ones have the property that each leaf of $\mathcal{F}|M^\circ$ is dense in M. (For C^1 foliations, the situation is more complicated due to the existence of Denjoy type foliations.)

Our main theorem is,

Theorem 1.2. *There is a finite set of nonoverlapping, closed, convex, polyhedral cones* $\mathfrak{C}_1, \mathfrak{C}_2, \dots, \mathfrak{C}_r \subset H^1(M;\mathbb{R})$ *with common vertex at the origin, such that the* C^0 *ambient isotopy classes of codimension* 1 *foliations* $\mathcal{F}$ *almost without holonomy of* M *are in natural one-to-one correspondence with the rays* $\langle\mathcal{F}\rangle$ *out of the origin in the interiors of these cones. Furthermore,*

(1) *the rational rays* (*those that meet nontrivial elements of the integer lattice* $H^1(M;\mathbb{Z})$) *correspond exactly to the depth* 1 *foliations*;

(2) *the irrational rays* (*those that do not meet nontrivial elements of the integer lattice* $H^1(M;\mathbb{Z})$) *correspond exactly to the dense leaved type.*

Here, the assertion that the cones do not overlap means that they have disjoint interiors.

Theorem 1.2 is closely analogous to the classification of smooth foliations without holonomy transverse to ∂M. These foliations are either fibrations of M over S^1 or they are dense leaved. A well known theorem of W. P. Thurston [35] shows that, if M has any such foliations, certain top dimensional faces of the "Thurston ball" (a convex polyhedron which is the unit ball of the Thurston norm) subtend cones $\mathfrak{C}_1, \mathfrak{C}_2, \dots, \mathfrak{C}_r \subset H^1(M;\mathbb{R})$ such that the rational rays in the interior of these cones correspond one-to-one to the isotopy classes of fibrations. Furthermore, combining the Laudenbach–Blank theorem [26] with a theorem of the authors [10], the irrational rays in the interiors of the Thurston cones correspond one-to-one to the C^0 isotopy classes of dense leaved foliations without holonomy.

Remark. Thurston's set of cones is invariant under multiplication by -1, but ours is not. This is due to the fact that ours classify only those foliations whose transverse orientations agree with that of $R(\gamma)$.

Remark. Thurston's paper [35] is the main motivation for our research and we assume familiarity with it.

Remark. Recently, I. Altman [1] has shown that, with some important restrictions on the sutured manifold (M, γ), our cones are subtended by certain top dimensional faces of the dual Juhász polytope [25], the unit ball for a *nonsymmetric* norm defined via sutured Floer homology.

Remark. Our work depends on the Schwartzmann–Sullivan theory of foliation currents [34, 33] and the Handel–Miller classification of endperiodic automorphisms of surfaces (unpublished). The first of these is well understood and rigorous. The second has been largely folklore and not well understood at all. In fact, to the best of our knowledge, the only published accounts of the Handel–Miller theory are some short sketches without proofs in the early work of S. Fenley [16, 17]. Consequently, our earlier work on foliation cones [8, 9] did not have a rigorous foundation and, in fact, contained serious errors due to our imperfect understanding of the Handel–Miller theory. In [5], we have tried to remedy this situation by putting the Handel–Miller theory on a rigorous axiomatic foundation and using these axioms to prove new theorems essential to the construction and analysis of the foliation cones. This makes it possible to recover the key ideas of [8] in a new and rigorous setting and to extend the classification theory there to include all taut foliations almost without holonomy, not just those of depth 1.

This paper can be read independently of [8] with one exception. The proof that there are only finitely many foliation cones, given in [8], is rigorous and we see no need to reproduce it here.

§2. Monodromy

We consider a depth 1 foliation $\mathcal{F}$ of M. Let $\mathcal{L}$ be a smooth, 1-dimensional foliation of M everywhere transverse to $\mathcal{F}$ and let L be a noncompact leaf of $\mathcal{F}$. Then the first return map $f\colon L \to L$ defined by flowing along $\mathcal{L}$ is the *monodromy* of L defined by $\mathcal{L}$. From now on, when emphasizing the monodromy f, we denote the 1-dimensional foliation by $\mathcal{L}_f$. Varying f by a smooth isotopy f_t corresponds to varying $\mathcal{L}$ by a smooth homotopy $\mathcal{L}_{f_t}$.

Generally, $\mathcal{L}$ will be smooth and so the monodromy will be a diffeomorphism. Serious smoothness problems arise when we try to isotope the monodromy to have the "tightest" dynamics in its isotopy class. This is the Handel–Miller theory which we have analyzed in great detail in [5]. The smoothness issue is resolved there. In Section 5 we sketch the Handel–Miller theory as developed in [5] and state the main theorems we need for this paper.

Suppose $L \supset V_1 \supset \ldots \supset V_n \supset \overline{V}_{n+1} \supset V_{n+1} \supset \ldots$ with the V_n open, connected, $\bigcap_{n=1}^{\infty} V_n = \emptyset$, and $\overline{V}_n \smallsetminus V_n$ compact. Then the nested sequence of sets $\{V_n\}$ defines an *end* of L.

If $\{V_n\}$ and $\{U_n\}$ define ends of L, then $\{V_n\}$ is said to be equivalent to $\{U_n\}$ if for every n there exists an m such that $V_n \supset U_m$ and for every m there exists an n such that $U_m \supset V_n$. The equivalence classes, $e = [\{V_n\}]$, are called the ends of L and the set $\mathcal{E}(L)$ of equivalence classes is called the *endset* of L.

Let $\mathcal{T}$ be the topology on L, that is $\mathcal{T}$ is the family of open sets in L. For $V \in \mathcal{T}$ let,

$$\widehat{V} = V \cup \{e = [\{V_n\}] \in \mathcal{E}(L) \mid \text{there exists an } n \text{ with } V \supset V_n\}.$$

Then it is well known that $\widehat{\mathcal{B}} = \mathcal{T} \cup \{\widehat{V} \mid V \in \mathcal{T}\}$ is a basis for a compact, separable metrizable topology $\widehat{\mathcal{T}}$ on $L \cup \mathcal{E}(L)$ which restricts to a totally disconnected topology on the closed set $\mathcal{E}(L)$.

If $e \in \mathcal{E}(L)$, we will say that $U \subset L$ is a neighborhood of the end e if $\widehat{U}$ is a neighborhood of e in the space $(L \cup \mathcal{E}(L), \widehat{\mathcal{T}})$.

It is well known that f induces a homeomorphism $\widehat{f} \colon \mathcal{E}(L) \to \mathcal{E}(L)$. By the well understood way in which the leaf L winds in on $\partial_\tau M$, the monodromy diffeomorphism is "endperiodic".

This term needs explanation. Let $f \colon L \to L$ be a homeomorphism of a noncompact surface. If an end e is periodic under iterations of $\widehat{f}$, we let p_e denote the period of e, the smallest positive integer such that $\widehat{f}^{p_e}(e) = e$.

Definition 2.1. *Let $f \colon L \to L$ be a homeomorphism and let e be a periodic end of L. The end is* attracting (*or a* positive end) *if there is a closed, connected neighborhood U of e such that $L \smallsetminus U$ is connected and*

(1) $f^{p_e}(U) \subset U$;
(2) $\bigcap_{n=0}^{\infty} f^{np_e}(U) = \emptyset$.

The end is repelling (*or a* negative end) *if the parallel assertions hold with p_e replaced by $-p_e$.*

Definition 2.2. *A homeomorphism $f\colon L \to L$ of a noncompact surface is endperiodic if L has only finitely many periodic ends, each of which is attracting or repelling. The homeomorphism f is also called an endperiodic automorphism of L.*

Remark. In the literature (e.g., [3, 17]) and in some earlier work of the authors, it is required in Definition 2.1 that $f^{p_e}(U) \subset \operatorname{int} U$. (This was only implicit in [3] where the term "endperiodic" does not occur. The pertinent discussion is in Section 8.4 of that reference.) This excludes some natural examples of endperiodic homeomorphisms. The definitions given above are to be taken as the "canonical" ones. For further discussion of these definitions and their consequences, cf. [5, pp. 3–11].

The following is well known and elementary.

Theorem 2.3. *Every monodromy map $f\colon L \to L$ of a noncompact leaf of a depth 1 foliation is endperiodic and L has only finitely many ends.*

We also need the following.

Theorem 2.4. *Suppose the surface L has finitely many ends. Every endperiodic automorphism $f\colon L \to L$ occurs as the monodromy of a noncompact leaf of a taut depth 1 foliation $\mathcal{F}$ of a compact manifold M, and f is the first return map defined by a transverse, 1-dimensional foliation $\mathcal{L}_f$. If f is a diffeomorphism, M, $\mathcal{F}$ and $\mathcal{L}_f$ are smooth.*

In fact, with care, one can show that $(M, \mathcal{F})$ can be given a smooth structure, but not $\mathcal{L}_f$, even if f is only a homeomorphism. However we do not need this. For a sketch of the construction proving this theorem, see [5, Section 12.3].

Example 2.5. An endperiodic automorphism f is depicted in Figure 1. Here, the ends e_1 and e_2 are periodic and negative, with $p_{e_2} = p_{e_1} = 2$. The end e is periodic and positive, with $p_e = 1$. As the arrows indicate, f exchanges the shaded neighborhoods U_i of e_i, $i = 1, 2$, and f^2 shifts to the right the compact segments of these neighborhoods that are cut off by the circles. The shaded neighborhood U of e is invariant under f and its segments are also shifted to the right as indicated. In this example, the stronger requirement that $f(U) \subset \operatorname{int} U$ is fulfilled and, similarly, $f^{-2}(U_i) \subset \operatorname{int} U_i$, $i = 1, 2$. While there are infinitely many ways to define f in the compact unshaded region, one can see intuitively that it can be defined there to have only one periodic point. This is the point p of intersection of the two boldfaced curves and is fixed by f. These curves are each f-invariant and all points outside

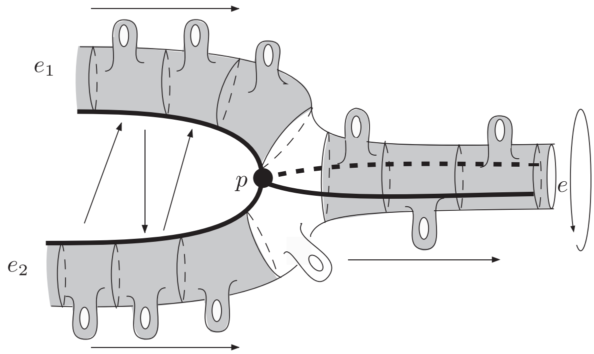

Fig. 1. An example with two negative ends

their union approach e under iterations of f and cluster at e_1 and e_2 under iterations of f^{-1}. In one of these curves, the points other than p approach e under iteration of f and converge to p under iteration of f^{-1}. In the other, the points other than p cluster at e_1 and e_2 under iterations of f^{-1} and converge to p under iteration of f. When realizing this endperiodic automorphism as the monodromy of a noncompact leaf of a depth 1 foliated manifold $(M, \mathcal{F})$, $\partial M = \partial_\tau M$ and consists of two surfaces of genus 2.

Example 2.6. We describe a 2-ended surface L and an endperiodic automorphism f. The surface L will be formed by cutting pairs of pants P_i along certain essential, properly imbedded subarcs and then pasting the resulting disks P'_i to one another along these subarcs. The endperiodic map $f\colon L \to L$ will take P'_i to P'_{i+1}.

In Figure 2, we depict the typical pair of pants and the essential arcs with transverse orientation. After cutting, P_i becomes a disk P'_i, with A_i and D_i split and indexed as indicated in Figure 3. Here i varies over the integers. The index i on A^+_i, D^+_i indicates that these arcs are identified with the original A_i and D_i, while the index on A^-_{i+2} indicates that it is to be attached to A^+_{i+2}, forming a single arc to be labeled A_{i+2}, and the index on D^-_{i+3} indicates that it is to be attached to D^+_{i+3}, forming a single arc to be labeled D_{i+3}. In this way, a connected, 2-ended surface

$$L = \bigcup_{-\infty<i<\infty} P'_i$$

is assembled.

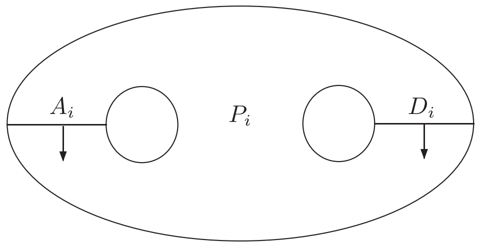

Fig. 2. A pair of pants P_i

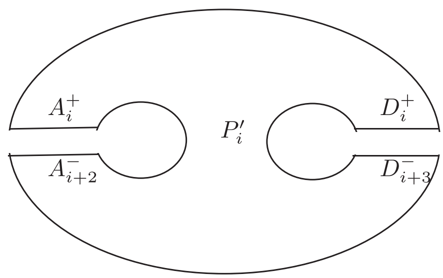

Fig. 3. Arcs $A^\pm, D^\pm$ with indices

For the closed, connected neighborhood of the positive end take

$$U = \bigcup_{i=0}^{\infty} P_i'.$$

One notes that $f(U) \subset U$ and

$$\partial U = D_0 \cup D_1 \cup D_2 \cup A_0 \cup A_1$$

while

$$\partial f(U) = D_1 \cup D_2 \cup D_3 \cup A_1 \cup A_2.$$

In this example, therefore, $f(U) \not\subset \operatorname{int} U$. Once again this is easily realized as the monodromy of a noncompact leaf of a depth 1 foliated manifold M. In this case, $M = P \times I$, where P is a pair of pants, and $\mathcal{L}_f$ is just the foliation by the I-factors.

Remark. Example 2.6 provides an example of an endperiodic map where a natural choice of U does not satisfy the condition "$f(U) \subset \operatorname{int} U$".

Remark. It should be noted that in both of these examples the neighborhood U can be partitioned into compact submanifolds F_i, $0 \leq i < \infty$, such that F_i is attached to finitely many F_j's, $j > i$, along some common boundary components. If $\partial L = \emptyset$ (the situation envisioned in much of the literature), F_i is exactly attached to F_{i+1}. This is the case in Example 2.5. But example 2.6 shows that the situation can be much more complex. In all cases, however, U can be chosen so that $f(F_i) = F_{i+1}$, $0 \leq i < \infty$. In particular, this implies that ∂U and $f(\partial U)$ intersect, if at all, only in common components. Similar remarks hold for neighborhoods of negative ends. For a careful discussion of this, see [5, Section 2.4].

§3. The asymptotic cycles

Let $(M, \mathcal{F})$ be a taut, depth 1 foliated manifold, let L be a noncompact leaf and $f\colon L \to L$ the endperiodic monodromy defined by a transverse, 1-dimensional foliation $\mathcal{L}_f$. Throughout this section, f and $\mathcal{L}_f$ are fixed, so we will denote the 1-dimensional foliation simply by $\mathcal{L}$.

It will not be necessary to assume that $\mathcal{L}$ is smooth, only that it is integral to a nonsingular C^0 vector field v.

Definition 3.1. *The core lamination $\mathcal{X} \subset \mathcal{L}$ is the set of leaves of $\mathcal{L}$ that do not meet $\partial_\tau M$.*

We emphasize that $\mathcal{L}$ is oriented by the transverse orientation of $\mathcal{F}$ and so $\mathcal{X}$ is also an oriented lamination.

The only way that $\mathcal{X}$ can be empty is if $M = F \times I$, $\mathcal{F}$ is transverse to the interval fibers $\{x\} \times I$, and these are the leaves of $\mathcal{L}$. We call $(M, \mathcal{F})$ a *foliated product*. In this case, $f\colon L \to L$ is a translation and this whole scenario is of very limited topological interest.

From now on we suppose that $(M, \mathcal{F})$ is not a foliated product.

The union $|\mathcal{X}|$ of the leaves of the core lamination is called the *support* of the lamination and is clearly compact. (We say that the lamination is compact.) Sullivan's theory of foliation cycles [34] works perfectly well for compact laminations. It is only required that none of the leaves have boundary. When applied to $\mathcal{X}$, it produces closed de Rham 1-currents in M which are the asymptotic cycles introduced by S. Schwartzmann in [33].

3.1. Forms and currents

We give a fairly detailed sketch of the theory of de Rham currents. For more details, see [3, Section 10.1] and for the definitive treatment see [13]. In this section, M can be a compact manifold of arbitrary dimension.

Slightly modifying the notation of [13] so as to keep track of the degrees of forms and currents, we set $\mathcal{D}_p = \mathcal{D}_p(M)$, the locally convex topological vector space of p-forms of class C^∞. The underlying vector space is $A^p(M)$ and the topology $\mathbb{T}$ is generated by the increasing union of the topologies $\mathbb{T}_k$ defined by the C^k norm $\|\cdot\|_k$, $0 \le k < \infty$. More precisely, fix a choice of finite C^∞ atlas $\{U_i, x_i\}_{1\le i\le m}$ on M. For each p-form $\varphi \in A^p(M)$, set $\|\varphi\|_k$ equal to the maximum value of the function obtained by summing the absolute values of the mixed partials of the coefficients of $\varphi|U_i$ of order $\le k$ (including the 0th-order partials), computed relative to the coordinates x_i, and then summing over $i = 1, 2, \dots, m$. This is the C^k-norm and defines a locally convex, Hausdorff topology $\mathbb{T}_k$ on $A^p(M)$. While the norm depends on the choice of C^∞ atlas, the topology is independent of that choice and $\mathbb{T}_k \subseteq \mathbb{T}_{k+1}$, $k \ge 0$. The union of these topologies generates a topology $\mathbb{T}$ which makes $A^p(M)$ into a locally convex, topological vector space $\mathcal{D}_p$.

Definition 3.2. *A subset $S \subset \mathcal{D}_p$ is bounded if, for each $k \ge 0$, it is bounded relative to the C^k-norm. A linear map $\theta\colon \mathcal{D}_p \to \mathcal{D}_r$ is bounded if it takes bounded sets to bounded sets, and similarly for linear functionals $\theta\colon \mathcal{D}_p \to \mathbb{R}$.*

Lemma 3.3. *A bounded linear map $\theta\colon \mathcal{D}_p \to \mathcal{D}_r$ is continuous. A continuous linear functional $\theta\colon \mathcal{D}_p \to \mathbb{R}$ is bounded.*

Corollary 3.4. *Exterior differentiation*

$$d\colon \mathcal{D}_p \to \mathcal{D}_{p+1}$$

is a continuous linear map.

Definition 3.5. *The strong dual $\mathcal{D}'_p$ of $\mathcal{D}_p$ is the space of continuous linear functionals on $\mathcal{D}_p$. The elements of $\mathcal{D}'_p$ are called (de Rham) p-currents on M.*

Example 3.6. Any tangent vector $v \in T_x(M)$ is a 1-current. Indeed, $v\colon \mathcal{D}_1 \to \mathbb{R}$ is defined by $v(\varphi) = \varphi_x(v)$. This is easily shown to be a continuous linear functional, called a Dirac current.

Example 3.7. A probability measure μ on M is a 0-current. Indeed, $\mathcal{D}_0 = C^\infty(M)$, the space of real valued C^∞ functions on M with the C^∞-topology. One defines $\mu\colon \mathcal{D}_0 \to \mathbb{R}$ by $\mu(f) = \int_M f\,d\mu$, a continuous linear functional.

Example 3.8. A smooth singular p-chain c on M is a p-current. Indeed, for $\varphi \in \mathcal{D}_p$, set $c(\varphi) = \int_c \varphi$, again obtaining a continuous linear functional $c\colon \mathcal{D}_p \to \mathbb{R}$.

The space of p-currents carries a natural topology also.

Definition 3.9. *If $B \subset \mathcal{D}_p$ is bounded and $\varepsilon > 0$, let*

$$V_{B,\varepsilon} \subset \mathcal{D}'_p$$

be the set of p-currents ψ such that $\psi(B) \subset [-\varepsilon, \varepsilon]$. A subset $S_0 \subset \mathcal{D}'_p$ is a neighborhood of 0 if $V_{B,\varepsilon} \subseteq S_0$, for some such B and ε. If ψ is a p-current, the neighborhoods of ψ are sets $S_\psi = \psi + S_0$, where S_0 is a neighborhood of 0. A subset $W \subseteq \mathcal{D}'_p$ is open if each of its points has a neighborhood in W.

This makes $\mathcal{D}'_p$ into a locally convex, topological vector space. Both $\mathcal{D}_p$ and $\mathcal{D}'_p$ are strong duals of one another [13, p. 89, Théorème 13].

All of these spaces are Montel, meaning that every bounded subset is precompact. For the case $p = 0$, this is proven in [32, p. 70, Théorèm VII and p. 74, Théorème XII], the general case being similar.

Since the exterior derivative $d\colon \mathcal{D}_p \to \mathcal{D}_{p+1}$ is continuous, its adjoint $\partial\colon \mathcal{D}'_{p+1} \to \mathcal{D}'_p$ is also continuous. Since $d^2 = 0$, we see that $\partial^2 = 0$. The homology of the chain complex $(\mathcal{D}'_*, \partial)$ gives the dual space $H_p(M)$ to the de Rham cohomology $H^p(M)$, for each $p \geq 0$. One calls $H_p(M)$ the de Rham homology. By the de Rham theorem, these spaces are canonically the same as the real singular cohomology and homology of M, respectively.

We denote the kernel of $\partial\colon \mathcal{D}'_p \to \mathcal{D}'_{p-1}$ by $\mathcal{Z}_p$, called the space of de Rham p-cycles, and the image of $\partial\colon \mathcal{D}_{p+1} \to \mathcal{D}_p$ by $\mathcal{B}_p$, the space of de Rham p-boundaries. These are closed subspaces of $\mathcal{D}_p$ and $H_p(M) = \mathcal{Z}_p/\mathcal{B}_p$.

3.2. The 1-currents and cones associated to $\mathcal{X}$

We return to the core lamination $\mathcal{X}$ associated to $\mathcal{L}$.

Definition 3.10. *A Dirac current for $\mathcal{X}$ is a positively oriented, nontrivial tangent vector to $\mathcal{X}$. The closure in $\mathcal{D}'_1$ of the union of all positive linear combinations of Dirac currents is a closed, convex cone $\mathcal{C}_{\mathcal{X}} \subset \mathcal{D}'_1$, called the cone of* asymptotic currents.

This cone lies on one side of a hyperplane $H = \omega^{-1}(0)$, where $\omega\colon \mathcal{D}'_1 \to \mathbb{R}$ is a 1-form such that $\omega(v) > 0$ on $|\mathcal{X}|$. (Recall that we assume that $\mathcal{L}$ is integral to a continuous, nonsingular vector field v.) Such a form is easily produced. Indeed, a small perturbation of v produces a nonsingular C^∞ field v' which, together with a Riemannian metric, defines a 1-form $\omega = \langle v', \cdot \rangle$ which is everywhere positive on $v|\mathcal{X}$ as desired.

The base $\widehat{\mathcal{C}}_{\mathcal{X}} = \mathcal{C}_{\mathcal{X}} \cap \omega^{-1}(1)$ of the cone $\mathcal{C}_{\mathcal{X}}$ is compact (cf. [3, Lemma 10.2.3]. The cited proof goes through by the compactness of $|\mathcal{X}|$ and the fact that our spaces are Montel.) Those continuous linear functionals $\eta\colon \mathcal{D}_1' \to \mathbb{R}$ which are strictly positive on $\widehat{\mathcal{C}}_{\mathcal{X}}$ are exactly the smooth 1-forms on M which are transverse to $\mathcal{X}$ (meaning that they take a positive value on each Dirac current). Sullivan applies the Hahn–Banach theorem, using compactness of the base, to produce interesting 1-forms that are transverse to $\mathcal{X}$ (see [3, Subsection 10.2]).

The cone $\mathcal{C}_{\mathcal{X}} \cap \mathcal{Z}_1$ of *asymptotic cycles* is also a closed, convex cone with compact base. There is a natural continuous linear surjection of $\mathcal{Z}_1$ onto $H_1(M)$, carriying the cone of asymptotic cycles onto a convex cone $\mathfrak{C}'_{\mathcal{X}} \subset H_1(M)$ with compact base. Compactness of the base implies that this cone is closed. There is a dual closed, convex cone in $H^1(M)$ defined by

$$\mathfrak{C}_{\mathcal{X}} = \{[\eta] \in H^1(M) \mid [\eta]([z]) \geq 0,\, \forall [z] \in \mathfrak{C}'_{\mathcal{X}}\}.$$

Definition 3.11. *The homology cone* $\mathfrak{C}'_{\mathcal{X}}$ *and the cohomology cone* $\mathfrak{C}_{\mathcal{X}}$ *will be called Sullivan cones.*

Generally, the Sullivan cone $\mathfrak{C}_{\mathcal{X}}$ does not have a compact base, as the following example shows.

Example 3.12. In Example 2.5, every point of L except p escapes under forward and/or backward iterations of f to ends of L. Consequently, in the depth 1 foliated manifold M in which f is the monodromy of a depth 1 leaf L, the leaves of $\mathcal{L}_f$ (oriented by the transverse orientation of the foliation) that do not pass through p all meet $\partial_\tau M$. The leaf σ through p is an oriented loop, hence a singular 1-cycle. This loop is all of $\mathcal{X}$ and one can prove that it is an asymptotic cycle. All asymptotic cycles are obtained by adding elements of $\mathcal{B}_1$ to nonnegative multiples of σ. Thus, $\mathfrak{C}'_{\mathcal{X}}$ reduces to the single ray $\{a[\sigma]\}_{a\geq 0} \subset H_1(M)$. It follows that $\mathfrak{C}_{\mathcal{X}}$ is a whole half-space in $H^1(M)$, hence does not have compact base.

Examples of asymptotic cycles are nonnegative, transverse, holonomy invariant measures μ on $\mathcal{X}$ that are finite on (transverse) compact sets. By Sullivan (cf. [3, Theorem 10.2.12]), these are the only ones.

Theorem 3.13. *The asymptotic cycles for* $\mathcal{X}$ *are exactly the nonnegative, transverse, holonomy invariant measures on* $\mathcal{X}$ *that are finite on compact sets.*

By a well known theorem of J. F. Plante [29] and the fact that the leaves of $\mathcal{X}$, being 1-dimensional, are either compact or have linear growth, we obtain the following.

Lemma 3.14. *There are nontrivial asymptotic cycles for $\mathcal{X}$.*

Lemma 3.15. *No nontrivial asymptotic cycle bounds.*

Proof. Recall that $\mathcal{F}|M^\circ$ defines a fibration $\pi\colon M^\circ \to S^1$. If $d\theta$ is the canonical closed, nonsingular 1-form on S^1, let $\omega = \pi^*(d\theta)$, a closed, nonsingular 1-form on M° transverse to $\mathcal{X}$. A small deformation retraction of M into M° pulls ω back to a closed form ω' on M which is also transverse to $\mathcal{X}$. Thus, ω' takes positive values on all nontrivial asymptotic cycles which, therefore, cannot bound. Q.E.D.

Theorem 3.16. *The interior of $\mathfrak{C}_{\mathcal{X}}$ is nonempty and consists exactly of those classes $[\eta] \in H^1(M)$ that are represented by closed 1-forms η transverse to $\mathcal{X}$.*

For the proof, see [3, Lemma 10.2.8].

3.3. Homology directions

It will be important to characterize a particularly simple spanning set of $\mathfrak{C}'_{\mathcal{X}}$, the so called "homology directions" of Fried [18, p. 260]. Parametrize $\mathcal{L}|M^\circ$ as a nonsingular flow Φ_t, preserving $\mathcal{F}|M^\circ$, such that Φ_1 sends each leaf of $\mathcal{F}|M^\circ$ to itself. Select a point $x \in \mathcal{X}$ and let Γ (depending on x) denote the Φ-orbit of x. If this is a closed orbit, it defines an asymptotic cycle which we will denote by $\overline{\Gamma}$. If it is not a closed orbit, let $\Gamma_\tau = \{\Phi_t(x) \mid 0 \le t \le \tau\}$. Let $\tau_k \uparrow \infty$ and set $\Gamma_k = \Gamma_{\tau_k}$. The singular chain $(1/\tau_k)\Gamma_k$ is called a "long, almost closed orbit" of $\mathcal{X}$. After passing to a subsequence, we obtain an asymptotic current as the limit

$$\overline{\Gamma} = \lim_{k\to\infty} \frac{1}{\tau_k} \int_{\Gamma_k}$$

in the topological vector space $\mathcal{D}_1$.

Lemma 3.17. *The currents $\overline{\Gamma}$ are asymptotic cycles.*

Proof. The endpoints of Γ_k lie in the compact set $|\mathcal{X}|$, hence Γ_k can be closed by adding a curve s_k in M of length uniformly bounded independently of k. This gives a singular cycle $\Gamma'_k = \Gamma_k + s_k$ which can be viewed as a closed de Rham current. Since the s_k's have uniformly bounded length, the cycles $(1/\tau_k)\Gamma'_k$ also limit on $\overline{\Gamma}$. Since $\mathcal{Z}_1$ is closed in $\mathcal{D}_1$, $\overline{\Gamma} \in \mathcal{Z}_1$. Q.E.D.

Another proof can be given by appealing to Stokes' theorem.

Definition 3.18. *The asymptotic cycles* $\overline{\Gamma}$ *and their homology classes* (*also denoted by* $\overline{\Gamma}$) *will be called "homology directions". Both* $(1/\tau_k)\Gamma_k$ *and* $(1/\tau_k)\Gamma'_k$ *will be called long, almost closed orbits.*

An elementary application of ergodic theory proves the following (see [34, Proposition II.25] and [3, Proposition 10.3.11]).

Lemma 3.19. *Any asymptotic cycle* $\mu \in \mathcal{Z}_1$ *can be arbitrarily well approximated by finite*, *nonnegative linear combinations* $\sum_{i=1}^{r} a_i \overline{\Gamma}_i$ *of homology directions. If* $\mu \neq 0$, *the coefficients* a_i *are strictly positive and their sum is bounded below by a constant* $b_\mu > 0$ *depending only on the cycle* μ.

§4. Properties of Sullivan cones

We establish some important properties of the Sullivan cones and their relations to foliations.

4.1. Independence of the choice of $\mathcal{L}_f$

The Sullivan cones $\mathfrak{C}_{\mathcal{X}}$ and $\mathfrak{C}'_{\mathcal{X}}$ seem to depend not merely on the monodromy $f\colon L \to L$, but on the choice of the transverse 1-dimensional foliation $\mathcal{L} = \mathcal{L}_f$ which defines f as first return map. Our first task will be to show that these cones depend only on f. After that, we will denote them by $\mathfrak{C}_f$ and $\mathfrak{C}'_f$.

In [8], the following elementary theorem was deduced as a corollary of a much deeper result (Lemma 4.10 in that reference) which we attempted to deduce from results of M. E. Hamstrom [21] [22] [23]. A correct proof of that lemma needs a deep result of T. Yagasaki [37], but we omit this because we do not need it.

Theorem 4.1. *Let* L *be a leaf of* $\mathcal{F}|M^\circ$, *and let* $\mathcal{L}$ *and* $\mathcal{L}_\sharp$ *be* 1-*dimensional foliations of* M *transverse to* $\mathcal{F}$, *having respective core laminations* $\mathcal{X}$ *and* $\mathcal{X}_\sharp$, *and inducing the same endperiodic monodromy* $f\colon L \to L$. *Then* $\mathfrak{C}'_{\mathcal{X}} = \mathfrak{C}'_{\mathcal{X}_\sharp}$ *and* $\mathfrak{C}_{\mathcal{X}} = \mathfrak{C}_{\mathcal{X}_\sharp}$.

We will show that the homology directions of $\mathcal{X}$ are exactly the same as the ones of $\mathcal{X}_\sharp$ and the theorem will follow.

Our proof of Theorem 4.1 depends on Lemma 4.2 and its corollary below. In this lemma and its corollary, we consider an arbitrary connected surface L that is either compact with strictly negative Euler characteristic or is noncompact, noncontractible and not homotopy equivalent to the circle. It will not matter in the proofs of the lemma or its corollary whether L is a leaf of a foliation, let alone a leaf at depth 1.

Let I be the compact interval $[0,1]$ and consider the product $L \times I$. (One obtains such a product, for instance, by cutting M apart along the depth 1 leaf L and taking as the interval fibers the resulting segments of the leaves of $\mathcal{L}$.) For each $x \in L$, denote by I_x the interval fiber with endpoints $\{x\} \times \{0,1\}$. Consider a second fibration of $L \times I$ by intervals J_x, requiring that the endpoints of J_x coincide with those of I_x, for all $x \in L$. (In using the lemma and its corollary to prove Theorem 4.1, this second fibration arises by cutting apart along L and using the segments of leaves of $\mathcal{L}_\sharp$ as the fibers J_x.) For each $x \in L$, let α_x denote the loop in $L \times I$ obtained by following I_x from $(x,0)$ to $(x,1)$ and then following J_x from $(x,1)$ to $(x,0)$. Finally, if $p : L \times I \to L$ is the canonical projection, let $\beta_x = p \circ \alpha_x$, a loop in L.

For the following two results, fix the hypothesis that L is an arbitrary connected surface that is either compact with strictly negative Euler characteristic or is noncompact, noncontractible and not homotopy equivalent to the circle.

Lemma 4.2. *Let $x_0 \in L$ and set $\delta = \beta_{x_0}$. If $\gamma(s)$, $0 \le s \le 1$, is any other closed curve in L based at x_0, then $\gamma \cdot \delta = \delta \cdot \gamma$ in $\pi_1(L, x_0)$.*

Proof. Define $F(s,t) = \beta_{\gamma(s)}(t)$. Then $F(s,0) = \gamma(s) = F(s,1)$. Also $F(0,t) = F(1,t) = \beta_{x_0}(t) = \delta(t)$. Because of this last, we can view F as a map from the cylinder $S^1 \times [0,1]$ into L. The curve obtained by following $F(0,t)$, $0 \le t \le 1$, followed by $F(s,1)$, $0 \le s \le 1$, and then $F(0,1-t)$, $0 \le t \le 1$, is the composite loop $\delta \cdot \gamma \cdot \delta^{-1}$. We show how to deform this curve continuously to γ, keeping the basepoint x_0 fixed throughout the deformation.

Let σ_t be the curve obtained by following $F(0,\tau)$, $0 \le \tau \le t$, followed by $F(s,t)$, $0 \le s \le 1$, followed by $F(0, t-\tau)$, $0 \le \tau \le t$. Since $\sigma_0 = \gamma$ and $\sigma_1 = \delta \cdot \gamma \cdot \delta^{-1}$, we have the desired deformation. Q.E.D.

Corollary 4.3. *If there exists an $x_0 \in L$ so that J_{x_0} cannot be deformed into I_{x_0} keeping the endpoints fixed, then L is either compact with nonnegative Euler characteristic, or is contractible, or has the homotopy type of the circle.*

Proof. The hypothesis implies that α_{x_0} is essential in $L \times I$, and so β_{x_0} is essential in L and thus is a nontrivial element of $\pi_1(L, x_0)$. By Lemma 4.2, every element of $\pi_1(L, x_0)$ commutes with β_{x_0}. Thus, if L is compact, $\chi(L) \ge 0$. If L is not compact and not contractible, L is homotopically equivalent to a bouquet B of circles. The only bouquet of circles that contains a nontrivial element of $\pi_1(B, *)$ that commutes with every other element of $\pi_1(B, *)$ is one circle. Q.E.D.

Proof of Theorem 4.1. Under our ongoing hypotheses (see the second page of this article), our depth 1 leaves are obviously noncompact, noncontractible and do not have the homotopy type of the circle.

Let $\overline{\Gamma} \in H_1(M;\mathbb{R})$ be a homology direction for $\mathcal{X}$.

First assume that $\overline{\Gamma}$ is not represented by a closed orbit in $\mathcal{X}$ and write

$$\overline{\Gamma} = \lim_{k\to\infty} \frac{1}{\tau_k}[\Gamma'_k],$$

a limit in $H_1(M)$. Recall that Γ'_k is the singular cycle obtained by closing Γ_k by adding an arc s_k of length uniformly bounded independently of k. The numbers τ_k are the "lengths" of Γ_k (measured by the transverse, invariant measure for $\mathcal{F}|M^\circ$) and increase to ∞ with k. Thus, except for a uniformly bounded arc in L, Γ'_k is a sequence of segments, $\sigma_1,\dots,\sigma_{n_k}$ of an orbit in $\mathcal{X}$, each starting and ending in L. There is a corresponding sequence $\sigma'_1,\dots,\sigma'_{n_k}$ of segments of an orbit in $\mathcal{X}_\sharp$ such that σ_i and σ'_i have the same endpoints and the same lengths, $1 \le i \le n_k$. By Corollary 4.3, these respective segments are homotopic by a homotopy that keeps their endpoints fixed. Thus, we see that $\overline{\Gamma}$ is also a homology direction for $\mathcal{X}_\sharp$. In the case that $\overline{\Gamma}$ is represented by a closed orbit, the argument adapts and is simpler. Finally, the roles of $\mathcal{X}$ and $\mathcal{X}_\sharp$ can be interchanged, proving that the two laminations have the same homology directions. Q.E.D.

4.2. Invariance under certain isotopies

Generally, the Sullivan cones change as the monodromy f is varied by an isotopy. In fact, one of the primary goals of this paper is to find an endperiodic automorphism h, isotopic to f, so that $\mathfrak{C}'_h \subseteq \mathfrak{C}'_g$ for all endperiodic automorphisms g isotopic to f. This Sullivan cone is, therefore, minimal and the dual cone $\mathfrak{C}_h$ is maximal for the isotopy class of f.

We will see, however, that these cones do not change if f is varied by what we will call a "strong isotopy".

Definition 4.4. *A leafwise isotopy is an ambient isotopy φ_t of M such that $\varphi_0 = \mathrm{id}$ and φ_t carries each leaf of $\mathcal{F}$ to itself,* $0 \le t \le 1$.

We think of a leafwise isotopy as "sliding along the leaves of $\mathcal{F}$".

Proposition 4.5. *Let $\mathcal{L}$ and $\mathcal{L}_\sharp$ be two 1-dimensional foliations transverse to $\mathcal{F}$. Suppose that the respective core laminations $\mathcal{X}$ and $\mathcal{X}_\sharp$ are isotopic by a leafwise isotopy φ_t, $\mathcal{X} = \varphi_0(\mathcal{X})$, $\mathcal{X}_\sharp = \varphi_1(\mathcal{X})$. Then $\mathfrak{C}'_{\mathcal{X}} = \mathfrak{C}'_{\mathcal{X}_\sharp}$ and $\mathfrak{C}_{\mathcal{X}} = \mathfrak{C}_{\mathcal{X}_\sharp}$.*

Proof. Parametrize the two foliations as flows using the same transverse invariant measure θ for $\mathcal{F}$. Since $\mathcal{F}$ is leafwise invariant under the isotopy, the flow parameter is preserved and the singular cycles Γ'_k giving rise to a homology direction for $\mathcal{X}$ are isotoped to those for $\mathcal{X}_\sharp$. Notice that the lengths τ_k given by the transverse invariant measure are invariant by such an isotopy and the arcs s_k also remain uniformly bounded. Homotopic singular cycles are homologous and the assertions follow. Q.E.D.

Definition 4.6. *Homeomorphisms $f, g\colon L \to L$ are strongly isotopic if there is an ambient isotopy $\varphi_t\colon L \to L$, $\varphi_0 = \mathrm{id}$, such that $g = \varphi_1 \circ f \circ \varphi_1^{-1}$.*

Corollary 4.7. *Let $f, g\colon L \to L$ be endperiodic first return homeomorphisms induced on a leaf L of $\mathcal{F}$ by transverse 1-dimensional foliation $\mathcal{L}_f$ and $\mathcal{L}_g$ of class C^0. If f and g are strongly isotopic, then $\mathfrak{C}'_f = \mathfrak{C}'_g$.*

Proof. Let N be a closed normal neighborhood of L in M° which is a foliated product with leaves the leaves of $\mathcal{F}$ meeting N and normal fibers the arcs of $\mathcal{L}\cap N$. Write $N = L\times[-\varepsilon, \varepsilon]$ and consider each arc ℓ_x of a leaf of $\mathcal{L}$ issuing in the positive direction from $(x, \varepsilon) \in L\times\{\varepsilon\}$ and first returning to N at $(f(x), -\varepsilon)$. In N, replace each arc $\tau_y = \{y\} \times [-\varepsilon, \varepsilon]$ of $\mathcal{L} \cap N$ with an arc $\sigma_y\colon [-1, 1] \to N$ defined by

$$\sigma_y(t) = (\varphi_{t+1}(y), \varepsilon t), \quad -1 \le t \le 0,$$
$$\sigma_y(t) = (\varphi_t^{-1}(\varphi_1(y)), \varepsilon t), \quad 0 \le t \le 1.$$

Notice that this still connects $(y, -\varepsilon)$ to (y, ε). We construct an ambient leaf-preserving isotopy ψ, supported in N and carrying each τ_y to σ_y, by

$$\psi_s(y, \varepsilon t) = (\varphi_{s(t+1)}(y), \varepsilon t), \quad -1 \le t \le 0,\ 0 \le s \le 1,$$
$$\psi_s(y, \varepsilon t) = (\varphi_{st}^{-1}(\varphi_s(y)), \varepsilon t), \quad 0 \le t \le 1,\ 0 \le s \le 1.$$

We obtain $\mathcal{L}'$ from $\mathcal{L}$ by replacing τ_y with σ_y, $\forall y \in L$, observing that the monodromy induced by $\mathcal{L}'$ on $L = L \times \{0\}$ is $\varphi_1 \circ f \circ \varphi_1^{-1}$. The assertion follows by Proposition 4.5. Q.E.D.

4.3. Foliated forms

In this subsection we consider the general situation in which $\mathcal{L}$ is assumed to be transverse to a foliation $\mathcal{F}$ almost without holonomy which is possibly dense leaved in M°. We will need that $\mathcal{L}\smallsetminus\mathcal{X}$ is smooth and that $\mathcal{L}$ itself is integral to a nonsingular C^0. vector field. Our discussion will be valid for all dimensions ≥ 3 and without restrictions on the topology of leaves.

Definition 4.8. *A 1-form $\eta \in A^1(M^\circ)$ is a foliated form if it is closed, nowhere vanishing and becomes unbounded at $\partial_\tau M$ in such a way that the corresponding foliation $\mathcal{F}^\circ_\eta$ that it defines on M° extends by adjunction of $\partial_\tau M$ to a C^∞ foliation $\mathcal{F}_\eta$ of M, C^∞-flat at $\partial_\tau M$.*

We will prove the following.

Theorem 4.9. *The open cone* $\operatorname{int}\mathfrak{C}_{\mathcal{X}}$ *consists of classes in $H^1(M)$ that can be represented by foliated forms transverse to $\mathcal{L}|M^\circ$.*

Remark that foliated forms only live in M°, not in M, but $H^1(M) = H^1(M^\circ)$ and any form representing a class in M° can be taken to be equal to that representing the class in M outside of any small neighborhood of $\partial_\tau M$.

The rays out of the origin in $\operatorname{int}\mathfrak{C}_{\mathcal{X}}$ correspond to smooth foliations almost without holonomy. The rational rays correspond to foliations defined by forms η with period group infinite cyclic, defining depth 1 foliations of M. The rest of the rays in $\operatorname{int}\mathfrak{C}_{\mathcal{X}}$ consist of classes having period group dense in $\mathbb{R}$ and so define foliations that are dense leaved in M°.

Proof of Theorem 4.9. Fix a class $[\eta] \in \operatorname{int}\mathfrak{C}_{\mathcal{X}}$, the 1-form $\eta \in [\eta]$ being defined on M and transverse to $\mathcal{X}$ (Theorem 3.16). Select a neighborhood U of $|\mathcal{X}|$ such that $\eta \pitchfork (\mathcal{L}|U)$. We need to show that η is cohomologous to a foliated form.

Given $x \in M^\circ \setminus |\mathcal{X}|$, let $s(t)$ be the smooth trajectory along $\mathcal{L}$ in that set, smoothly reparametrized so that $x = s(0)$ and the trajectory either reaches an outwardly oreiented boundary leaf at time $t = 1$ or an inwardly oriented one at time $t = -1$, or both. Let F_+ denote the union of outwardly oriented components of $\partial_\tau M$ and F_- the union of the inwardly oriented ones. For definiteness, consider the case $s(-1) \in F_-$. Define a tubular neighborhood $V_x = D \times [-1, 3/4)$ of s so that $s(t) = (0, t)$ and $\{z\} \times [-1, 3/4)$ is an arc in $\mathcal{L}$, $\forall z \in D$. Here, D is the open unit $(n-1)$-ball with polar coordinates $(r, \theta_1, \dots, \theta_{n-2})$, $0 \le r < 1$. This gives cylindrical coordinates $(t, r, \theta_1, \dots, \theta_{n-2})$ on V_x. On V_x, define a smooth, real valued function

$$\ell_x(t, r, \theta_1, \dots, \theta_{n-1}) = \ell_x(t, r) = \ell(t)\lambda(r),$$

where $\ell(t) = t - 1$, $-1 \le t \le 1/2$, and damps off to 0 smoothly and with positive derivative as $t \to 3/4$, and $\lambda(r) \equiv 1$, $0 \le r \le 1/2$, and damps off to 0 smoothly through positive values as $r \to 1$. Thus, $\ell_x(t, r)$ vanishes outside of V_x and $d\ell_x$ is transverse to $\mathcal{L}$ in V_x. Let $V'_x \subset V_x$ be the neighborhood of x defined by $-1 \le t < 1/2$ and $0 \le r < 1/2$. Perform an analogous construction for trajectories out of x with $s(1) \in F_+$. If the trajectory through x goes from $s(-1) \in F_-$ to $s(1) \in F_+$, the cylinder $V_x = D \times [-1,1]$ and the construction of ℓ_x is simpler.

A suitable choice of these open cylinders gives an open cover

$$\{U, V'_x\}_{x \in M^\circ}$$

of M. Using the local compactness, pass to a finite subcover

$$\{U, V'_{x_1}, V'_{x_2}, \dots, V'_{x_r}\}.$$

For suitable choices of positive constants c_i, set $\ell = \sum_{i=1}^r c_i \ell_{x_i}$, a smooth function, supported in $M \smallsetminus |\mathcal{X}|$, with $d\ell \pitchfork \mathcal{L}$ outside of a compact neighborhood of $|\mathcal{X}|$ in U. Since η is transverse to $\mathcal{L}$ in U, we can choose the coefficients $c_i > 0$ large enough that $\eta' = \eta + d\ell$ is a closed form in M, cohomologous to η and transverse to $\mathcal{L}$. This form might be badly behaved at $\partial_\tau M$, hence we must modify it by adding on a suitable exact form supported in a neighborhood of the boundary leaves.

Let $V = F_- \times [0,1)$ be a normal neighborhood of $F_- \equiv F_- \times \{0\}$ in M, the fibers being arcs in leaves of $\mathcal{L}$. Let λ be a smooth function on the deleted normal neighborhood $V \smallsetminus F_-$, depending only on the normal parameter t, and having $\lambda'(t) \geq 0$, with $\lambda'(t) = e^{1/t^2}$ near F_-. Make a similar construction near F_+. Now $\widetilde{\eta} = \eta' + d\lambda$ is everywhere transverse to $\mathcal{L}$, hence nonsingular on M°, it is cohomologous to η' and it becomes unbounded at $\partial_\tau M$. We must show that $\ker \widetilde{\eta}$ extends C^∞-smoothly to a plane field on M by adding on the tangent planes to $\partial_\tau M$. For this, set $\overline{\eta} = \widetilde{\eta}/\lambda' = \eta'/\lambda' + dt$, a form defined on a small enough deleted neighborhood of F_-. This form is no longer closed but satisfies $\ker \overline{\eta} = \ker \widetilde{\eta}$ in that neighborhood. Since η' is bounded on M, it is clear that $\overline{\eta}$ approaches dt in the C^∞ topology as $t \to 0$ and that the resulting foliation of M is of class C^∞ which is C^∞-trivial at $\partial_\tau M$. After a similar construction in a normal neighborhood of F_+, we obtain a foliated form, again denoted by $\widetilde{\eta}$, transverse to $\mathcal{L}$. The proof of Theorem 4.9 is complete. Q.E.D.

Remark. Thus, $\mathfrak{C}_{\mathcal{X}}$ can be called a foliation cone, but we will reserve this term for the maximal cone which will be produced using the Handel–Miller theory.

Definition 4.10. *A ray out of the origin in $H^1(M;\mathbb{R})$ containing a class represented by a foliated form will be called a* foliated ray.

§5. The Handel–Miller theory

For the reader's convenience, we give here a sketch of the main features of the Handel–Miller theory, as established in great detail in

[5]. This theory was developed in analogy with the Nielsen–Thurston theory which we will also need, so we begin with a sketch of that.

5.1. Nielsen–Thurston theory

We will state the central theorem and then offer some explanations and heuristics.

Theorem 5.1. *Let N be a compact, connected, oriented surface of negative Euler characteristic, possibly with boundary, and let $f\colon N \to N$ be a homeomorphism. Then f is isotopic to a homeomorphism h such that one of the following holds:*

(1) $h^n = \mathrm{id}$, *some integer $n > 0$;*
(2) *h is pseudo-Anosov;*
(3) *there is a finite collection $\{s_1, \dots, s_k\}$ of disjoint simple closed curves such that h permutes these curves and, in fact, permutes open, annular regular neighborhoods V_i of s_i. Let $S_1, \dots, S_m$ be the components of $N \smallsetminus \bigcup_{i=1}^{k} V_i$ and n_j the least positive integer such that $h^{n_j}(S_j) = S_j$. Then $h^{n_j}|S_j$ satisfies* (1) *or* (2).

The third case is called the reducible case. Briefly, then, the isotopy class of every automorphism f of N has a representative that is either periodic, pseudo-Anosov or reducible.

A proof was sketched in [36] and given in full detail in [15]. A very readable account will be found in [24], but with a weaker version of (1), asserting only that h^n is isotopic to the identity. See also [2]. The stronger assertion was made by Nielsen who gave a false proof. It was established by Thurston using Smith theory and his deep analysis of the boundary of Teichmüller space.

It is necessary to explain (2). For the homeomorphism h to be pseudo-Anosov, there must be a pair of measured, h-invariant, geodesic laminations Λ^s (the *stable* lamination) and Λ^u (the *unstable* lamination). The measures are transverse, holonomy invariant measures μ_s and μ_u, respectively, and there is a constant $\lambda > 1$ such that, under the action of h, μ_s is multiplied by $1/\lambda$ and μ_u by λ. Thus, under h, the stable lamination contracts transversely and the unstable lamination expands transversely.

There is a real difficulty, at least for our purposes, with the pseudo-Anosov case. While the periodic automorphism h in (1) can be taken to be a diffeomorphism, there are serious obstructions to smoothing the automorphism in (2). In order to remedy this problem, at least partially, one "blows down" the laminations to singular foliations $\mathcal{F}^s$ and $\mathcal{F}^u$. The laminations are transversely Cantor and the blow down mimics the standard way of mapping a Cantor subset of $\mathbb{R}$ onto an

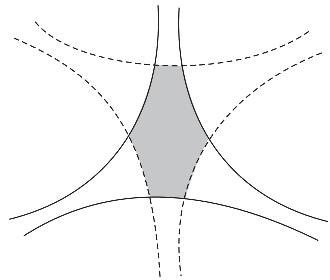

Fig. 4. Dual principal regions and nucleus

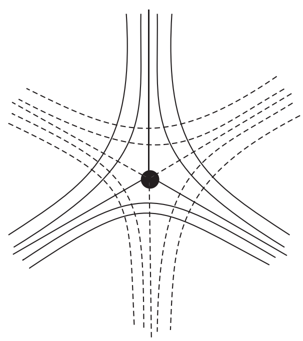

Fig. 5. A 3-pronged common singularity

interval by collapsing each of its gaps to a point. The singularities of these foliations are very important and will now be described.

The complement of Λ^s consists of finitely many components, called *principal regions*, which outside of a bounded region look like infinitely long strips. In Figure 4 a principal region is depicted. A dual principal region in Λ^u is indicated by dashed lines. The shaded intersection of these dual principal regions is their common *nucleus*. The principal regions of the two laminations all come in such dual pairs. The blow down collapses the nucleus to a point, this being a common singularity for the foliations $\mathcal{F}^s$ and $\mathcal{F}^u$. The picture of the foliations near this singularity is given in Figure 5. Such a singularity is called "p-pronged". In our figure, $p = 3$, there being 3 prongs issuing from the singularity which are leaves of $\mathcal{F}^s$ and 3 which are leaves of $\mathcal{F}^u$.

The measures survive the blowdown and the pseudo-Anosov map passes to a homeomorphism, again denoted by h and called pseudo-Anosov, preserving each of the foliations and again multiplying μ_s by $1/\lambda$ and μ_u by λ. These measures define a coordinate atlas on N which is smooth except at the singularities and which makes h a diffeomorphism except at the singularities. For more details, see [10, Appendix], where the structure of the foliations at the boundary of N is also described and the smoothness of h there is established. This latter fact was obscured in the literature (cf. [15, pp. 216–217, erratum, p. 286]).

Theorem 5.2. *The dynamical system generated by the pseudo-Anosov map h admits a Markov partition.*

For a definition and quick proof, see [2, Corollary 6.5.1]. For more detail, see [15, pp. 191–204]. This implies that h is semi-conjugate to a 2-ended subshift of finite type. We will need this fact in proving that our cones are polyhedral.

Finally, a word about the construction of Λ^s and Λ^u. To begin with, one is given the automorphism $f\colon N \to N$. Fix a choice of hyperbolic metric on N, choose any simple closed geodesic s and apply f to s. Tighten $f(s)$ to a closed geodesic and again apply f. Iterate this procedure, ad infinitum, producing a sequence of simple, closed geodesics which, to the naked eye, begin to look more and more like laminations. In fact, a subsequence converges to a geodesic lamination in the Hausdorff metric. This is not generally the right lamination, but after some more work, one produces Λ^s. One produces Λ^u similarly by using f^{-1} instead of f

Remark 5.3. The homeomorphism h of Theorem 5.1 can be taken to be a diffeomorphism except at the finitely many singularities. With a slight and common abuse of terminology, we will call it a (Nielsen–Thurston) diffeomorphism. In our application of this theorem, another mild problem arises. Our compact surface N will have piecewise smooth boundary anf f will be a diffeomorphism in a small open annular neighborhood V of ∂N, where $\partial\overline{V} \smallsetminus \partial N$ is smooth. A preliminary isotopy allows us to assume that f preserves $N' = N \smallsetminus V$. One applies Theorem 5.1 to $f|N'$, extending f over V to agree with its original definition near ∂N.

5.2. A heuristic example for the Handel–Miller theory

The example (which we learned from [16]) sketched here is a little too simple to illustrate all the subtleties of the Handel–Miller theory, but has enough complexity to give the general idea. In Figure 6, an

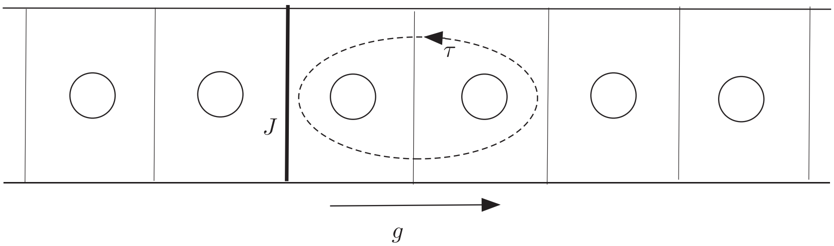

Fig. 6. $f = \tau \circ g$ is the composition of a translation and a Dehn twist

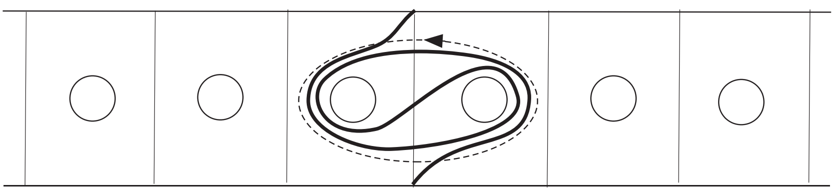

Fig. 7. $f(J)$ tightened to a geodesic with the same endpoints

endperiodic automorphism $f\colon L \to L$ is indicated, where g is a simple translation, τ is a Dehn twist in the dotted oval curve, and $f = \tau \circ g$.

We fasten our attention on the arc J and observe that $f(J)$ looks roughly like the curve pictured in Figure 7. In fact, we should view this curve as the tightening of $f(J)$ to a geodesic relative to a choice of hyperbolic metric on L. In this tightening, the endpoints remain fixed.

Finally, in Figure 8, we depict the geodesic tightening of $f^2(J)$. It is intuitively plausible that this process converges in some reasonable sense to a geodesic lamination Λ_+. This is, in fact true. The proof uses standard methods in hyperbolic geometry and is not very deep. It is hard to draw a picture of the limit lamination, but fairly easy to draw the "train track" that carries it. We do this in Figure 9. The lamination is transversely Cantor, so each segment of the track represents a Cantor-like packet of arcs. The dotted track carries the geodesic lamination Λ_- obtained by a similar process using f^{-1} rather than f and a new choice of J on the opposite side of the circle in which the Dehn twist is performed. At the switches, the Cantor packets split at a gap, one packet heading off the main line into a side track, the other continuing on the main line. Finally, one shows that there is an endperiodic homeomorphism h, isotopic to f, which preserves these laminations.

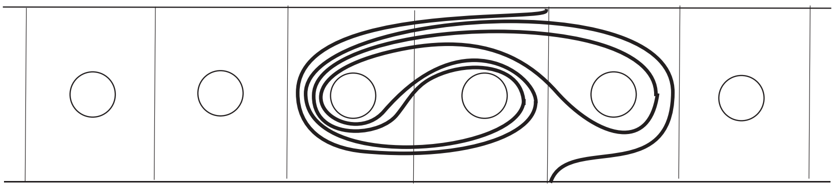

Fig. 8. $f^2(J)$ tightened to a geodesic with the same endpoints

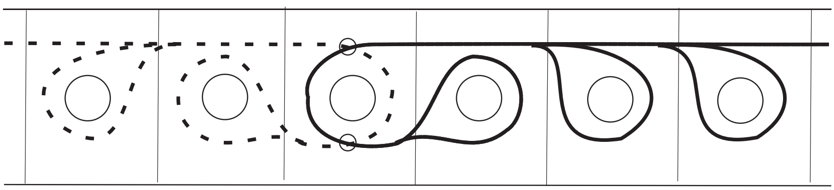

Fig. 9. Traintracks carrying the laminations

The laminations Λ_+ and Λ_- are analogues of Λ^s and Λ^u, respectively, in the Nielsen–Thurston theory. The analogy is not complete. For example, the transverse measures do not generally have full support. There is still a weak sense in which Λ_+ contracts transversely and Λ_- expands transversely. Our choice of notation emphasizes the fact that Λ_+ lives entirely outside a neighborhood of the negative end and Λ_- outside a neighborhood of the positive end. In [5], it seemed more important to emphasize this distinction than the notions of "stable" and "unstable".

Very important for our purposes is the Cantor set $X_0 = \Lambda_+ \cap \Lambda_-$. This set is h-invariant and the dynamical system generated by $h|X_0$ admits a Markov partition. When L is realized as a noncompact leaf of a depth 1 foliation with monodromy h, the $\mathcal{L}_h$-saturation of X_0 will be the core lamination $\mathcal{X}$.

Remark. Another example, but way oversimple, is given by Example 2.5. The two circles bounding the unshaded region on the left can play the role of J in our example. Repeatedly applying f and tightening limits on the boldface curve outside a neighborhood of the negative ends e_1, e_2. The similar iteration process, using f^{-1} and the right boundary circle of the unshaded region, limits on the boldface curve outside a neighborhood of the positive end e. Thus, each of the laminations Λ_+ and Λ_- consists of a single geodesic and X_0 degenerates to a single point fixed by h. As earlier observed, $\mathcal{X}$ is just the closed leaf of $\mathcal{L} = \mathcal{L}_h$ through this point.

5.3. A brief sketch of the Handel–Miller theory

Let L be a complete hyperbolic surface with finitely many ends and geodesic boundary. Let $f\colon L \to L$ be an endperiodic automorphism. It is necessary to assume that no end has a neighborhood homeomorphic to $K \times [0, \infty)$, K being a circle or compact interval. In this paper, the assumption that each compact leaf has negative Euler characteristic guarantees this hypothesis. It is also necessary that the hyperbolic surface contains no isometrically imbedded hyperbolic half-planes. Because L is to be a leaf of a depth 1 foliation, this hypothesis is easily guaranteed.

In their original work, Handel and Miller proceeded, in analogy with our heuristic discussion, to generate geodesic laminations Λ_+, disjoint from a neighborhood of the negative ends, and Λ_-, disjoint from a neighborhood of the positive ends. The compact 1-manifold J to which iterates of f are applied is the union of components of ∂U as U ranges over the neighborhoods of positive ends given in Definition 2.1. Some caution is needed here. As earlier remarked, one must choose U carefully so that ∂U and $f(\partial U)$ intersect, if at all, only in common connected components (cf. [5, Proposition 2.30]). The lamination Λ_- is constructed similarly and an endperiodic automorphism h, isotopic to f and preserving the laminations is constructed. For details on the construction of $\Lambda_\pm$, see [5, Section 4.7]. For the construction of h, See Section 9 of that reference.

It is prohibitively inconvenient for many purposes to require the laminations to be geodesic. By axiomatizing the structure of the laminations in [5, Section 4.3], we relax this condition while keeping the theory on a rigorous foundation. We will not give these axioms here, but describe the general setup and state the principal theorems proven in the above reference.

Definition 5.4. *A endperiodic map $h\colon L \to L$ which preserves $\Lambda_\pm$ and is isotopic to f wil be called a* Handel–Miller automorphism *associated to f.*

Recall that, if $\partial L = \emptyset$, the universal cover $\widetilde{L}$, together with the lifted metric, is the hyperbolic plane. We use the Poincaré disk model. This is the open unit disk Δ with the standard hyperbolic metric. The unit circle S^1_∞, while not part of the hyperbolic plane, plays a fundamental role. It is called the circle at infinity or the ideal boundary. If $\partial L \neq \emptyset$, $\widetilde{L}$ with its metric can be viewed as imbedded in Δ with boundary an infinite family of complete geodesics. As is standard, these geodesics have ideal endpoints in S^1_∞ and the closure of the set of such endpoints in S^1_∞ is a Cantor set $\mathfrak{K}$. This is called the ideal boundary of $\widetilde{L}$. In general, then, the term "ideal boundary" refers either to S^1_∞, if $\partial L = \emptyset$, or to $\mathfrak{K}$ otherwise.

As is standard, every complete geodesic in L without endpoints has lifts to geodesics in $\widetilde{L}$ with well defined ideal endpoints in the ideal boundary of $\widetilde{L}$. The only other geodesics are properly imbedded geodesics with one or two endpoints in ∂L and they lift to geodesics in $\widetilde{L}$ having either two endpoints in $\partial\widetilde{L}$, or one endpoint there and the other in the ideal boundary.

Definition 5.5. *A pseudo-geodesic is a curve in L whose lifts have two distinct, well defined endpoints either in S^1_∞, if $\partial L = \emptyset$, or in $\mathfrak{K} \cup \partial\widetilde{L}$ otherwise, except that both endpoints may not be in a component of $\partial\widetilde{L}$.*

Remark. This definition is a bit more general than [5, Definition 4.5]. For completeness, we allow the case of a pseudo-geodesic with one end on ∂L. Remark that any essential closed curve in L is a pseudo-geodesic, its lifts having endpoints in the ideal boundary. Also, every pseudo-geodesic is associated to a unique geodesic to which it is homotopic.

Our axioms stipulate that the laminations $\Lambda_\pm$ have all leaves pseudo-geodesics with lifts connecting two points of the ideal boundary. It is also required that a leaf of the lifted lamination $\widetilde{\Lambda}_+$ can meet a leaf of $\widetilde{\Lambda}_-$ in at most one point. This prevents the leaves of Λ_+ and Λ_- from having intersections that form "digons". It is well known that, in a hyperbolic surface, there can be no geodesic digons. The axioms are all in this spirit, reflecting a few key properties of the Handel–Miller construction, but a detailed account would take us too far afield. Here are some important theorems.

The following is [5, Theorem 8.1].

Theorem 5.6. *If $\Lambda_\pm$ are pseudo-geodesic laminations satisfying the axioms and $\Lambda'_\pm$ are the Handel–Miller geodesic laminations, then there is a homeomorphism $\varphi\colon L \to L$, isotopic to the identity, such that $\varphi(\Lambda_+) = \Lambda'_+$ and $\varphi(\Lambda_-) = \Lambda'_-$.*

Once this has been established, we can use φ to conjugate a choice of the Handel–Miller automorphism h for the geodesic laminations to obtain a choice of h for the pseudo-geodesic laminations. We fix such a choice.

Let X be the set of points of L which do not cluster at any ends of L under forward or backward iteration of h. This is clearly the maximal compact, h-invariant set. It may have nonempty interior. An important subset, also compact and h-invariant, is $X_0 = \Lambda_+ \cap \Lambda_-$. This is totally disconnected (typically, but not always, a Cantor set).

The following result [5, Theorem 10.3] is key to proving that our foliation cones are polyhedral.

Theorem 5.7. *The dynamical system $h\colon X_0 \to X_0$ admits a Markov partition.*

The following [5, Theorem 11.2] is key to showing that $\mathcal{L}_h$ has sufficient regularity to allow application of the theory of asymptotic cycles to the core lamination $\mathcal{X}$.

Theorem 5.8. *The choice of pseudo-geodesic laminations $\Lambda_\pm$ satisfying the axioms, and of the endperiodic automorphism isotopic to f and preserving the laminations, can be made so that the laminations are smooth and h is a diffeomorphism.*

The final result we will cite [5, Theorem 12.8] is key to showing that the foliation cones corresponding to a monodromy map h which is Handel–Miller are maximal. More precisely, there is no foliated ray on the boundary of such a cone.

Recall the notion of monodromy for depth 1 foliations discussed in Section 2.

Definition 5.9. *Let $\mathcal{F}$ be a depth* 1 *foliation of a compact* 3*-manifold, $\mathcal{L}$ a transverse* 1*-dimensional foliation. If the monodromy h induced on a depth* 1 *leaf L is a Handel–Miller automorphism, we will say that the monodromy is Handel–Miller.*

Theorem 5.10. *Let $\mathcal{F}$ be a depth* 1 *foliation and $\mathcal{L}_h$ a transverse,* 1*-dimensional foliation inducing Handel–Miller monodromy on some noncompact leaf. Let $\mathcal{F}'$ be a depth* 1 *foliation transverse to $\mathcal{L}_h$. Then $\mathcal{L}_h$ induces Handel–Miller monodromy on each noncompact leaf of $\mathcal{F}'$. In particular, $\mathcal{L}_h$ induces Handel–Miller monodromy on* every *noncompact leaf of $\mathcal{F}$.*

Indeed, the proof of Theorem 5.10 uses the local projections along $\mathcal{L}_h$ of small neighborhoods on L to small neighborhoods on a noncompact leaf L' of $\mathcal{F}'$ to transfer the laminations locally. An easy continuation argument allows us to transfer the laminations. The monodromy maps $h\colon L \to L$ and $h'\colon L' \to L'$ are both first return maps along $\mathcal{L}_h$ and one must show that the truth of the axioms for the system $(\Lambda_\pm, h)$ implies their truth for $(\Lambda'_\pm, h')$. The proof is delicate.

Remark that Theorem 5.8 is almost certainly false (except for some trivial cases) for geodesic laminations and Theorem 5.10 wouldn't even make sense if it were required that the laminations be geodesic.

5.4. Principal regions and tight Handel–Miller automorphisms

In the Nielsen–Thurston theory, the components of $N \setminus \Lambda^s$ and of $N \setminus \Lambda^u$ are called principal regions and were described above. In the Handel–Miller theory, the situation is more complicated. The complement $L \setminus \Lambda_+$ consists of two disjoint, open sets, $\mathcal{U}_-$ and $\mathcal{P}_+$. The first of these consists of all points which escape to negative ends of L under iterations of h^{-1}. The second, which might be empty, has at most finitely many connected components, called positive principal regions. Similarly, $L \setminus \Lambda_-$ decomposes into $\mathcal{U}_+$ and $\mathcal{P}_-$, the positive escaping set and the union of negative principal regions, respectively. The principal regions again come in dual pairs, a positive principal region being paired with a negative one. Figure 4 is only roughly accurate for the Handel–Miller theory, the shaded region (called the nucleus N of the dual principal regions) is connected and may be topologically complicated. Its boundary may not be connected and so, out of each boundary component there radiates a family of arms for the positive principal region and a family of arms for the negative one. Since $\mathcal{P}_\pm$ is h-invariant, h permutes the positive principal regions, the negative principal regions, and the nuclei.

By Theorem 5.8, we may assume that h is a diffeomorphism and that the laminations are smooth.

If a nucleus N is a disk or an annulus, we will not be concerned with it, but if $\chi(N) < 0$, we will change h by an isotopy supported in $\operatorname{int} N$, using the Nielsen–Thurston theory.

Suppose first that $h(N) = N$. Then the remark at the end of Section 5.1 tells us exactly how to make the modification. If $n \geq 2$ is the smallest integer such that $h^n(N) = N$, then a little fussing is necessary. Set $N_i = h^i(N)$ and $h_i = h|N_i$, $0 \leq i < n$. Note that $N_0 = N$. Let $\varphi \colon N_0 \to N_0$ be the Nielsen–Thurston representative of $h^n|N_0$ and define $h_0' \colon N_0 \to N_1$ by

$$h_0' = h_1^{-1} \circ h_2^{-1} \circ \cdots \circ h_{n-1}^{-1} \circ \varphi.$$

Replace h_0 with h_0', do not modify h_i, $1 \leq i < n$, and check that this new definition of h has $h^n|N_i$ a Nielsen–Thurston diffeomorphism, $0 \leq i < n$.

Definition 5.11. *If the Handel–Miller automorphism h, which has been modified as above for every h-cycle of nuclei of principal regions, except for those of non-negative Euler characteristic, is a diffeomorphism, it will be called a* tight Handel–Miller automorphism. *Monodromy of a depth 1 leaf which is a tight Handel–Miller automorphism is called tight Handel–Miller monodromy.*

In Section 6.4, we will sketch how to modify the proof of Theorem 5.10 to obtain the following.

Theorem 5.12. *Let $\mathcal{F}$ be a depth 1 foliation and $\mathcal{L}_h$ a transverse, 1-dimensional foliation inducing tight Handel–Miller monodromy on some noncompact leaf. Let $\mathcal{F}'$ be a depth 1 foliation transverse to $\mathcal{L}_h$. Then $\mathcal{L}_h$ induces tight Handel–Miller monodromy on each noncompact leaf of $\mathcal{F}'$. In particular, $\mathcal{L}_h$ induces tight Handel–Miller monodromy on* every *noncompact leaf of $\mathcal{F}$.*

Recall that, in abuse of terminology, our "diffeomorphisms" are diffeomorphisms except at finitely many singularities. See Remark 5.3.

§6. The Handel–Miller cones

We consider a smooth depth 1 foliation of a compact 3-manifold M. Let $h\colon L \to L$ be a tight Handel–Miller representative of the isotopy class of the monodromy f of L. One difficulty is that there are infinitely many such representatives h. We will show, however, that $\mathfrak{C}_h$ is independent of the choice of tight Handel–Miller monodromy h and is, indeed, the maximal Sullivan cone $\mathfrak{C}_f$ as f ranges over the isotopy class of h. We need to investigate the asymptotic cycles for the core lamination $\mathcal{X} = \mathcal{X}_h$ more carefully.

Definition 6.1. *The Sullivan cones $\mathfrak{C}'_h$ and $\mathfrak{C}_h$, defined by a tight Handel–Miller automorphism h, will be called Handel–Miller cones.*

By definition, h is an honest diffeomorphism except at a possibly nonempty but finite set of p-pronged singularities, hence the proof of Theorem 2.4 shows that $(M, \mathcal{F}, \mathcal{L}_h)$ can be chosen to be smooth except at the finitely many closed orbits passing through the singularities. Happily, the well understood properties of the Nielsen–Thurston singularities will ensure that M has a differentiable structure agreeing with the one already constructed outside of the singular orbits and that $\mathcal{F}$ is smooth in this structure and $\mathcal{L}_h$ remains smooth away from these orbits and is integral to a nonsingular, C^0 vector field everywhere. This allows us to use the Schwartzmann–Sullivan theory of asymptotic cycles and Theorem 4.9. For details, see [10, Appendix, Part B].

6.1. The invariant set and generating cycles of $\mathfrak{C}'_h$

The lamination $\mathcal{X} = \mathcal{X}_h$ is the $\mathcal{L}_h$-saturation of the invariant set $X = L \smallsetminus (\mathcal{U}_+ \cup \mathcal{U}_-)$, the set of points that do not escape to ends of L under forward or backward iteration of h. We recall that h leaves invariant a pair of pseudo-geodesic laminations $\Lambda_\pm$ and that the totally disconnected set $X_0 = \Lambda_+ \cap \Lambda_-$ is also, therefore, h-invariant. We will call $X_0 \subseteq X$ the *meager* invariant set. Generally, $X_0 \neq X$, X being the union of X_0, the nuclei of the principal regions, the union of components of $\Lambda_+ \cap \mathcal{P}_-$ and the union of components of $\Lambda_- \cap \mathcal{P}_+$. These latter are (possibly infinite) collections of compact arcs in the arms of principal regions with endpoints in X_0.

Similarly, we define $\mathcal{X}_0 \subseteq \mathcal{X}$ as the saturation of X_0.

Definition 6.2. *A closed, convex cone K in a real vector space is said to be spanned by a subset $\Sigma \subset K$ if K is the closure of the set of finite linear combinations of elements of Σ with positive coefficients.*

Lemma 6.3. *Suppose the core lamination $\mathcal{X}$ is expressed as a not necessarily disjoint union, $\mathcal{X} = \bigcup_{\alpha \in \mathfrak{A}} \mathcal{X}_\alpha$, of not necessarily closed sublaminations, and let $\mathfrak{C}'_\alpha$ denote the homology cone spanned by the homology directions detemined by orbits (leaves) of $\mathcal{X}_\alpha$. Then $\mathfrak{C}'_h$ is spanned by the set $\bigcup_{\alpha \in \mathfrak{A}} \mathfrak{C}'_\alpha$.*

Proof. Indeed, every homology direction in $\mathfrak{C}'_h$ corresponds to a single orbit in $\mathcal{X}$, hence in some $\mathcal{X}_\alpha$. We now appeal to Lemma 3.19. Q.E.D.

Our goal is to prove the following.

Theorem 6.4. *The cone $\mathfrak{C}'_{\mathcal{X}}$ is spanned by finitely many of its homology directions.*

The proof of theorem 6.4 will be given in the rest of Section 6.1 and Section 6.2. We decompose the invariant set $\mathcal{X}$ into finitely masny parts, prove that each part contributes either no new generators to $\mathfrak{C}'_h$ or contributes a finite set of generators, and then use Lemma 6.3 to verify that the cone $\mathfrak{C}'_h$ is thus spanned by finitely many of its homolgy directions.

Lemma 6.5. *The $\mathcal{L}_h$-saturation of a component σ of $\Lambda_\pm \cap \mathcal{P}_\mp$, contributes no classes to $\mathfrak{C}'_h$ not already contributed by $\mathfrak{C}'_{\mathcal{X}_0}$.*

Proof. A long, almost closed orbit of a point of σ is clearly homologous to a long, almost closed orbit through an endpoint of σ (the number τ_k in Definition 3.18 can be measured by a transverse, invariant measure for $\mathcal{F}|M^\circ$, hence is independent of the point on σ). Since the endpoints of σ are in X_0, the assertion follows. Q.E.D.

Lemma 6.6. *If the nucleus N of a principal region is a disk, the $\mathcal{L}_h$-saturation of N contributes no classes to $\mathfrak{C}'_h$ not already contributed by $\mathfrak{C}'_{\mathcal{X}_0}$.*

Indeed, the vertices of ∂N are in X_0, hence the proof closely mimics that of Lemma 6.5.

The nucleus of a principal region might be an annulus. There are at most finitely many of these. The set of such annuli is permuted by h. When the nucleus has negative Euler characteristic, the Nielsen–Thurston theory provides a finite, h-invariant family of annuli (part (3) of Theorem 5.1).

In the following discussion, the claim that a sublamination of $\mathcal{X}$ contributes only finitely many generators to $\mathfrak{C}'_{\mathcal{X}}$ means that the set of homology directions contributed by this sublamination consists of positive linear combinations of a finite subset.

Lemma 6.7. *The $\mathcal{L}_h$-saturation of the finite, h-invariant family of annuli provides only finitely many generators of $\mathfrak{C}'_{\mathcal{X}}$.*

Proof. Let $\mathbb{T}$ be the $\mathcal{L}_h$-saturation of one of these annuli. This is a thickened torus, hence $H_1(\mathbb{T}) = \mathbb{R}^2$ and the cone of asymptotic cycles in this space is 2-dimensional with compact base. Such a cone is the set of positive linear combinations of two of its elements. Since there are only finitely many of these annuli, they contribute at most finitely many generators to $\mathfrak{C}'_{\mathcal{X}}$. Q.E.D.

Let N be a nucleus of a principal region, $\chi(N) < 0$, and $n \geq 1$ the smallest integer such that $h^n(N) = N$. Let N' be the subsurface cut off by the annular neighborhood of ∂N as in the remark at the end of Section 5.1.

Lemma 6.8. *If $h^n\colon N' \to N'$ is periodic, its $\mathcal{L}_h$-saturation contributes at most one generator to $\mathfrak{C}'_h$.*

Proof. The $\mathcal{L}_h$-orbits through points of N' are all closed. If ℓ and ℓ' are two of these orbits, one easily sees that, for suitable choices of positive integers p and q, the singular cycles $p\ell$ and $q\ell'$ are homologous. The assertion follows. Q.E.D.

If $h^n|N$ is reducible, the same argument applies to the periodic components of the reduction. In order to prove Theorem 6.4, we need to show that any pseudo-Anosov component of the reduction contributes only finitely many generators and that $\mathfrak{C}'_{\mathcal{X}_0}$ has a finite spanning set. This uses the Markov system for pseudo-Anosov maps (Theorem 5.2) and for $h\colon X_0 \to X_0$ (Theorem 5.7).

6.2. Markov partitions and homology directions

Let $X_1, X_2, \ldots, X_q$ be pseudo-Anosov pieces of nuclei of principal regions, one for each h-orbit of such pieces. Of course, X_0 continues to denote the meager invariant set. There are minimal positive integers m_j, $1 \le j \le q$, such that

$$h^{m_j} : X_j \to X_j$$

and these are pseudo-Anosov. Let us concentrate first on $h\colon X_0 \to X_0$.

As in [5, Section 10], the Markov partition implies that the dynamical system generated by $h|X_0$ is conjugate to a 2-ended subshift of finite type. More precisely, each point of X_0 is encoded by a periodic, bi-infinite sequence $\iota = (i_k)_{k\in\mathbb{Z}}$ of finitely many letters, say $1, 2, \dots, r$. There is an $r \times r$ matrix $A = [a_{ij}]$ of 0's and 1's encoding which letter can follow which. That is, j can follow i if and only if $a_{ij} = 1$. The set of all allowable sequences is denoted by $\mathcal{S}_A$ and the so-called subshift of finite type $\sigma_A\colon \mathcal{S}_A \to \mathcal{S}_A$ shifts each sequence one step to the right. There is a compact, discrete (usually Cantor) topology on $\mathcal{S}_A$ and $h|X_0$ is semi-conjugate to σ_A as we now describe.

The letters $i = 1, 2, \dots, r$ each label a rectangle $R_i \subset L$ of a Markov partition. (As in [5, Section 10], we allow degenerate rectangles, either intervals or points.) These rectangles cover X_0 and do not properly overlap. An element $\iota = (i_k)_{k\in\mathbb{Z}} \in \mathcal{S}_A$ represents a unique point $x_\iota \in R_{i_0} \cap X_0$ such that $h^k(x_\iota) \in R_{i_k}$, $\forall k \in \mathbb{Z}$. In terms of the lamination $\mathcal{X}_0$, this means that the leaf issuing from x_ι meets L successively in $R_{i_0}, R_{i_1}, \dots, R_{i_k}, \dots$ in forward time, with a corresponding statement for backward time. While each $\iota \in \mathcal{S}_A$ encodes a unique point of X_0, some points of X_0 may have finitely many such representatives. The problem is that distinct Markov rectangles in $\mathcal{N}'$ may meet along parts of their boundaries. Thus the map $\iota \mapsto x_\iota$ is finite to one, defining a semi-conjugacy of σ_A to $h|X_0$.

Remark. Actually, in the proof of Theorem 5.7, we were able to make the rectangles disjoint, but this is not generally possible in the pseudo-Anosov case. For ease in carrying our discussion over to that case, we do not require disjointness here.

The periodic elements of $\mathcal{S}_A$ are those carried to themselves by some power σ_A^q, $q \ge 1$. These correspond to closed orbits in $\mathcal{X}_0$. The substring $(i_0, i_1, \dots, i_{q-1})$ of a periodic sequence ι, $\sigma_A^q(\iota) = \iota$, where $q \ge 1$ is minimal, will be called the period of ι. The substring $(i_0, i_1, \dots, i_{q-1}, i_0)$ will be called a periodic string. If no proper substring of a period is a periodic string, we say that the period is *minimal*. Since there are only finitely many distinct entries occurring in the sequences $\iota \in \mathcal{S}_A$, it is evident that there are only finitely many minimal periods.

Definition 6.9. *Those finitely many closed leaves γ of $\mathcal{X}_0$ that correspond to minimal periods in the symbolic system will be called the minimal loops in $\mathcal{X}_0$.*

Let Φ_t denote the flow on M that stabilizes $\partial_\tau M$ pointwise, has flow lines in M° coinciding with the leaves of $\mathcal{L}_h$ and is parametrized so as to preserve $\mathcal{F}|M^\circ$ and so that $\Phi_1|L = h$.

Let $\iota = (i_k)_{k=-\infty}^{\infty} \in \mathcal{S}_A$ and suppose that $i_q = i_0$ for some $q > 0$. Let $x \in R_\iota = \bigcap_{j=-\infty}^{\infty} h^{-j}(R_{i_j})$. Then there is a corresponding singular cycle Γ_q formed from the orbit segment $\gamma_q = \{\Phi_t(x)\}_{0\le t\le q}$ and an arc $\tau \subset R_{i_0}$ from $\Phi_q(x) = h^q(x)$ to x. Also, since $i_q = i_0$, there is a periodic element $\iota' \in \mathcal{S}_A$ with period $(i_0, \dots, i_{q-1})$ and a corresponding closed leaf $\Gamma_{\iota'} = \Gamma'$ of $\mathcal{X}_0$. The loop Γ' is the orbit segment $\{\Phi_t(y)\}_{0\le t\le q}$, for a periodic point

$$y \in R_{i_0} \cap h^{-1}(R_{i_1}) \cap \cdots \cap h^{-q}(R_{i_q}) = R'.$$

Lemma 6.10. *The singular cycle Γ_q and closed leaf Γ', obtained as above, are homologous in M. In particular, the homology class of Γ_q depends only on the periodic element ι'.*

Proof. Remark that $x \in R'$ also. Let τ' be an arc in the rectangle R' from x to y and set $\tau'' = h^q(\tau')$, an arc in $h^q(R')$ from $h^q(x)$ to y. Since $i_q = i_0$, $h^q(R') \subset R_{i_0}$ and the cycle $\tau + \tau' - \tau''$ in the rectangle R_{i_0} is homologous to 0. That is, we can replace the cycle $\Gamma_q = \gamma_q + \tau$ by the homologous cycle $\gamma_q - \tau' + \tau''$. Finally, a homology between this cycle and Γ' is given by the map

$$H\colon [0,1] \times [0,q] \to M,$$

defined by parametrizing τ' on $[0,1]$ and setting

$$H(s,t) = \Phi_t(\tau'(s)).$$

Q.E.D.

Corollary 6.11. *Every closed leaf Γ of $\mathcal{X}_0$ is homologous in M to a linear combination of the minimal loops in $\mathcal{X}_0$ with non-negative integer coefficients.*

Proof. The closed leaf Γ corresponds to a period $(i_0, \dots, i_{q-1})$. If this period is minimal, we are done. Otherwise, after a cyclic permutation, we can assume that the period is of the form $(i_0, i_1, \dots, i_p = i_0, i_{p+1}, \dots, i_{q-1})$. We then see that Γ is homologous to the sum of two loops, one being the arc γ of Γ corresponding to the periodic string $(i_0, \dots, i_p = i_0)$ followed by an arc τ from the endpoint of γ to its initial point, and one being $-\tau+\gamma'$, where γ' is the subarc of Γ corresponding to the periodic string $(i_0, i_{p+1}, \dots, i_{q-1}, i_q = i_0)$. By Lemma 6.10, both $\gamma+\tau$ and $-\tau+\gamma'$ are homologous to closed orbits corresponding to periods strictly shorter than $(i_0, \dots, i_{q-1})$. Thus, finite iteration of this procedure proves the corollary. Q.E.D.

Let $\iota = (i_k)_{k=-\infty}^{\infty}$, $\iota' = (i'_k)_{k=-\infty}^{\infty} \in \mathcal{S}_A$ and suppose that

$$(i_0, i_1, \dots, i_q) = (i'_0, i'_1, \dots, i'_q),$$

not necessarily a period. Let $x = x_\iota$ and $x' = x_{\iota'}$. Both of these points are in

$$R' = R_{i_0} \cap h^{-1}(R_{i_1}) \cap \cdots \cap h^{-q}(R_{i_q}).$$

Choose a path τ in R_{i_0} from x' to x and a path $\tau' \subset R_{i_q}$ from $h^q(x')$ to $h^q(x)$. Consider the orbit segments $\Gamma = \{\Phi_t(x)\}_{t=0}^{q}$ and $\Gamma' = \{\Phi_t(x')\}_{t=0}^{q}$. Let K be an upper bound of the diameters of R_i, $1 \le i \le r$. Then the paths τ and τ' can always be chosen to have length less than K.

The following is proven analogously to Lemma 6.10.

Lemma 6.12. *The singular chains Γ' and $\tau+\Gamma-\tau'$ are homologous. In particular, for each closed 1-form η on M,*

$$\int_{\Gamma'} \eta = \int_{\tau+\Gamma-\tau'} \eta.$$

Here, the paths τ and τ' have length less than K.

Proposition 6.13. *Every homology direction in the cone $\mathfrak{C}'_{X_0}$ can be arbitrarily well approximated by nonnegative linear combinations of the minimal loops.*

Proof. Let $\Gamma = \{\Phi(t)(x)\}_{t=-\infty}^{\infty}$ be an orbit and suppose that x corresponds to the symbol $\iota = \{i_r\}_{r=-\infty}^{\infty}$. By a suitable shift, we can assume that i_0 occurs infinitely often in forward time in this symbol. Consequently, for each index i in ι, there is a positive integer k_i such the the (i, i_0)-entry in A^{k_i} is strictly positive. Let k be the largest of the k_i. Thus, given a substring $(i_0, i_1, \dots, i_p)$ of ι, there is a periodic element $\iota' \in \mathcal{S}_A$ with period $(i_0, i_1, \dots, i_p, i_{p+1}, \dots, i_{p+s})$, where $s \le k$. Let Γ'_p denote the corresponding periodic orbit. If we parametrize the flow Φ_t by the invariant measure for $\mathcal{F}$ of period 1, then the length of the segment Γ_p of Γ corresponding to the string $(i_0, i_1, \dots, i_p)$ is p. Choosing a suitable sequence $p \uparrow \infty$, we obtain a homology direction

$$\mu = \lim_{p\to\infty} \frac{1}{p} \int_{\Gamma_p} \cdot$$

Passing to a subsequence, we also obtain a cycle

$$\mu' = \lim_{p\to\infty} \frac{1}{p} \int_{\Gamma'_p} .$$

Since s is bounded independently of p, Lemma 6.12 implies that μ and μ' agree on all closed 1-forms, and so μ' is a cycle homologous to μ in $(\mathcal{D}'_*, \partial)$. Corollary 6.11 then implies the assertion. Q.E.D.

At this point, we have proven the following.

Proposition 6.14. *The cone $\mathfrak{C}'_{X_0}$ is spanned by the finitely many homology directions corresponding to the minimal loops in $\mathcal{X}_0$* (Definition 6.9).

For $1 \le j \le q$, $h^{m_j} : X_j \to X_j$ is the pseudo-Anosov first return map induced by the sublamination $\mathcal{X}_j \subset \mathcal{X}$ obtained as the $\mathcal{X}$-saturation of X_j. Since pseudo-Anosov homeomorphisms have a Markov partition, the above discussion applies and gives

Proposition 6.15. *The cone $\mathfrak{C}'_{X_j}$ is spanned by the finitely many homology directions corresponding to the minimal loops in $\mathcal{X}_j$, $1 \le j \le q$.*

These propositions, together with the discussion in the previous section, establish Theorem 6.4. This, in turn, gives the following important result.

Theorem 6.16. *Each Handel–Miller cone $\mathfrak{C}_h \subset H^1(M)$ is polyhedral.*

Indeed, the finite spanning set $\theta_1, \theta_2, \dots, \theta_\ell$ of $\mathfrak{C}'_h$ defines finitely many linear equalities

$$\theta_i \ge 0, \quad 1 \le i \le \ell,$$

on the vector space $H^1(M)$. These inequalities exactly define $\mathfrak{C}_h$. Of course, the homology cone $\mathfrak{C}'_h$ is also polyhedral.

6.3. Uniqueness of the Handel–Miller cone

Each periodic piece in the Nielsen–Thurston decomposition of the nuclei contributes just a ray of homology classes to $\mathfrak{C}'_h$. Each pseudo-Anosov piece contributes a closed, convex subcone. Although the choice of the restriction of h to the pseudo-Anosov pieces X_i is not unique, any two choices h and h' are related by $(h')^{m_i} = \varphi \circ h^{m_i} \circ \varphi^{-1}$, where $\varphi\colon L \to L$ is a homeomorphism isotopic to the identity and supported on the pseudo-Anosov piece (cf. [36]). This also holds, by the same reference, for the periodic pieces. By an argument entirely analogous to the proof of Corollary 4.7, we obtain the following.

Proposition 6.17. *The Handel–Miller cone $\mathfrak{C}'_h = \mathfrak{C}'_{h^p}$, $p \geq 1$, is independent of the allowable choices of h on the pseudo-Anosov and periodic pieces in the nuclei.*

Remark. In particular, not only are the cones independent of the allowable choices of h but also the homology directions associated to the map h^p span the same cone as those of h.

Theorem 6.18. *The Handel–Miller cone $\mathfrak{C}'_h \subset H_1(M)$ is independent of the choice of the tight Handel–Miller representative of the isotopy class of the endperiodic monodromy of L.*

Proof. Let h and h' be two such choices. By Theorem 5.6, the laminations $\Lambda_\pm$ associated to h and the laminations $\Lambda'_\pm$ associated to h' are simultaneously ambiently isotopic. That is $\Lambda'_\pm = \varphi(\Lambda_\pm)$ where $\varphi\colon L \to L$ is a homeomorphism isotopic to the identity. Denote by X' and X'_0 the sets $\varphi(X)$ and $\varphi(X_0)$, respectively.

While the choices of h and h' associated to these laminations is not unique, the restrictions $h|X_0$ and $h'|X'_0$ are unique. Combined with Proposition 6.17, this allows us to assume that $\varphi^{-1} \circ h' \circ \varphi = h$ on the union of X_0 and the pseudo-Anosov and periodic pieces of the nuclei. By Corollary 4.7, $\mathfrak{C}'_h = \mathfrak{C}'_{h'}$. Q.E.D.

Remark. Because of this theorem, we will denote the cone $\mathfrak{C}'_h$ by $\mathfrak{C}'_{\mathcal{F}}$ and the dual cohomology cone by $\mathfrak{C}_{\mathcal{F}}$.

Definition 6.19. *The cone $\mathfrak{C}_{\mathcal{F}}$ will be called a Handel–Miller foliation cone or simply a foliation cone.*

6.4. Maximality of the Handel–Miller foliation cones

Our next goal is to show that, if g is an endperiodic map in the isotopy class of h, then $\mathfrak{C}_g \subseteq \mathfrak{C}_{\mathcal{F}}$. Recall Theorem 5.12, the mild generalization of Theorem 5.10 to the case of tight Handel–Miller monodromy. In this theorem, $\mathcal{L}_h$ is smooth except at the finitely many closed orbits through the p-pronged singularities. Thus, the smooth laminations on L transfer to smooth laminations on L' and $h'\colon L' \to L'$, the first return map along $\mathcal{L}_h$, is a diffeomorphism except at finitely many points. The proof of Theorem 5.10 given in [5, Theorem 12.8] shows that dual pairs of principal regions in L transfer to dual pairs of principal regions in L', hence nuclei transfer to nuclei. (Since the transfer is not globally defined, several principal regions might transfer to the same one and/or a principal region might have several transfers to distinct principal regions.) The periodic pieces in nuclei clearly transfer to periodic pieces. So do the pseudo-Anosov pieces. Indeed, the singular foliations transfer just as the Handel–Miller laminations did. The measures are also easy to transfer and since in L they are multiplied by λ and $1/\lambda$, respectively, by the first return map h, the same is true for h'. Finally, the transferred singularities remain p-pronged. We believe that this adequately sketches how to modify the proof of Theorem 5.10 given in [5, Theorem 12.8] so as to prove Theorem 5.12.

We will use the notation $\langle \mathcal{G} \rangle$ for the foliated ray in $H^1(M)$ which represents the foliation (almost without holonomy) $\mathcal{G}$. Recall that, if this foliated ray meets a nonzero element of the integer lattice $G = H^1(M; \mathbb{Z})$, it is said to be rational and represents a depth 1 foliation. The following is proven in [7].

Proposition 6.20. *A depth* 1 *foliation* $\mathcal{F}_\omega$ *is uniquely determined up to isotopy by the associated rational foliated ray, the isotopy being ambient, smooth in* M° *and* C^0 *on* M*, pointwise fixing* $\partial_\tau M$*.*

Proposition 6.21. *The rational foliated ray* $\langle \mathcal{G} \rangle$ *lies in* $\operatorname{int} \mathfrak{C}_\mathcal{F}$ *if and only if* $\mathfrak{C}_\mathcal{G} = \mathfrak{C}_\mathcal{F}$*.*

Proof. Suppose that $\langle \mathcal{G} \rangle \subset \operatorname{int} \mathfrak{C}_\mathcal{F}$. By Proposition 6.20 and Theorem 4.9, an isotopy makes the foliation $\mathcal{G}$ transverse to $\mathcal{L}_h$. By Theorem 5.12, $\mathcal{L}_h$ induces tight Handel–Miller monodromy g on each leaf of $\mathcal{G}|M^\circ$. Thus, the cones $\mathfrak{C}_\mathcal{G}$ and $\mathfrak{C}_\mathcal{F}$ are determined by the same core lamination $\mathcal{X}_g = \mathcal{X}$ and so are identical. For the converse, if $\mathfrak{C}_\mathcal{G} = \mathfrak{C}_\mathcal{F}$, for rational foliated rays $\langle \mathcal{G} \rangle$ and $\langle \mathcal{F} \rangle$, clearly $\langle \mathcal{G} \rangle \subset \operatorname{int} \mathfrak{C}_\mathcal{F}$. Q.E.D.

Corollary 6.22. *No rational foliated ray is contained in* $\partial \mathfrak{C}_\mathcal{F}$*.*

Proof. If there is a rational foliated ray $\langle \mathcal{G} \rangle \subset \partial \mathfrak{C}_\mathcal{F}$, then $\operatorname{int} \mathfrak{C}_\mathcal{G} \cap \operatorname{int} \mathfrak{C}_\mathcal{F} \neq \emptyset$. Since the union of the rational rays in $\mathfrak{C}_\mathcal{F}$ is dense in that cone, there is a rational foliated ray $\langle \mathcal{H} \rangle \subset \operatorname{int} \mathfrak{C}_\mathcal{G} \cap \operatorname{int} \mathfrak{C}_\mathcal{F}$. By Proposition 6.21, we see that $\mathfrak{C}_\mathcal{G} = \mathfrak{C}_\mathcal{H} = \mathfrak{C}_\mathcal{F}$. That is, $\langle \mathcal{G} \rangle \subset \operatorname{int} \mathfrak{C}_\mathcal{F}$, contrary to our hypothesis. Q.E.D.

The boundary $\partial \mathfrak{C}_\mathcal{F}$ is made up of r codimension 1 faces $F_1, \dots, F_r$, where F_i is a convex, polyhedral cone with nonempty (relative) interior in the hyperplane $R_i \subset H^1(M)$ defined by a linear equation of the form $[\gamma_i] = 0$.

Lemma 6.23. *Each* F_i *contains a dense family of rays that meet nontrivial points of the integer lattice* G*.*

Proof. Since the spanning vectors of $\mathfrak{C}'_h$, given by the proof of Theorem 6.16 are represented by closed loops in $\mathcal{X}$, they are integral cohomology classes and F_i meets the integer lattice $G \subset H^1(M)$ in a sublattice of full rank. Q.E.D.

Theorem 6.24. *If g is a monodromy map (endperiodic) for a depth 1 foliation of M, then either $(\operatorname{int}\mathfrak{C}_g) \cap \mathfrak{C}_{\mathcal{F}} = \emptyset$, or $\mathfrak{C}_g \subseteq \mathfrak{C}_{\mathcal{F}}$. In particular, $\mathfrak{C}_{\mathcal{F}} = \mathfrak{C}_h$ is the maximal foliation cone for monodromies in the isotopy class of h and two maximal foliation cones either coincide or have disjoint interiors.*

Proof. If $(\operatorname{int}\mathfrak{C}_g) \cap \mathfrak{C}_{\mathcal{F}} \neq \emptyset$ and $\mathfrak{C}_g \not\subseteq \mathfrak{C}_{\mathcal{F}}$, then Lemma 6.23 implies that there is a rational foliated ray in $\partial\mathfrak{C}_{\mathcal{F}}$, contradicting Corollary 6.22. Q.E.D.

Remark. Correspondingly, the dual homology cone $\mathfrak{C}'_{\mathcal{F}} = \mathfrak{C}'_h$ is the minimal $\mathfrak{C}'_g$ for all monodromies g isotopic to h. In this sense, we can say that the tight Handel–Miller monodromy has the "tightest" dynamics in its isotopy class.

6.5. Finiteness of the foliation cones

We refer the reader to [8, Theorem 6.4] for the proof of the following.

Theorem 6.25. *There are only finitely many foliation cones in $H^1(M)$.*

The idea is to produce a finite family of branched surfaces $\Sigma_1, \dots, \Sigma_q$ such that every depth 1 foliation is carried by some Σ_i. The cone of depth 1 foliations carried by any Σ_i is contained in a single foliation cone, hence the number of such cones must be finite.

Remark. An original goal of this work was to quantify the depth 1 foliations $\mathcal{F}$ of sutured manifolds M constructed by Gabai's process of disk decomposition [19]. If the decomposing disks all live in M from the start (a *simple* disk decomposition), the foliation cones can be read off of the disks. This involves relating the disks in the simple decomposition to the Markov process induced by the Handel–Miller monodromy. See [8, Section 7] and [11, Section 5] for examples.

Remark. The key idea to use Handel–Miller monodromy to define the foliation cones was suggested to us by D. Fried's use of pseudo-Anosov monodromy of fibrations to determine the fibered faces of the Thurston ball in hyperbolic 3-manifolds [18].

§7. Foliations of dense leaved type

Our goal in this section is to prove the following serious extension of Proposition 6.20 which was conjectured in [7].

Theorem 7.1. *The foliated rays in the interior of a foliation cone determine a foliation almost without holonomy uniquely up to a C^0 ambient isotopy.*

We will only assume that we are given a foliated ray, hence will not need to work in the maximal foliation cones. Thus, the Handel–Miller theory will not be used. Our ongoing assumption that all compact leaves have strictly negative Euler characteristic is, therefore, no longer needed. Proposition 6.20, which was proven in [7], proves Theorem 7.1 for proper foliations that are almost without holonomy. Thus, in Section 7, we will restrict our attention to the dense leaved type of foliations which are almost without holonomy. To reduce wordiness, we will call a codimension 1 foliation almost without holonomy of dense leaved type an AWHD foliation. The rest of Section 7 is devoted to the proof of Theorem 7.1 for irrational foliated rays.

Remark. The fact that the isotopy in Proposition 6.20 is ambient will be critically important for us. This was not emphasized in [7], but it is clear that all isotopies employed in that paper are ambient.

7.1. Invariant measures and isotopy

In this subsection, there is no restriction on $\dim M = n \geq 3$. The notion of an AWHD foliation makes sense in arbitrary dimension.

Remark. Save mention to the contrary, no smoothness is required either for the AWHD foliations $\mathcal{F}$ or the transverse 1-dimensional foliation $\mathcal{L}$.

By a well known theorem of Sacksteder [31], an AWHD foliation $\mathcal{F}$ of M admits a strictly positive, continuous, transverse invariant measure μ, finite on compact $\mathcal{F}$-transverse intervals. Continuity of the measure implies that it is non-atomic. At $\partial_\tau M$, the measure becomes unbounded. We call this a Sacksteder measure.

As an example, if $\mathcal{F}$ is defined by a foliated form ω, that form defines the desired measure in an obvious way. By "Sacksteder's trick", if $\mathcal{F}$ is smooth, the measure μ, even if it is not smooth (i.e., a differential form), can be used to put a new differentiable structure on M in which μ is a smooth, closed 1-form ω. We will not employ this trick.

In a standard way, μ can be integrated along compact C^0 paths σ in M°, this line integral being determined by the homotopy class (modulo the endpoints) of σ. It is not required that σ be transverse to $\mathcal{F}$. The line integral clearly vanishes on commutators in $\pi_1(M)$, hence μ defines a homomorphism

$$[\mu]\colon H_1(M;\mathbb{Z}) \to \mathbb{R}$$

which we view as a cohomology class in $H^1(M) = H^1(M^\circ)$. The period group $P(\mu) \subset \mathbb{R}$ of μ is the image of this homomorphism. For more detail, see [10, Section 2], which is also the source for details of the following discussion. In that reference, we assumed $\partial M = \emptyset$, but the adaptation to our more general context is straightforward.

We fix a 1-dimensional foliation $\mathcal{L}$ of M transverse to $\partial_\tau M$ and tangent to $\partial_\pitchfork M$, and assume that $\mathcal{F}$ is an AWHD foliation of M, transverse to $\mathcal{L}$ and having Sacksteder measure μ. Let Φ_t be the C^0 flow on M defined by μ, pointwise fixing $\partial_\tau M$ and having the leaves of $\mathcal{L}|M^\circ$ as flow lines.

Lemma 7.2. *The period group $P(\mu)$ is the set of $t \in \mathbb{R}$ such that Φ_t carries each leaf of $\mathcal{F}$ onto itself.*

Remark. If $\mathcal{F}$ is defined by a foliated form ω, the measure μ is defined by integrating ω and the period group is denoted by $P(\omega)$.

Corollary 7.3. *A compact path σ in M° has endpoints in the same leaf of $\mathcal{F}$ if and only if $\int_\sigma d\mu \in P(\mu)$.*

Suppose now that $\mathcal{F}_0$ and $\mathcal{F}_1$ are AWHD foliations of M, transverse to $\mathcal{L}$ and having respective Sacksteder measures μ_0 and μ_1, with $[\mu_0] = [\mu_1]$. Finally, let Φ^0 and Φ^1 be the respective C^0 flows defined by the measures and having the leaves of $\mathcal{L}$ as flow lines.

For $0 \le t \le 1$, $\nu_t = t\mu_1 + (1-t)\mu_0$ is a continuous, strictly positive measure along $\mathcal{L}$ and has line integrals

$$\int_\sigma d\nu_t = t\int_\sigma d\mu_1 + (1-t)\int_\sigma d\mu_0.$$

This defines a cohomology class and $[\mu_0] = [\nu_t] = [\mu_1]$, $0 \le t \le 1$. It is natural to suspect that ν_t is a transverse, invariant measure for an AWHD foliation $\mathcal{H}_t$ of M transverse to $\mathcal{L}$, that ν_t blows up at $\partial_\tau M$ (obvious), $0 \le t \le 1$, and that this *homotopy* of $\mathcal{H}_0 = \mathcal{F}_0$ with $\mathcal{H}_1 = \mathcal{F}_1$ is actually a continuous isotopy in M. The following discussion confirms this.

Fix a leaf L of $\mathcal{F}_0$ and a basepoint $x \in L$. For fixed but arbitrary $t \in [0,1]$, and $z \in L$, choose a path σ in L from x to z and let $a_t(z) = \int_\sigma d\nu_t$.

Lemma 7.4. *The number $a_t(z)$ depends only on z and t, not on the choice of σ, and the function $a_t \colon L \to \mathbb{R}$ is continuous.*

The measure ν_t defines a flow ${}^t\Psi$ on M° with flow lines the leaves of $\mathcal{L}|M^\circ$. Define

$$\varphi_t \colon L \to M^\circ,$$
$$\varphi_t(z) = {}^t\Psi_{-a_t(z)}(z).$$

Lemma 7.5. *If ℓ is a leaf of $\mathcal{L}$ (with its 1-dimensional manifold topology), then φ_t restricts to an order preserving bijection of the dense subset $\ell \cap L$ onto a dense subset of ℓ.*

The order on ℓ, of course, is that induced by the transverse orientation of $\mathcal{F}_0$. In the case that $\ell \cong S^1$, this is a cyclic order.

It follows that, for each leaf ℓ of $\mathcal{L}|M^\circ$, φ_t extends canonically to a homeomorphism of ℓ onto itself. Furthermore, $\varphi_t \colon L \to M^\circ$ will be an injective topological immersion extending canonically to a homeomorphism $\varphi_t \colon M^\circ \to M^\circ$ which continuously extends by the identity on $\partial_\tau M$. The continuous dependence of this homeomorphism on $t \in [0, 1]$ is also elementary. We have,

Lemma 7.6. *The map*

$$\varphi \colon M \times [0, 1] \to M,$$

defined by $\varphi(z, t) = \varphi_t(z)$ on $M^\circ \times [0, 1]$ and by $\varphi(z, t) = z$ for $z \in \partial_\tau M$, is an isotopy of φ_1 to $\varphi_0 = \mathrm{id}$.

Proposition 7.7. *If $\mathcal{F}_0$ and $\mathcal{F}_1$ are foliations of M with transverse invariant measures μ_0 and μ_1 respectively such that $[\mu_0] = [\mu_1]$ and with $\mathcal{F}_0$, $\mathcal{F}_1$ both transverse to the same 1-dimensional foliation $\mathcal{L}$, then $\mathcal{F}_0$ and $\mathcal{F}_1$ are topologically ambiently isotopic.*

Proof. For each $t \in [0, 1]$, we obtain a topological foliation $\mathcal{H}_t = \varphi_t(\mathcal{F}_0)$ which we view as an ambient isotopy of $\mathcal{F}_0 = \mathcal{H}_0$ to $\mathcal{H}_1$. Remark that φ_t pushes the invariant measure μ_0 forward to the measure ν_t. Proposition 7.7 then follows by continuity of the measures and the fact that L is dense in M°. Thus, ν_t is a holonomy invariant measure for $\mathcal{H}_t$, $0 \le t \le 1$, with $\nu_1 = \mu_1$ defining $\mathcal{H}_1 = \mathcal{F}_1$. Q.E.D.

Lemma 7.8. *If $\mathcal{F}$ is AWHD, it is C^0 ambiently isotopic to an AWHD foliation defined by a foliated form.*

Proof. Let μ be a Sacksteder measure for $\mathcal{F}$ and let $\mathcal{L}$ be a smooth, transverse 1-dimensional foliation with core lamination $\mathcal{X}$. Evidently, $[\mu]$ is strictly positive on every nontrivial asymptotic cycle of $\mathcal{X}$ so $[\mu] \in \operatorname{int} \mathfrak{C}_{\mathcal{X}}$. Thus, by Theorem 4.9, $[\mu] = [\omega]$, where ω is a foliated form transverse to $\mathcal{L}$ and, by Proposition 7.7, $\mathcal{F}$ and $\mathcal{F}_\omega$ are C^0 ambiently isotopic. Q.E.D.

Finally, the following result, essentially due to J. Moser [27], is basic to the proof of Theorem 7.1 for AWHD foliations. It can be thought of as a smooth version of Proposition 7.7.

Proposition 7.9. *If foliated forms ω_0 and ω_1 on M° are cohomologous and transverse to a common* 1*-dimensional foliation $\mathcal{L}$ which is integral to a nonsingular C^0 vector field, then the foliations $\mathcal{F}_{\omega_0}$ and $\mathcal{F}_{\omega_1}$ are smoothly ambiently isotopic in M° and continuously in M, keeping $\partial_\tau M$ pointwise fixed. Hence the foliated forms are smoothly isotopic.*

Proof. By hypothesis, $\omega_1 - \omega_0 = df$ for a smooth function $f\colon M^\circ \to \mathbb{R}$. Set

$$\omega_t = t\omega_1 + (1-t)\omega_0.$$

itself a foliated form transverse to $\mathcal{L}$. On $M^\circ \times [0,1]$, define

$$\Omega = \omega_t + f\,dt,$$

where t denotes the $[0,1]$ coordinate. Then Ω is a nonsingular 1-form, transverse to $\overline{\mathcal{L}}$, the 1-dimensional foliation with restriction $\mathcal{L} \times \{t\}$ on $M^\circ \times \{t\}$. An easy computation shows that $d\Omega = 0$. This form also blows up nicely at $\partial_\tau M \times [0,1]$, and so can be thought of as a foliated form on the open, saturated subset $M^\circ \times [0,1]$ of the foliated $(n+1)$-manifold $(M \times [0,1], \mathcal{F} \times [0,1])$. Let $\mathcal{H}$ denote the foliation defined by Ω on $M^\circ \times [0,1]$. Since $\mathcal{H}_t = \mathcal{H}|(M^\circ \times \{t\}) = \mathcal{F}_{\omega_t}$, this gives a smooth integrable homotopy of $\mathcal{F}_{\omega_0}$ to $\mathcal{F}_{\omega_1}$ on M°. Since M is compact, the integrable homotopy (extended continuously to $\partial_\tau M$) is an ambient isotopy. One can then view Ω itself as a smooth isotopy of ω_0 to ω_1 on M°. Q.E.D.

7.2. Regular coverings

Proofs of assertions in this subsection will be found in [6, Section 3] and are simpler under our current hypotheses. The setting in [6] was for foliations without holonomy of open $\mathcal{F}$-saturated subsets U of M, not necessarily with compact transverse completion $\widehat{U}$. Here, $U = M^\circ$ and $\widehat{U} = M$. Again, there is no restriction on $\dim M \geq 3$, but we will make no use of that fact.

Let $\mathcal{F}$ be an AWHD foliation. Assume, by Lemma 7.8, that $\mathcal{F}$ is defined by a foliated form ω and write $\mathcal{F} = \mathcal{F}_\omega$. Let $\mathcal{F}_{\omega'}$ be another foliation of M defined by a foliated form ω', not necessarily of dense type.

Definition 7.10. *We say that $\mathcal{F}_\omega$ covers $\mathcal{F}_{\omega'}$ if*

(1) *$\mathcal{F}_\omega$ and $\mathcal{F}_{\omega'}$ are transverse to a common,* 1*-dimensional foliation $\mathcal{L}$ of M;*

(2) $\ker(\omega\colon H_1(M;\mathbb{Z}) \to \mathbb{R}) \subseteq \ker(\omega'\colon H_1(M;\mathbb{Z}) \to \mathbb{R})$.

Recall that ω parametrizes $\mathcal{L}$ to produce an $\mathcal{F}_\omega$-preserving flow

$$\Phi\colon M \times \mathbb{R} \to M,$$

fixing $\partial_\tau M$ pointwise and having the leaves of $\mathcal{L}|M^\circ$ as flow lines in M°.

Proposition 7.11. *If $\mathcal{F}_\omega$ covers $\mathcal{F}_{\omega'}$, if $L \in \mathcal{F}_\omega$ and $L' \in \mathcal{F}_{\omega'}$ are leaves, and if $x_0 \in L \cap L'$, there is a unique continuous function $\tau\colon L \to \mathbb{R}$ such that*

(1) $\tau(x_0) = 0$;
(2) $\Phi_{\tau(x)}(x) \in L'$, *for each* $x \in L$.
(3) *The map* $p\colon L \to L'$ *defined by* $p(x) = \Phi_{\tau(x)}(x)$ *is a regular covering with covering group* $G = \{\Phi_{\tau(x)-\tau(y)} \mid p(x) = p(y)\}$.

Proposition 7.12. *Every AWHD foliation $\mathcal{F}_\omega$ covers a depth 1 foliation $\mathcal{F}_{\omega'}$ with $[\omega']$ in any preassigned neighborhood of $[\omega]$ in $H^1(M)$.*

7.3. Proof of Theorem 7.1 for irrational foliated rays

The restriction to $\dim M = 3$ is needed in this subsection.

We suppose that the AWHD foliations $\mathcal{F}$ and $\mathcal{G}$ correspond to the same foliated ray $\langle\mathcal{F}\rangle = \langle\mathcal{G}\rangle$. By Lemma 7.8, these foliations are C^0-isotopic to foliations $\mathcal{F}_\omega$ and $\mathcal{F}_\eta$, respectively, where ω and η are foliated forms with $[\omega] = [\eta]$ and the isotopy fixes $\partial_\tau M$ pointwise. We will find a C^0 isotopy between $\mathcal{F}_\eta$ and $\mathcal{F}_\omega$, smooth on M° and the identity on $\partial_\tau M$.

There exist foliations $\mathcal{F}_{\omega'}$ and $\mathcal{F}_{\eta'}$, covered by $\mathcal{F}_\omega$ and $\mathcal{F}_\eta$, respectively, with $[\omega'] = [\eta']$ and having rational period group. Indeed, let $\mathcal{L}$ be a 1-dimensional foliation transverse to $\mathcal{F}_\omega$ as usual and $\mathcal{L}^*$ a 1-dimensional foliation transverse to $\mathcal{F}_\eta$. Let $\mathcal{X}$ and $\mathcal{X}^*$ be the corresponding core laminations. Then $\mathfrak{C}_{\mathcal{X}} \cap \mathfrak{C}_{\mathcal{X}^*}$ has nonempty interior U containing $[\omega] = [\eta]$ and, by Proposition 7.12, we find a class $[\omega'] = [\eta'] \in U$ with rational periods, simultaneously representing a depth 1 foliation $\mathcal{F}_{\omega'}$ transverse to $\mathcal{L}$ and covered by $\mathcal{F}_\omega$ as well as a depth 1 foliation $\mathcal{F}_{\eta'}$ transverse to $\mathcal{L}^*$ and covered by $\mathcal{F}_\eta$.

Proposition 6.20 gives an isotopy of $\mathcal{F}_{\omega'}$ to $\mathcal{F}_{\eta'}$ which is smooth in M°. This is an ambient isotopy, dragging $\mathcal{F}_\omega$ to an AWHD foliation $\mathcal{F}_{\widetilde{\omega}}$ covering $\mathcal{F}_{\eta'}$ and $\mathcal{L}$ to a 1-dimensional foliation $\widetilde{\mathcal{L}}$ transverse to $\mathcal{F}_{\eta'}$ and $\mathcal{F}_{\widetilde{\omega}}$.

We can now simplify (and abuse) notation, denoting $\mathcal{F}_{\omega'} = \mathcal{F}_{\eta'}$ by $\mathcal{F}'$, $\mathcal{F}_{\widetilde{\omega}}$ by $\mathcal{F}_\omega$, and $\widetilde{\mathcal{L}}$ by $\mathcal{L}$ so that $\mathcal{L}$ is transverse both to $\mathcal{F}_\omega$ and $\mathcal{F}'$ and $\mathcal{L}^*$ is transverse both to $\mathcal{F}_\eta$ and $\mathcal{F}'$. We keep this notation for the rest of Section 7.3.

Lemma 7.13. *A smooth, transverse foliation $\mathcal{L}$ to any foliation $\mathcal{F}$ is unique up to a smooth homotopy through foliations transverse to $\mathcal{F}$.*

Proof. Indeed, let $\mathcal{L}_0$ and $\mathcal{L}_1$ be transverse to $\mathcal{F}$ and let v_0 be a smooth, nonsingular vector field tangent to $\mathcal{L}_0$ and oriented by the transverse orientation of $\mathcal{F}$, v_1 such a vector field for $\mathcal{L}_1$. Then $v_s =$

$sv_1 + (1-s)v_0$ integrates to give an $\mathcal{F}$-transverse flow with underlying foliation $\mathcal{L}_s$, $0 \le s \le 1$. Q.E.D.

In particular, the two 1-dimensional foliations $\mathcal{L}$, $\mathcal{L}^*$ are connected by a homotopy through 1-dimensional foliations transverse to $\mathcal{F}'$. Remark that this homotopy can be viewed as a 1-dimensional foliation $\overline{\mathcal{L}}$ on the 4-manifold $\overline{M} = M \times [0,1]$ which is transverse to the codimension one foliation $\overline{\mathcal{F}'} = \mathcal{F}' \times [0,1]$. The restriction of $\overline{\mathcal{L}}$ to each 3-dimensional slice $M \times \{s\}$ is $\mathcal{L}_s$, $0 \le s \le 1$. The core lamination $\overline{\mathcal{X}}$ of $\overline{\mathcal{L}}$ is defined as usual and restricts on $M \times \{s\}$ to the core lamination $\mathcal{X}_s$ of $\mathcal{L}_s$, $0 \le s \le 1$. The homology cone $\mathfrak{C}'_{\overline{\mathcal{X}}}$ and the cohomology cone $\mathfrak{C}_{\overline{\mathcal{X}}}$ are as defined in Section 3.2, where the dimension of the ambient manifold was allowed to be ≥ 3.

Remark 7.14. In case $\partial_\tau M = \emptyset$, the core lamination coincides with the whole transverse 1-dimensional foliation. The associated cones can still be defined as usual, but in this case the maximal cones correspond to the Nielsen–Thurston monodromy rather than to Handel–Miller. Proofs are essentially unchanged. These maximal cones can also be defined (and originally were) as cones subtended by the fibered faces of the Thurston ball. The idea of determining the fibered faces by the use of structure cycles for the transverse flow is due to D. Fried [18] and was the inspiration of our work on foliation cones.

Lemma 7.15. *Under the identifications* $H^1(\overline{M})$ *with* $H_1(M \times \{s\}) = H_1(M)$, $0 \le s \le 1$, *induced by the projection* $\pi\colon \overline{M} \to M$, *we have*

$$\mathfrak{C}'_{\overline{\mathcal{X}}} = \overline{\bigcup_{0 \le s \le 1} \mathfrak{C}'_{\mathcal{X}_s}},$$

$$\mathfrak{C}_{\overline{\mathcal{X}}} = \bigcap_{0 \le s \le 1} \mathfrak{C}_{\mathcal{X}_s}.$$

Proof. The second equality is an easy consequence of the first. For the first, it is clear that

$$\bigcup_{0 \le s \le 1} \mathfrak{C}'_{\mathcal{X}_s} \subseteq \mathfrak{C}'_{\overline{\mathcal{X}}}$$

and, the cone $\mathfrak{C}'_{\overline{\mathcal{X}}}$ being closed, the closure of the union of the cones $\mathfrak{C}'_{\mathcal{X}_s}$ is also contained in $\mathfrak{C}'_{\overline{\mathcal{X}}}$. For the reverse inclusion, recall that $\mathfrak{C}'_{\overline{\mathcal{X}}}$ is the closure of the union of nonnegative linear combinations of homology directions. Each homology direction of $\overline{\mathcal{X}}$ is a homology direction of some $\mathcal{X}_s$, proving the assertion. Q.E.D.

Corollary 7.16. *Under the identifications $H^1(\overline{M})$ with $H_1(M)$ induced by the projection $\pi\colon \overline{M} \to M$, the cohomology class $[\pi^*\omega] = [\omega] \in \operatorname{int}\mathfrak{C}_{\overline{\mathcal{X}}}$.*

Proof. Recall that the rational ray in $\operatorname{int}\mathfrak{C}_{\mathcal{X}_0}$ corresponding to $\mathcal{F}'$ is defined by the class $[\omega'] = [\eta']$. Under the identification $H^1(M) = H^1(\overline{M})$, the rational ray in $\operatorname{int}\mathfrak{C}_{\mathcal{X}_0}$ corresponding to $\mathcal{F}'$ and the one in $\operatorname{int}\mathfrak{C}_{\overline{\mathcal{X}}}$ corresponding to $\overline{\mathcal{F}'}$ are identified. Let $[\overline{\omega}] = [\pi^*\omega] \in H^1(\overline{M})$ denote tyhe cohomology class identified to $[\omega] \in H^1(M)$. We must show that $[\overline{\omega}]$ takes strictly positive values on $\mathfrak{C}'_{\overline{\mathcal{X}}}$. Indeed, if $[\overline{\omega'}]$ is the class defining $\overline{\mathcal{F}'}$, the fact that $\mathcal{F}_\omega$ covers $\mathcal{F}'$ implies that

$$\ker([\overline{\omega}]) = \ker([\omega]) \subset \ker([\omega']) = \ker([\overline{\omega'}]),$$

where these are the kernels in $H_1(M;\mathbb{Z}) = H_1(\overline{M};\mathbb{Z})$. Thus, $[\overline{\omega}]$ must take strictly nonzero values on the homology cone $\mathfrak{C}'_{\overline{\mathcal{X}}} \subset H^1(\overline{M})$. This cone is connected, hence its image under $[\overline{\omega}]$ is a connected subset of $\mathbb{R}$ not containing 0. Since $[\omega]$ takes strictly positive values on $\mathfrak{C}'_{\mathcal{X}_0} \subset \mathfrak{C}'_{\overline{\mathcal{X}}}$, these values are positive on all of $\mathfrak{C}'_{\overline{\mathcal{X}}}$. Q.E.D.

Lemma 7.17. *There is a codimension one foliation $\overline{\mathcal{F}}_\omega$ of $\overline{M}$, transverse to $\overline{\mathcal{L}}$, hence to $M \times \{s\}$, $0 \le s \le 1$, which induces $\mathcal{F}_\omega$ on $M \times \{0\}$ and covers $\overline{\mathcal{F}'}$. This foliation can be interpreted as a smooth ambient isotopy in M° of $\mathcal{F}_\omega = \mathcal{F}^0_\omega$ to a foliation $\mathcal{F}^1_\omega$ that covers $\mathcal{F}'$ and is transverse to $\mathcal{L}_1 = \mathcal{L}^*$. The isotopy extends continuously to M, fixing $\partial_\tau M$ pointwise.*

Proof. Recall from the proof of Theorem 4.9 that $\omega = \alpha + df$, where α is defined on M and hence is bounded near points of $\partial_\tau M$. Since ω "blows up nicely" at the tangential boundary $\partial_\tau M$, so does df. That is, the 2-planes that are defined by $df = 0$ converge in the C^∞ topology to the tangent planes to $\partial_\tau M$. Let

$$\pi\colon \overline{M} \to M$$

be the projection along the $[0,1]$-factors and set $\overline{\alpha} = \pi^*(\alpha)$. This represents the cohomology class $[\pi^*\omega]$ and so, by Corollary 7.16 and Theorem 4.9, we find an exact form dh on M° such that $\zeta = \overline{\alpha} + dh$ is a foliated form transverse to $\overline{\mathcal{L}}$. Again by the proof of Theorem 4.9, dh can be chosen to blow up nicely at $\partial_\tau(M \times [0,1])$. The covering property being purely homological, we see that the foliation defined by ζ covers $\overline{\mathcal{F}'}$. If this form agreed with ω on $M^\circ \times \{0\}$, we would be done with the proof of the first assertion of the lemma. We need to further modify ζ.

Let $\lambda\colon [0,1] \to [0,1]$ be smooth with $\lambda(s) = 0$, $0 \le s \le \varepsilon/2$, $\lambda(s) = 1$, $\varepsilon \le s \le 1$, and $0 < \lambda(s) < 1$, for $\varepsilon/2 < s < \varepsilon$. Set

$$\overline{\omega} = \overline{\alpha} + d(\lambda(s)h + (1 - \lambda(s))f).$$

(Here we indulge in the notational abuse $f = \pi^*(f)$.) For $0 \le s \le \varepsilon/2$, this form agrees with $\pi^*\omega$. In $M^\circ \times [0, \varepsilon/2]$, $\overline{\omega} = \pi^*\omega$. In $M^\circ \times [\varepsilon, 1]$, $\overline{\omega} = \zeta$. In $M^\circ \times (\varepsilon/2, \varepsilon)$, remark that

$$d(\lambda(s)h + (1 - \lambda(s))f) = (h - f)\lambda'(s)\,ds + \lambda(s)\,dh + (1 - \lambda(s))\,df$$

and in this same range, both $\overline{\alpha}+df$ and $\overline{\alpha}+dh$ are transverse to $\overline{\mathcal{L}}$. Since $\lambda'(s)\,ds$ clearly vanishes along $\overline{\mathcal{L}}$, $\overline{\omega}$ is transverse to $\overline{\mathcal{L}}|(M^\circ \times (\varepsilon/2, \varepsilon))$. Indeed, $\lambda(s)\overline{\alpha}+\lambda(s)\,dh$ and $(1-\lambda(s))\overline{\alpha}+(1-\lambda(s))\,df$ are transverse there. Now, df and dh blow up nicely at the tangential boundary $\partial_\tau(M\times[0,1])$, hence so does $\lambda(s)\,dh + (1 - \lambda(s))\,df$. It remains to investigate the behavior of $(h-f)\lambda'(s)\,ds$ at $\partial_\tau(M\times[0,1])$. In the proof of Theorem 4.9 we chose f (respectively, h) to be the sum of a function bounded near the tangential boundary and an antiderivative of e^{1/t^2}, where t is the normal coordinate in a normal neighborhood of the tangential boundary. Thus, we can assume that the form $(h-f)\lambda'(s)\,ds$ remains bounded near $\partial_\tau(M \times [0,1])$, completing the proof that $\overline{\omega}$ blows up at the tangential boundary so as to define a smooth foliation with trivial infinitesimal holonomy along the tangential boundary. Thus, $\overline{\omega}$ is a foliated form.

We turn to the isotopy assertion. The foliation $\overline{\mathcal{F}}_\omega$ defined by $\overline{\omega}$ clearly defines an integrable homotopy in M° of $\mathcal{F}_\omega = \mathcal{F}^0_\omega$ to $\mathcal{F}^1_\omega$. To see that this integral homotopy is actually an isotopy, even though M° is not compact, observe that the vector field $\partial/\partial t$ tangent to the I-factors has nonzero component tangent to $\overline{\mathcal{F}}_\omega$. The semiflow that it generates can be reparametrized by $0 \le t \le 1$ and so serves as the track of the ambient isotopy in M°. Viewing $\overline{\mathcal{F}}_\omega$ as a foliation of $M \times [0,1]$, we see that this isotopy extends continuously to M, fixing $\partial_\tau M$ pointwise. Q.E.D.

Proof of Theorem 7.1. Lemma 7.17 produces a smooth ambient isotopy of $\mathcal{F}_\omega$ in M°, moving it to a foliation $\mathcal{F}^1_\omega$ transverse to the 1-dimensional foliation $\mathcal{L}^*$. The isotopy extends continuously to M, fixing $\partial_\tau M$ pointwise. Since $\mathcal{F}^1_\omega$ and $\mathcal{F}_\eta$ are both transverse to the same 1-dimensional foliation $\mathcal{L}^*$, an appeal to Proposition 7.9 completes the isotopy. The proof of Theorem 7.1 is complete. Q.E.D.

Remark. The arguments in this section are simplified if $\partial_\tau M = \emptyset$, hence we have proven that foliations without holonomy are determined up to isotopy by the corresponding rays meeting the interior of a fibered face of the Thurston ball. Our argument in the smooth case recovers the

Laudenbach–Blank theorem [26] that cohomologous closed, nonsingular 1-forms are smoothly isotopic. This proof is significantly simpler than the one we offered in [10]. In particular, no appeal is made to the simple connectivity of $\mathrm{Diff}_0(F)$ [14, 20], where F is a compact surface. We do not, however, achieve the full generality of the Laudenbach–Blank theorem which allowed nonempty tangential boundary. In this case a closed, nonsingular 1-form is necessarily exact and their proof showed that such foliations are isotopic to the product foliation of $M = F \times [0,1]$. This yielded an independent proof of the theorem of J. Cerf [12]. Subsequent attempts [30, 10] at simpler proofs of Laudenbach–Blank assumed Cerf's theorem. Our present proof also assumes that theorem since it is key to the proof that a fibration $\pi\colon M \to S^1$ is uniquely determined up to isotopy by the cohomology class $\pi^*[d\theta]$.

References

[1] I. Altman, The sutured Floer polytope and taut depth-one foliations, *Algebr. Geom. Topol.* **14** (2014), 1881–1923.

[2] S. A. Bleiler and A. J. Casson, *Automorphisms of Surfaces After Nielsen and Thurston*, London Mathematical Society Student Texts **9**, Cambridge Univ. Press, Cambridge, 1988.

[3] A. Candel and L. Conlon, *Foliations.* I, Amer. Math. Soc., Providence, RI, 2000.

[4] A. Candel and L. Conlon, *Foliations.* II, Amer. Math. Soc., Providence, RI, 2003.

[5] J. Cantwell and L. Conlon, *Endperiodic automorphisms of surfaces and foliations*, arXiv:1006.4525v4.

[6] J. Cantwell and L. Conlon, Tischler fibrations of open, foliated sets, *Ann. Inst. Fourier* (*Grenoble*) **31** (1981), 113–135.

[7] J. Cantwell and L. Conlon, Isotopy of depth one foliations, in *Geometric study of foliations* (*Tokyo, 1993*), World Sci. Publ., River Edge, NJ, 1993, 153–173.

[8] J. Cantwell and L. Conlon, Foliation cones, *Proceedings of the Kirbyfest*, Geometry & Topology Monographs **2**, Geometry & Topology Publications, Coventry, 1999, 35–86.

[9] J. Cantwell and L. Conlon, Foliation cones; a correction, *Proceedings of the Kirbyfest*, Geometry & Topology Monographs **2**, Geometry & Topology Publications, Coventry, 1999, 571–576.

[10] J. Cantwell and L. Conlon, Isotopies of foliated 3-manifolds without holonomy, *Adv. Math.* **144** (1999), 13–49.

[11] J. Cantwell and L. Conlon, The sutured Thurston norm, in *Foliations 2012*, World Sci. Publ., Hackensack, NJ, 2013, 41–66.

[12] J. Cerf, *Sur les Difféomorphismes de la Sphère de Dimension Trois* ($\Gamma_4 = 0$), Lecture Notes in Mathematics **53**, Springer, Berlin, 1968.

[13] G. de Rham, *Variétés Différentiables*, 2de éd., Hermann, Paris, 1960.

[14] C. J. Earle and A. Schatz, Teichmüller theory for surfaces with boundary, *J. Differential Geometry* **4** (1970), 169–185.

[15] A. Fathi, F. Laudenbach and V. Poénaru, *Travaux de Thurston sur les Surfaces*, 2de éd., Astérisque **66**–**67**, 1991.

[16] S. R. Fenley, *Depth-One Foliations in Hyperbolic 3-Manifolds*, ProQuest LLC, Ann Arbor, MI, 1989.

[17] S. R. Fenley, End periodic surface homeomorphisms and 3-manifolds, *Math. Z.* **224** (1997), 1–24.

[18] D. Fried, *Fibrations Over S^1 with pseudo-Anosov Monodromy*, Astérisque **66**–**67** (1991), 251–266.

[19] D. Gabai, Foliations and genera of links, *Topology* **23** (1984), 381–394.

[20] A. Gramain, Le type d'homotopie du groupe des difféomorphismes d'une surface compacte, *Ann. Sci. École Norm. Sup.* (4) **6** (1973), 53–66.

[21] M.-E. Hamstrom, Some global properties of the space of homeomorphisms on a disc with holes, *Duke Math. J.* **29** (1962), 657–662.

[22] M.-E. Hamstrom, The space of homeomorphisms on a torus, *Illinois J. Math.* **9** (1965), 59–65.

[23] M.-E. Hamstrom, Homotopy groups of the space of homeomorphisms on a 2-manifold, *Illinois J. Math.* **10** (1966), 563–573.

[24] M. Handel and W. P. Thurston, New proofs of some results of Nielsen, *Adv. in Math.* **56** (1985), 173–191.

[25] A. Juhász, The sutured Floer homology polytope, *Geom. Topol.* **14** (2010), 1303–1354.

[26] F. Laudenbach and S. Blank, Isotopie de formes fermées en dimension trois, *Invent. Math.* **54** (1979), 103–177.

[27] J. Moser, On the volume elements on a manifold, *Trans. Amer. Math. Soc.* **120** (1965), 286–294.

[28] S. P. Novikov, The topology of foliations, *Trudy Moskov. Mat. Obšč.* **14** (1965), 248–278.

[29] J. F. Plante, Foliations with measure preserving holonomy, *Ann. of Math.* (2) **102** (1975), 327–361.

[30] Ngô Van Quê and R. Roussarie, Sur l'isotopie des formes fermées en dimension 3, *Invent. Math.* **64** (1981), 69–87.

[31] R. Sacksteder, Foliations and pseudogroups, *Amer. J. Math.* **87** (1965), 79–102.

[32] L. Schwartz, *Théorie des distributions*, Hermann, Paris, 1966.

[33] S. Schwartzman, Asymptotic cycles, *Ann. of Math.* (2) **66** (1957), 270–284.

[34] D. Sullivan, Cycles for the dynamical study of foliated manifolds and complex manifolds, *Invent. Math.* **36** (1976), 225–255.

[35] W. P. Thurston, A norm for the homology of 3-manifolds, *Mem. Amer. Math. Soc.* **59** (1986), 99–130.

[36] W. P. Thurston, On the geometry and dynamics of diffeomorphisms of surfaces, *Bull. Amer. Math. Soc.* (*N.S.*) **19** (1988), 417–431.
[37] T. Yagasaki, Homotopy types of homeomorphism groups of noncompact 2-manifolds, *Topology Appl.* **108** (2000), 123–136.

John Cantwell
St. Louis University
St. Louis, MO 63103
E-mail address: CANTWELLJC@SLU.EDU

Lawrence Conlon
Washington University
St. Louis, MO 63130
E-mail address: LC@MATH.WUSTL.EDU