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The mathematical works of Shigefumi Mori

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Shigefumi Mori has been one of the most influential mathematicians of the past three decades. A proper survey of his achievements and influence—showing how his work fits in with that of his contemporaries and detailing the results proved by other mathematicians following the paths he laid out—would be a complete history of higher dimensional birational geometry, and thus far beyond the scope of this short summary.

The following overview groups his works according to themes rather than focusing on his main theorems, with the aim of laying out the fundamental ideas more clearly. However, this approach does result in some seemingly odd choices: one of the main topics (Mori's program) does not correspond to any of his publications and several important papers are omitted or mentioned only in passing.

I have had the privilege of knowing and working with Shigefumi Mori since 1983. This collaboration has been a fundamental influence on my mathematical point of view and development, thus I cannot pretend to offer an unbiased survey of Mori's contributions to algebraic geometry. Those who prefer to form their own opinion would be well-rewarded by reading his papers, especially [1] and [2]. These fundamentally important and enjoyable works illustrate the depth and originality of his ideas, as well as the clarity and precision of his writing.

§ Bend-and-break

It is a rare pleasure in mathematics to be surprised by a theorem whose proof is simple, yet its applications are profound. *Bend-and-break* is among them. The proof consists of three observations.

(i) Let C be a smooth, projective curve and $f_t : C \to X$ a 1-parameter family of non-constant morphisms. If the genus of C is ≥ 1

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and $f_t(c)$ stays fixed for some point $c \in C$, then a suitable limit of the 1-cycles $f_t(C)$ becomes reducible and acquires a rational component.

(ii) If $C \cong \mathbb{P}^1$, then one gets a reducible limit cycle if $f_t(c_i)$ stays fixed for at least 2 points $c_1, c_2 \in C$.

(iii) If $\deg f_0^* T_X > 0$, then one can construct such deformations in positive characteristic, reduce the degree of the resulting rational curves using (ii) and then lift the low-degree rational curves back to characteristic 0.

The bend-and-break method has several applications. The first, given in [1], characterizes the projective space $\mathbb{P}^n$ as the unique n-dimensional smooth, projective variety whose tangent bundle is ample. This solved conjectures of Frankel and Hartshorne.

Since we assume that the tangent bundle is ample, the condition $\deg f_0^* T_X > 0$ is automatic and the resulting rational curves turn out to be the lines in $\mathbb{P}^n$. Once we have the lines, the rest is relatively straightforward. These ideas have been used repeatedly to describe varieties whose tangent bundle is close to being ample; the conditions becoming more general every year.

A subsequent paper [9] shows that one can get low-degree rational curves by *one* application of a variant of (i) and (iii), making it possible to use the method for many singular varieties.

Another application, presented in [2] shows that the cone of curves of a Fano variety is polyhedral. This led to a classification of smooth, 3-dimensional Fano varieties [3, 4, 21] and is the cornerstone of most results on higher-dimensional Fano varieties.

§ **The cone of curves and extremal rays**

Let X be a projective variety, for simplicity over $\mathbb{C}$. Every algebraic curve $C \subset X$ has a homology class $[C] \in H_2(X, \mathbb{R})$. The *cone of curves* $\overline{NE}(X)$ is the closure of the cone generated by them. This cone was studied by Hironaka and Kleiman, but it was Mori who realized that if X is smooth, then the part of the cone lying in the open half-space where the intersection number with the canonical class is negative is locally finitely generated [2]. These generators are called the *extremal rays* of X or of $\overline{NE}(X)$.

Mori used bend-and-break to show that every extremal ray is spanned by (the homology class of) a rational curve. Then he showed, in dimension 3, that for every extremal ray $R \subset \overline{NE}(X)$ there is a unique morphism $g_R : X \to Z$, called the *contraction of* R such that an irreducible curve $C \subset X$ is mapped to a point by g_R iff $[C] \in R$. He also gave a complete description of all possible contractions.

This resulted in a completely different way of thinking about morphisms of varieties. Before [2], people were thinking in terms of linear systems; since then everyone imagines an extremal ray or face of a cone.

Later, cohomological methods were used to extend the existence results to all dimensions and to log terminal varieties and pairs, but the basic picture established by Mori remains unchanged.

An even more important consequence of [2] is that, as we discuss next, it led to a whole new approach to algebraic varieties.

§ Mori's program

Mori is, and probably will remain, best known for developing what is now known as *Mori's program, Mori theory* or *minimal model program.* Interestingly, he never wrote down the program and there is no published reference that one can point to.

It was obvious to experts that [2] gave the beginning steps of a program, but the continuation was not immediately clear. It seems that Mori had the whole plan by 1980 but he did not write it down; thus Reid and Tsunoda both discovered various parts independently. By 1982 the details were well-known to a rapidly growing group of researchers.

The program provides a recipe that allows one to write any birational contraction as a composite of two types of elementary maps: *divisorial contractions* and *flips.* As an end result, every smooth projective variety is birational either to a variety (with terminal singularities) whose canonical class is nef or to the total space of a family of Fano varieties. More generally, for many questions, the program allows us to find a birational model whose global and local geometry match optimally.

This viewpoint was new even for surfaces and it led to predictions about higher dimensions that were not even contemplated before. It determined the main direction of research in birational geometry for decades.

After a few years, two basic approaches started to diverge rapidly from each other. The schools of Kawamata and of Shokurov aimed at establishing methods that would work in all dimensions, culminating in the results of Hacon and McKernan. These results give existence theorems but rarely produce explicit descriptions of flips or of minimal models.

Mori's aim was to develop a deep and complete understanding of the 3-dimensional situation. As a consequence of his work, now we see that for 3-folds it is possible to understand and control every detail of the minimal model program; we discuss some of it next. By contrast, even the classification of 4-dimensional terminal singularities seems unlikely.

§ Extremal curve neighborhoods

As [2] showed, the contraction of an extremal ray of a smooth 3-fold sometimes results in a singular 3-fold. In order to continue with Mori's program, it was necessary to extend the theory of [2] to 3-folds with terminal singularities.

At the beginning the most important part was to understand flipping contractions. For later applications the key class turned out to be extremal contractions of 3-folds $f : X \to Y$ whose fiber dimension is ≤ 1. If X has terminal singularities and $C \subset X$ is a fiber, its neighborhood (formal or analytic) is called an *extremal curve neighborhood.*

Mori's greatest technical achievement was the development of a method to understand extremal curve neighborhoods. The full method—which should give a complete classification of extremal curve neighborhoods—was never written down; it exists only in his head and in thousands of pages of hand-written notes. The classification is exceedingly lengthy and it is not clear that every detail of it is needed in applications. Thus the published parts of the method give partial results, stopping when enough knowledge is assembled for the question at hand. This project, started in 1980, is still unfinished.

The paper [7] is preliminary; it gives a classification of terminal 3-fold singularities. The first, and most important, application of the method is the proof of the existence of 3-fold flips given in [10], completing the minimal model program in dimension 3.

Later uses of the method include the classification of 3-dimensional, irreducible, flipping curve neighborhoods [15, 20] and a (hopefully soon to be complete) classification of divisorial contractions with 1-dimensional fibers and $\mathbb{Q}$-conic bundles [22, 23, 24, 25].

§ Rationally connected varieties

Rational curves are the simplest among algebraic curves and rational surfaces form the simplest class among algebraic surfaces. Max Noether knew that rational 3-folds are too special and for over a century there was no agreement, nor even a sensible candidate, for what should constitute the correct class of "simplest varieties."

In characteristic 0, the definition of this class, called *rationally connected varieties,* was given in [14], together with numerous characterizations and good properties. Roughly speaking, a variety X is rationally connected if it has as many rational curves as rational varieties have, but we do not assume anything about the existence of rational surfaces on X.

Other papers of the same series [12, 13, 19] prove that smooth (and certain singular) Fano varieties are rationally connected. As a byproduct, they establish boundedness theorems for Fano varieties.

Since then the study of rationally connected varieties became a significant subfield of birational geometry. Most geometric questions have been solved in characteristic 0 but there are many open problems concerning the arithmetic of rationally connected varieties.

§ Miscellanea

The study of Fano 3-folds is intimately connected with the study of their hyperplane sections, which are K3 surfaces. Understanding curves on K3 surfaces turned out to be useful in proving the unirationality of the moduli of curves of genus 11 and in understanding space curves [5, 6].

One of the problems of dealing with minimal models is that many of the invariants that are positive integers for smooth varieties end up being positive rational numbers. In applications it is important to know that these are bounded away from 0 or, more generally, that they satisfy the descending chain condition. By now many existence results are known but [18] remains almost the only known effective estimate in this area.

The paper [16] shows that the geometric quotient of an algebraic space by a groupoid with finite stabilizer is again an algebraic space. This finished a problem that has been investigated since the sixties.

§ Books and surveys

For many of the younger generation, their introduction to Mori's works came through his books [11, 17] or his survey paper [8]. These still are the best introductions to higher dimensional birational geometry.

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