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Dual cones of varieties of minimal rational tangents

Jun-Muk Hwang

Abstract.

The varieties of minimal rational tangents play an important role in the geometry of uniruled projective manifolds. The goal of this paper is to exhibit their role in the symplectic geometry of the cotangent bundles of uniruled projective manifolds. More precisely, let X be a uniruled projective manifold satisfying the assumption that the VMRT at a general point is smooth. We show that the total family of dual cones of the varieties of minimal rational tangents is a coisotropic subvariety in $T^*(X)$. Furthermore, the closure of a general leaf of the null foliation of this coisotropic subvariety is an immersed projective space of dimension $\delta + 1$ where δ is the dual defect of the variety of minimal rational tangents at a general point. When $\delta = 0$, the symplectic reduction of the coisotropic variety can be realized as a subbundle of the cotangent bundle $T^*(\mathcal{K})$ of the parameter space $\mathcal{K}$ of the rational curves.

§1. Introduction

The cotangent bundle of a complex projective manifold carries a natural holomorphic symplectic form, rendering it a (non-compact holomorphic) symplectic manifold (cf. Definitions 2.1 and 2.2). The symplectic geometry of these non-compact complex manifolds has not been studied very much, except for a very few special classes of projective manifolds. Although these special classes are quite limited, their cotangent bundles exhibit remarkably rich geometry. In the two examples below, we collect (very selective and limited) samples of results on symplectic geometry of the cotangent bundles of some of these projective manifolds.

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Example 1.1. Let X be the moduli scheme $\mathcal{SU}_C(r)$ of stable vector bundles over a curve C of genus ≥ 2, of rank r with determinant isomorphic to a fixed line bundle of degree coprime to r. Then X is a Fano manifold of dimension $(r^2-1)(g-1)$. There exists a morphism $h : T^*(X) \to \mathbb{C}^{(r^2-1)(g-1)}$, called the Hitchin map (cf.[Hi]). Let $\mathcal{H} \subset T^*(X)$ be the discriminant hypersurface of h, i.e., the union of singular fibers of h. Then $\mathcal{H}$ is covered by a family of $\mathbb{P}^1$'s. The images of these $\mathbb{P}^1$'s in X are 'Hecke curves' describing family of vector bundles on C arising from certain analogue of Hecke correspondence (cf.[HR]).

Example 1.2. Let G/P be a rational homogeneous space where G is a semisimple complex Lie group and P is a parabolic subgroup. The group G acts on the cotangent bundle $T^*(G/P)$ as symplectic automorphisms. This induces a morphism $\mu : T^*(G/P) \to \mathfrak{g}^*$ to the dual of the Lie algebra of G by the moment map construction (cf. Section 1.4 of [CG]). The image of μ is the closure of a nilpotent orbit in $\mathfrak{g}^*$ (e.g. Proposition 1.5 of [Fu]). The exceptional locus $E \subset T^*(G/P)$ of m is covered by certain projective spaces contracted by μ.

The projective manifolds in the above examples, $\mathcal{SU}_C(r)$ and G/P, are Fano manifolds. This suggests that there may exist some interesting symplectic geometry in the cotangent bundles of Fano manifolds. Of course, one cannot expect that the cotangent bundle of an arbitrary Fano manifold has a geometric structure as rich as those of the above examples. Yet we will show that some features of Examples 1.1 and 1.2 can be recovered in the cotangent bundles of all Fano manifolds (in fact, all uniruled projective manifolds), satisfying one technical assumption.

To state our result and the technical assumption, we need to recall the concept of VMRT. For a uniruled projective manifold X, we have the notion of a minimal dominating family $\bar{\mathcal{K}}$ of rational curves (cf. Definition 3.3). The tangent vectors to the members of such a family $\bar{\mathcal{K}}$ of rational curves define a distinguished subvariety $\mathcal{C} \subset \mathbb{P}T(X)$ called the total family of VMRT (cf. Definition 4.5). The technical assumption we require is

Assumption 1.3. For a general point $x \in X$, the subvariety $\mathcal{C}_x := \mathcal{C} \cap \mathbb{P}T_x(X)$ is nonsingular.

This assumption holds for a large class of examples of $(X, \bar{\mathcal{K}})$. It is equivalent to the normality of $\mathcal{C}_x$ by Corollary 1 of [HM]. For this reason, when working with uniruled projective manifolds, it is often harmless to make this assumption.

Now consider the dual cone $\mathcal{D}_x := \hat{\mathcal{C}}^*_x \subset T^*_x(X)$ of $\mathcal{C}_x \subset \mathbb{P}T_x(X)$ in the sense of Definition 4.1. Let $\mathcal{D} \subset T^*(X)$ be the closure of the union of $\mathcal{D}_x$'s as x varies over general points. This irreducible subvariety of the cotangent bundle is called the total family of the dual cones of VMRT (cf. Definition 4.5), associated to the given minimal dominating family. Our main result is the following. We refer to Definition 2.1 for the terminology in symplectic geometry.

Theorem 1.4. *Let X be a uniruled projective manifold and choose a minimal dominating family of rational curves satisfying Assumption* 1.3. *Then the total family of dual cones $\mathcal{D} \subset T^*(X)$ is a coisotropic subvariety. The closure of a general leaf of the null foliation on this coisotropic subvariety $\mathcal{D}$ is an immersed projective space $\mathbb{P}^{\delta+1}$ where $\delta \geq 0$ is the dual defect of the VMRT $\mathcal{C}_x \subset \mathbb{P}T_x(X)$ at a general point $x \in X$.*

Examples 1.1 and 1.2 can be interpreted in terms of Theorem 1.4. In fact, Hecke curves in Example 1.1 give a minimal dominating family in $\mathcal{SU}_C(r)$ and the Hitchin discriminant $\mathcal{H}$ is precisely $\mathcal{D}$, as proved in [HR]. In Example 1.2, the lines in the projective spaces contracted by μ are sent to rational curves in G/P which give a minimal dominating family.

When $\delta = 0$, i.e., when $\mathcal{D}$ is a hypersurface in $T^*(X)$, which holds in many examples, we can given an explicit description of the symplectic reduction (cf. Definition 2.3) of $\mathcal{D}$:

Theorem 1.5. *In the situation of Theorem* 1.4, *assume that $\delta = 0$. Then there exists a vector subbundle $\mathcal{R} \subset T^*(\mathcal{K})$ over a Zariski dense open subset $\mathcal{K} \subset \bar{\mathcal{K}}$, which is generically symplectic in the sense of Definition* 2.2, *together with a natural dominant rational map $\mathcal{D} \dashrightarrow \mathcal{R}$ which provides a symplectic reduction of $\mathcal{D}$.*

The following example is a well-known case of Theorem 1.5. In fact, it served as a motivation for Theorem 1.5.

Example 1.6. Let X be the smooth hyperquadric $\mathbb{Q}^n \subset \mathbb{P}^{n+1}, n \geq 3$. Choose $\mathcal{K}$ as the space of lines lying on $\mathbb{Q}^n$. The variety of minimal rational tangents $\mathcal{C}_x \subset \mathbb{P}T_x(X)$ at each $x \in X$ is a hyperquadric of dimension $n-2$. Its dual cone $\mathcal{D}_x \subset T^*_x(X)$ is an affine hyperquadric. The moment map associated to the $SO(n+2)$-action on X and $T^*(X)$ is a morphism $\mu : T^*(X) \to \mathfrak{g}^*, \mathfrak{g} := \mathfrak{so}(n+2)$, which sends $\mathcal{D}$ to the minimal nilpotent orbit $\mathcal{O}_{min} \subset \mathfrak{g}^*$. The fibers of $\mu : \mathcal{D} \to \mathcal{O}_{min}$ are $\mathbb{P}^1$'s. This morphism μ is the symplectic reduction given by the canonical symplectic form ω_X on $T^*(X)$. Furthermore, the projectivization $\mathbb{P}\mathcal{O}_{min}$

is isomorphic to the space $\mathcal{K}$ of lines. In fact, the $\mathbb{C}^\times$-bundle $\mathcal{O}_{min} \to \mathcal{K}$ is the symplectifcation of the contact manifold $\mathcal{K} = \mathbb{P}\mathcal{O}_{min}$ (e.g. (2.6) in [Be]). The contact structure on $\mathcal{K}$ is given by a line subbundle $\mathcal{R} \subset T^*(\mathcal{K})$ and $\mathcal{O}_{min}$ is isomorphic to the complement of the zero section in this line bundle.

The proofs of Theorems 1.4 and 1.5 are obtained by combining results from the theory of VMRT's with some standard arguments in symplectic geometry. Beside their intrinsic interest, we expect our results and methods will be useful in the application of VMRT technics to the geometry of uniruled projective manifolds.

§2. Definitions and results from symplectic geometry

In this section, we collect some basic definitions and facts from symplectic geometry.

Definition 2.1. Let M be a complex manifold equipped with a closed holomorphic 2-form ω. For $z \in M$, let

$$\mathrm{Null}_z(M) := \{v \in T_z(M), \omega(v, w) = 0 \text{ for all } w \in T_z(M)\}.$$

This defines a distribution, called the *null distribution* on a Zariski open subset of M. If $\mathrm{Null}_z(M) = 0$ for every point $z \in M$, we say that ω is a *symplectic form* and M is a *symplectic manifold.* Given an irreducible subvariety $Z \subset M$ in a symplectic manifold (M, ω), consider the restriction of ω to the smooth locus $Sm(Z) \subset Z$. The rank of the null distribution of $\omega|_{Sm(Z)}$ cannot be bigger than the codimension of $Z \subset M$. If this rank is equal to the codimension of $Z \subset M$, we say that the variety Z is *coisotropic.* The null distribution on Z defines a foliation on a Zariski open subset $Z^o \subset Z$, which is called the *null foliation* of ω on Z.

The most important example of a symplectic manifold is the cotangent bundle of a complex manifold:

Definition 2.2. For a complex manifold Y, its cotangent bundle $T^*(Y)$ has a natural 1-form λ_Y whose value on $T_\gamma(T^*(Y))$ at a point $\gamma \in T^*(Y)$ is given by the composition

$$T_\gamma(T^*(Y)) \stackrel{d\pi_Y}{\to} T_y(Y) \stackrel{\gamma}{\to} \mathbb{C}$$

where $\pi_Y : T^*(Y) \to Y$ is the natural projection and $y = \pi_Y(\gamma)$. The 2-form $\omega_Y := d\lambda_Y$ is a symplectic form, rendering $T^*(Y)$ a symplectic manifold. A vector subbundle $W \subset T^*(Y)$ is said to be *generically* *symplectic* if the restriction of ω_Y to a Zariski open subset $W^o \subset W$ is symplectic so that $(W^o, \omega_Y|_{W^o})$ is a symplectic manifold.

For our purpose, the following definition of symplectic reduction will be sufficient.

Definition 2.3. In Definition 2.1, assume that M is a quasi-projective algebraic variety and $Z \subset M$ is a coisotropic subvariety which is also quasi-projective. Then a quasi-projective symplectic manifold (B, ω') is called the *symplectic reduction* of the coisotropic subvariety Z, if there exists a Zariski open subset $Z^o \subset Z$ equipped with a smooth morphism

$$\mathrm{red} : Z^o \to B$$

such that $\omega|_{Z^o} = \mathrm{red}^* \omega'$.

The following result is useful in checking the coisotropicity of certain subvarieties. Although this result must have been known to experts, we can not find a good reference. We will apply it to $\mathbb{S} = \mathbb{P}^{\delta+1}$. When $\mathbb{S} = \mathbb{P}^1$, this is Proposition 4.5 of [Dr].

Proposition 2.4. *Let* $\mathbb{S} := G/P$ *be a rational homogeneous space, as in Example* 1.2, *with* $\dim \mathbb{S} = \delta + 1 \geq 1$ *for a non-negative integer* δ. *Let* M *be a quasi-projective symplectic manifold of dimension* $2n$. *Let* $Z \subset M$ *be a subvariety of dimension* $2n - \delta - 1 \geq 1$. *Suppose that for each general point* $z \in Z$, *there exists a morphism* $\kappa : \mathbb{S} \to Z$ *with* $z \in \kappa(\mathbb{S})$ *and* $\dim \kappa(\mathbb{S}) = \dim \mathbb{S}$. *Then* Z *is coisotropic and a leaf of the null foliation through* z *is an open set in the image of* κ.

Proof. From the countability of the number of components of $\mathrm{Hom}(\mathbb{S}, Z)$, the assumption implies that there exist a quasi-projective manifold U and a dominant morphism $\mu : \mathbb{S} \times U \to Z$ such that for each $u \in U$, the image of $\mathbb{S}_u := \mathbb{S} \times \{u\}$ under μ has dimension equal to $\dim \mathbb{S}$. It follows that a general point $u \in U$ and a general point $y \in \mathbb{S}_u$ satisfy the following conditions.

(i) The image $z := \mu(y)$ is a smooth point of Z.
(ii) The differential $(d\mu)_y : T_y(\mathbb{S} \times U) \to T_z(Z)$ is surjective.
(iii) For $\mu_u := \mu|_{\mathbb{S}_u}$, its differential $(d\mu_u)_y : T_y(\mathbb{S}_u) \to T_z(Z)$ is injective.
(iv) The image $P_u := \mu_u(\mathbb{S}_u)$ is smooth at z.

Using the choice of (u, y) as above and the morphism $\mu_u : \mathbb{S}_u \to M$, we have the following three claims.

Claim 1. Given a vector $w \in T_z(Z)$, there exists $\tilde{w} \in H^0(\mathbb{S}_u, \mu_u^* T(M))$ satisfying $\tilde{w}_y = w$.

Proof. We can find $w' \in T_y(\mathbb{S} \times U)$ satisfying $d\mu_y(w') = w$ by the condition (ii) in the choice of y. Since the vector bundle $T(\mathbb{S} \times U)$ is globally generated on $\mathbb{S}_u$, we have a section

$$\hat{w} \in H^0(\mathbb{S}_u, T(\mathbb{S} \times U)|_{\mathbb{S}_u})$$

satisfying $\hat{w}_y = w'$. Then under the homomorphism $d\mu : T(\mathbb{S} \times U) \to \mu^* T(M)$, the section

$$\tilde{w} := d\mu(\hat{w}) \in H^0(\mathbb{S}_u, \mu_u^* T(M))$$

satisfies $\tilde{w}_y = w$. Q.E.D.

Claim 2. Given a vector $v \in T_z(P_u) \subset T_z(M)$, there exists $\tilde{v} \in H^0(\mathbb{S}_u, \mu_u^* T(M))$ satisfying $\tilde{v}_y = v$ and $\tilde{v}$ vanishes at some point of $\mathbb{S}_u$.

Proof. Using the isomorphism $T_z(P_u) = T_y(\mathbb{S}_u)$ from the conditions (iii) and (iv), we have a vector $v' \in T_y(\mathbb{S}_u)$ such that $d\mu(v') = v$. Recall that $T(\mathbb{S}_u)$ is generated by sections and any vector field on $\mathbb{S}$ vanishes at some point. Thus we can find a section

$$\hat{v} \in H^0(\mathbb{S}_u, T(\mathbb{S}_u))$$

that vanishes at a point of $\mathbb{S}_u$ and satisfies $\hat{v}_y = v'$. Then $\tilde{v} := d\mu(\hat{v})$ satisfies the required conditions. Q.E.D.

Claim 3. $\omega(T_z(P_u), T_z(Z)) = 0$, in other words, $T_z(P_u) \subset \mathrm{Null}_z(Z)$.

Proof. The pull-back of ω by the morphism $\mu_u : \mathbb{S}_u \to M$ defines a homomorphism

$$\omega' : \bigwedge^2 \left((\mu_u)^* T(M)\right) \to \mathcal{O}_{\mathbb{S}_u}.$$

For any $v \in T_z(P_u)$ and $w \in T_z(Z)$, let $\tilde{v}$ and $\tilde{w}$ be as in Claim 1 and Claim2. Then $\omega'(\tilde{v}, \tilde{w})$ is a holomorphic function on $\mathbb{S}_u$. Since $\tilde{v}$ vanishes at a point of $\mathbb{S}_u$, this holomorphic function must be zero. It follows that

$$\omega'(\tilde{v}, \tilde{w})_u = \omega(v, w) = 0.$$

This proves the claim. Q.E.D.

From the generality of y, the inclusion $T_z(P_u) \subset \mathrm{Null}_z(Z)$ of Claim 3 holds for any general point z of P_u. Note that $\dim T_z(P_u) = \delta + 1$, which is equal to the codimension of $Z \subset M$. This implies that $T_z(P_u) = \mathrm{Null}_z(Z)$ and Z is coisotropic. Moreover subvarieties of the form P_u

given by general points $u \in U$ are tangent to the null foliation on an open subset of Z.

To complete the proof of Proposition 2.4, just note that the null foliation is uniquely defined at a smooth point of Z by $\omega|_Z$. Thus the germ of the image of κ through a general point z must coincide with that of some P_u defined above, which is the germ of the leaf of the null foliation through z. Q.E.D.

§3. Vector bundles associated to a minimal dominating family of rational curves

In this section, we recall some basic notions regarding a minimal dominating family of rational curves and introduce a number of vector bundles associated to such a family. It is convenient to use the following.

Definition 3.1. A subvariety Y in a complex manifold M is *an immersed* $\mathbb{P}^d$ if the normalization Y^{norm} is biholomorphic to projective space $\mathbb{P}^d$ and the normalization morphism $Y^{norm} \to Y \subset M$ is an immersion.

Definition 3.2. A rational curve $C \subset M$ on a complex manifold M is *standard* if under a normalization $h : \tilde{C} \to C \subset M$,

$$h^*T(M) \cong \mathcal{O}(2) \oplus \mathcal{O}(1)^p \oplus \mathcal{O}^{\dim M - 1 - p}$$

for some nonnegative integer p. In this case, $C \subset M$ is an immersed $\mathbb{P}^1$ and the $\mathcal{O}(2)$-factor in $h^*T(M)$ corresponds to $T(\tilde{C})$.

Definition 3.3. Let X be a nonsingular projective variety. An irreducible component $\bar{\mathcal{K}}$ of the normalized space $\mathrm{RatCurves}^n(X)$ of rational curves on X (cf. II.2.11 of [Ko]) is a *minimal dominating family* if for a general point $x \in X$, the subscheme $\bar{\mathcal{K}}_x \subset \bar{\mathcal{K}}$ consisting of members passing through x is non-empty and complete. It is well-known that every uniruled projective manifold has a minimal dominating family (cf. IV.2 of [Ko] where the term 'generically unsplit family' is used).

Remark 3.4. In [Hw] or [HM], we used the notation $\mathcal{K}$ for a minimal dominating family, while we use $\bar{\mathcal{K}}$ in Definition 3.3. In this paper, we will use $\mathcal{K}$ to denote an open subset in $\bar{\mathcal{K}}$ defined in the next proposition.

Proposition 3.5. *Given a minimal dominating family $\bar{\mathcal{K}}$ on a projective manifold X, the following holds.*

(i) *The Zariski open subset $\mathcal{K} \subset \bar{\mathcal{K}}$ consisting of members of $\bar{\mathcal{K}}$ which are standard is nonempty and nonsingular.*

(ii) *Let $\rho : \mathcal{U} \to \mathcal{K}$ and $\mu : \mathcal{U} \to X$ be the universal family morphisms, i.e., ρ is a $\mathbb{P}^1$-bundle such that for a member $[C] \in \mathcal{K}$, we have $C = \mu(\rho^{-1}([C]))$. For each point $y \in \mathcal{U}$, define $T^\rho_y :=$ $\mathrm{Ker}(d\rho_y)$ and $T^\mu_y := \mathrm{Ker}(d\mu_y)$. Then μ is a smooth morphism and $T^\rho_y \cap T^\mu_y = 0$ for each $y \in \mathcal{U}$.*

(iii) *The tangent morphism $\tau : \mathcal{U} \to \mathbb{P}T(X)$ sending $y \in \mathcal{U}$ to $\mathbb{P}\big((d\mu)_y(T^\rho)\big)$ is a generically injective immersion factoring μ:*

$$\begin{array}{ccccc} \mathcal{K} & \xleftarrow{\rho} & \mathcal{U} & \xrightarrow{\tau} & \mathbb{P}T(X) \\ & & \mu \downarrow & & \downarrow \pi \\ & & X & = & X \end{array}$$

where π is the natural projection.

(iv) *For a point $x \in \mu(\mathcal{U}) \subset X$, the restriction of τ to the fiber $\mathcal{U}_x := \mu^{-1}(x)$*

$$\tau_x : \mathcal{U}_x \to \mathbb{P}T_x(X)$$

is an immersion. It is generically injective if x is a general point of X.

Proof. (i) is Corollary IV.2.9 of [Ko] or Theorem 1.2 of [Hw]. The smoothness of μ in (ii) is by the same reasoning as II.3.11.5 of [Ko]. Since a standard rational curve is an immersed $\mathbb{P}^1$, we have $(d\mu)_y(T^\rho) \neq$ 0 for any $y \in \mathcal{U}$, proving (ii). The morphisms τ and τ_x in (iii) and (iv) are immersions by Proposition 1.4 of [Hw]. The morphism τ is generically injective from Theorem 1 of [HM] and so is τ_x for a general $x \in X$. Q.E.D.

From Proposition 3.5, the manifold $\mathcal{U}$ has two natural distributions T^ρ and T^μ satisfying $T^\rho \cap T^\mu = 0$. It is convenient to introduce an additional distribution $\mathcal{P} \subset T(\mathcal{U})$, which plays an important role in the geometry of VMRT (cf. p. 54 of [HM]).

Definition 3.6. In the setting of Proposition 3.5, for each $y \in \mathcal{U}$ and $x = \mu(y)$, since the morphism $\tau_x : \mathcal{U}_x \to \mathbb{P}T_x(X)$ is an immersion by Proposition 3.5 (iv), we have the affine tangent space $\hat{T}_y(\mathcal{U}_x) \subset T_x(X)$ to the image of the germ of $\mathcal{U}_x$ at y. Using the differential $(d\mu)_y : T_y(\mathcal{U}) \to$ $T_x(X)$, define the subspace $\mathcal{P}_y \subset T_y(\mathcal{U})$ by

$$\mathcal{P}_y := (d\mu)_y^{-1}(\hat{T}_y(\mathcal{U}_x)).$$

This defines a vector subbundle $\mathcal{P}$ of $T(\mathcal{U})$ with the property $T^\rho \oplus T^\mu \subset$ $\mathcal{P}$.

Proposition 3.7. *In the setting of Definition* 3.6, *let $\mathcal{I} \subset T^*(\mathcal{U})$ be the annihilator of $\mathcal{P}$. There are canonical inclusions $\mu^*T^*(X) \subset T^*(\mathcal{U})$ and $\rho^*T^*(\mathcal{K}) \subset T^*(\mathcal{U})$ such that $\mathcal{I} \subset \mu^*T^*(X) \cap \rho^*T^*(\mathcal{K})$.*

Proof. Since μ and ρ are smooth morphisms from Proposition 3.5 (ii), we have natural inclusions $\mu^*T^*(X) \subset T^*(\mathcal{U})$ and $\rho^*T^*(\mathcal{K}) \subset T^*(\mathcal{U})$. Since $\mathcal{I}$ annihilates $\mathcal{P} \subset T(\mathcal{U})$ which contains both $T^\mu = \mathrm{Ker}(d\mu)$ and $T^\rho = \mathrm{Ker}(d\rho)$, we have the inclusion $\mathcal{I} \subset \mu^*T^*(X) \cap \rho^*T^*(\mathcal{K})$. Q.E.D.

Definition 3.8. In the setting of Proposition 3.7, denote by $\chi : \mathcal{I} \to T^*(X)$ the composition of the inclusion $\mathcal{I} \subset \mu^*T^*(X)$ and the natural collapsing morphism $\mu^*T^*(X) \to T^*(X)$ and by $\eta : \mathcal{I} \to T^*(\mathcal{K})$ the composition of the inclusion $\mathcal{I} \subset \rho^*T(\mathcal{K})$ and the natural collapsing $\rho^*T(\mathcal{K}) \to T^*(\mathcal{K})$. Recall from Definition 2.2 that

$$\pi_{\mathcal{K}} : T^*(\mathcal{K}) \to \mathcal{K},\ \pi_{\mathcal{U}} : T^*(\mathcal{U}) \to \mathcal{U} \text{ and } \pi_X : T^*(X) \to X$$

denote the natural projections. Defining $\varpi := \pi_{\mathcal{U}}|_{\mathcal{I}}$, we have the commutative diagram

$$\begin{array}{ccccc} T^*(\mathcal{K}) & \xleftarrow{\eta} & \mathcal{I} & \xrightarrow{\chi} & T^*(X) \\ \pi_{\mathcal{K}} \downarrow & & \downarrow \varpi & & \downarrow \pi_X \\ \mathcal{K} & \xleftarrow{\rho} & \mathcal{U} & \xrightarrow{\mu} & X. \end{array}$$

Definition 3.9. Using the notation of Definition 3.2, let $h : \tilde{C} \to C \subset M$ be a normalization of a standard rational curve in M. We denote by $h^*T(M)^+ \subset h^*T(M)$ the vector subbundle corresponding to $\mathcal{O}(2) \oplus \mathcal{O}(1)^p$ and $h^*T^*(M)^0 \subset h^*T^*(M)$ the vector subbundle corresponding to $\mathcal{O}^{\dim M-1-p}$ in

$$h^*T^*(M) \cong \mathcal{O}(-2) \oplus \mathcal{O}(-1)^p \oplus \mathcal{O}^{\dim M-1-p}.$$

For each $y \in \tilde{C}$, the fiber $(h^*T^*(M)^0)_y \subset T^*_{h(y)}(M)$ is the annihilator of $(h^*T(M)^+)_y \subset T_{h(y)}(M)$.

Recall the following well-known fact from the classical deformation theory of rational curves (see e.g. p. 58 of [HM]).

Proposition 3.10. *In the setting of Proposition* 3.5, *for a given $[C] \in \mathcal{K}$, write $\tilde{C} := \rho^{-1}([C])$ and denote by*

$$h : \tilde{C} \to C \subset X$$

the restriction of μ, which is a normalization of C. Then there is a canonical isomorphism

$$T_{[C]}(\mathcal{K}) = H^0(\tilde{C}, h^*T(X))/H^0(\tilde{C}, T(\tilde{C})) \ \big(\cong H^0(\mathbb{P}^1, \mathcal{O}(1)^p \oplus \mathcal{O}^{\dim X - 1 - p}) \big).$$

Furthermore, for a point $y \in \tilde{C} \subset \mathcal{U}$, there is a canonical isomorphism

$$T_y(\mathcal{U}) = H^0(\tilde{C}, h^*T(X))/H^0(\tilde{C}, T(\tilde{C}) \otimes \mathbf{m}_y),$$

where $\mathbf{m}_y$ is the maximal ideal of $\mathcal{O}_{\tilde{C}}$ at $y \in \tilde{C}$.

Definition 3.11. We define a vector subbundle $\mathcal{R} \subset T^*(\mathcal{K})$ by setting its fiber $\mathcal{R}_{[C]}$ at $[C]$ to be the annihilator of

$$H^0(\tilde{C}, h^*T(X)^+)/H^0(\tilde{C}, T(\tilde{C})) \ \subset \ T_{[C]}(\mathcal{K})$$

under the description of $T_{[C]}(\mathcal{K})$ in Proposition 3.10. We will regard the vector bundle $\mathcal{R}$ as a quasi-projective variety.

Proposition 3.12. *In the setting of Definition* 3.11, *there exist natural isomorphisms*

$$(h^*T^*(X)^0)_y = H^0(\tilde{C}, h^*T^*(X)^0) = H^0(\tilde{C}, h^*T^*(X))$$

for each $y \in \tilde{C}$ and a canonical injection

$$\epsilon : H^0(\tilde{C}, h^*T^*(X)) \to T^*_{[C]}(\mathcal{K})$$

such that $\mathcal{R}_{[C]} = \mathrm{Im}(\epsilon)$.

Proof. From the splitting type of $h^*T^*(X)$ in Definition 3.9, we have natural isomorphisms

$$H^0(\tilde{C}, h^*T^*(X)) = H^0(\tilde{C}, h^*T^*(X)^0) = (h^*T^*(X)^0)_y.$$

The natural homomorphism

$$H^0(\tilde{C}, h^*T^*(X)) \to H^0(\tilde{C}, h^*T(X))^*$$

factors through a homomorphism

$$\epsilon : H^0(\tilde{C}, h^*T^*(X)) \to \big(H^0(\tilde{C}, h^*T(X))/H^0(\tilde{C}, T(\tilde{C}))\big)^* = T^*_{[C]}(\mathcal{K})$$

where the last equality is from Proposition 3.10. From the splitting type of $h^*T^*(X)$ in Definition 3.9, the homomorphism ϵ must be injective and its image annihilates

$$H^0(\tilde{C}, h^*T(X)^+)/H^0(\tilde{C}, T(\tilde{C})).$$

Thus $\mathrm{Im}(\epsilon) \subset \mathcal{R}_{[C]}$. By the comparison of dimensions, we have $\mathrm{Im}(\epsilon) = \mathcal{R}_{[C]}$. Q.E.D.

Proposition 3.13. *In the setting of Proposition* 3.10, *for a point* $y \in \mathcal{U}$ *and* $x = \mu(y)$, *the subspace* $\mathcal{I}_y \subset \mu^* T^*_x(X)$ *in Proposition* 3.7 *coincides with the subspace* $(h^*T^*(X)^0)_y \subset T^*_x(X)$. *In particular, the vector subbundle* $\mathcal{I} \subset \mu^* T^*(X)$ *restricted to* $\tilde{C}$ *coincides with* $h^*T^*(X)^0$.

Proof. It is well-known (e.g. Proposition 2.3 of [Hw]) that the affine tangent space $\hat{T}_y(\mathcal{U}_x) \subset T_x(X)$ to the image of the germ of $\mathcal{U}_x$ at y under τ_x satisfies

$$h^*\hat{T}_y(\mathcal{U}_x) = (h^*T(X)^+)_y$$

where the right hand side denotes the fiber of the vector bundle $h^*T(X)^+$ on $\tilde{C}$ at y. As $\mathcal{I}_y$ is defined as the annihilator of the left hand side, while $(h^*T^*(X)^0)_y$ is the annihilator of the right hand side as mentioned in Definition 3.9, we have the equality $\mathcal{I}_y = (h^*T^*(X)^0)_y$. Q.E.D.

Proposition 3.14. *In the setting of Proposition* 3.13, *regard* $\rho^*\mathcal{R}$ *as a subbundle of* $T^*(\mathcal{U})$ *via the submersion* $\rho : \mathcal{U} \to \mathcal{K}$. *Then the subbundle* $\mathcal{I} \subset T^*(\mathcal{U})$ *coincides with* $\rho^*\mathcal{R}$. *In particular, for the morphism* $\eta : \mathcal{I} \to T^*(\mathcal{K})$ *in Definition* 3.8, *we have* $\mathcal{R} = \eta(\mathcal{I})$.

Proof. By Proposition 3.13, the bundle $\mathcal{I}$ is trivial along the fibers of ρ. Thus it suffices to show that for each $y \in \mathcal{U}$ with $[C] = \rho(y)$, the image of the inclusion $\eta_y : \mathcal{I}_y \to T^*(\mathcal{K})$ obtained from Proposition 3.7 is $\mathcal{R}_{[C]}$. By Proposition 3.12, the subspace $\mathcal{R}_{[C]} \subset T^*_{[C]}(\mathcal{K})$ is the image of the natural homomorphism

$$\epsilon : H^0(\tilde{C}, h^*T^*(X)) \to T^*_{[C]}(\mathcal{K}).$$

By Proposition 3.13, $\mathcal{I}_y$ coincides with $(h^*T^*(X)^0)_y$. From the definition of η, the image of η_y must coincide with that of

$$(h^*T^*(X)^0)_y = H^0(\tilde{C}, h^*T^*(X)) \xrightarrow{\epsilon} T^*_{[C]}(\mathcal{K}).$$

This implies the proposition. Q.E.D.

§4. Proof of Theorem 1.4

Let us start with recalling some basic facts on dual varieties. A good reference is Example 15.22 of [Ha]. For our purpose, it is more convenient to look at the dual cones.

Definition 4.1. Let V be a (finite dimensional complex) vector space and let V^* be its dual vector space. For a point $z \in \mathbb{P}V$, denote by $\hat{z} \subset V$ the 1-dimensional subspace over z. For a projective subvariety

$Z \subset \mathbb{P}V$ (which may have finitely many irreducible components), the homogenous cone over Z will be denoted by $\hat{Z} \subset V$. Fix a projective variety $Z \subset \mathbb{P}V$ and a smooth point $z \in Z$. The *affine tangent space* of Z at z is the tangent space to the homogeneous cone $\hat{Z} \subset V$ at a point of $\hat{z} \setminus \{0\}$ and is denoted by $\hat{T}_z(Z) \subset V$. Its annihilator in V^* is denoted by $\hat{T}_z^{\perp}(Z) \subset V^*$. The *incidence variety* $I_Z \subset \mathbb{P}V \times V^*$ is the quasi-projective subvariety defined by

$$I_Z := \text{ the closure of } \{(z,u) \in \mathbb{P}V \times V^*,\ z \in Sm(Z), u \in \hat{T}_z^{\perp}(Z)\}.$$

The image of I_Z in V^* under the projection to the second factor $\mathbb{P}V \times V^* \to V^*$ is the *dual cone* of Z, to be denoted by $\hat{Z}^* \subset V^*$.

The following is immediate from the definition.

Lemma 4.2. *Let $Z = Z_1 \cup \ldots \cup Z_k$ be the decomposition into irreducible components. Then $I_Z = I_{Z_1} \cup \cdots \cup I_{Z_k}$ and $\hat{Z}^* = \hat{Z}_1^* \cup \cdots \cup \hat{Z}_k^*$, set-theoretically.*

Definition 4.3. For an irreducible variety $Z \subset \mathbb{P}V$, the fiber dimension of the morphism $I_Z \to \hat{Z}^*$ is the *dual defect* of Z and is denoted by δ_Z.

The next proposition is a direct consequence of the duality theorem (e.g. Theorem 15.24 of [Ha]).

Proposition 4.4. *For an irreducible variety $Z \subset \mathbb{P}V$, the fiber of $I_Z \to \hat{Z}^*$ over a smooth point of $\hat{Z}^*$ is biregular to $\mathbb{P}^{\delta_Z}$. This fiber is sent to a linear subspace $\mathbb{P}^{\delta_Z} \subset Z \subset \mathbb{P}V$ under the projection $\mathbb{P}V \times V^* \to \mathbb{P}V$ to the first factor.*

Now we are ready to define our main object of study.

Definition 4.5. Assume the setting of Proposition 3.5. The closure of the image of τ in Proposition 3.5 (iii) is called the *total family of VMRT* (*varieties of minimal rational tangents*) of $\mathcal{K}$ and denoted by $\mathcal{C} \subset \mathbb{P}T(X)$. The closure of the image of τ_x in Proposition 3.5 (iv) is called the *VMRT at* x and denoted by $\mathcal{C}_x \subset \mathbb{P}T_x(X)$. Its dual cone will be denoted by $\mathcal{D}_x := \hat{\mathcal{C}}_x^* \subset T_x^*(X)$. The *total family of dual cones of VMRT* is the subvariety $\mathcal{D} \subset T^*(X)$ defined as the closure of the union of $\mathcal{D}_x$'s as x varies over general points of X.

Proposition 4.6. *In Definition* 4.5, *let*

$$\mathcal{C}_x = \mathcal{C}_1 \cup \cdots \cup \mathcal{C}_k$$

be the decomposition into distinct irreducible components, as in Lemma 4.2. Then the dual defects satisfy $\delta_{\mathcal{C}_1} = \cdots = \delta_{\mathcal{C}_k}$*, which we will call the dual defect of* $\mathcal{C}_x$*. In particular,*

$$\widehat{\mathcal{C}_x}^* = \widehat{\mathcal{C}_1}^* \cup \cdots \cup \widehat{\mathcal{C}_k}^*$$

has k *distinct components.*

Proof. Since $\mathcal{C}$ is irreducible, so is $\mathcal{D}$. Thus for a general $x \in X$, all components of $\mathcal{C}_x$ have the same dimension and the same dual defect. By the duality theorem (e.g. Theorem 15.24 of [Ha]), this implies that $\widehat{\mathcal{C}_x}^*$ has k distinct components. Q.E.D.

Proposition 4.7. *In the setting of Definition* 3.8*, the closure of the image* $\chi(\mathcal{I})$ *coincides with* $\mathcal{D} \subset T^*(X)$ *in Definition* 4.5.

Proof. Note that for a general point $x \in X$, the VMRT $\mathcal{C}_x$ is the closure of $\tau_x(\mathcal{U}_x)$. For any $y \in \mathcal{U}_x$ and $z = \tau_x(y)$, we have the equality $\hat{T}_z(\mathcal{C}_x) = \hat{T}_y(\mathcal{U}_x)$, which implies

$$\hat{T}_z^{\perp}(\mathcal{C}_x) = \mathcal{I}_y$$

when we regard $\mathcal{I}_y$ as a subspace of $T_x(X)$ via Proposition 3.7. It follows that

$$\begin{aligned} I_{\mathcal{C}_x} &= \text{the closure of } \{(z,u) \in \mathbb{P}T_x(X) \times T_x^*(X),\ z \in Sm(\mathcal{C}_x), \\ &\qquad\qquad u \in \hat{T}_z^{\perp}(\mathcal{C}_x)\} \\ &= \text{the closure of } \{(\tau_x(y),u) \in \mathbb{P}T_x(X) \times T_x^*(X),\ y \in \mathcal{U}_x, u \in \mathcal{I}_y\}. \end{aligned}$$

By definition, $\mathcal{D}_x \subset T_x^*(X)$ is the closure of the image of $I_{\mathcal{C}_x} \subset \mathbb{P}T_x(X) \times T_x^*(X)$ under the projection to $T_x^*(X)$. Consequently, the closure of the image $\chi(\mathcal{I})$ must coincide with $\mathcal{D}$. Q.E.D.

Now we impose Assumption 1.3 of Section 1.

Proposition 4.8. *Let* X *be a uniruled projective manifold with a minimal dominating family of rational curves satisfying Assumption* 1.3 *in Section* 1*. Let* $x \in X$ *be a general point. In the notation of Proposition* 3.5*, the subvariety* $\mathcal{U}_x = \mu^{-1}(x)$ *is complete and the morphism* $\tau_x : \mathcal{U}_x \to \mathbb{P}T_x(X)$ *is an embedding.*

Proof. Let $\bar{\mathcal{K}}_x \subset \bar{\mathcal{K}}$ be the subscheme of members of $\bar{\mathcal{K}}$ through x. To show that $\mathcal{U}_x = \mu^{-1}(x)$ is complete, it suffices to show that all members of $\bar{\mathcal{K}}_x$ are standard. Let $\bar{\mathcal{K}}_x^{norm}$ be the normalization of $\bar{\mathcal{K}}_x$. The morphism τ_x in Proposition 3.5 can be extended to a morphism $\tilde{\tau}_x : \bar{\mathcal{K}}_x^{norm} \to \mathbb{P}T_x(X)$ by [Ke] which gives a normalization of

$\mathcal{C}_x = \tilde{\tau}_x(\bar{\mathcal{K}}_x^{norm})$ by Corollary 1 of [HM]. By the same argument as in Proposition 1.4 of [Hw], it is easy to see that if a member of $\bar{\mathcal{K}}_x$ is not standard, the morphism $\tilde{\tau}_x$ must be ramified at the corresponding point. Thus $\mathcal{C}_x$ must be singular at the image point, a contradiction to Assumption 1.3. This proves the completeness of $\mathcal{U}_x$. Then τ_x is a normalization of $\mathcal{C}_x$ by Proposition 3.5 (iv) and it is an embedding of $\mathcal{U}_x$ from Assumption 1.3. Q.E.D.

Proposition 4.9. *In the setting of Proposition* 4.8, *regard $\mathcal{I}_y$ as a subspace of $T_x^*(X)$ for each $y \in \mathcal{U}_x$ via Proposition* 3.7. *Then we have the equality*

$$I_{\mathcal{C}_x} = \{(\tau_x(y), u) \in \mathbb{P}T_x(X) \times T_x^*(X),\ y \in \mathcal{U}_x, u \in \mathcal{I}_y\}.$$

Proof. In the proof of Proposition 4.7, we have already seen that $I_{\mathcal{C}_x}$ is the closure of the right hand side of the equality. But the right hand side is closed, because $\tau_x : \mathcal{U}_x \to \mathcal{C}_x \subset \mathbb{P}T_x(X)$ is an embedding of a smooth projective variety by Proposition 4.8. Q.E.D.

Proposition 4.10. *Assume the setting of Proposition* 4.8. *Let us use the notation of Definition* 3.8 *and recall that $\chi(\mathcal{I})$ is a dense subset of $\mathcal{D}$ by Proposition* 4.7. *Let F be a general fiber of $\chi : \mathcal{I} \to \mathcal{D}$, contained in $\mathcal{I}_x = (\mu \circ \varpi)^{-1}(x)$ for a general point $x \in X$. Then*

(i) *F is biregular to $\mathbb{P}^\delta$ where δ is the dual defect of the VMRT $\mathcal{C}_x$ at x in the sense of Proposition* 4.6.

(ii) *The morphism $\varpi|_F$ is an embedding. The morphisms $(\rho \circ \varpi)|_F$ and $\eta|_F$ are immersions.*

(iii) *Let $\varrho : S \to F$ be the $\mathbb{P}^1$-bundle defined by the pull-back of the $\mathbb{P}^1$-bundle ρ via the immersion $(\rho \circ \varpi)|_F$. Denote by $E \subset S$ the distinguished section of ϱ given by $\varpi(F)$. Then S is biregular to the blow-up of $\mathbb{P}^{\delta+1}$ at one point and $E \subset S$ corresponds to the exceptional divisor of the blow-up.*

Proof. By Proposition 4.8, we know that $\mathcal{U}_x$ is projective and biregular to $\mathcal{C}_x$. By Proposition 4.9, the fiber F corresponds to a general fiber of $I_{\mathcal{C}_x} \to \mathcal{D}_x$. Thus (i) is a consequence of Proposition 4.4 and Proposition 4.6.

The morphism $(\tau \circ \varpi)|_F : F \to \mathbb{P}T_x(X)$ is a linear embedding by Proposition 4.4. It follows that $\varpi|_F$ is an embedding. Its image $\varpi(F)$ is a submanifold in $\mathcal{U}_x = \mu^{-1}(x)$. Since $T^\rho \cap T^\mu = 0$, we see that $(\rho \circ \varpi)|_F = (\pi_{\mathcal{K}} \circ \eta)|_F$ is an immersion. Consequently, the morphism $\eta|_F$ is an immersion, too. This proves (ii).

To prove (iii), it suffices to show that the normal bundle of the section E in S is isomorphic to $\mathcal{O}(-1)$ on $E \cong \mathbb{P}^\delta$. For a general $x \in X$, the line bundle $L_x := T^\rho|_{\mathcal{U}_x}$ on $\mathcal{U}_x$ agrees with $\tau_x^* \mathcal{O}(-1)_{\mathbb{P}T_x(X)}$ where $\mathcal{O}(-1)_{\mathbb{P}T_x(X)}$ is the tautological line bundle of $\mathbb{P}T_x(X)$. The normal bundle of $E \subset S$ is isomorphic to $(\varpi \circ \varrho|_E)^* L_x$. Since the image of F in $\mathbb{P}T_x(X)$ is a linear subspace by Proposition 4.4, the line bundle $\varpi^* L_x$ is isomorphic to $\mathcal{O}(-1)$ on $\mathbb{P}^\delta \cong F \subset \mathcal{U}_x$. This proves (iii). Q.E.D.

Notation 4.11. In the setting of Proposition 4.10, since $\varrho : S \to F$ is the pull-back of $\rho : \mathcal{U} \to \mathcal{K}$ by the immersion $\xi := (\rho \circ \varpi)|_F : F \to \mathcal{K}$, we have a natural immersion $\zeta : S \to \mathcal{U}$ with a commutative diagram

$$\begin{array}{ccc} S & \stackrel{\zeta}{\longrightarrow} & \mathcal{U} \\ \varrho \downarrow & & \downarrow \rho \\ F & \stackrel{\xi}{\longrightarrow} & \mathcal{K}. \end{array}$$

The inclusion $F \subset \mathcal{I}$ in Proposition 4.10 as a general fiber of χ determines a natural section $\tilde{E} \subset (\zeta^* \mathcal{I})|_E$ of the pull-back bundle $\zeta^* \mathcal{I}$ restricted to $E \subset S$. Since the bundle $\zeta^* \mathcal{I}$ is trivial along fibers of ϱ from Proposition 3.14, the section $\tilde{E}$ over E extends to a section $\tilde{S}$ of $\zeta^* \mathcal{I}$ over S. (Equivalently, $F \subset \mathcal{I}$ determines a section of $\xi^* \mathcal{R}$ which gives rise to a section $\tilde{S}$ of

$$\varrho^* \xi^* \mathcal{R} = \zeta^* \rho^* \mathcal{R} = \zeta^* \mathcal{I}$$

via Proposition 3.14.) Denote by $\iota : S \to \mathcal{I}$ the composition

$$S \cong \tilde{S} \subset \zeta^* \mathcal{I} \to \mathcal{I}$$

and by $\theta : S \to \mathbb{P}^{\delta+1}$ the blow-down of E. Since $\chi \circ \iota : S \to \mathcal{D}$ contracts E, we can write $\chi \circ \iota = \chi' \circ \theta$ for some $\chi' : \mathbb{P}^{\delta+1} \to \mathcal{D}$. Write $\varpi' = \varpi \circ \iota$. Since $\mu \circ \varpi' : S \to X$ contracts E, there exists a morphism $\gamma : \mathbb{P}^{\delta+1} \to X$ satisfying $\mu \circ \varpi' = \gamma \circ \theta$. In conclusion, we have the following commutative diagram

$$\begin{array}{ccccc} \mathcal{U} & \stackrel{\varpi'}{\longleftarrow} & S & \stackrel{\iota}{\longrightarrow} & \mathcal{I} \\ \mu \downarrow & & \downarrow \theta & & \downarrow \chi \\ X & \stackrel{\gamma}{\longleftarrow} & \mathbb{P}^{\delta+1} & \stackrel{\chi'}{\longrightarrow} & \mathcal{D}. \end{array}$$

Proposition 4.12. *In Notation* 4.11, *we have* $\gamma = \pi_X \circ \chi'$ *for the projection* $\pi_X : T^*(X) \to X$

Proof. Under the inclusion $\mathcal{I} \subset \mu^* T^*(X)$ of Proposition 3.7, the image $\iota(s)$ of a point $s \in S$ should take value in $T_x^*(X)$ for $x = \gamma \circ \theta(s)$. From our definition of $\iota(s)$, we have $x = \mu \circ \zeta(s)$. Since the latter is equal to $\pi_X \circ \chi \circ \iota(s)$, we obtain the proposition. Q.E.D.

Proposition 4.13. *In Notation* 4.11, *the morphisms* γ *and* χ' *are immersions. In particular,* $\gamma(\mathbb{P}^{\delta+1})$ *and* $\chi'(\mathbb{P}^{\delta+1})$ *are immersed* $\mathbb{P}^{\delta+1}$*'s.*

Proof. The lines in $\mathbb{P}^{\delta+1}$ through the points $\theta(E)$ are images of fibers of $\varrho : S \to F$ under the blow-down $\theta : S \to \mathbb{P}^{\delta+1}$. Since the fibers of ϱ come from fibers of ρ, the morphism $\gamma : \mathbb{P}^{\delta+1} \to X$ sends the lines through $\theta(E)$ to members of $\mathcal{K}_x \subset \bar{\mathcal{K}}$, the subscheme parametrizing members of $\mathcal{K}$ through the point $x \in X$. Since $\bar{\mathcal{K}}$ is an irreducible component of $\mathrm{RatCurves}^n(X)$, all lines of $\mathbb{P}^{\delta+1}$ are sent to members of $\bar{\mathcal{K}}$ by γ. Suppose that γ is not an immersion and ramified at some point $Q \in \mathbb{P}^{\delta+1}$. Choose a line $\ell_0 \subset \mathbb{P}^{\delta+1}$ passing through Q and $\theta(E)$. Since γ is ramified at Q, we can find a non-trivial family of lines

$$\{\ell_t \subset \mathbb{P}^{\delta+1},\ t \in \Delta,\ Q \in \ell_t\}$$

such that $\gamma(\ell_t)$ are tangent to $\gamma(\ell_0)$ at $\gamma(Q)$ for all $t \in \Delta$. It follows that $\{\gamma(\ell_t), t \in \Delta\}$ is a deformation of $\gamma(\ell_0)$ in X fixing a point and a tangent vector at that point. This is a contradiction to the fact that $\gamma(\ell_0)$ is standard, unless $\gamma(\ell_t) = \gamma(\ell_0)$ for all small $|t|$. But the latter implies that $\gamma : \mathbb{P}^{\delta+1} \to X$ contracts some curve in $\mathbb{P}^{\delta+1}$, which implies that $\gamma(\mathbb{P}^{\delta+1})$ is a point, a contradiction. It follows that γ is an immersion and so is χ' from Proposition 4.12. Q.E.D.

Proof of Theorem 1.4. From Proposition 4.13, we see that a general point of $\mathcal{D}$ lies on an immersed $\mathbb{P}^{\delta+1}$ in $T^*(X)$ contained in $\mathcal{D} \subset T^*(X)$. This implies Theorem 1.4 by Proposition 2.4 with $\mathbb{S} = \mathbb{P}^{\delta+1}$. Q.E.D.

§5. Proof of Theorem 1.5

The following proposition is crucial for the proof of Theorem 1.5.

Proposition 5.1. *In the setting of Definition* 3.8, *denoting by* $\omega_{\mathcal{K}}$ *the canonical symplectic form on* $T^*(\mathcal{K})$ *and by* ω_X *the canonical symplectic form on* $T^*(X)$, *we have the equality of two holomorphic 2-forms* $\eta^*\omega_{\mathcal{K}} = \chi^*\omega_X$ *on* $\mathcal{I}$.

Proof. Recall the notation of Definition 2.2. Let $\pi_X : T^*(X) \to X$ (resp. $\pi_{\mathcal{K}} : T^*(\mathcal{K}) \to \mathcal{K}$) be the vector bundle projection. Let λ_X (resp. $\lambda_{\mathcal{K}}$) be the canonical 1-form on $T^*(X)$ (resp. $T^*(\mathcal{K})$) whose value at $\alpha \in T^*(X)$ (resp. $\beta \in T^*(\mathcal{K})$) is given by the composition

$$T_\alpha(T^*(X)) \xrightarrow{d\pi_X} T_{\pi_X(\alpha)}(X) \xrightarrow{\alpha} \mathbb{C}$$

$$\big(\text{resp. } T_\beta(T^*(\mathcal{K})) \xrightarrow{d\pi_{\mathcal{K}}} T_{\pi_{\mathcal{K}}(\beta)}(\mathcal{K}) \xrightarrow{\beta} \mathbb{C}\big).$$

Then $\omega_X = d\lambda_X$ and $\omega_{\mathcal{K}} = d\lambda_{\mathcal{K}}$. We claim that $\eta^*\lambda_{\mathcal{K}} = \chi^*\lambda_X$ which implies the proposition. We will check this claim at a given point $s \in \mathcal{I}_y$ over a point $y \in \mathcal{U}$ through the following commuting diagram obtained from Definition 3.8, Proposition 4.7 and Proposition 3.14.

$$\begin{array}{ccccccccc} T^*(\mathcal{K}) & \supset & \mathcal{R} & \stackrel{\eta}{\leftarrow} & \mathcal{I} & \stackrel{\chi}{\rightarrow} & \mathcal{D} & \subset & T^*(X) \\ \pi_{\mathcal{K}} \downarrow & & & & \downarrow \varpi & & & & \downarrow \pi_X \\ \mathcal{K} & = & \mathcal{K} & \stackrel{\rho}{\leftarrow} & \mathcal{U} & \stackrel{\mu}{\rightarrow} & X & = & X. \end{array}$$

In the notation of Proposition 3.13, a point of $\mathcal{I}_y$ can be identified with an element $s \in H^0(\tilde{C}, h^*T^*(X)^0)$ where $[C] = \rho(y)$. Let us denote by $s_{[C]} \in T^*_{[C]}(\mathcal{K})$ the cotangent vector determined by s via the isomorphism in Propositin 3.12

$$H^0(\tilde{C}, h^*T^*(X)^0) = \mathcal{R}_{[C]} \subset T^*_{[C]}(\mathcal{K}).$$

Denote by $s_y \in T^*_{h(y)}(X)$ the evaluation of s at y. To prove the claim, we will check the equality

$$\eta^*\lambda_{\mathcal{K}}(v) = \chi^*\lambda_X(v)$$

for any $v \in T_s(\mathcal{I})$. From Proposition 3.10, $d\varpi(v) \in T_y(\mathcal{U})$ is represented by an element $w \in H^0(\tilde{C}, h^*T(X))$ modulo $H^0(\tilde{C}, T(\tilde{C}) \otimes \mathbf{m}_y)$. Then

$$\eta^*\lambda_{\mathcal{K}}(v) = \lambda_{\mathcal{K}}(d\eta(v)) = \langle s_{[C]}, d(\pi_{\mathcal{K}} \circ \eta)(v)\rangle = \langle s_{[C]}, d(\rho \circ \varpi)(v)\rangle = \langle s, w\rangle$$

where the first two pairings $\langle , \rangle$ denote the one induced by

$$T^*_{[C]}(\mathcal{K}) \otimes T_{[C]}(\mathcal{K}) \to \mathbb{C},$$

while the last pairing denotes the one coming from

$$H^0(\tilde{C}, h^*T^*(X)) \otimes H^0(\tilde{C}, h^*T(X)) \to \mathbb{C}.$$

On the other hand,

$$\chi^*\lambda_X(v) = \lambda_X(d\chi(v)) = \langle s_y, d(\pi_X \circ \chi)(v)\rangle = \langle s_y, d(\mu \circ \varpi)(v)\rangle$$

where the pairings denote the one induced by

$$T^*_{h(y)}(X) \otimes T_{h(y)}(X) \to \mathbb{C}.$$

Note that $d(\mu \circ \varpi)(v) = w_y$ where w_y denotes the evaluation of $w \in H^0(\tilde{C}, h^*T(X))$ at the point $y \in \tilde{C}$. But we have the equality

$$\langle s, w\rangle = \langle s_y, w_y\rangle$$

because the left hand side is a holomorphic function on $\tilde{C} \cong \mathbb{P}^1$. This proves the claim that $\eta^*\lambda_{\mathcal{K}} = \chi^*\lambda_X$. Q.E.D.

Now we are ready to prove the following more precise version of Theorem 1.5.

Theorem 5.2. *We assume the setting of Theorem* 1.4. *Let* $\mathcal{R} \subset T^*(\mathcal{K})$ *be the vector subbundle defined in Definition* 3.11. *Then* $\mathcal{R}$ *is a generically symplectic subundle in the sense of Definition* 2.2 *if and only if* $\delta = 0$. *When* $\delta = 0$, *the map* $\chi : \mathcal{I} \to \mathcal{D}$ *is birational and the rational map*

$$\eta \circ \chi^{-1} : \mathcal{D} \dashrightarrow \mathcal{R}$$

is a symplectic reduction of $(\mathcal{D}, \omega_X|_{\mathcal{D}})$ *to* $(\mathcal{R}, \omega_{\mathcal{K}}|_{\mathcal{R}})$.

Proof. When $\delta = 0$, the morphism $\chi : \mathcal{I} \to \mathcal{D}$ is birational from Proposition 4.4 and Proposition 4.9. The closures of the general leaves of the null foliation on $\mathcal{D}$ must agree with the χ-images of the fibers of η by Proposition 2.4. Thus $\eta \circ \chi^{-1}$ gives the symplectic reduction and the symplectic form induced by ω_X on the reduction $\mathcal{R}$ must be the restriction of $\omega_{\mathcal{K}}$ by Proposition 5.1. This also shows that $\mathcal{R}$ is generically symplectic if $\delta = 0$.

Finally, if $\mathcal{R}$ is generically symplectic with respect to $\omega_{\mathcal{K}}$, then by Proposition 5.1, the null distribution of $\chi^*\omega_X$ on $\mathcal{I}$ (in the sense of Definition 2.1) must be given by the fibers of η. Thus $\delta = 0$ from Theorem 1.4. Q.E.D.

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Korea Institute for Advanced Study
Hoegiro 87
Seoul, 130-722
Korea
E-mail address: `jmhwang@kias.re.kr`