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A survey on nonlinear Schrödinger equation with growing nonlocal nonlinearity

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Abstract.

We consider nonlinear Schrödinger equation with nonlocal nonlinearity which is described by a growing interaction potential. This model contains low-dimensional Schrödinger–Poisson system. We briefly survey recent progress on this subject and then show existence of ground state in a specific model.

§1. Introduction

This article surveys several aspects on the following nonlinear Schrödinger equation

$$\text{(NLS)} \qquad \begin{cases} iu_t + \dfrac{1}{2}\Delta u = N_V(u)u, & (x,t) \in \mathbb{R}^d \times \mathbb{R}, \\ u(0) = u_0, \end{cases}$$

where the nonlinearity N_V is given, with a real-valued function V on $\mathbb{R}_+$, as follows:

$$(1) \qquad N_V(u)(x) = (V(|\cdot|) * |u|^2)(x) = \int_{\mathbb{R}^d} V(|x-y|)|u(y)|^2 dy.$$

In this article, we are interested in the case where the nonlinear potential grows at the spatial infinity. Hence, we suppose that the interaction potential V satisfies a growing condition

$$(2) \qquad |V(r)| \to \infty \text{ as } r \to \infty.$$

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For simplicity, in this article, we have only a logarithmic potential

$$V(r) = \pm \log r \tag{3}$$

and a power type potential

$$V(r) = \pm r^{\gamma}, \qquad \gamma > 0 \tag{4}$$

in mind. The "$-$" sign corresponds to the *defocusing* case and the "$+$" sign to the *focusing* case. This type nonlinearities appear when we consider Schrödinger–Poisson systems in dimensions less than three, and their generalizations (see [6] for details). For N_V making sense, it is natural to assume that

$$\int_{\mathbb{R}^d} |V(|y|)||u_0(y)|^2 dy < \infty.$$

We denote this condition by $u_0 \in \sqrt{|V|}^{-1} L^2$.

This article is organized as follows: We give a short survey on this equation in Sections 2 and 3. Our main result is in Section 4.

§2. Formulation and well-posedness

It have been turned out that the nonlinear effect caused by this type nonlinearity is different from which the usual power type nonlinearity $\pm|u|^{p-1}u$ or the Hartree type nonlinearity $\pm(|x|^{-\gamma} * |u|^2)u$ produce. The feature of N_V under (2) is that the N_V itself also grows at the spatial infinity no matter how fast u decays. Hence, an integral equation

$$u(t) = e^{\frac{it\Delta}{2}} u_0 - i \int_0^t e^{\frac{i(t-s)\Delta}{2}} (N_V(u)u)(s) ds \tag{5}$$

is *not* a good formulation since this equation requires that linear operator $e^{\frac{i(t-s)\Delta}{2}}$ (and time integration) recovers the decay rate of u spoiled by multiplication by a growing term N_V, which seems hard to occur.

A recipe for dealing with this nonlinearity is the following decomposition:

$$\begin{aligned} N_V(u) = {} & \|u_0\|_{L^2}^2 V(|x|) + \int_{\mathbb{R}^d} (V(|x-y|) - V(|x|))|u(y)|^2 dy \\ =: {} & \|u_0\|_{L^2}^2 V(|x|) + \tilde{N}_V(u) \end{aligned}$$

where we have applied conservation of mass $\|u(t)\|_{L^2}^2 = \|u_0\|_{L^2}^2$ which is a common property as long as the concerned nonlinearity is real-valued.

We regard the first term as a *linear potential*[1], which is the key idea, and try to solve the following equation

$$u(t) = e^{-itA}u_0 - i\int_0^t e^{-i(t-s)A}(\tilde{N}_V(u)u)(s)ds, \tag{6}$$

where $A = -\frac{1}{2}\Delta + \|u_0\|_{L^2}^2 V(|x|)$ is the new linear part. When the condition

(7) $V(|x-y|) - V(|x|)$ does not grow at the spatial infinity

is satisfied, the new nonlinearity $\tilde{N}_V$ becomes harmless and so the integral equation (6) can be actually solved by a standard argument. We can summarize as follows: N_V grows at the spatial infinity, but, under (7), the growing part, which turns out to be time-independent, is successfully extracted and handled as a linear potential. The logarithmic potential (3) can be handled by this argument (see [5]). The situation is the same for the power type potential (4) as long as $\gamma \leqslant 1$ (see [2], [9], [5], [6]). The condition (7) can be rephrased as "V is sub-linear."

On the other hand, the above argument is not sufficient for the potential which does not satisfy the condition (7) (for example, (4) with $\gamma > 1$ is such a potential). So, we refine the condition (7) as follows:

> There exists an $\mathbb{R}^d$-valued function W such that $V(|x-y|) - V(|x|) - W(x)\cdot y$ does not grow at the spatial infinity (with respect to x),

where $\cdot$ stands for the usual inner product on $\mathbb{R}^d$. We refer to this condition as *sub-quadratic* condition. Notice that (7) is included as a special case $W = 0$. As for (4), this condition is satisfied with $W(x) = \mp\gamma|x|^{\gamma-2}x$, provided $0 < \gamma \leqslant 2$. Now, we introduce

$$\tilde{\tilde{N}}_V(u) = \int_{\mathbb{R}^d}(V(|x-y|) - V(|x|) - W(x)\cdot y)|u(y)|^2dy,$$

which is harmless under the sub-quadratic condition, and obtain the following decomposition

$$N_V(u) = \|u_0\|_{L^2}^2 V(|x|) + \tilde{\tilde{N}}_V(u) + W(x)\cdot\int_{\mathbb{R}^d} y|u(y)|^2dy.$$

[1]Strictly speaking, $V(|x|)$ would have a singularity at the origin. Since our purpose is to extract the growing part near the spatial infinity, in that case, it suffices to choose $\|u_0\|_{L^2}^2 V(|x|+1)$, $\|u_0\|_{L^2}^2 V(\langle x\rangle)$, or $\|u_0\|_{L^2}^2 \chi(|x|)V(|x|)$ as the linear potential instead of $\|u_0\|_{L^2}^2 V(|x|)$, where χ is a suitable smooth cut-off.

The third term is new, and the others are treatable. The quantity $\int_{\mathbb{R}^d} y|u(y)|^2 dy$ is known as *center of mass*. The key is to nail this vector to zero by introducing a *center-of-mass frame*. Let us be more precise. Just as the mass conservation law, as long as the nonlinear potential is real-valued, we can expect that

$$\frac{d}{dt}\int_{\mathbb{R}^d} y|u(y)|^2 dy = \operatorname{Im}\int_{\mathbb{R}^d} \overline{u(y)}\nabla u(y) dy$$

holds. Similarly, since our model (NLS) does not involve any external force, which is usually described by a linear potential or a magnetic field, the momentum conservation

$$\frac{d}{dt}\operatorname{Im}\int_{\mathbb{R}^d} \overline{u(y)}\nabla u(y) dy = 0$$

would also hold[2]. As a result, we see that $\int_{\mathbb{R}^d} y|u(y)|^2 dy$ is a straight line

$$\int_{\mathbb{R}^d} y|u(y)|^2 dy = \int_{\mathbb{R}^d} y|u_0|^2 dy + t\operatorname{Im}\int_{\mathbb{R}^d} \overline{u_0}\nabla u_0 dy. \tag{8}$$

Let us now use the Galilean transform. If $u(t,x)$ is a solution to (NLS), then for any $a, b \in \mathbb{R}^d$, $u_{a,b}(t,x) := e^{-\frac{i}{2}a\cdot(at+b)}e^{ix\cdot a}u(t, x-at-b)$ is also a solution. One verifies, in light of mass conservation, that

$$\int_{\mathbb{R}^d} y|u_{a,b}(y)|^2 dy = \int_{\mathbb{R}^d} y|u(y)|^2 dy + \|u_0\|_{L^2}^2 (at+b). \tag{9}$$

Thus, if we choose

$$\begin{aligned} a_0 &= -\frac{1}{\|u_0\|_{L^2}^2}\operatorname{Im}\int_{\mathbb{R}^d} \overline{u_0}\nabla u_0 dy, \\ b_0 &= -\frac{1}{\|u_0\|_{L^2}^2}\int_{\mathbb{R}^d} y|u_0|^2 dy, \end{aligned}$$

then u_{a_0,b_0} is a solution of (NLS) with $\int_{\mathbb{R}^d} y|u_{a_0,b_0}(y)|^2 \equiv 0$. For such u_{a_0,b_0}, the decomposition

$$N_V(u_{a_0,b_0}) = \|u_0\|_{L^2}^2 V(|x|) + \tilde{\tilde{N}}_V(u_{a_0,b_0})$$

[2]This is just the first law of motion "the velocity of a body remains constant unless the body is acted upon by an external force."

is available. Hence, the equation for u_{a_0,b_0} is

$$(10)\quad u_{a_0,b_0}(t) = e^{-itA}u_{a_0,b_0}(0) - i\int_0^t e^{-i(t-s)A}(\tilde{\tilde{N}}_V(u_{a_0,b_0})u_{a_0,b_0})(s)ds,$$

where $A = -\frac{1}{2}\Delta + \|u_0\|_{L^2}^2 V(|x|)$ is as above. By a standard perturbation argument[3], we obtain a solution of (10). Once u_{a_0,b_0} is obtained, the inverse Galilean transform, which is the Galilean transform with signs of a and b being opposite, gives the desired solution u.

Using the above argument, we obtain the following well-posedness results.

Theorem 1 ([5, 6]). *Suppose $V(x)$ is either $\pm\log r$ or $\pm r^\gamma$ with $0 < \gamma \leqslant 2$. Then,* (NLS) *is globally well-posed in $H^1 \cap \sqrt{|V|}^{-1}L^2$.*

§3. On scattering

A natural next question would be global behavior of the solution. As seen in Theorem 1, blow-up does not occur in this model even in the focusing case. In the defocusing case, one may expect a kind of scattering occur. However, it seems that the solution cannot behave like a free solution. This is because growing nonlocal nonlinearity creates a linear potential and so the linear part of the equation is not $i\partial_t + \frac{1}{2}\Delta$ any more but $i\partial_t - (-\frac{1}{2}\Delta + \|u_0\|_{L^2}^2 V(|x|))$.

The obstacle is analysis of $(i\partial_t - (-\frac{1}{2}\Delta + \|u_0\|_{L^2}^2 V(|x|)))u_{\text{lin}} = 0$. So for, beside few exceptional cases, we have no knowledge on global behavior, global dispersive estimate, or global Strichartz estimates on solutions to a linear Schrödinger equation, provided the linear potential grows at the spatial infinity. One exceptional case is $V(r) = -r^2$ (cf. harmonic oscillator). In this case, we know an explicit representation of the solution not only of the linear Schrödinger equation but also of the full nonlinear equation (NLS) (see [6, 7]). This exception suggests that solutions to (NLS) would behave like u_{lin} with phase modification (cf. long range scattering) and Galilean transform. However, we would be far from complete understanding.

§4. On standing waves

Another typical behavior of the solution is standing waves. This happens when we work with a focusing nonlinearity. For $V(r) = \log r$ in

[3]To use the Strichartz estimates for e^{-itA}, we need $\partial^\alpha V \in L^\infty$ for $|\alpha| \geqslant 2$. This is essentially the same condition as the sub-quadratic condition above.

two dimensions and $V(r) = r$ in one dimension, existence of the ground state is known ([1, 3, 10]). In [7], the special case $V(r) = r^2$ is considered and explicit examples of ground states and all excited states are given thanks to the explicit representation of the solution. Our aim here is to prove that the ground states exist and are radially symmetric for the focusing power type potential $V(r) = r^\gamma$ for $0 < \gamma < 2$. In what follows, we concentrate our attention to

$$\begin{cases} iu_t + \frac{1}{2}\Delta u = (|x|^\gamma * |u|^2)u, & (x,t) \in \mathbb{R}^d \times \mathbb{R}, \\ u(0) = u_0 \in \Sigma^{1,\gamma/2}, \end{cases}$$

where $\Sigma^{1,\gamma/2} = \{f \in H^1 \mid |x|^{\gamma/2} f \in L^2\}$. We follow the paper of Choquard and Stubbe [1] in which the case $\gamma = 1$ is treated. Set

$$\begin{aligned} E(u) &:= \frac{1}{4}\int_{\mathbb{R}^d} |\nabla u|^2\,dx + \frac{1}{4}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |x-y|^\gamma |u(x)|^2 |u(y)|^2\,dx\,dy \\ &=: T(u) + P(u), \\ M(u) &:= \frac{1}{2}\int_{\mathbb{R}^d} |u(x)|^2\,dx. \end{aligned}$$

We minimize the energy E under the constraint of the mass M, that is, we consider the minimizing problem

$$e_0(\mu) = \inf\{E(u) \mid u \in \Sigma^{1,\gamma/2}, M(u) = \mu\}.$$

By the Lagrange multiplier method, a minimizer w satisfies $-\frac{1}{2}\Delta w - (|x|^\gamma * |w|^2)w + \omega w$ for some $\omega \in \mathbb{R}$, which implies $e^{i\omega t}w(t,x)$ is a standing-wave solution.

Theorem 2. *For any $\mu > 0$, there exists $u_\mu \in \Sigma^{1,\gamma/2}$ such that $M(u_\mu) = \mu$ and $e_0(\mu) = E(u_\mu)$. Further, such $u_\mu \in \Sigma^{1,\gamma/2}$ is spherically symmetric decreasing modulo translation and phase.*

Proof of Theorem 2. Let $\{u_n\} \subset \Sigma^{1,\gamma/2}$ be a minimizing sequence for $e_0(\mu)$, that is, $M(u_n) = \mu$ and $E(u_n) \to e_0(\mu)$ as $n \to \infty$. Since C_0^∞ is dense in $\Sigma^{1,\gamma/2}$, we can further assume that $u_n \in C_0^\infty$. Let u_n^* be the spherically symmetric-decreasing rearrangement of u_n. Then, by [1, Lemma 3.2] and the rearrangement inequality (see [4]), it holds that

$$M(u_n^*) = M(u_n) = \mu, \; T(u_n^*) \le T(u_n), \; P(u_n^*) \le P(u_n).$$

For any fixed $y \in \mathbb{R}^d$, by using [4, Theorem 3.4], we have

$$\int_{\mathbb{R}^d} \max\{0, m - |x-y|^\gamma\}|u_n(x)|^2\,dx \le \int_{\mathbb{R}^d} \max\{0, m - |x|^\gamma\}|u_n^*(x)|^2\,dx.$$

Therefore, multiplying -1, adding $m\mu$ to the both sides, and then taking the limit $m \to \infty$, one verifies

$$\int_{\mathbb{R}^d} |x-y|^\gamma |u_n(x)|^2\,dx \geq \int_{\mathbb{R}^d} |x|^\gamma |u_n^*(x)|^2\,dx.$$

Thus,

$$E(u_n) \geq T(u_n^*) + \frac{\eta\mu}{2}\int_{\mathbb{R}^d} |x|^\gamma |u_n^*(x)|^2 dx.$$

This implies $u_n^* \in \Sigma^{1,\gamma/2}$. Since $\{u_n^*\}$ satisfies the same condition as $\{u_n\}$ and since $E(u_n^*) \leqslant E(u_n)$, we can replace u_n by u_n^*. Now, by the compact embedding (see [8, Theorem XIII.65]), there exists u^* such that $u_n^* \rightharpoonup u^*$ weakly in $\Sigma^{1,\gamma/2}$ and $u_n^* \to u^*$ strongly in $L^2(\mathbb{R}^d)$. One then sees that $M(u^*) = \mu$, $T(u^*) \leqslant \liminf_{n\to\infty} T(u_n^*)$, and u^* is radial. Hence, to see that u^* is a minimizer, it suffices to show that

$$P(u^*) \leq \liminf_{n\to\infty} P(u_n^*).$$

Let χ be a smooth cut-off such that $\chi(s) = 1$ for $0 \leq s \leq 1$ and $\chi(s) = 0$ for $s > 1$ and set $\chi_R(s) = \chi(s/R)$. Then, by the monotone convergence theorem, we have

$$\begin{aligned}\lim_{R\to\infty} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} &\chi_R(|x-y|)|x-y|^\gamma |u^*(x)|^2 |u^*(y)|^2\,dxdy \\ &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |x-y|^\gamma |u^*(x)|^2 |u^*(y)|^2\,dxdy.\end{aligned}$$

Therefore, for any $\epsilon > 0$, there exists $R > 0$ such that

$$\begin{aligned}&\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |x-y|^\gamma |u^*(x)|^2 |u^*(y)|^2\,dxdy \\ &\leq \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \chi_R(|x-y|)|x-y|^\gamma |u^*(x)|^2 |u^*(y)|^2\,dxdy + \epsilon \\ &= \liminf_{n\to\infty} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \chi_R(|x-y|)|x-y|^\gamma |u_n^*(x)|^2 |u_n^*(y)|^2\,dxdy + \epsilon \\ &\leq \liminf_{n\to\infty} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |x-y|^\gamma |u_n^*(x)|^2 |u_n^*(y)|^2\,dxdy + \epsilon.\end{aligned}$$

Since ϵ was arbitrary, we conclude that $P(u^*) \leq \liminf_{n\to\infty} P(u_n^*)$. Therefore, u^* is the minimizer. If we can pick up $u \in \Sigma^{1,\gamma/2}$ such that $u^* \notin \{e^{i\theta}u(x-a) : \theta \in \mathbb{R}^d, a \in \mathbb{R}^d\}$, then the strict version of rearrangement inequality leads us to a contradiction. Q.E.D.

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