Advanced Studies in Pure Mathematics 63, 2012
Galois–Teichmüller Theory and Arithmetic Geometry
pp. 163–187

Geometric interpretation of double shuffle relation for multiple L-values

Hidekazu Furusho

Abstract.

This paper gives a geometric interpretation of the generalized (including the regularization relation) double shuffle relation for multiple L-values. Precisely it is proved that Enriquez' mixed pentagon equation implies the relations. As a corollary, an embedding from his cyclotomic analogue of the Grothendieck–Teichmüller group into Racinet's cyclotomic double shuffle group is obtained. It cyclotomically extends the result of our previous paper [F3] and the project of Deligne and Terasoma which are the special case $N = 1$ of our result.

§0. Introduction

Multiple L-values $L(k_1, \cdots, k_m; \zeta_1, \cdots, \zeta_m)$ are the complex numbers defined by the following series

$$(0.1) \qquad L(k_1, \cdots, k_m; \zeta_1, \cdots, \zeta_m) := \sum_{0<n_1<\cdots<n_m} \frac{\zeta_1^{n_1} \cdots \zeta_m^{n_m}}{n_1^{k_1} \cdots n_m^{k_m}}$$

for m, $k_1, \ldots, k_m \in \mathbf{N}(= \mathbf{Z}_{>0})$ and $\zeta_1, \ldots, \zeta_m \in \mu_N$(: the group of N-th roots of unity in $\mathbf{C}$). They converge if and only if $(k_m, \zeta_m) \neq (1, 1)$. Multiple zeta values are regarded as a special case for $N = 1$. These values have been discussed in several papers [AK, BK, G, R] etc. Multiple L-values appear as coefficients of the cyclotomic Drinfel'd associator Φ_{KZ}^N (1.5) in $U\mathfrak{F}_{N+1}$: the non-commutative formal power series ring with $N + 1$ variables A and $B(a)$ ($a \in \mathbf{Z}/N\mathbf{Z}$).

The mixed pentagon equation (1.3) is a geometric equation introduced by Enriquez [E]. The series Φ_{KZ}^N satisfies the equation, which yields non-trivial relations among multiple L-values. The generalized double shuffle relation (the double shuffle relation and the regularization relation) is a combinatorial relation among multiple L-values. It

Received April 2, 2011.
Revised February 24, 2012.
2010 *Mathematics Subject Classification.* Primary 11M32; Secondary 11G55.

is formulated as (2.2) for $h = \Phi^N_{KZ}$. It is Zhao's remark [Z] that for specific N's the generalized double shuffle relation does not provide all the possible relations among multiple L-values.

Our main theorem is an implication of the generalized double shuffle relation from the mixed pentagon equation.

Theorem 0.1. *Let $U\mathfrak{F}_{N+1}$ be the universal enveloping algebra of the free Lie algebra $\mathfrak{F}_{N+1}$ with variables A and $B(a)$ $(a \in \mathbf{Z}/N\mathbf{Z})$. Let h be a group-like element in $U\mathfrak{F}_{N+1}$ with $c_{B(0)}(h) = 0$ satisfying the mixed pentagon equation* (1.3) *with a group-like series $g \in U\mathfrak{F}_2$. Then h also satisfies the generalized double shuffle relation* (2.2).

As a consequence we get an embedding from Enriquez' cyclotomic associator set $Pseudo(N, \mathbf{Q})$ (Definition 1.4) defined by the mixed pentagon (1.3) and the octagon (1.4) equations into Racinet's double shuffle set $DMR(N, \mathbf{Q})$ (Definition 2.1) defined by the generalized double shuffle relation (2.2).

Theorem 0.2. *For $N > 0$, $Pseudo(N, \mathbf{Q})$ is embedded into $DMR(N, \mathbf{Q})$. In more detail, we have embeddings of torsors*

$$Pseudo_{(a,\mu)}(N, \mathbf{Q}) \subset DMR_{(a,\mu)}(N, \mathbf{Q})$$

for $(a, \mu) \in (\mathbf{Z}/N\mathbf{Z})^\times \times \mathbf{Q}$ and of pro-algebraic groups

$$GRTM_{(\bar{1},1)}(N, \mathbf{Q}) \subset DMR_{(\bar{1},0)}(N, \mathbf{Q})$$

(for $GRTM_{(\bar{1},1)}(N, \mathbf{Q})$ see Definition 1.6).

It might be worthy to note that we do not use the octagon equation to show $Pseudo(N, \mathbf{Q}) \subset DMR(N, \mathbf{Q})$. By adding distribution relations (1.7) to both sides, we also get the inclusion $Psdist_{(a,\mu)}(N, \mathbf{Q}) \subset DMRD_{(a,\mu)}(N, \mathbf{Q})$ (for their definitions see Remark 1.7 and 2.2).

The motivic fundamental group of the algebraic curve $\mathbf{G}_m \backslash \mu_N$ is constructed and discussed in [DG]. It gives a pro-object of the tannakian category of mixed Tate motives (constructed in loc.cit.) over $\mathbf{Z}[\mu_N, 1/N]$, which yields an action of the motivic Galois group (: the *Galois group* of the tannakian category) $Gal^M(\mathbf{Z}[\mu_N, 1/N])$ on $\mathfrak{F}_{N+1}$. It was shown that the action is faithful for $N = 1$ in [Br] and $N = 2, 3, 4, 8$ in [De08]. The image of its unipotent part in $Aut\mathfrak{F}_{N+1}$ is contained in $GRTMD_{(\bar{1},1)}(N, \mathbf{Q})$ and $DMRD_{(\bar{1},0)}(N, \mathbf{Q})$, which follows from a geometric interpretation of the defining equations of $GRTMD_{(\bar{1},1)}(N, \mathbf{Q})$. It is a basic problem to ask if they are equal or not.

The contents of the article are as follows: We recall the mixed pentagon equation in §1 and the generalized double shuffle relation in §2. In

§3 we calculate Chen's reduced bar complex for the Kummer coverings of the moduli spaces $\mathcal{M}_{0,4}$ and $\mathcal{M}_{0,5}$. Two variable cyclotomic multiple polylogarithms and their associated bar elements are introduced in §4. By using them, we prove theorem 0.1 in §5. §6 proves several auxiliary lemmas which are essential to prove Theorem 0.1.

§1. Mixed pentagon equation

This section is to recall Enriquez' mixed pentagon equation and his cyclotomic analogue of the associator set [E].

Let us fix notations: For $n \geqslant 2$, the Lie algebra $\mathfrak{t}_n$ of infinitesimal pure braids is the completed $\mathbf{Q}$-Lie algebra with generators t^{ij} $(i \neq j$, $1 \leqslant i, j \leqslant n)$ and relations

$$t^{ij} = t^{ji}, [t^{ij}, t^{ik} + t^{jk}] = 0 \text{ and } [t^{ij}, t^{kl}] = 0 \text{ for all distinct } i,\ j,\ k,\ l.$$

We note that $\mathfrak{t}_2$ is the 1-dimensional abelian Lie algebra generated by t^{12}. The element $z_n = \sum_{1 \leqslant i < j \leqslant n} t^{ij}$ is central in $\mathfrak{t}_n$. Put $\mathfrak{t}_n^0$ to be the Lie subalgebra of $\mathfrak{t}_n$ with the same generators except t^{1n}. Then we have $\mathfrak{t}_n = \mathfrak{t}_n^0 \oplus \mathbf{Q} \cdot z_n$. Especially when $n = 3$, $\mathfrak{t}_3^0$ is a free Lie algebra $\mathfrak{F}_2$ of rank 2 with generators $A := t^{12}$ and $B = t^{23}$. For a partially defined map $f : \{1, \ldots, m\} \to \{1, \ldots, n\}$, the Lie algebra morphism $\mathfrak{t}_n \to \mathfrak{t}_m$: $x \mapsto x^f = x^{f^{-1}(1), \ldots, f^{-1}(n)}$ is uniquely defined by

$$(t^{ij})^f = \sum_{i' \in f^{-1}(i), j' \in f^{-1}(j)} t^{i'j'}.$$

Definition 1.1 ([Dr]). The associator set $M(\mathbf{Q})$ is defined to be the set of pairs $(\mu, g) \in \mathbf{Q} \times \exp \mathfrak{F}_2 = \exp \mathfrak{t}_3^0$ satisfying the *pentagon equation* in $\exp \mathfrak{t}_4^0$

$$g^{1,2,34} g^{12,3,4} = g^{2,3,4} g^{1,23,4} g^{1,2,3}. \tag{1.1}$$

and *two hexagon equations* in $\exp \mathfrak{F}_2 = \exp \mathfrak{t}_3^0$

$$\begin{aligned} &g(A, B) g(B, A) = 1, \\ &\exp\{\frac{\mu A}{2}\} g(C, A) \exp\{\frac{\mu C}{2}\} g(B, C) \exp\{\frac{\mu B}{2}\} g(A, B) = 1 \end{aligned} \tag{1.2}$$

with $C = -A - B$. For $\mu \in \mathbf{Q}$, the set $M_\mu(\mathbf{Q})$ is the subset of $M(\mathbf{Q})$ with $(\mu, g) \in M(\mathbf{Q})$. Particularly the set $GRT_1(\mathbf{Q})$ means $M_0(\mathbf{Q})$.

For any field $\mathbf{k}$ of characteristic 0, $M(\mathbf{k})$ and $GRT(\mathbf{k})$ are defined in the same way by replacing $\mathbf{Q}$ by $\mathbf{k}$.

By our notation, the equation (1.1) can be read as

$$g(t^{12},t^{23}+t^{24})g(t^{13}+t^{23},t^{34}) = g(t^{23},t^{34})g(t^{12}+t^{13},t^{24}+t^{34})g(t^{12},t^{23}).$$

In [Dr] it is shown that $GRT_1(\mathbf{Q})$ forms a group, called the *Grothendieck–Teichmüller group*, and the associator set $M_\mu(\mathbf{Q})$ with $\mu \in \mathbf{Q}^\times$ forms a $GRT_1(\mathbf{Q})$-torsor.

Remark 1.2. It is shown in [F2] that the two hexagon equations (1.2) are consequences of the pentagon equation (1.1).

Remark 1.3. The *Drinfel'd associator* $\Phi_{KZ} = \Phi_{KZ}(A,B) \in \mathbf{C}\langle\langle A,B\rangle\rangle$ is defined to be the quotient $\Phi_{KZ} = G_1(z)^{-1}G_0(z)$ where G_0 and G_1 are the solutions of the *formal KZ (Knizhnik–Zamolodchikov) equation*

$$\frac{d}{dz}G(z) = (\frac{A}{z}+\frac{B}{z-1})G(z)$$

such that $G_0(z) \approx z^A$ when $z \to 0$ and $G_1(z) \approx (1-z)^B$ when $z \to 1$ (cf.[Dr]). The series has the following expression

$$\Phi_{KZ}(A,B) = 1+\sum(-1)^m\zeta(k_1,\cdots,k_m)A^{k_m-1}B\cdots A^{k_1-1}B \\ +\text{(regularized terms)}$$

and the regularized terms are explicitly calculated to be linear combinations of multiple zeta values $\zeta(k_1,\cdots,k_m) = L(k_1,\ldots,k_m;1,\ldots,1)$ in [F1] Proposition 3.2.3 by Le–Murakami's method [LM]. It is shown in [Dr] that the series belongs to $M_\mu(\mathbf{C})$ with $\mu = 2\pi\sqrt{-1}$. This is achieved by considering monodromy in the real interval $(0,1)$ and the upper half plane in $\mathcal{M}_{0,4}$ (see §3) and the pentagon formed by the divisors $y=0$, $x=1$, the exceptional divisor of the blowing-up at $(1,1)$, $y=1$ and $x=0$ in $\mathcal{M}_{0,5}$ (see §3).

For $n \geqslant 2$ and $N \geqslant 1$, the Lie algebra $\mathfrak{t}_{n,N}$ is the completed $\mathbf{Q}$-Lie algebra with generators

$$t^{1i}\ (2 \leqslant i \leqslant n), \quad t(a)^{ij}\ (i \neq j, 2 \leqslant i,j \leqslant n, a \in \mathbf{Z}/N\mathbf{Z})$$

and relations

$$t(a)^{ij} = t(-a)^{ji}, \quad [t(a)^{ij}, t(a+b)^{ik}+t(b)^{jk}] = 0,$$
$$[t^{1i}+t^{1j}+\sum_{c\in\mathbf{Z}/N\mathbf{Z}} t(c)^{ij}, t(a)^{ij}] = 0, \qquad [t^{1i}, t^{1j}+\sum_{c\in\mathbf{Z}/N\mathbf{Z}} t(c)^{ij}] = 0,$$
$$[t^{1i}, t(a)^{jk}] = 0 \quad \text{and} \quad [t(a)^{ij}, t(b)^{kl}] = 0$$

for all a, $b \in \mathbf{Z}/N\mathbf{Z}$ and all distinct i, j, k, l $(2 \leqslant i,j,k,l \leqslant n)$. We note that $\mathfrak{t}_{n,1}$ is equal to $\mathfrak{t}_n$ for $n \geqslant 2$. We have a natural injection $\mathfrak{t}_{n-1,N} \hookrightarrow \mathfrak{t}_{n,N}$. The Lie subalgebra $\mathfrak{f}_{n,N}$ of $\mathfrak{t}_{n,N}$ generated by t^{1n} and $t(a)^{in}$ $(2 \leqslant i \leqslant n-1,\ a \in \mathbf{Z}/N\mathbf{Z})$ is free of rank $(n-2)N+1$ and forms an ideal of $\mathfrak{t}_{n,N}$. Actually it shows that $\mathfrak{t}_{n,N}$ is a semi-direct product of $\mathfrak{f}_{n,N}$ and $\mathfrak{t}_{n-1,N}$. The element $z_{n,N} = \sum_{1 \leqslant i<j \leqslant n} t^{ij}$ with $t^{ij} = \sum_{a \in \mathbf{Z}/N\mathbf{Z}} t(a)^{ij}$ $(2 \leqslant i < j \leqslant n)$ is central in $\mathfrak{t}_{n,N}$. Put $\mathfrak{t}^0_{n,N}$ to be the Lie subalgebra of $\mathfrak{t}_{n,N}$ with the same generators except t^{1n}. Then we have $\mathfrak{t}_{n,N} = \mathfrak{t}^0_{n,N} \oplus \mathbf{Q}\cdot z_{n,N}$. Occasionally we regard $\mathfrak{t}^0_{n,N}$ as the quotient $\mathfrak{t}_{n,N}/\mathbf{Q}\cdot z_{n,N}$. Especially when $n=3$, $\mathfrak{t}^0_{3,N}$ is free Lie algebra $\mathfrak{F}_{N+1}$ of rank $N+1$ with generators $A := t^{12}$ and $B(a) = t(a)^{23}$ $(a \in \mathbf{Z}/N\mathbf{Z})$.

For a partially defined map $f : \{1,\dots,m\} \to \{1,\dots,n\}$ such that $f(1) = 1$, the Lie algebra morphism $\mathfrak{t}_{n,N} \to \mathfrak{t}_{m,N}$ $x \mapsto x^f = x^{f^{-1}(1),\dots,f^{-1}(n)}$ is uniquely defined by

$$(t(a)^{ij})^f = \sum_{i' \in f^{-1}(i), j' \in f^{-1}(j)} t(a)^{i'j'} \quad (i \neq j, 2 \leqslant i,j \leqslant n)$$

and

$$\begin{aligned}(t^{1j})^f = &\sum_{j' \in f^{-1}(j)} t^{1j'} + \frac{1}{2} \sum_{j',j'' \in f^{-1}(j)} \sum_{c \in \mathbf{Z}/N\mathbf{Z}} t(c)^{j'j''} \\ &+ \sum_{i' \neq 1 \in f^{-1}(1), j' \in f^{-1}(j)} \sum_{c \in \mathbf{Z}/N\mathbf{Z}} t(c)^{i'j'} \quad (2 \leqslant j \leqslant n).\end{aligned}$$

Again for a partially defined map $g : \{2,\dots,m\} \to \{1,\dots,n\}$, the Lie algebra morphism $\mathfrak{t}_n \to \mathfrak{t}_{m,N}$ $x \mapsto x^g = x^{g^{-1}(1),\dots,g^{-1}(n)}$ is uniquely defined by

$$(t^{ij})^g = \sum_{i' \in g^{-1}(i), j' \in g^{-1}(j)} t(0)^{i'j'} \qquad (i \neq j, 1 \leqslant i,j \leqslant n).$$

Definition 1.4 ([E]). The cyclotomic associator set $Pseudo(N,\mathbf{Q})$ is defined to be the collection of $Pseudo_{(a,\mu)}(N,\mathbf{Q})$ for $(a,\mu) \in \mathbf{Z}/N\mathbf{Z} \times \mathbf{Q}$ which is defined to be the set of pairs (g,h) with $g \in \exp\mathfrak{F}_2$ and $h = \sum_{W:\text{word}} c_W(h)W \in \exp\mathfrak{F}_{N+1}$ satisfying $g \in M_\mu(\mathbf{Q})$, $c_{B(0)}(h) = 0$, the *mixed pentagon equation* in $\exp\mathfrak{t}^0_{4,N}$

$$h^{1,2,34}h^{12,3,4} = g^{2,3,4}h^{1,23,4}h^{1,2,3} \tag{1.3}$$

and the *octagon equation* in $\exp\mathfrak{F}_{N+1}$

$$h\bigl(A,B(a),B(a+1),\dots,B(a+N-1)\bigr)^{-1}\exp\{\frac{\mu B(a)}{2}\}\cdot \tag{1.4}$$
$$h\bigl(C,B(a),B(a-1),\dots,B(a+1-N)\bigr)\exp\{\frac{\mu C}{N}\}\cdot$$
$$h\bigl(C,B(0),B(N-1),\dots,B(1)\bigr)^{-1}\exp\{\frac{\mu B(0)}{2}\}\cdot$$
$$h\bigl(A,B(0),B(1),\dots,B(N-1)\bigr)\exp\{\frac{\mu A}{N}\}=1$$

with $A+\sum_{a\in\mathbf{Z}/N\mathbf{Z}}B(a)+C=0$.

By our notation, each term in the equation (1.3) can be read as

$$\begin{aligned}
h^{1,2,34}&=h(t^{12},t^{23}(0)+t^{24}(0),t^{23}(1)+t^{24}(1),\dots,\\
&\qquad t^{23}(N-1)+t^{24}(N-1)),\\
h^{12,3,4}&=h(t^{13}+\sum_c t^{23}(c),t^{34}(0),t^{34}(1),\dots,t^{34}(N-1)),\\
g^{2,3,4}&=g(t^{23}(0),t^{34}(0)),\\
h^{1,23,4}&=h(t^{12}+t^{13}+\sum_c t^{23}(c),t^{24}(0)+t^{34}(0),\dots,\\
&\qquad t^{24}(N-1)+t^{34}(N-1)),\\
h^{1,2,3}&=h(t^{12},t^{23}(0),t^{23}(1),\dots,t^{23}(N-1)).
\end{aligned}$$

Remark 1.5. In loc.cit. the cyclotomic analogue $\Phi_{KZ}^N\in\exp\mathfrak{F}_{N+1}(\mathbf{C})$ of the Drinfel'd associator is introduced to be the renormalized holonomy from 0 to 1 of the KZ-like differential equation

$$\frac{d}{dz}H(z)=\Bigl(\frac{A}{z}+\sum_{a\in\mathbf{Z}/N\mathbf{Z}}\frac{B(a)}{z-\zeta_N^a}\Bigr)H(z)$$

with $\zeta_N=\exp\{\frac{2\pi\sqrt{-1}}{N}\}$, i.e., $\Phi_{KZ}^N=H_1^{-1}H_0$ where H_0 and H_1 are the solutions such that $H_0(z)\approx z^A$ when $z\to 0$ and $H_1(z)\approx(1-z)^{B(0)}$ when $z\to 1$ (cf.[E]). There appear multiple L-values (0.1) in each of its coefficient;

$$\begin{aligned}
&\Phi_{KZ}^N \\
&=1+\sum(-1)^m L(k_1,\cdots,k_m;\xi_1,\dots,\xi_m)A^{k_m-1}B(a_m)\cdots A^{k_1-1}B(a_1)\\
&\qquad+(\text{regularized terms})
\end{aligned}\tag{1.5}$$

with $\xi_1 = \zeta_N^{a_2-a_1}, \ldots, \xi_{m-1} = \zeta_N^{a_m-a_{m-1}}$ and $\xi_m = \zeta_N^{-a_m}$, where the regularized terms can be explicitly calculated to combinations of multiple L-values by the method of Le–Murakami [LM]. In [E] it is shown that (Φ_{KZ}, Φ_{KZ}^N) belongs to $Pseudo_{(-1,2\pi\sqrt{-1})}(N, \mathbf{C})$. This is achieved by considering monodromy in the pentagon formed by the divisors $y = 0$, $x = 1$, the exceptional divisor of the blowing-up at $(1,1)$, $y = 1$ and $x = 0$ in $\mathcal{M}_{0,5}^{(N)}$ (see §3) to get (1.3) and in the octagon formed by 0, 1, ∞ and ζ_N in $\mathcal{M}_{0,4}^{(N)} = \mathbf{G}_m \backslash \mu_N$ to get (1.4).

Definition 1.6 ([E])**.** The set $GRTM_{(\bar{1},1)}(N, \mathbf{Q})$ means the subset of $Pseudo_{(\bar{1},0)}(N, \mathbf{Q})$ satisfying the *special action condition* in $\exp \mathfrak{t}_{3,N}^0$

$$(1.6) \quad A + \sum_{a \in \mathbf{Z}/N\mathbf{Z}} Ad(\tau_a h^{-1})(B(a)) \\ + Ad\Big(h^{-1} \cdot h\big(C, B(0), B(N-1), \ldots, B(1)\big)\Big)(C) = 0$$

where τ_a ($a \in \mathbf{Z}/N\mathbf{Z}$) is the automorphism defined by $A \mapsto A$ and $B(c) \mapsto B(c+a)$ for all $c \in \mathbf{Z}/N\mathbf{Z}$.

In loc.cit. it is shown that $GRTM_{(\bar{1},1)}(N, \mathbf{Q})$ forms a group and $Pseudo_{(a,\mu)}(N, \mathbf{Q})$ with $(a, \mu) \in (\mathbf{Z}/N\mathbf{Z})^\times \times \mathbf{Q}^\times$ forms a $GRTM_{(\bar{1},1)}(N, \mathbf{Q})$-torsor. In the case of $N = 1$ we have $g = h$ and then $M_\mu(\mathbf{Q}) = Pseudo_{(1,\mu)}(N, \mathbf{Q})$ and $GRT_1(\mathbf{Q}) = GRTM_{(\bar{1},1)}(N, \mathbf{Q})$. It is not known for general N whether $GRTM_{(\bar{1},1)}(N, \mathbf{Q})$ is equal to $Pseudo_{(\bar{1},0)}(N, \mathbf{Q})$ or not.

Let $N, N' \geqslant 1$ with $N'|N$. Put $d = N/N'$. The morphism

$$\pi_{NN'} : \mathfrak{t}_{n,N} \to \mathfrak{t}_{n,N'}$$

is defined by $t^{1i} \mapsto dt^{1i}$ and $t^{ij}(a) \mapsto t^{ij}(\bar{a})$ ($i \neq j$, $2 \leqslant i, j \leqslant n$, $a \in \mathbf{Z}/N\mathbf{Z}$), where $\bar{a} \in \mathbf{Z}/N'\mathbf{Z}$ means the image of a under the map $\mathbf{Z}/N\mathbf{Z} \to \mathbf{Z}/N'\mathbf{Z}$. The morphism

$$\delta_{NN'} : \mathfrak{t}_{n,N} \to \mathfrak{t}_{n,N'}$$

is defined by $t^{1i} \mapsto t^{1i}$ and $t^{ij}(a) \mapsto t^{ij}(a/d)$ if $d|a$ and $t^{ij}(a) \mapsto 0$ if $d \nmid a$ ($i \neq j$, $2 \leqslant i, j \leqslant n$, $a \in \mathbf{Z}/N\mathbf{Z}$). The morphism $\pi_{NN'}$ (resp. $\delta_{NN'}$) : $\mathfrak{t}_{n,N} \to \mathfrak{t}_{n,N'}$ induces the morphisms $Pseudo_{(a,\mu)}(N, \mathbf{Q}) \to Pseudo_{(\bar{a},\mu)}(N', \mathbf{Q})$ which we denote by the same symbol $\pi_{NN'}$ (resp. $\delta_{NN'}$). Here $\bar{a}$ means the image of a by the natural map $\mathbf{Z}/N\mathbf{Z} \to \mathbf{Z}/N'\mathbf{Z}$.

Remark 1.7. In [E], the *distribution relation* in $\exp \mathfrak{t}_{3,N'}^0$

$$(1.7) \qquad \pi_{NN'}(h) = \exp\{c_{B(0)}(\pi_{NN'}(h)) B(0)\} \delta_{NN'}(h)$$

is also discussed and $Psdist_{(a,\mu)}(N,\mathbf{Q})$ (resp. $GRTMD_{(\bar{1},1)}(N,\mathbf{Q})$) is defined to be the subset of $Pseudo_{(a,\mu)}(N,\mathbf{Q})$ (resp. $GRTM_{(\bar{1},1)}(N,\mathbf{Q})$) by adding the distribution relation (1.7) in $\exp \mathfrak{t}^0_{3,N'}$ for all $N'|N$. In loc.cit. it is shown that $GRTMD_{(\bar{1},1)}(N,\mathbf{Q})$ forms a group and $Psdist_{(a,\mu)}(N,\mathbf{Q})$ with $(a,\mu) \in (\mathbf{Z}/N\mathbf{Z})^\times \times \mathbf{Q}^\times$ forms a $GRTMD_{(\bar{1},1)}(N,\mathbf{Q})$-torsor and the pair (Φ_{KZ},Φ^N_{KZ}) belongs to it with $(a,\mu)=(-1,2\pi\sqrt{-1})$.

Remark 1.8. In [EF] it is proved that the mixed pentagon equation (1.3) implies the distribution relation (1.7) for $N'=1$ and that the octagon equation (1.4) follows from the mixed pentagon equation (1.3) and the special action condition for $N=2$. It is also shown that the duality relation shown in [B] is compatible with the torsor structure of $Psdist(2,\mathbf{Q})$ and a new subtorsor $Psdist^+(2,\mathbf{Q})$ is discussed in [EF].

§2. Double shuffle relation

This section is to recall the generalized double shuffle relation and Racinet's associated torsor [R].

Let us fix notations: Let $\mathfrak{F}_{Y_N}$ be the completed graded Lie $\mathbf{Q}$-algebra generated by $Y_{n,a}$ ($n \geqslant 1$ and $a \in \mathbf{Z}/N\mathbf{Z}$) with $\deg Y_{n,a}=n$. Put $U\mathfrak{F}_{Y_N}$ its universal enveloping algebra: the non-commutative formal series ring with free variables $Y_{n,a}$ ($n \geqslant 1$ and $a \in \mathbf{Z}/N\mathbf{Z}$).

Let

$$\pi_Y : U\mathfrak{F}_{N+1} \to U\mathfrak{F}_{Y_N}$$

be the $\mathbf{Q}$-linear map between non-commutative formal power series rings that sends all the words ending in A to zero and the word $A^{n_m-1}B(a_m)\cdots A^{n_1-1}B(a_1)$ ($n_1,\dots,n_m \geqslant 1$ and $a_1,\dots,a_m \in \mathbf{Z}/N\mathbf{Z}$) to

$$(-1)^m Y_{n_m,-a_m} Y_{n_{m-1},a_m-a_{m-1}} \cdots Y_{n_1,a_2-a_1}.$$

Define the coproduct Δ_* of $U\mathfrak{F}_{Y_N}$ by

$$\Delta_*(Y_{n,a}) = \sum_{k+l=n,b+c=a} Y_{k,b}\otimes Y_{l,c} \qquad (n \geqslant 0 \text{ and } a \in \mathbf{Z}/N\mathbf{Z})$$

with $Y_{0,a}:=1$ if $a=0$ and 0 if $a\neq 0$. For $h=\sum_{W:\text{word}} c_W(h)W \in U\mathfrak{F}_{N+1}$, define the series shuffle regularization

$$h_* = h_{\text{corr}}\cdot \pi_Y(h)$$

with the correction term

$$h_{\text{corr}} = \exp\left(\sum_{n=1}^{\infty} \frac{(-1)^n}{n} c_{A^{n-1}B(0)}(h) Y_{1,0}^n\right). \tag{2.1}$$

Definition 2.1 ([R]). For $N \geqslant 1$, the set $DMR(N, \mathbf{Q})$ is defined to be the set of series $h = \sum_{W:\mathrm{word}} c_W(h)W \in \exp \mathfrak{F}_{N+1}$ satisfying $c_A(h) = c_{B(0)}(h) = 0$ and the *generalized double shuffle relation*

$$\Delta_*(h_*) = h_* \widehat{\otimes} h_*. \tag{2.2}$$

For $(a, \mu) \in \mathbf{Z}/N\mathbf{Z} \times \mathbf{Q}$, the set $DMR_{(a,\mu)}(N, \mathbf{Q})$ is defined to be the subset of $DMR(N, \mathbf{Q})$ defined by

$$c_{B(ka)}(h) - c_{B(-ka)}(h) = \frac{N-2k}{N-2}\{c_{B(a)}(h) - c_{B(-a)}(h)\} \tag{2.3}$$

for $1 \leqslant k \leqslant N/2$ and

$$\begin{cases} c_{AB(0)}(h) = \frac{\mu^2}{24} & \text{if } N = 1, 2, \\ c_{B(a)}(h) - c_{B(-a)}(h) = -\frac{(N-2)\mu}{2N} & \text{if } N \geqslant 3. \end{cases} \tag{2.4}$$

In loc.cit. it is shown that $DMR_{(\bar{1},0)}(N, \mathbf{Q})$ forms a group and $DMR_{(a,\mu)}(N, \mathbf{Q})$ with $(a, \mu) \in (\mathbf{Z}/N)^\times \times \mathbf{Q}^\times$ forms a $DMR_{(\bar{1},0)}(N, \mathbf{Q})$-torsor.

Remark 2.2. In [R], $DMRD(N, \mathbf{Q})$ (resp. $DMRD_{(a,\mu)}(N, \mathbf{Q})$) is introduced to be the subset of $DMR(N, \mathbf{Q})$ (resp. $DMR_{(a,\mu)}(N, \mathbf{Q})$) by adding the distribution relation (1.7) in $\exp \mathfrak{t}^0_{3,N'} = \exp \mathfrak{F}_{N'+1}$ for all $N'|N$. The series Φ^N_{KZ} belongs to $DMRD_{(a,\mu)}(N, \mathbf{Q})$ with $(a, \mu) = (-\bar{1}, 2\pi\sqrt{-1})$ because regularized multiple L-values satisfy the double shuffle relation and the distribution relation (loc.cit). It is shown by Zhao [Z] that for specific N's all the defining equations of $DMRD_{(a,\mu)}(N, \mathbf{Q})$ do not provide all the possible relations among multiple L-values.

§3. Bar constructions

This section reviews the notion of the reduced bar construction and calculates its 0-th cohomology for $\mathcal{M}^{(N)}_{0,4}$ and $\mathcal{M}^{(N)}_{0,5}$.

We recall the notion of Chen's reduced bar construction [C]. Let $(A^\bullet = \oplus_{q=0}^\infty A^q, d)$ be a differential graded algebra (DGA). The reduced bar complex $\bar{B}^\bullet(A)$ is the tensor algebra $\oplus_{r=0}^\infty (\bar{A}^\bullet)^{\otimes r}$ with $\bar{A}^\bullet = \oplus_{i=0}^\infty \bar{A}^i$ where $\bar{A}^0 = A^1/dA^0$ and $\bar{A}^i = A^{i+1}$ $(i > 0)$. We denote $a_1 \otimes \cdots \otimes a_r$ $(a_i \in \bar{A}^\bullet)$ by $[a_1|\cdots|a_r]$. The degree of elements in $\bar{B}^\bullet(A)$ is given by the total degree of $\bar{A}^\bullet$. Put $Ja = (-1)^{p-1}a$ for $a \in \bar{A}^p$. Define

$$d'[a_1|\cdots|a_k] = \sum_{i=1}^{k} (-1)^i [Ja_1|\cdots|Ja_{i-1}|da_i|a_{i+1}|\cdots|a_k]$$

and

$$d''[a_1|\cdots|a_k]=\sum_{i=1}^{k}(-1)^{i-1}[Ja_1|\cdots|Ja_{i-1}|Ja_i\cdot a_{i+1}|a_{i+2}|\cdots|a_k].$$

Then $d'+d''$ forms a differential. The differential and the shuffle product (loc.cit.) give $\bar{B}^\bullet(A)$ a structure of commutative DGA. Actually it also forms a Hopf algebra, whose coproduct Δ is given by

$$\Delta([a_1|\cdots|a_r])=\sum_{s=0}^{r}[a_1|\cdots|a_s]\otimes[a_{s+1}|\cdots|a_r].$$

For a smooth complex manifold $\mathcal{M}$, $\Omega^\bullet(\mathcal{M})$ means the de Rham complex of smooth differential forms on $\mathcal{M}$ with values in $\mathbf{C}$. We denote the 0-th cohomology of the reduced bar complex $\bar{B}^\bullet(\Omega(\mathcal{M}))$ with respect to the differential by $H^0\bar{B}(\mathcal{M})$.

Let $\mathcal{M}_{0,4}$ be the moduli space $\{(x_1,\cdots,x_4)\in(\mathbf{P}^1_{\mathbf{C}})^4|x_i\neq x_j(i\neq j)\}/PGL_2(\mathbf{C})$ of 4 different points in $\mathbf{P}^1$. It is identified with $\{z\in\mathbf{P}^1_{\mathbf{C}}|z\neq 0,1,\infty\}$ by sending $[(0,z,1,\infty)]$ to z. Denote its Kummer N-covering

$$\mathbf{G}_m\backslash\mu_N=\{z\in\mathbf{P}^1_{\mathbf{C}}|z^N\neq 0,1,\infty\}$$

by $\mathcal{M}_{0,4}^{(N)}$. The space $H^0\bar{B}(\mathcal{M}_{0,4}^{(N)})$ is generated by

$$\omega_0:=d\log(z)\text{ and }\omega_\zeta:=d\log(z-\zeta)\quad(\zeta\in\mu_N).$$

We have an identification $H^0\bar{B}(\mathcal{M}_{0,4}^{(N)})$ with the graded $\mathbf{C}$-linear dual of $U\mathfrak{F}_{N+1}$,

$$H^0\bar{B}(\mathcal{M}_{0,4}^{(N)})\simeq U\mathfrak{F}_{N+1}^*\otimes\mathbf{C},$$

by $\operatorname{Exp}\Omega_4^{(N)}:=\sum X_{i_m}\cdots X_{i_1}\otimes[\omega_{i_m}|\cdots|\omega_{i_1}]\in U\mathfrak{F}_{N+1}\widehat{\otimes}_{\mathbf{Q}}H^0\bar{B}(\mathcal{M}_{0,4}^{(N)})$.

Here the sum is taken over $m\geqslant 0$ and $i_1,\cdots,i_m\in\{0\}\cup\mu_N$ and $X_0=A$ and $X_\zeta=B(a)$ when $\zeta=\zeta_N^a$. It is easy to see that the identification is compatible with Hopf algebra structures. We note that the product $l_1\cdot l_2\in H^0\bar{B}(\mathcal{M}_{0,4}^{(N)})$ for l_1, $l_2\in H^0\bar{B}(\mathcal{M}_{0,4}^{(N)})$ is given by $l_1\cdot l_2(f):=\sum_i l_1(f_1^{(i)})l_2(f_2^{(i)})$ for $f\in U\mathfrak{F}_{N+1}\otimes\mathbf{C}$ with $\Delta(f)=\sum_i f_1^{(i)}\otimes f_2^{(i)}$. Occasionally we regard $H^0\bar{B}(\mathcal{M}_{0,4}^{(N)})$ as the regular function ring of $F_{N+1}(\mathbf{C})=\{g\in U\mathfrak{F}_{N+1}\otimes\mathbf{C}|g:\text{group-like}\}=\{g\in U\mathfrak{F}_{N+1}\otimes\mathbf{C}|g(0)=1,\Delta(g)=g\otimes g\}$.

Let $\mathcal{M}_{0,5}$ be the moduli space $\{(x_1,\cdots,x_5)\in(\mathbf{P}^1_{\mathbf{C}})^5|x_i\neq x_j(i\neq j)\}/PGL_2(\mathbf{C})$ of 5 different points in $\mathbf{P}^1$. It is identified with $\{(x,y)\in$

$\mathbf{G}_m^2|x \neq 1, y \neq 1, xy \neq 1\}$ by sending $[(0, xy, y, 1, \infty)]$ to (x, y). Denote its Kummer N^2-covering

$$\{(x, y) \in \mathbf{G}_m^2 | x^N \neq 1, y^N \neq 1, (xy)^N \neq 1\}$$

by $\mathcal{M}_{0,5}^{(N)}$. It is identified with $W_N/\mathbf{C}^\times$ by $(x, y) \mapsto (xy, y, 1)$ where

$$W_N = \{(z_2, z_3, z_4) \in \mathbf{G}_m | z_i^N \neq z_j^N (i \neq j)\}.$$

The space $H^0\bar{B}(\mathcal{M}_{0,5}^{(N)})$ is a subspace of the tensor coalgebra generated by

$$\omega_{1,i} := d\log z_i \text{ and } \omega_{i,j}(a) := d\log(z_i - \zeta_N^a z_j) \quad (2 \leqslant i, j \leqslant 4, a \in \mathbf{Z}/N).$$

Proposition 3.1. *We have an identification*

$$H^0\bar{B}(\mathcal{M}_{0,5}^{(N)}) \simeq (U\mathbf{t}_{4,N}^0)^* \otimes \mathbf{C}.$$

Proof. By [K], $H^0\bar{B}(W_N)$ can be calculated to be the 0-th cohomology $H^0\bar{B}^\bullet(S)$ of the reduced bar complex of the Orlik–Solomon algebra $S^\bullet$. The algebra $S^\bullet$ is the (trivial-)differential graded $\mathbf{C}$-algebra $S^\bullet = \oplus_{q=0}^\infty S^q$ defined by generators

$$\omega_{1,i} = d\log z_i \text{ and } \omega_{i,j}(a) = d\log(z_i - \zeta_N^a z_j) \quad (2 \leqslant i, j \leqslant 4, a \in \mathbf{Z}/N\mathbf{Z})$$

in degree 1 and relations

$$\begin{aligned}
&\omega_{i,j}(a) = \omega_{j,i}(-a), \qquad \omega_{ij}(a) \wedge \{\omega_{ik}(a+b) + \omega_{jk}(b)\} = 0, \\
&\{\omega_{1i} + \omega_{1j} + \sum_{c \in \mathbf{Z}/N\mathbf{Z}} \omega(c)_{ij}\} \wedge \omega(a)_{ij} = 0, \\
&\omega_{1i} \wedge \{\omega_{1j} + \sum_{c \in \mathbf{Z}/N\mathbf{Z}} \omega(c)_{ij}\} = 0, \\
&\omega_{1i} \wedge \omega(a)_{jk} = 0 \quad \text{and} \quad \omega(a)_{ij} \wedge \omega(b)_{kl} = 0
\end{aligned}$$

for all a, $b \in \mathbf{Z}/N\mathbf{Z}$ and all distinct i, j, k, l $(2 \leqslant i, j, k, l \leqslant n)$. By direct calculation, the element

$$\sum_{i=2}^{4} t_{1i} \otimes \omega_{1i} + \sum_{2 \leqslant i < j \leqslant 4, a \in \mathbf{Z}/N\mathbf{Z}} t_{ij}(a) \otimes \omega_{ij}(a) \in (\mathbf{t}_{4,N})^{\deg=1} \otimes S^1$$

yields a Hopf algebra identification of $H^0\bar{B}(W_N)$ with $(U\mathbf{t}_{4,N})^* \otimes \mathbf{C}$ since both are quadratic.

By the long exact sequence of cohomologies induced from the $\mathbf{G}_m$-bundle $W_N \to \mathcal{M}_{0,5}^{(N)} = W_N/\mathbf{C}^\times$, we get

$$0 \to H^1(\mathcal{M}_{0,5}^{(N)}) \to H^1(W_N) \to H^1(\mathbf{G}_m) \to 0$$

and

$$H^i(\mathcal{M}_{0,5}^{(N)}) \simeq H^i(W_N) \qquad (i \geqslant 2).$$

It yields the identification of the subspace $H^0\bar{B}(\mathcal{M}_{0,5}^{(N)})$ of $H^0\bar{B}(W_N)$ with $(U\mathfrak{t}_{4,N}^0)^* \otimes \mathbf{C}$. Q.E.D.

The above identification is induced from

$$\operatorname{Exp} \Omega_5^{(N)} := \sum t_{J_m} \cdots t_{J_1} \otimes [\omega_{J_m}|\cdots|\omega_{J_1}] \in U\mathfrak{t}_{4,N}^0 \widehat{\otimes}_{\mathbf{Q}} H^0\bar{B}(\mathcal{M}_{0,5}^{(N)})$$

where the sum is taken over $m \geqslant 0$ and $J_1, \cdots, J_m \in \{(1,i)|2 \leqslant i \leqslant 4\} \cup \{(i,j,a)|2 \leqslant i < j \leqslant 4, a \in \mathbf{Z}/N\mathbf{Z}\}$.

Especially the identification between degree 1 terms is given by

$$\Omega_5^{(N)} = \sum_{i=2}^{4} t_{1i} d\log z_i + \sum_{2\leqslant i<j\leqslant 4} \sum_{a\in\mathbf{Z}/N\mathbf{Z}} t_{i,j}(a) d\log(z_i - \zeta_N^a z_j)$$
$$\in \mathfrak{t}_{4,N}^0 \otimes H_{DR}^1(\mathcal{M}_{0,5}^{(N)}).$$

In terms of the coordinate (x,y),

$$\begin{aligned}
\Omega_5^{(N)} &= t_{12} d\log(xy) + t_{13} d\log y + \sum_a t_{23}(a) d\log y(x - \zeta_N^a) \\
&\quad + \sum_a t_{24}(a) d\log(xy - \zeta_N^a) + \sum_a t_{34}(a) d\log(y - \zeta_N^a) \\
&= t_{12} d\log x + \sum_a t_{23}(a) d\log(x - \zeta_N^a) + (t_{12} + t_{13} + t_{23}) d\log y \\
&\quad + \sum_a t_{34}(a) d\log(y - \zeta_N^a) + \sum_a t_{24}(a) d\log(xy - \zeta_N^a).
\end{aligned}$$

It is easy to see that the identification is compatible with Hopf algebra structures. We note again that the product $l_1 \cdot l_2 \in H^0\bar{B}(\mathcal{M}_{0,5}^{(N)})$ for $l_1, l_2 \in H^0\bar{B}(\mathcal{M}_{0,5}^{(N)})$ is given by $l_1 \cdot l_2(f) := \sum_i l_1(f_1^{(i)}) l_2(f_2^{(i)})$ for $f \in U\mathfrak{t}_{4,N}^0 \otimes \mathbf{C}$ with $\Delta(f) = \sum_i f_1^{(i)} \otimes f_2^{(i)}$ (Δ: the coproduct of $U\mathfrak{t}_{4,N}^0$). Occasionally we also regard $H^0\bar{B}(\mathcal{M}_{0,5}^{(N)})$ as the regular function ring of $K_4^N(\mathbf{C}) = \{g \in U\mathfrak{t}_{4,N}^0 \otimes \mathbf{C} | g : \text{group-like}\}$.

By a generalization of Chen's theory [C] to the case of tangential basepoints, especially for $\mathcal{M}=\mathcal{M}_{0,4}^{(N)}$ or $\mathcal{M}_{0,5}^{(N)}$, we have an isomorphism

$$\rho : H^0\bar{B}(\mathcal{M}) \simeq I_o(\mathcal{M})$$

as algebras over $\mathbf{C}$ which sends $\sum_{I=(i_m,\cdots,i_1)} c_I[\omega_{i_m}|\cdots|\omega_{i_1}]$ $(c_I \in \mathbf{C})$ to $\sum_I c_I \mathrm{It}\int_o \omega_{i_m}\circ\cdots\circ\omega_{i_1}$. Here $\sum_I c_I \mathrm{It}\int_o \omega_{i_m}\circ\cdots\circ\omega_{i_1}$ means the iterated integral defined by

(3.1)

$$\sum_I c_I \int_{0<t_1<\cdots<t_{m-1}<t_m<1} \omega_{i_m}(\gamma(t_m))\cdot\omega_{i_{m-1}}(\gamma(t_{m-1}))\cdots\cdot\omega_{i_1}(\gamma(t_1))$$

for all analytic paths $\gamma:(0,1)\to\mathcal{M}(\mathbf{C})$ starting from the tangential basepoint o (defined by $\frac{d}{dz}$ for $\mathcal{M}=\mathcal{M}_{0,4}^{(N)}$ and defined by $\frac{d}{dx}$ and $\frac{d}{dy}$ for $\mathcal{M}=\mathcal{M}_{0,5}^{(N)}$) at the origin in $\mathcal{M}$ (for its treatment see also [De89]§15) and $I_o(\mathcal{M})$ stands for the $\mathbf{C}$-algebra generated by all such homotopy invariant iterated integrals with $m \geqslant 1$ and $\omega_{i_1},\ldots,\omega_{i_m}\in H^1_{DR}(\mathcal{M})$.

§4. Two variable cyclotomic multiple polylogarithms

We introduce cyclotomic multiple polylogarithms, $Li_{\mathbf{a}}(\bar{\zeta}(z))$ and $Li_{\mathbf{a},\mathbf{b}}(\bar{\zeta}(x),\bar{\eta}(y))$, and their associated bar elements, $l_{\mathbf{a}}^{\bar{\zeta}}$ and $l_{\mathbf{a},\mathbf{b}}^{\bar{\zeta}(x),\bar{\eta}(y)}$, which play important roles to prove our main theorems.

For a pair $(\mathbf{a},\bar{\zeta})$ with $\mathbf{a}=(a_1,\cdots,a_k)\in\mathbf{Z}^k_{>0}$ and $\bar{\zeta}=(\zeta_1,\ldots,\zeta_k)$ with $\zeta_i\in\mu_N$: the group of roots of unity in $\mathbf{C}$ $(1\leqslant i\leqslant k)$, its weight and its depth are defined to be $wt(\mathbf{a},\bar{\zeta})=a_1+\cdots+a_k$ and $dp(\mathbf{a},\bar{\zeta})=k$ respectively. Put $\bar{\zeta}(x)=(\zeta_1,\ldots,\zeta_{k-1},\zeta_k x)$. Put $z\in\mathbf{C}$ with $|z|<1$. Consider the following complex analytic function, *one variable cyclotomic multiple polylogarithm*

$$Li_{\mathbf{a}}(\bar{\zeta}(z)) := \sum_{0<m_1<\cdots<m_k} \frac{\zeta_1^{m_1}\cdots\zeta_{k-1}^{m_{k-1}}(\zeta_k z)^{m_k}}{m_1^{a_1}\cdots m_{k-1}^{a_{k-1}} m_k^{a_k}}. \tag{4.1}$$

It satisfies the following differential equation

$$\frac{d}{dz}Li_{\mathbf{a}}(\bar{\zeta}(z)) = \begin{cases} \frac{1}{z}Li_{(a_1,\cdots,a_{k-1},a_k-1)}(\bar{\zeta}(z)) & \text{if } a_k\neq 1,\\ \frac{1}{\zeta_k^{-1}-z}Li_{(a_1,\cdots,a_{k-1})}(\zeta_1,\ldots,\zeta_{k-2},\zeta_{k-1}z) & \text{if } a_k=1, k\neq 1,\\ \frac{1}{\zeta_1^{-1}-z} & \text{if } a_k=1, k=1.\end{cases}$$

It gives an iterated integral starting from o, which lies on $I_o(\mathcal{M}_{0,4}^{(N)})$. Actually by the map ρ it corresponds to an element of the $\mathbf{Q}$-structure

$U\mathfrak{F}^*_{N+1}$ of $H^0\bar{B}(\mathcal{M}^{(N)}_{0,4})$ denoted by $l^{\bar\zeta}_{\mathbf{a}}$. It is expressed as
(4.2)
$$l^{\bar\zeta}_{\mathbf{a}}=(-1)^k[\underbrace{\omega_0|\cdots|\omega_0}_{a_k-1}|\omega_{\zeta_k^{-1}}|\underbrace{\omega_0|\cdots|\omega_0}_{a_{k-1}-1}|\omega_{\zeta_k^{-1}\zeta_{k-1}^{-1}}|\omega_0|\cdots\cdots|\omega_0|\omega_{\zeta_k^{-1}\cdots\zeta_1^{-1}}].$$

By the standard identification $\mu\simeq\mathbf{Z}/N\mathbf{Z}$ sending $\zeta_N=\exp\{\frac{2\pi\sqrt{-1}}{N}\}\mapsto 1$, for a series $\varphi=\sum_{W:\text{word}}c_W(\varphi)W$ it is calculated by
$$l^{\bar\zeta}_{\mathbf{a}}(\varphi)=(-1)^k c_{A^{a_k-1}B(-e_k)A^{a_{k-1}-1}B(-e_k-e_{k-1})\cdots A^{a_1-1}B(-e_k-\cdots-e_1)}(\varphi)$$
with $\zeta_i=\zeta_N^{e_i}$ $(e_i\in\mathbf{Z}/N\mathbf{Z})$.

For $\mathbf{a}=(a_1,\cdots,a_k)\in\mathbf{Z}^k_{>0}$, $\mathbf{b}=(b_1,\cdots,b_l)\in\mathbf{Z}^l_{>0}$, $\bar\zeta=(\zeta_1,\ldots,\zeta_k)$, $\bar\eta=(\eta_1,\ldots,\eta_l)$ with $\zeta_i,\eta_j\in\mu_N$ and $x,y\in\mathbf{C}$ with $|x|<1$ and $|y|<1$, consider the following complex function, the *two variables multiple polylogarithm*

(4.3) $Li_{\mathbf{a},\mathbf{b}}(\bar\zeta(x),\bar\eta(y))$
$$:=\sum_{\substack{0<m_1<\cdots<m_k\\<n_1<\cdots<n_l}}\frac{\zeta_1^{m_1}\cdots\zeta_{k-1}^{m_{k-1}}(\zeta_kx)^{m_k}\cdot\eta_1^{n_1}\cdots\eta_{l-1}^{n_{l-1}}(\eta_ly)^{n_l}}{m_1^{a_1}\cdots m_{k-1}^{a_{k-1}}m_k^{a_k}\cdot n_1^{b_1}\cdots n_{l-1}^{b_{l-1}}n_l^{b_l}}.$$

It satisfies the following differential equations.
$$\frac{d}{dx}Li_{\mathbf{a},\mathbf{b}}(\bar\zeta(x),\bar\eta(y))$$
$$=\begin{cases}\frac{1}{x}Li_{(a_1,\cdots,a_{k-1},a_k-1),\mathbf{b}}(\bar\zeta(x),\bar\eta(y)) & \text{if } a_k\neq1,\\ \frac{1}{\zeta_k^{-1}-x}Li_{(a_1,\cdots,a_{k-1}),\mathbf{b}}(\zeta_1,\ldots,\zeta_{k-2},\zeta_{k-1}x,\bar\eta(y))-\left(\frac{1}{x}+\frac{1}{\zeta_k^{-1}-x}\right)\cdot & \\ \quad Li_{(a_1,\cdots,a_{k-1},b_1),(b_2,\cdots,b_l)}(\zeta_1,\ldots\zeta_{k-1},\zeta_k\eta_1x,\eta_2,\ldots,\eta_{l-1},\eta_ly) & \\ & \text{if } a_k=1,k\neq1,l\neq1,\\ \frac{1}{\zeta_1^{-1}-x}Li_{\mathbf{b}}(\eta(y))-\left(\frac{1}{x}+\frac{1}{\zeta_1^{-1}-x}\right)Li_{(b_1),(b_2,\cdots,b_l)}(\zeta_1\eta_1x,\eta_2,\ldots,\eta_{l-1},\eta_ly) & \\ & \text{if } a_k=1,k=1,l\neq1,\\ \frac{1}{\zeta_k^{-1}-x}Li_{(a_1,\cdots,a_{k-1}),b_1}(\zeta_1,\ldots,\zeta_{k-1}x,\eta_1y)-\left(\frac{1}{x}+\frac{1}{\zeta_k^{-1}-x}\right)\cdot & \\ \quad Li_{(a_1,\cdots,a_{k-1},b_1)}(\zeta_1,\ldots,\zeta_{k-1},\zeta_k\eta_1xy) & \text{if } a_k=1,k\neq1,l=1,\\ \frac{1}{\zeta_1^{-1}-x}Li_{b_1}(\eta_1y)-\left(\frac{1}{x}+\frac{1}{\zeta_1^{-1}-x}\right)Li_{b_1}(\zeta_1\eta_1xy) & \text{if } a_k=1,k=1,l=1,\end{cases}$$
$$\frac{d}{dy}Li_{\mathbf{a},\mathbf{b}}(\bar\zeta(x),\bar\eta(y))$$
$$=\begin{cases}\frac{1}{y}Li_{\mathbf{a},(b_1,\cdots,b_{l-1},b_l-1)}(\bar\zeta(x),\bar\eta(y)) & \text{if } b_l\neq1,\\ \frac{1}{\eta_l^{-1}-y}Li_{\mathbf{a},(b_1,\cdots,b_{l-1})}(\bar\zeta(x),\eta_1,\ldots,\eta_{l-2},\eta_{l-1}y) & \text{if } b_l=1,l\neq1,\\ \frac{1}{\eta_1^{-1}-y}Li_{\mathbf{a}}(\bar\zeta(\eta_1xy)) & \text{if } b_l=1,l=1.\end{cases}$$

By analytic continuation, the functions $Li_{\mathbf{a},\mathbf{b}}(\bar{\zeta}(x),\bar{\eta}(y))$, $Li_{\mathbf{b},\mathbf{a}}(\bar{\eta}(y),\bar{\zeta}(x))$, $Li_{\mathbf{a}}(\bar{\zeta}(x))$, $Li_{\mathbf{a}}(\bar{\zeta}(y))$ and $Li_{\mathbf{a}}(\bar{\zeta}(xy))$ give iterated integrals starting from o, which lie on $I_o(\mathcal{M}_{0,5}^{(N)})$. They correspond to elements of the $\mathbf{Q}$-structure $(U\mathfrak{t}^0_{4,N})^*$ of $H^0\bar{B}(\mathcal{M}_{0,5}^{(N)})$ by the map ρ denoted by $l_{\mathbf{a},\mathbf{b}}^{\bar{\zeta}(x),\bar{\eta}(y)}$, $l_{\mathbf{b},\mathbf{a}}^{\bar{\eta}(y),\bar{\zeta}(x)}$, $l_{\mathbf{a}}^{\bar{\zeta}(x)}$, $l_{\mathbf{a}}^{\bar{\eta}(y)}$ and $l_{\mathbf{a}}^{\bar{\zeta}(xy)}$ respectively. Note that they are expressed as

$$\sum_{I=(i_m,\cdots,i_1)} c_I[\omega_{i_m}|\cdots|\omega_{i_1}] \tag{4.4}$$

for some $m\in\mathbf{N}$ with $c_I\in\mathbf{Q}$ and $\omega_{i_j}\in\{\frac{dx}{x},\frac{dx}{\zeta-x},\frac{dy}{y},\frac{dy}{\zeta-y},\frac{xdy+ydx}{\zeta-xy}(\zeta\in\mu_N)\}$.

§5. Proofs of main Theorems

This section gives proofs of Theorem 0.1 and Theorem 0.2.

Proof of Theorem 0.1. Let $\mathbf{a}=(a_1,\dots,a_k)\in\mathbf{Z}^k_{>0}$, $\mathbf{b}=(b_1,\dots,b_l)\in\mathbf{Z}^l_{>0}$, $\bar{\zeta}=(\zeta_1,\dots,\zeta_k)$ and $\bar{\eta}=(\eta_1,\dots,\eta_l)$ with $\zeta_i,\eta_j\in\mu_N\subset\mathbf{C}$ $(1\leqslant i\leqslant k$ and $1\leqslant j\leqslant l)$. Put $\bar{\zeta}(x)=(\zeta_1,\dots,\zeta_{k-1},\zeta_k x)$ and $\bar{\eta}(y)=(\eta_1,\dots,\eta_{l-1},\eta_l y)$. Recall that multiple polylogarithms satisfy the following analytic identity, the series shuffle formula in $I_o(\mathcal{M}_{0,5}^{(N)})$:

$$Li_{\mathbf{a}}(\bar{\zeta}(x))\cdot Li_{\mathbf{b}}(\bar{\eta}(y))=\sum_{\sigma\in Sh^{\leqslant}(k,l)} Li_{\sigma(\mathbf{a},\mathbf{b})}^{\sigma(\bar{\zeta}(x),\bar{\eta}(y))}.$$

Here $Sh^{\leqslant}(k,l):=\cup_{N=1}^{\infty}\{\sigma:\{1,\cdots,k+l\}\to\{1,\cdots,N\}|\sigma$ is onto, $\sigma(1)<\cdots<\sigma(k),\sigma(k+1)<\cdots<\sigma(k+l)\}$, $\sigma(\mathbf{a},\mathbf{b}):=(c_1,\cdots,c_N)$ with

$$c_i=\begin{cases}a_s+b_{t-k} & \text{if } \sigma^{-1}(i)=\{s,t\} \text{ with } s<t,\\ a_s & \text{if } \sigma^{-1}(i)=\{s\} \quad \text{with } s\leqslant k,\\ b_{s-k} & \text{if } \sigma^{-1}(i)=\{s\} \quad \text{with } s>k,\end{cases}$$

and $\sigma(\bar{\zeta}(x),\bar{\eta}(y)):=(z_1,\dots,z_N)$ with

$$z_i=\begin{cases}x_s y_{t-k} & \text{if } \sigma^{-1}(i)=\{s,t\} \text{ with } s<t,\\ x_s & \text{if } \sigma^{-1}(i)=\{s\} \quad \text{with } s\leqslant k,\\ y_{s-k} & \text{if } \sigma^{-1}(i)=\{s\} \quad \text{with } s>k,\end{cases}$$

for $x_i=\zeta_i$ $(i\neq k)$, $\zeta_k x$ $(i=k)$ and $y_j=\eta_j$ $(j\neq l)$, $\eta_j y$ $(j=l)$. Since ρ is an embedding of algebras, the above analytic identity immediately implies the algebraic identity, the series shuffle formula in the $\mathbf{Q}$-structure

$(U\mathfrak{t}^0_{4,N})^*$ of $H^0\bar{B}(\mathcal{M}^{(N)}_{0,5})$

$$l^{\bar{\zeta}(x)}_{\mathbf{a}} \cdot l^{\bar{\eta}(y)}_{\mathbf{b}} = \sum_{\sigma \in Sh^{\leqslant}(k,l)} l^{\sigma(\bar{\zeta}(x),\bar{\eta}(y))}_{\sigma(\mathbf{a},\mathbf{b})}. \tag{5.1}$$

Let (g,h) be a pair in Theorem 0.1. By the group-likeness of h, i.e. $h \in \exp \mathfrak{F}_{N+1}$, the product $h^{1,23,4}h^{1,2,3}$ is group-like, i.e. belongs to $\exp \mathfrak{t}^0_{4,N}$. Hence $\Delta(h^{1,23,4}h^{1,2,3}) = (h^{1,23,4}h^{1,2,3})\widehat{\otimes}(h^{1,23,4}h^{1,2,3})$, where Δ is the standard coproduct of $U\mathfrak{t}^0_{4,N}$. Therefore

$$\begin{aligned} l^{\bar{\zeta}(x)}_{\mathbf{a}} \cdot l^{\bar{\eta}(y)}_{\mathbf{b}}(h^{1,23,4}h^{1,2,3}) =&(l^{\bar{\zeta}(x)}_{\mathbf{a}} \widehat{\otimes} l^{\bar{\eta}(y)}_{\mathbf{b}})(\Delta(h^{1,23,4}h^{1,2,3})) \\ =& l^{\bar{\zeta}(x)}_{\mathbf{a}}(h^{1,23,4}h^{1,2,3}) \cdot l^{\bar{\eta}(y)}_{\mathbf{b}}(h^{1,23,4}h^{1,2,3}). \end{aligned}$$

Evaluation of the equation (5.1) at the group-like element $h^{1,23,4}h^{1,2,3}$ gives the series shuffle formula

$$l^{\bar{\zeta}}_{\mathbf{a}}(h) \cdot l^{\bar{\eta}}_{\mathbf{b}}(h) = \sum_{\sigma \in Sh^{\leqslant}(k,l)} l^{\sigma(\bar{\zeta},\bar{\eta})}_{\sigma(\mathbf{a},\mathbf{b})}(h) \tag{5.2}$$

for admissible pairs[1] $(\mathbf{a},\bar{\zeta})$ and $(\mathbf{b},\bar{\eta})$ by Lemma 6.1 and Lemma 6.2 below because the group-likeness and (1.3) for h implies $c_0(h) = 1$ and $c_A(h) = 0$.

By putting $l^{1,S}_1(h) := -T$ and $l^{\bar{\zeta},S}_{\mathbf{a}}(h) := l^{\bar{\zeta}}_{\mathbf{a}}(h)$ for all admissible pairs $(\mathbf{a},\bar{\zeta})$, the series regularized value $l^{\bar{\zeta},S}_{\mathbf{a}}(h)$ in $\mathbf{Q}[T]$ (T: a parameter which stands for $\log z$. cf. [R]) for a non-admissible pair $(\mathbf{a},\bar{\zeta})$ is uniquely determined in such a way (cf.[AK]) that the above series shuffle formulae remain valid for $l^{\bar{\zeta},S}_{\mathbf{a}}(h)$ with all pairs $(\mathbf{a},\bar{\zeta})$.

Define the integral regularized value $l^{\bar{\zeta},I}_{\mathbf{a}}(h)$ in $\mathbf{Q}[T]$ for all pairs $(\mathbf{a},\bar{\zeta})$ by $l^{\bar{\zeta},I}_{\mathbf{a}}(h) = l^{\bar{\zeta}}_{\mathbf{a}}(e^{TB(0)}h)$. Equivalently $l^{\bar{\zeta},I}_{\mathbf{a}}(h)$ for any pair $(\mathbf{a},\bar{\zeta})$ can be uniquely defined in such a way that the iterated integral shuffle formulae (loc.cit) remain valid for all pairs $(\mathbf{a},\bar{\zeta})$ with $l^{1,I}_1(h) := -T$ and $l^{\bar{\zeta},I}_{\mathbf{a}}(h) := l^{\bar{\zeta}}_{\mathbf{a}}(h)$ for all admissible pairs $(\mathbf{a},\bar{\zeta})$ because they hold for admissible pairs by the group-likeness of h (cf. loc.cit).

[1] A pair $(\mathbf{a},\bar{\zeta})$ with $\mathbf{a} = (a_1,\cdots,a_k)$ and $\bar{\zeta} = (\zeta_1,\ldots,\zeta_k)$ is called *admissible* if $(a_k,\zeta_k) \neq (1,1)$.

Let $\mathbb{L}$ be the $\mathbf{Q}$-linear map from $\mathbf{Q}[T]$ to itself defined via the generating function:

$$\mathbb{L}(\exp Tu) = \sum_{n=0}^{\infty} \mathbb{L}(T^n)\frac{u^n}{n!} = \exp\left\{-\sum_{n=1}^{\infty} l_n^{1,I}(h)\frac{u^n}{n}\right\} \tag{5.3}$$
$$\left(= \exp\left\{Tu - \sum_{n=1}^{\infty} l_n^{1}(h)\frac{u^n}{n}\right\}\right).$$

Proposition 5.1. *Let h be an element as in Theorem 0.1. Then the regularization relation holds, i.e. $l_{\mathbf{a}}^{\bar{\zeta},S}(h) = \mathbb{L}\big(l_{\mathbf{a}}^{\bar{\zeta},I}(h)\big)$ for all pairs $(\mathbf{a},\bar{\zeta})$.*

Proof. We may assume that $(\mathbf{a},\bar{\zeta})$ is non-admissible because the proposition is trivial if it is admissible. Put $1^n = \underbrace{(1,1,\cdots,1)}_{n}$. When $\mathbf{a} = 1^n$ and $\bar{\zeta} = \bar{1}^n$, the proof is given by the same argument to [F3] as follows: By the series shuffle formulae,

$$\sum_{k=0}^{m}(-1)^k l_{k+1}^{\bar{1},S}(h)\cdot l_{1^{m-k}}^{\bar{1}^{m-k},S}(h) = (m+1)l_{1^{m+1}}^{\bar{1}^{m+1},S}(h)$$

for $m \geqslant 0$. Here we put $l_{\emptyset}^{\emptyset,S}(h) = 1$. This means

$$\sum_{k,l\geqslant 0}(-1)^k l_{k+1}^{\bar{1},S}(h)\cdot l_{1^l}^{\bar{1}^l,S}(h)u^{k+l} = \sum_{m\geqslant 0}(m+1)l_{1^{m+1}}^{\bar{1}^{m+1},S}(h)u^m.$$

Put $f(u) = \sum_{n\geqslant 0} l_{1^n}^{\bar{1}^n,S}(h)u^n$. Then the above equality can be read as

$$\sum_{k\geqslant 0}(-1)^k l_{k+1}^{\bar{1},S}(h)u^k = \frac{d}{du}\log f(u).$$

Integrating and adjusting constant terms gives

$$\begin{aligned}\sum_{n\geqslant 0} l_{1^n}^{\bar{1}^n,S}(h)u^n &= \exp\left\{-\sum_{n\geqslant 1}(-1)^n l_n^{\bar{1},S}(h)\frac{u^n}{n}\right\}\\ &= \exp\left\{-\sum_{n\geqslant 1}(-1)^n l_n^{\bar{1},I}(h)\frac{u^n}{n}\right\}\end{aligned}$$

because $l_n^{\bar{1},S}(h) = l_n^{\bar{1},I}(h) = l_n^{1}(h)$ for $n > 1$ and $l_1^{\bar{1},S}(h) = l_1^{\bar{1},I}(h) = -T$. Since $l_{1^m}^{\bar{1}^m,I}(h) = \frac{(-T)^m}{m!}$, we get $l_{1^m}^{\bar{1}^m,S}(h) = \mathbb{L}\big(l_{1^m}^{\bar{1}^m,I}(h)\big)$.

When $(\mathbf{a},\bar{\zeta})$ is of the form $(\mathbf{a}'1^l,\bar{\zeta}'\bar{1}^l)$ with $(\mathbf{a}',\bar{\zeta}')$ admissible, the proof is given by the following induction on l. By (5.1),

$$l_{\mathbf{a}'}^{\bar{\zeta}'(x)}(h')\cdot l_{1^l}^{\bar{1}^l(y)}(h')=\sum_{\sigma\in Sh^{\leqslant}(k,l)} l_{\sigma(\mathbf{a}',1^l)}^{\sigma(\bar{\zeta}'(x),\bar{1}^l(y))}(h')$$

for $h'=e^{T\{t^{23}(0)+t^{24}(0)+t^{34}(0)\}}h^{1,23,4}h^{1,2,3}$ with $k=dp(\mathbf{a}')$. The group-likeness and (1.3) for h implies $c_0(h)=1$ and $c_A(h)=0$ and the group-likeness and our assumption $c_{B(0)}(h)=0$ implies $c_{B(0)^n}(h)=0$ for $n\in\mathbf{Z}_{>0}$. Hence by Lemma 6.3 and Lemma 6.4,

$$l_{\mathbf{a}'}^{\bar{\zeta}'}(h)\cdot l_{1^l}^{\bar{1}^l,I}(h)=\sum_{\sigma\in Sh^{\leqslant}(k,l)} l_{\sigma(\mathbf{a}',1^l)}^{\sigma(\bar{\zeta}',\bar{1}^l),I}(h).$$

Then by our induction assumption, taking the image by the map $\mathbb{L}$ gives

$$l_{\mathbf{a}'}^{\bar{\zeta}'}(h)\cdot l_{1^l}^{\bar{1}^l,S}(h)=\mathbb{L}\big(l_{\mathbf{a}'1^l}^{\bar{\zeta}'\bar{1}^l,I}(h)\big)+\sum_{\sigma\neq id\in Sh^{\leqslant}(k,l)} l_{\sigma(\mathbf{a}',1^l)}^{\sigma(\bar{\zeta}',\bar{1}^l),S}(h).$$

Since $l_{\mathbf{a}'}^{\bar{\zeta}',S}(h)$ and $l_{1^l}^{\bar{1}^l,S}(h)$ satisfy the series shuffle formula, $\mathbb{L}\big(l_{\mathbf{a}}^{\bar{\zeta},I}(h)\big)$ must be equal to $l_{\mathbf{a}}^{\bar{\zeta},S}(h)$, which concludes Proposition 5.1. Q.E.D.

Embed $U\mathfrak{F}_{Y_N}$ into $U\mathfrak{F}_{N+1}$ by sending $Y_{m,a}$ to $-A^{m-1}B(-a)$. Then by the above proposition,

$$\begin{aligned}
l_{\mathbf{a}}^{\bar{\zeta},S}(h)&=\mathbb{L}(l_{\mathbf{a}}^{\bar{\zeta},I}(h))=\mathbb{L}(l_{\mathbf{a}}^{\bar{\zeta}}(e^{TB(0)}h))=l_{\mathbf{a}}^{\bar{\zeta}}\left(\mathbb{L}(e^{TB(0)}\pi_Y(h))\right)\\
&=l_{\mathbf{a}}^{\bar{\zeta}}(\exp\left\{-\sum_{n=1}^{\infty} l_n^{1,I}(h)\frac{B(0)^n}{n}\right\}\cdot\pi_Y(h))\\
&=l_{\mathbf{a}}^{\bar{\zeta}}(\exp\left\{-TY_{1,0}+\sum_{n=1}^{\infty}\frac{(-1)^n}{n}c_{A^{n-1}B(0)}(h)Y_{1,0}^n\right\}\cdot\pi_Y(h))\\
&=l_{\mathbf{a}}^{\bar{\zeta}}(e^{-TY_{1,0}}h_*)
\end{aligned}$$

for all $(\mathbf{a},\bar{\zeta})$ because $l_1^1(h)=0$. As for the third equality we use

$$(\mathbb{L}\otimes_{\mathbf{Q}} id)\circ(id\otimes_{\mathbf{Q}} l_{\mathbf{a}}^{\bar{\zeta}})=(id\otimes_{\mathbf{Q}} l_{\mathbf{a}}^{\bar{\zeta}})\circ(\mathbb{L}\otimes_{\mathbf{Q}} id) \text{ on } \mathbf{Q}[T]\otimes_{\mathbf{Q}} U\mathfrak{F}_{N+1}.$$

All $l_{\mathbf{a}}^{\bar{\zeta},S}(h)$'s satisfy the series shuffle formulae (5.2), so the $l_{\mathbf{a}}^{\bar{\zeta}}(e^{-TY_{1,0}}h_*)$'s do also. By putting $T=0$, we get that $l_{\mathbf{a}}^{\bar{\zeta}}(h_*)$'s also satisfy the series shuffle formulae for all $\mathbf{a}$. Therefore $\Delta_*(h_*)=h_*\widehat{\otimes}h_*$. This completes the proof of Theorem 0.1. Q.E.D.

Proof of Theorem 0.2. The first statement follows from Theorem 0.1.

Let $(g,h) \in Pseudo_{(a,\mu)}(N,\mathbf{Q})$ with $(a,\mu) \in (\mathbf{Z}/N\mathbf{Z})^{\times} \times \mathbf{Q}$. By comparing the coefficient of $B(a)$ in the octagon equation (1.4),

$$-c_{B(0)}(h)+\frac{\mu}{2}-c_A(h)+c_{B(0)}(h)-\frac{\mu}{N}+c_A(h)-c_{B(-a)}(h)+c_{B(a)}(h)=0.$$

Thus $c_{B(a)}(h) - c_{B(-a)}(h) = (\frac{1}{N}-\frac{1}{2})\mu$.

Next by comparing the coefficient of $B(ka)$ in (1.4) for $2 \leqslant k \leqslant N/2$,

$$\begin{aligned} &- \quad c_{B((k-1)a)}(h) - c_A(h) + c_{B(-(k-1)a)}(h) \\ &- \quad \frac{\mu}{N} + c_A(h) - c_{B(-ka)}(h) + c_{B(ka)}(h) = 0. \end{aligned}$$

Thus $c_{B(ka)}(h) - c_{B(-ka)}(h) = c_{B((k-1)a)}(h) - c_{B(-(k-1)a)}(h) + \frac{\mu}{N}$.

By combining these equations we get (2.3) and (2.4) for $N \geqslant 3$. Since we have $c_{AB}(g) = \frac{\mu^2}{24}$ for $g \in M_\mu(\mathbf{Q})$, we have (2.4) for $N=1,2$ by $c_{AB}(g) = c_{AB(0)}(h)$. Q.E.D.

§6. Auxiliary lemmas

We prove all Lemmas which are required to prove Theorem 0.1 in the previous section.

Lemma 6.1. *Let $h \in U\mathfrak{F}_{N+1}$ with $c_0(h)=1$* [2] *and $c_A(h)=0$. Then*

$$l_{\mathbf{a}}^{\bar{\zeta}(x)}(h^{1,23,4}h^{1,2,3}) = l_{\mathbf{a}}^{\bar{\zeta}}(h),$$

$$l_{\mathbf{a}}^{\bar{\zeta}(y)}(h^{1,23,4}h^{1,2,3}) = l_{\mathbf{a}}^{\bar{\zeta}}(h),$$

$$l_{\mathbf{a}}^{\bar{\zeta}(xy)}(h^{1,23,4}h^{1,2,3}) = l_{\mathbf{a}}^{\bar{\zeta}}(h),$$

$$l_{\mathbf{a},\mathbf{b}}^{\bar{\zeta}(x),\bar{\eta}(y)}(h^{1,23,4}h^{1,2,3}) = l_{\mathbf{ab}}^{\bar{\zeta}\bar{\eta}}(h)$$

for any pairs $(\mathbf{a},\bar{\zeta})$ and $(\mathbf{b},\bar{\eta})$.

Proof. Put $U\mathfrak{t}_{4,N}^0$ the universal enveloping algebra of $\mathfrak{t}_{4,N}^0$. Consider the map $\mathcal{M}_{0,5}^{(N)} \to \mathcal{M}_{0,4}^{(N)}$ induced from $\mathcal{M}_{0,5} \to \mathcal{M}_{0,4} : [(x_1,\cdots,x_5)] \mapsto [(x_1,x_2,x_3,x_5)]$. This yields the projection $p_4 : U\mathfrak{t}_{4,N}^0 \twoheadrightarrow U\mathfrak{F}_{N+1}$ sending $t^{14}, t^{24}(a), t^{34}(a) \mapsto 0$, $t^{12} \mapsto A$ and $t^{23}(a) \mapsto B(a)$ $(a \in \mathbf{Z}/N\mathbf{Z})$. Express $l_{\mathbf{a}}^{\bar{\zeta}}$ as (4.2). Since $(p_4 \otimes id)(\mathrm{Exp}\Omega_5^{(N)}) = \mathrm{Exp}\Omega_4^{(N)}(x) \in$

[2]The symbol $c_0(h)$ stands for the constant term of h.

$U\mathfrak{F}_{N+1}\widehat{\otimes}_{\mathbf{Q}}H^0\bar{B}(\mathcal{M}_{0,5}^{(N)}) \simeq H^0\bar{B}(\mathcal{M}_{0,4}^{(N)})^*\widehat{\otimes}_{\mathbf{C}}H^0\bar{B}(\mathcal{M}_{0,5}^{(N)})$, it induces the map

$$p_4^* : H^0\bar{B}(\mathcal{M}_{0,4}^{(N)}) \to H^0\bar{B}(\mathcal{M}_{0,5}^{(N)})$$

which gives $p_4^*([\frac{dz}{z}]) = [\frac{dx}{x}]$ and $p_4^*([\frac{dz}{\zeta_N^a - z}]) = [\frac{dx}{\zeta_N^a - x}]$. Hence

$$p_4^*(l_{\mathbf{a}}^{\bar{\zeta}}) = l_{\mathbf{a}}^{\bar{\zeta}(x)}.$$

Then $l_{\mathbf{a}}^{\bar{\zeta}(x)}(h^{1,23,4}h^{1,2,3}) = l_{\mathbf{a}}^{\bar{\zeta}}(p_4(h^{1,23,4}h^{1,2,3})) = l_{\mathbf{a}}^{\bar{\zeta}}(h)$ because $p_4(h^{1,23,4}) = 0$ by our assumption $c_A(h) = 0$.

Next consider the map $\mathcal{M}_{0,5}^{(N)} \to \mathcal{M}_{0,4}^{(N)}$ induced from $\mathcal{M}_{0,5} \to \mathcal{M}_{0,4}$: $[(x_1, \cdots, x_5)] \mapsto [(x_1, x_3, x_4, x_5)]$. This induces the projection $p_2 : U\mathfrak{t}_{4,N}^0 \twoheadrightarrow U\mathfrak{F}_{N+1}$ sending $t^{12}, t^{23}(a), t^{24}(a) \mapsto 0$, $t^{12} + t^{13} + t^{23} \mapsto A$ and $t^{34}(a) \mapsto B(a)$ $(a \in \mathbf{Z}/N\mathbf{Z})$. Since $(p_2 \otimes id)(\mathrm{Exp}\Omega_5^{(N)}) = \mathrm{Exp}\Omega_4^{(N)}(y) \in U\mathfrak{F}_{N+1}\widehat{\otimes}_{\mathbf{Q}}H^0\bar{B}(\mathcal{M}_{0,5}^{(N)}) \simeq H^0\bar{B}(\mathcal{M}_{0,4}^{(N)})^*\widehat{\otimes}_{\mathbf{C}}\, H^0\bar{B}(\mathcal{M}_{0,5}^{(N)})$, it induces the map

$$p_2^* : H^0\bar{B}(\mathcal{M}_{0,4}^{(N)}) \to H^0\bar{B}(\mathcal{M}_{0,5}^{(N)})$$

which gives $p_2^*([\frac{dz}{z}]) = [\frac{dy}{y}]$ and $p_2^*([\frac{dz}{\zeta_N^a - z}]) = [\frac{dy}{\zeta_N^a - y}]$. Hence

$$p_2^*(l_{\mathbf{a}}^{\bar{\zeta}}) = l_{\mathbf{a}}^{\bar{\zeta}(y)}.$$

Then $l_{\mathbf{a}}^{\bar{\zeta}(y)}(h^{1,23,4}h^{1,2,3}) = l_{\mathbf{a}}^{\bar{\zeta}}(p_2(h^{1,23,4}h^{1,2,3})) = l_{\mathbf{a}}^{\bar{\zeta}}(h)$ because $p_2(h^{1,2,3}) = 0$.

Similarly consider the map $\mathcal{M}_{0,5}^{(N)} \to \mathcal{M}_{0,4}^{(N)}$ induced from $\mathcal{M}_{0,5} \to \mathcal{M}_{0,4}$: $[(x_1, \cdots, x_5)] \mapsto [(x_1, x_2, x_4, x_5)]$. This induces the projection $p_3 : U\mathfrak{t}_{4,N}^0 \twoheadrightarrow U\mathfrak{F}_{N+1}$ sending $t^{13}, t^{23}(a), t^{34}(a) \mapsto 0$, $t^{12} \mapsto A$ and $t^{24}(a) \mapsto B(a)$ $(a \in \mathbf{Z}/N\mathbf{Z})$. Since $(p_3 \otimes id)(\mathrm{Exp}\Omega_5^{(N)}) = \mathrm{Exp}\Omega_4^{(N)}(xy) \in U\mathfrak{F}_{N+1}\widehat{\otimes}_{\mathbf{Q}}H^0\bar{B}(\mathcal{M}_{0,5}^{(N)}) \simeq H^0\bar{B}(\mathcal{M}_{0,4}^{(N)})^*\widehat{\otimes}_{\mathbf{C}}H^0\bar{B}(\mathcal{M}_{0,5}^{(N)})$, it induces the map

$$p_3^* : H^0\bar{B}(\mathcal{M}_{0,4}^{(N)}) \to H^0\bar{B}(\mathcal{M}_{0,5}^{(N)})$$

which gives $p_3^*([\frac{dz}{z}]) = [\frac{dx}{x} + \frac{dy}{y}]$ and $p_3^*([\frac{dz}{\zeta_N^a - z}]) = [\frac{xdy + ydx}{\zeta_N^a - xy}]$. Hence

$$p_3^*(l_{\mathbf{a}}^{\bar{\zeta}}) = l_{\mathbf{a}}^{\bar{\zeta}(xy)}.$$

Then $l_{\mathbf{a}}^{\bar{\zeta}(xy)}(h^{1,23,4}h^{1,2,3}) = l_{\mathbf{a}}^{\bar{\zeta}}(p_3(h^{1,23,4}h^{1,2,3})) = l_{\mathbf{a}}^{\bar{\zeta}}(h)$ because $p_3(h^{1,2,3}) = 0$ by our assumption $c_A(h) = 0$.

Consider the embedding of Hopf algebras $i_{1,2,3} : U\mathfrak{F}_{N+1} \hookrightarrow U\mathfrak{t}_{4,N}^0$ sending $A \mapsto t^{12}$ and $B(a) \mapsto t^{23}(a)$ along the divisor $\{y = 0\}$. Since

$(i_{1,2,3}\otimes id)(\mathrm{Exp}\Omega_4^{(N)}) = \mathrm{Exp}\Omega_4^{(N)}(z)^{1,2,3} \in U\mathfrak{t}^0_{4,N}\widehat{\otimes}_{\mathbf{Q}}H^0\bar{B}(\mathcal{M}_{0,4}^{(N)}) \simeq H^0\bar{B}(\mathcal{M}_{0,5}^{(N)})^*\widehat{\otimes}_{\mathbf{C}}H^0\bar{B}(\mathcal{M}_{0,4}^{(N)})$, it induces the map

$$i_{1,2,3}^* : H^0\bar{B}(\mathcal{M}_{0,5}^{(N)}) \to H^0\bar{B}(\mathcal{M}_{0,4}^{(N)})$$

which gives $i_{1,2,3}^*([\frac{dy}{y}]) = i_{1,2,3}^*([\frac{dy}{\zeta_N^a-y}]) = i_{1,2,3}^*([\frac{xdy+ydx}{\zeta_N^a-xy}]) = 0$. Express $l_{\mathbf{a},\mathbf{b}}^{\bar{\zeta}(x),\bar{\eta}(y)}$ and $l_{\mathbf{a}}^{\bar{\zeta}(xy)}$ as (4.4). In the expression each term contains at least one $\frac{dy}{y}$, $\frac{dy}{\zeta_N^a-y}$ or $\frac{xdy+ydx}{\zeta_N^a-xy}$. Therefore we have

$$i_{1,2,3}^*(l_{\mathbf{a},\mathbf{b}}^{\bar{\zeta}(x),\bar{\eta}(y)}) = 0 \text{ and } i_{1,2,3}^*(l_{\mathbf{a}}^{\bar{\zeta}(xy)}) = 0.$$

Thus $l_{\mathbf{a},\mathbf{b}}^{\bar{\zeta}(x),\bar{\eta}(y)}(h^{1,2,3}) = i_{1,2,3}^*(l_{\mathbf{a},\mathbf{b}}^{\bar{\zeta}(x),\bar{\eta}(y)})(h) = 0$ and $l_{\mathbf{a}}^{\bar{\zeta}(xy)}(h^{1,2,3}) = i_{1,2,3}^*(l_{\mathbf{a}}^{\bar{\zeta}(xy)})(h) = 0$.

Next consider the embedding of Hopf algebras $i_{1,23,4} : U\mathfrak{F}_{N+1} \hookrightarrow U\mathfrak{t}^0_{4,N}$ sending $A \mapsto t^{12}+t^{13}+t^{23}$ and $B(a) \mapsto t^{24}(a)+t^{34}(a)$ (geometrically caused by the divisor $\{x=1\}$.) Since $(i_{1,23,4}\otimes id)(\mathrm{Exp}\Omega_4^{(N)}) = \mathrm{Exp}\Omega_4^{(N)}(z)^{1,23,4} \in U\mathfrak{t}^0_{4,N}\widehat{\otimes}_{\mathbf{Q}}H^0\bar{B}(\mathcal{M}_{0,4}^{(N)}) \simeq H^0\bar{B}(\mathcal{M}_{0,5}^{(N)})^*\widehat{\otimes}_{\mathbf{C}}H^0\bar{B}(\mathcal{M}_{0,4}^{(N)})$, it induces the map

$$i_{1,23,4}^* : H^0\bar{B}(\mathcal{M}_{0,5}^{(N)}) \to H^0\bar{B}(\mathcal{M}_{0,4}^{(N)})$$

which gives $i_{1,23,4}^*([\frac{dx}{x}]) = 0$, $i_{1,23,4}^*([\frac{dx}{\zeta_N^a-x}]) = [\frac{dz}{\zeta_N^a-z}]$, $i_{1,23,4}^*([\frac{dy}{y}]) = [\frac{dz}{z}]$, $i_{1,23,4}^*([\frac{dy}{\zeta_N^a-y}]) = [\frac{dz}{\zeta_N^a-z}]$ and $i_{1,23,4}^*([\frac{xdy+ydx}{\zeta_N^a-xy}]) = [\frac{dz}{\zeta_N^a-z}]$. As is same to the proof of [F3] Lemma 5.1,

$$i_{1,23,4}^*(l_{\mathbf{a},\mathbf{b}}^{\bar{\zeta}(x),\bar{\eta}(y)}) = l_{\mathbf{ab}}^{\bar{\zeta}\bar{\eta}} \text{ and } i_{1,23,4}^*(l_{\mathbf{a}}^{\bar{\zeta}(xy)}) = l_{\mathbf{a}}^{\bar{\zeta}}$$

can be deduced by induction on weight. Thus $l_{\mathbf{a},\mathbf{b}}^{\bar{\zeta}(x),\bar{\eta}(y)}(h^{1,23,4}) = l_{\mathbf{ab}}^{\bar{\zeta}\bar{\eta}}(h)$. Let δ be the coproduct of $H^0\bar{B}(\mathcal{M}_{0,5}^{(N)})$. Express $\delta(l_{\mathbf{a},\mathbf{b}}^{\bar{\zeta}(x),\bar{\eta}(y)}) = \sum_i l_i'\otimes l_i''$ with $\deg l_i' = m_i'$ and $\deg l_i'' = m_i''$ for some m_i' and m_i'' such that $m_i'+m_i'' = wt(\mathbf{a},\bar{\zeta})+wt(\mathbf{b},\bar{\eta})$. If $m_i''\neq 0$, $l_i''(h^{1,2,3}) = 0$ because l_i'' is a combination of elements of the form $l_{\mathbf{c},\mathbf{d}}^{\bar{\lambda}(x),\bar{\mu}(y)}$ and $l_{\mathbf{e}}^{\bar{\nu}(xy)}$ for some pairs $(\mathbf{c},\bar{\lambda})$, $(\mathbf{d},\bar{\mu})$ and $(\mathbf{e},\bar{\nu})$. Since $\delta(l_{\mathbf{a},\mathbf{b}}^{\bar{\zeta}(x),\bar{\eta}(y)})(1\otimes h^{1,23,4}h^{1,2,3}) = \delta(l_{\mathbf{a},\mathbf{b}}^{\bar{\zeta}(x),\bar{\eta}(y)})(h^{1,23,4}\otimes h^{1,2,3})$, it follows that $l_{\mathbf{a},\mathbf{b}}^{\bar{\zeta}(x),\bar{\eta}(y)}(h^{1,23,4}h^{1,2,3}) = \sum_i l_i'(h^{1,23,4})\otimes l_i''(h^{1,2,3}) = l_{\mathbf{a},\mathbf{b}}^{\bar{\zeta}(x),\bar{\eta}(y)}(h^{1,23,4}) = l_{\mathbf{ab}}^{\bar{\zeta}\bar{\eta}}(h)$. For the second equality we use the assumption $c_0(h)=1$. Q.E.D.

Lemma 6.2. *Let $(g,h)\in U\mathfrak{F}_2\times U\mathfrak{F}_{N+1}$ be a pair satisfying $c_0(h)=1$, $c_A(h)=0$ and (1.3). Suppose that $(\mathbf{a},\bar\zeta)$ and $(\mathbf{b},\bar\eta)$ are admissible. Then*

$$l_{\mathbf{b},\mathbf{a}}^{\bar\eta(y),\bar\zeta(x)}(h^{1,23,4}h^{1,2,3})=l_{\mathbf{ba}}^{\bar\eta\bar\zeta}(h).$$

Proof. It follows $c_0(g)=1$ by our assumptions $c_0(h)=1$ and (1.3). Consider the embedding of Hopf algebra $i_{2,3,4}:U\mathfrak{F}_2\hookrightarrow U\mathfrak{t}_{4,N}^0$ sending $A\mapsto t^{23}(0)$ and $B\mapsto t^{34}(0)$ (geometrically caused by the exceptional divisor obtained by blowing up at $(x,y)=(1,1)$). Since $(i_{2,3,4}\otimes id)(\mathrm{Exp}\Omega_4^{(N)})=\mathrm{Exp}\Omega_4^{(N)}(z)^{2,3,4}\in U\mathfrak{t}_{4,N}^0\widehat{\otimes}_{\mathbf{Q}}H^0\bar B(\mathcal{M}_{0,4})\simeq H^0\bar B(\mathcal{M}_{0,5}^{(N)})^*\widehat{\otimes}_{\mathbf{C}}H^0\bar B(\mathcal{M}_{0,4})$, it induces the morphism

$$i_{2,3,4}^*:H^0\bar B(\mathcal{M}_{0,5}^{(N)})\to H^0\bar B(\mathcal{M}_{0,4})$$

which gives $i_{2,3,4}^*([\frac{dx}{x}])=0$, $i_{2,3,4}^*([\frac{dx}{1-x}])=[\frac{dz}{z}]$, $i_{2,3,4}^*([\frac{dx}{\zeta_N^a-x}])=0$ $(a\neq 0)$, $i_{2,3,4}^*([\frac{dy}{y}])=0$, $i_{2,3,4}^*([\frac{dy}{1-y}])=[\frac{dz}{1-z}]$ and $i_{2,3,4}^*([\frac{dy}{\zeta_N^a-y}])=0$ $(a\neq 0)$, $i_{2,3,4}^*([\frac{xdy+ydx}{\zeta_N^a-xy}])=0$ $(a\in\mathbf{Z}/N\mathbf{Z})$. In each term of the expression $l_{\mathbf{b},\mathbf{a}}^{\bar\eta(y),\bar\zeta(x)}=\sum_{I=(i_m,\cdots,i_1)}c_I[\omega_{i_m}|\cdots|\omega_{i_1}]$, the first component ω_{i_m} is always one of $\frac{dx}{x}$, $\frac{dy}{y}$, $\frac{dx}{\zeta_N^a-x}$ and $\frac{dy}{\zeta_N^a-y}$ for $a\neq 0$ because both $(\mathbf{a},\bar\zeta)$ and $(\mathbf{b},\bar\eta)$ are admissible. So $i_{2,3,4}^*(l_i')=0$ unless $m_i'=0$. Therefore

$$\begin{aligned}l_{\mathbf{b},\mathbf{a}}^{\bar\eta(y),\bar\zeta(x)}(g^{2,3,4}h^{1,23,4}h^{1,2,3})&=\sum_i l_i'(g^{2,3,4})\otimes l_i''(h^{1,23,4}h^{1,2,3})\\&=l_{\mathbf{b},\mathbf{a}}^{\bar\eta(y),\bar\zeta(x)}(h^{1,23,4}h^{1,2,3})\end{aligned}$$

by $c_0(g)=1$. So by our assumption,

$$\begin{aligned}l_{\mathbf{b},\mathbf{a}}^{\bar\eta(y),\bar\zeta(x)}(h^{1,23,4}h^{1,2,3})&=l_{\mathbf{b},\mathbf{a}}^{\bar\eta(y),\bar\zeta(x)}(g^{2,3,4}h^{1,23,4}h^{1,2,3})\\&=l_{\mathbf{b},\mathbf{a}}^{\bar\eta(y),\bar\zeta(x)}(h^{1,2,34}h^{12,3,4}).\end{aligned}$$

By the same arguments to the last two paragraphs of the proof of Lemma 6.1,

$$\text{(6.1)}\qquad i_{12,3,4}^*(l_{\mathbf{b},\mathbf{a}}^{\bar\eta(y),\bar\zeta(x)})=0,\qquad i_{12,3,4}^*(l_{\mathbf{a}}^{\bar\zeta(xy)})=0,$$

$$i_{1,2,34}^*(l_{\mathbf{b},\mathbf{a}}^{\bar\eta(y),\bar\zeta(x)})=l_{\mathbf{b},\mathbf{a}}^{\bar\eta\bar\zeta},\qquad i_{1,2,34}^*(l_{\mathbf{a}}^{\bar\zeta(xy)})=l_{\mathbf{a}}^{\bar\zeta}$$

for admissible pairs $(\mathbf{a},\bar\zeta)$ and $(\mathbf{b},\bar\eta)$, from which we can deduce

$$l_{\mathbf{b},\mathbf{a}}^{\bar\eta(y),\bar\zeta(x)}(h^{1,2,34}h^{12,3,4})=l_{\mathbf{ba}}^{\bar\eta\bar\zeta}(h).$$

Q.E.D.

Lemma 6.3. *Let $h \in U\mathfrak{F}_{N+1}$ with $c_0(h)=1$ and $c_A(h)=0$. Then*

$$l_{\mathbf{a}}^{\bar{\zeta}(x)}(e^{T\{t^{23}(0)+t^{24}(0)+t^{34}(0)\}}h^{1,23,4}h^{1,2,3}) = l_{\mathbf{a}}^{\bar{\zeta},I}(h),$$

$$l_{\mathbf{a}}^{\bar{\zeta}(y)}(e^{T\{t^{23}(0)+t^{24}(0)+t^{34}(0)\}}h^{1,23,4}h^{1,2,3}) = l_{\mathbf{a}}^{\bar{\zeta},I}(h),$$

$$l_{\mathbf{a}}^{\bar{\zeta}(xy)}(e^{T\{t^{23}(0)+t^{24}(0)+t^{34}(0)\}}h^{1,23,4}h^{1,2,3}) = l_{\mathbf{a}}^{\bar{\zeta},I}(h),$$

$$l_{\mathbf{a},\mathbf{b}}^{\bar{\zeta}(x),\bar{\eta}(y)}(e^{T\{t^{23}(0)+t^{24}(0)+t^{34}(0)\}}h^{1,23,4}h^{1,2,3}) = l_{\mathbf{ab}}^{\bar{\zeta}\bar{\eta},I}(h)$$

for any pairs $(\mathbf{a},\bar{\zeta})$ and $(\mathbf{b},\bar{\eta})$.

Proof. By the arguments in Lemma 6.1 and our assumption $c_A(h)=0$,

$$\begin{aligned} & l_{\mathbf{a}}^{\bar{\zeta}(x)}(e^{T\{t^{23}(0)+t^{24}(0)+t^{34}(0)\}}h^{1,23,4}h^{1,2,3}) \\ &= l_{\mathbf{a}}^{\bar{\zeta}}(p_4(e^{T\{t^{23}(0)+t^{24}(0)+t^{34}(0)\}}h^{1,23,4}h^{1,2,3})) = l_{\mathbf{a}}^{\bar{\zeta}}(e^{TB(0)}h) = l_{\mathbf{a}}^{\bar{\zeta},I}(h), \end{aligned}$$

$$\begin{aligned} & l_{\mathbf{a}}^{\bar{\eta}(y)}(e^{T\{t^{23}(0)+t^{24}(0)+t^{34}(0)\}}h^{1,23,4}h^{1,2,3}) \\ &= l_{\mathbf{a}}^{\bar{\eta}}(p_2(e^{T\{t^{23}(0)+t^{24}(0)+t^{34}(0)\}}h^{1,23,4}h^{1,2,3})) = l_{\mathbf{a}}^{\bar{\eta}}(e^{TB(0)}h) = l_{\mathbf{a}}^{\bar{\eta},I}(h), \end{aligned}$$

$$\begin{aligned} & l_{\mathbf{a}}^{\bar{\zeta}(xy)}(e^{T\{t^{23}(0)+t^{24}(0)+t^{34}(0)\}}h^{1,23,4}h^{1,2,3}) \\ &= l_{\mathbf{a}}^{\bar{\zeta}}(p_3(e^{T\{t^{23}(0)+t^{24}(0)+t^{34}(0)\}}h^{1,23,4}h^{1,2,3})) = l_{\mathbf{a}}^{\bar{\zeta}}(e^{TB(0)}h) = l_{\mathbf{a}}^{\bar{\zeta},I}(h). \end{aligned}$$

By $c_0(h)=1$,

$$\begin{aligned} & l_{\mathbf{a},\mathbf{b}}^{\bar{\zeta}(x),\bar{\eta}(y)}(e^{T\{t^{23}(0)+t^{24}(0)+t^{34}(0)\}}h^{1,23,4}h^{1,2,3}) \\ &= l_{\mathbf{a},\mathbf{b}}^{\bar{\zeta}(x),\bar{\eta}(y)}(e^{T\{t^{24}(0)+t^{34}(0)\}}h^{1,23,4}) = l_{\mathbf{ab}}^{\bar{\zeta}\bar{\eta}}(e^{TB(0)}h) = l_{\mathbf{ab}}^{\bar{\zeta}\bar{\eta},I}(h). \end{aligned}$$

As for the last equation, we use the following trick:

$$\begin{aligned} e^{Tt^{23}(0)}e^{T\{t^{24}(0)+t^{34}(0)\}}h^{1,23,4}h^{1,2,3} &= e^{T\{t^{24}(0)+t^{34}(0)\}}e^{Tt^{23}(0)}h^{1,23,4}h^{1,2,3} \\ &= e^{T\{t^{24}(0)+t^{34}(0)\}}h^{1,23,4}e^{Tt^{23}(0)}h^{1,2,3}. \end{aligned}$$

Q.E.D.

Lemma 6.4. *Let $(g,h) \in U\mathfrak{F}_2 \times U\mathfrak{F}_{N+1}$ be a pair satisfying $c_0(h) = 1$, $c_A(h) = c_{B(0)^n}(h) = 0$ for all $n \in \mathbf{N}$ and (1.3). Suppose that $(\mathbf{a},\bar{\zeta})$ is admissible. Then*

$$l_{\mathbf{b},\mathbf{a}}^{\bar{\eta}(y),\bar{\zeta}(x)}(e^{T\{t^{23}(0)+t^{24}(0)+t^{34}(0)\}}h^{1,23,4}h^{1,2,3}) = l_{\mathbf{ba}}^{\bar{\eta}\bar{\zeta}}(h).$$

Proof. Express $\delta(l_{\mathbf{b},\mathbf{a}}^{\bar{\eta}(y),\bar{\zeta}(x)}) = \sum_i l_i' \otimes l_i''$ with $\deg l_i' = m_i'$ and $\deg l_i'' = m_i''$ for some m_i' and m_i'' such that $m_i'+m_i'' = wt(\mathbf{a},\bar{\zeta})+wt(\mathbf{b},\bar{\eta})$. Since $(\mathbf{a},\bar{\zeta})$ is admissible, $i_{2,3,4}^*(l_i')$ is of the form $\alpha[\frac{dz}{1-z}|\cdots|\frac{dz}{1-z}]$ with $\alpha \in \mathbf{Q}$. But by our assumption $c_{B(0)^n}(h) = 0$, $i_{2,3,4}^*(l_i') = 0$ unless $m_i' = 0$. Thus

$$\begin{aligned} &l_{\mathbf{b},\mathbf{a}}^{\bar{\eta}(y),\bar{\zeta}(x)}(e^{T\{t^{23}(0)+t^{24}(0)+t^{34}(0)\}}h^{1,23,4}h^{1,2,3}) \\ &\quad = l_{\mathbf{b},\mathbf{a}}^{\bar{\eta}(y),\bar{\zeta}(x)}(g^{2,3,4}e^{T\{t^{23}(0)+t^{24}(0)+t^{34}(0)\}}h^{1,23,4}h^{1,2,3}). \end{aligned}$$

By (1.3),

$$\begin{aligned} &g^{2,3,4}e^{T\{t^{23}(0)+t^{24}(0)+t^{34}(0)\}}h^{1,23,4}h^{1,2,3} \\ &\quad = e^{T\{t^{23}(0)+t^{24}(0)+t^{34}(0)\}}g^{2,3,4}h^{1,23,4}h^{1,2,3} \\ &\quad = e^{T\{t^{23}(0)+t^{24}(0)+t^{34}(0)\}}h^{1,2,34}h^{12,3,4} \\ &\quad = e^{T\{t^{23}(0)+t^{24}(0)\}}e^{Tt^{34}(0)}h^{1,2,34}h^{12,3,4} \\ &\quad = e^{T\{t^{23}(0)+t^{24}(0)\}}h^{1,2,34}e^{Tt^{34}(0)}h^{12,3,4}. \end{aligned}$$

By (6.1),

$$\begin{aligned} &l_{\mathbf{b},\mathbf{a}}^{\bar{\eta}(y),\bar{\zeta}(x)}(e^{T\{t^{23}(0)+t^{24}(0)\}}h^{1,2,34}e^{Tt^{34}(0)}h^{12,3,4}) \\ &\quad = l_{\mathbf{b},\mathbf{a}}^{\bar{\eta}(y),\bar{\zeta}(x)}(e^{T\{t^{23}(0)+t^{24}(0)\}}h^{1,2,34}) \\ &\quad = l_{\mathbf{ba}}^{\bar{\eta}\bar{\zeta}}(e^{TB(0)}h) = l_{\mathbf{ba}}^{\bar{\eta}\bar{\zeta},I}(h) = l_{\mathbf{ba}}^{\bar{\eta}\bar{\zeta}}(h). \end{aligned}$$

The last equality follows from the admissibility of $(\mathbf{a},\bar{\zeta})$ Q.E.D.

References

[AK] T. Arakawa and M. Kaneko, On multiple L-values, J. Math. Soc. Japan, **56** (2004), 967–991.

[B] D. J. Broadhurst, Massive 3-loop Feynman diagrams reducible to SC* primitives of algebras of the sixth root of unity, Eur. Phys. J. C Part. Fields, **8** (1999), 313–333.

[BK] D. J. Broadhurst and D. Kreimer, Association of multiple zeta values with positive knots via Feynman diagrams up to 9 loops, Phys. Lett. B, **393** (1997), 403–412.

[Br] F. Brown, Mixed Tate motives over **Z**, Ann. of Math. (2), **175** (2012), 949–976.

[C] K. T. Chen, Iterated path integrals, Bull. Amer. Math. Soc., **83** (1977), 831–879.

[De89] P. Deligne, Le groupe fondamental de la droite projective moins trois points, In: Galois Groups over Q, Berkeley, CA, 1987, Math. S. Res. Inst. Publ., **16**, Springer-Verlag, 1989, pp. 79–297.

[De08] P. Deligne, Le groupe fondemental de $\mathbf{G}_m\backslash\mu_N$, pour $N = 2, 3, 4, 6$, ou 8, preprint, 2008, available from www.math.ias.edu/files/deligne/121108Fondamental.pdf.

[DG] P. Deligne and A. B. Goncharov, Groupes fondamentaux motiviques de Tate mixte, Ann. Sci. Ecole Norm. Sup. (4), **38** (2005), 1–56.

[Dr] V. G. Drinfel'd, On quasitriangular quasi-Hopf algebras and a group closely connected with Gal$(\overline{Q}/Q)$, Leningrad Math. J., **2** (1991), 829–860.

[E] B. Enriquez, Quasi-reflection algebras and cyclotomic associators, Selecta Math. (N.S.), **13** (2007), 391–463.

[EF] B. Enriquez and H. Furusho, Mixed Pentagon, octagon and Broadhurst duality equation, J. Pure Appl. Algebra, **216** (2012), 982–995.

[F1] H. Furusho, The multiple zeta value algebra and the stable derivation algebra, Publ. Res. Inst. Math. Sci., **39** (2003), 695–720.

[F2] H. Furusho, Pentagon and hexagon equations, Ann. of Math. (2), **171** (2010), 545–556.

[F3] H. Furusho, Double shuffle relation for associators, Ann. of Math. (2), **174** (2011), 341–360.

[G] A. B. Goncharov, The dihedral Lie algebras and Galois symmetries of $\pi_1^{(l)}(\mathbb{P}^1 - (\{0, \infty\} \cup \mu_N))$, Duke Math. J., **110** (2001), 397–487.

[K] T. Kohno, Bar complex of the Orlik-Solomon algebra, In: Arrangements in Boston: a Conference on Hyperplane Arrangements (1999), Topology Appl., **118**, Elsevier Science B.V., Amsterdam, 2002, pp. 147–157.

[LM] T. T. Q. Le and J. Murakami, Kontsevich's integral for the Kauffman polynomial, Nagoya Math. J., **142** (1996), 39–65.

[R] G. Racinet, Doubles melanges des polylogarithmes multiples aux racines de l'unite, Publ. Math. Inst. Hautes Études Sci., **95** (2002), 185–231.

[Z] J. Zhao, Multiple polylogarithm values at roots of unity, C. R. Math. Acad. Sci. Paris, **346** (2008), 1029–1032.

Graduate School of Mathematics
Nagoya University
Furo-cho, Chikusa-ku, Nagoya, 464-8602
Japan
E-mail address: `furusho@math.nagoya-u.ac.jp`