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On the incompressible Euler equations and the blow-up problem

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Abstract.

In this article we review some of the recent results on the Cauchy problem and the issue of finite time blow-up for the 3D incompressible Euler equations. We begin with the refinements Kato's local existence of classical solution and the finite time blow-up criterion by Beale, Kato and Majda, using the tools from harmonic analysis. Then, we present the developments of observation of depletion of nonlinearity of the vortex stretching term, first discovered by Constantin and Fefferman for the case of the Navier-Stokes equations. One consequence of this observation is the regularity control in terms of the direction field of the vorticity, and the other one is the regularity control in terms of the reduced number of vorticity components. We develop those idea both for the generalized Navier-Stokes equations and for the Euler equations. We also present some consequences of the dynamics of the deformation tensor. One of them leads to the spectral dynamics approaches to the Euler equations, which provides us with the local in time enstrophy growth and decay estimates, among others. The other one is characterizations of the set of points leading to the possible finite time singularities. Finally we present studies on the various model problems of the 3D Euler equations. More specifically, we introduce the 2D quasi-geostrophic equation and its one dimensional model equation, the 2D Boussinesq system and a modified Euler system. Finally we present recent result on the nonexistence of the self-similar blow-up.

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§1. Introduction

The motion of homogeneous incompressible ideal fluid in a domain $\Omega \subset \mathbb{R}^n$ is described by the following system of Euler equations.

$$\frac{\partial v}{\partial t} + (v \cdot \nabla) v = -\nabla p, \tag{1.1}$$

$$\text{div } v = 0, \tag{1.2}$$

$$v(x,0) = v_0(x), \tag{1.3}$$

where $v = (v^1, v^2, \cdots, v^n)$, $v^j = v^j(x,t)$, $j = 1, 2, \cdots, n$ is the velocity of the fluid flows, $p = p(x,t)$ is the scalar pressure, and $v_0(x)$ is a given initial velocity field satisfying div $v_0 = 0$. Here we use the standard notion of vector calculus, denoting

$$\nabla p = \left(\frac{\partial p}{\partial x_1}, \cdots, \frac{\partial p}{\partial x_n} \right), \quad (v \cdot \nabla) v^j = \sum_{k=1}^n v^k \frac{\partial v^j}{\partial x_k}, \quad \text{div } v = \sum_{k=1}^n \frac{\partial v^k}{\partial x_k}.$$

The first equation, (1.1) represents the balance of momentum for each portion of fluid, while the second equation, (1.2) represents the conservation of mass of fluid during its motion, combined with the homogeneity(constant density) assumption on the fluid. The equations (1.1)-(1.2) are first obtained by L. Euler in 1755([53]). In this article we are concerned on the Cauchy problem of the system (1.1)-(1.3) in the domain, $\Omega = \mathbb{R}^n$(whole domain of $\mathbb{R}^n$), or $\mathbb{R}^n/\mathbb{Z}^n$(periodic domain). We note that the Euler equation is obtained formally by setting formally viscosity$= 0$(or equivalently, Reynolds number$= \infty$) in the Navier-Stokes equations, which have the addition of the viscosity term $\nu \Delta v$ in the right hand side of (1.1) with viscosity constant $\nu > 0$. For detailed mathematical studies on the finite Reynolds number Navier-Stokes equations we refer e.g. [69, 89, 73, 33, 76]. In the study of the Euler equations the notion of vorticity, $\omega =$curl v, plays very important roles. In particular we can reformulate the Euler system in terms of the vorticity fields only as follows. We first rewrite (1.1) as

$$\frac{\partial v}{\partial t} - v \times \operatorname{curl} v = -\nabla (p + \frac{1}{2}|v|^2). \tag{1.4}$$

Taking curl of (1.4), and using elementary vector identities we obtain the following vorticity formulation:

$$\frac{\partial \omega}{\partial t} + (v \cdot \nabla) \omega = \omega \cdot \nabla v, \tag{1.5}$$

$$\text{div } v = 0, \quad \text{curl } v = \omega, \tag{1.6}$$

$$\omega(x,0) = \omega_0(x). \tag{1.7}$$

The linear elliptic system (1.6) for v can be solved explicitly in terms of ω to provides us with the Biot-Savart law

$$v(x,t) = \frac{1}{4\pi}\int_{\mathbb{R}^n} \frac{(x-y)\times\omega(y,t)}{|x-y|^3}dy. \tag{1.8}$$

Substituting this v into (1.5) formally, we obtain a integro-differential system for ω. The term in the right hand side of (1.5) is called the 'vortex stretching term', and is regarded as the main source of difficulties in the mathematical theory of the 3D Euler equations.

For given $m \in \mathbb{Z}_+ \cup \{0\}$ the Sobolev space, $H^m(\Omega)$ is the space of functions consists of functions $f \in L^2(\Omega)$ such that

$$\|f\|_{H^m} := \left(\sum_{k=0}^{m}\int_\Omega |D^k f(x)|^2 dx\right)^{\frac{1}{2}} < \infty,$$

where the derivatives are in the sense of distributions. The local in time solution of the Euler equations in the Sobolev space $H^m(\mathbb{R}^n)$ for $m > n/2+1, n = 2,3$ was obtained by Kato in [59, 89], and there are several other local well-posedness results after that, using various function spaces[60, 61, 62, 63, 31, 9, 10, 11, 93, 94]. One of the most outstanding open problem for the Euler equations is whether or not there exists any smooth initial data, say $v_0 \in C_0^\infty(\mathbb{R}^3)$, which evolves in finite time into a blowing up solution(breakdown of the initial data regularity). The answer to this question is tremendously important to understand the physics of turbulence. Even the results of numerical experiments are not yet conclusive(see e.g. [64, 55, 56]). For very interesting discussions of physical motivation of the study to answer this question and the interplay of multifaced approaches to it from the points of views of mathematics, numerics and physics we refer the articles [34, 35, 75], or the Chapter 5 of [76]. In this direction there is a celebrated criterion of the blow-up due to Beale, Kato and Majda(called the BKM criterion)[1], which states for $m > \frac{5}{2}$

$$\limsup_{t\nearrow T_*} \|v(t)\|_{H^m} = \infty \quad \text{if and only if} \quad \int_0^{T_*} \|\omega(t)\|_{L^\infty} dt = \infty. \tag{1.9}$$

See e.g. [65, 68, 9, 10, 11, 12, 15, 16] for the refinements of this result, replacing the L^∞ norm of the vorticity by weaker norms close to the L^∞ norm, or reducing the number of components of the vorticity. In this review article we survey some of the recent results on the 3D Euler equations concentrating on the local well-posedness and the finite time blow-up problem.

§2. Local Existence and the Blow-up Criterion in Terms of the Besov and the Triebel-Lizorkin Spaces

As remarked at the end of the previous section, the construction of local solution was obtained by many authors, using various function spaces. e.g. $L^{s,p}(\mathbb{R}^n) = (1-\Delta)^{-\frac{s}{2}}L^p(\mathbb{R}^n), s > \frac{n}{p}+1,\ 1 < p < \infty$(the Bessel potential space), $C^s(\mathbb{R}^n)$, $s>1$ (Zygmund class), $B^s_{p,q}$,(Besov space), $F^s_{p,q}$(Triebel-Lizorkin space). In particular we note that $L^{s,p}(\mathbb{R}^n) = F^s_{p,2}$. Hence, the results in the Triebel-Lizorkin space implies the corresponding results in the classical fractional Sobolev space. The BKM criterion is also refined by many authors replacing the L^∞ norm of the vorticity by the slightly weaker norm as BMO norm, and even weaker norm in $\dot{B}^0_{\infty,\infty}$. In this section we present essential ingredients of the a priori estimates in proving local existence in the Tribel-Lizorkin spaces, and the corresponding refinements of the BKM criterion, following mostly [10, 11]. The case of the Besov spaces is similar, and even simpler than that of the Triebel-Lizorkin spaces([9, 14]). We refer [6, 7, 8] for the studies of the Cauchy problem of the Navier-Stokes equations in the setting of the Besov and the Triebel-Lizorkin spaces. We first introduce definitions of the function spaces, following [91]. Given $f\in\mathcal{S}$ its Fourier transform $\hat{f}$ is defined by

$$\mathcal{F}(f) = \hat{f}(\xi) = \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)dx.$$

We consider $\varphi\in\mathcal{S}$ satisfying

(i) $\mathrm{Supp}\,\hat{\varphi}\subset\{\xi\in\mathbb{R}^n|\frac{1}{2}\le|\xi|\le 2\}$,
(ii) $\hat{\varphi}(\xi)\ge C>0$ if $\frac{2}{3}<|\xi|<\frac{3}{2}$,
(iii) $\sum_{j\in\mathbb{Z}}\hat{\varphi}_j(\xi)=1$, where $\hat{\varphi}_j=\hat{\varphi}(2^{-j}\xi)$.

The homogeneous Besov semi-norm $\|f\|_{\dot{B}^s_{p,q}}$ and the homogeneous Tribel-Lizorkin semi-norm $\|f\|_{\dot{F}^s_{p,q}}$ are defined by

$$\|f\|_{\dot{B}^s_{p,q}} = \left(\sum_{j\in\mathbb{Z}} 2^{jqs}\|\varphi_j * f\|^q_{L^p}\right)^{\frac{1}{q}},$$

$$\|f\|_{\dot{F}^s_{p,q}} = \left\|\left(\sum_{j\in\mathbb{Z}} 2^{jqs}|\varphi_j * f(\cdot)|^q\right)^{\frac{1}{q}}\right\|_{L^p}$$

respectively. We denote $\hat{S}_j(\xi) = \sum_{k\leq j} \hat{\varphi}_k(\xi)$. Given $f \in \mathcal{S}'$, we set

$$\Delta_j f = \begin{cases} \varphi_j * f & \text{if } j \geq 0 \\ S_{-1} * f & \text{if } j = -1 \\ 0 & \text{if } j \leq -2. \end{cases}$$

The inhomogeneous Besov norm $\|f\|_{B^s_{p.q}}$ and the inhomogeneous Tribel-Lizorkin norm $\|f\|_{F^s_{p.q}}$ are defined by

$$\|f\|_{B^s_{p.q}} = \left[\sum_{j=-1}^{\infty} 2^{jqs}\|\Delta_j f\|^q_{L^p}\right]^{\frac{1}{q}}, \quad \|f\|_{F^s_{p.q}} = \left\|\left(\sum_{j=-1}^{\infty} 2^{jqs}|\Delta_j f(\cdot)|^q\right)^{\frac{1}{q}}\right\|_{L^p}$$

respectively.
For the construction of the local solution in the Triebel-Lizorkin space we take operation, Δ_j on the both sides of (1.5), we have

$$\begin{aligned} \partial_t\Delta_j\omega + (S_{j-2}v\cdot\nabla)\Delta_j\omega = (S_{j-2}v\cdot\nabla)\Delta_j\omega \\ -\Delta_j((v\cdot\nabla)\omega) + \Delta_j((\omega\cdot\nabla)v). \end{aligned} \tag{2.1}$$

Consider particle trajectory mapping $\{X_j(\alpha,t)\}$ defined by

$$\begin{cases} \frac{\partial}{\partial t}X_j(\alpha,t) & = (S_{j-2}v)(X_j(\alpha,t),t) \\ X_j(\alpha,0) & = \alpha \end{cases}.$$

We note that div $S_{j-2}v = 0$ implies each $\alpha \mapsto X_j(\alpha,t)$ is a volume preserving mapping. Integration (2.1) along the trajectories, we obtain

$$\begin{aligned} |\Delta_j\omega(X_j(\alpha,t),t)| \leq |\Delta_j\omega_0(\alpha)| + \int_0^t |\Delta_j((\omega\cdot\nabla)v)(X_j(\alpha,\tau),\tau)|d\tau \\ + \int_0^t |[(S_{j-2}v\cdot\nabla)\Delta_j\omega - \Delta_j((v\cdot\nabla)\omega)](X_j(\alpha,\tau),\tau)|\, d\tau. \end{aligned} \tag{2.2}$$

Multiplying both sides by 2^{js}, and taking l^q norm, we deduce, using the Minkowski inequality,

$$
\begin{aligned}
\left(\sum_{j\in\mathbb{Z}} 2^{jqs}|\Delta_j\omega(X_j(\alpha,t),t)|^q\right)^{\frac{1}{q}} &\leq \left(\sum_{j\in\mathbb{Z}} 2^{jqs}|\Delta_j\omega_0(\alpha)|^q\right)^{\frac{1}{q}} \\
&+\int_0^t \left(\sum_{j\in\mathbb{Z}} 2^{jqs}\,|\Delta_j((\omega\cdot\nabla)v)(X_j(\alpha,\tau),\tau)|^q\right)^{\frac{1}{q}} d\tau \\
&+\int_0^t \left(\sum_{j\in\mathbb{Z}} 2^{jqs}\,|[(S_{j-2}v\cdot\nabla)\Delta_j\omega-\Delta_j((v\cdot\nabla)\omega)](X_j(\alpha,\tau),\tau)|^q\right)^{\frac{1}{q}} d\tau.
\end{aligned}
\tag{2.3}
$$

We take $L^p(\mathbb{R}^n)$ norm of the both sides of (2.3), then using the fact that $\alpha\mapsto X_j(\alpha,t)$ is volume preserving, we obtain again by use of the Minkowski inequality

$$
\begin{aligned}
\|\omega(t)\|_{\dot F^s_{p,q}} &\leq \|\omega_0\|_{\dot F^s_{p,q}} + \int_0^t \|((\omega\cdot\nabla)v)(\tau)\|_{\dot F^s_{p,q}} d\tau \\
&+\int_0^t \left\|\left(\sum_{j\in\mathbb{Z}} 2^{jqs}\,|[(S_{j-2}v\cdot\nabla)\Delta_j\omega-\Delta_j((v\cdot\nabla)\omega)]|^q\right)^{\frac{1}{q}}\right\|_{L^p} d\tau.
\end{aligned}
$$

We apply to this the Moser type of inequality,

$$
\|fg\|_{\dot F^s_{p,q}} \leq C(\|f\|_{L^\infty}\|g\|_{\dot F^s_{p,q}}+\|g\|_{L^\infty}\|f\|_{\dot F^s_{p,q}})
$$

for $s>0, (p,q)\in(1,\infty)^2$, and the commutator estimate (see [10]),

$$
\begin{aligned}
&\left\|\left(\sum_{j\in\mathbb{Z}} 2^{jqs}\,|[(S_{j-2}v\cdot\nabla)\Delta_j\omega-\Delta_j((v\cdot\nabla)\omega)]|^q\right)^{\frac{1}{q}}\right\|_{L^p} \\
&\qquad\qquad \leq C\|\nabla v\|_{L^\infty}\|\omega\|_{\dot F^s_{p,q}},
\end{aligned}
$$

respectively to obtain the homogeneous space estimate

$$
\|v(t)\|_{\dot F^s_{p,q}} \leq \|v_0\|_{\dot F^s_{p,q}} + C\int_0^t \|\nabla v(\tau)\|_{L^\infty}\|v(\tau)\|_{\dot F^s_{p,q}} d\tau.
$$

Combining this with the easily obtainable L^p estimate,

$$\|\omega(t)\|_{L^p} \le \|\omega_0\|_{L^p} + C\int_0^t \|\nabla v(\tau)\|_{L^\infty}\|\omega(\tau)\|_{L^p} d\tau,$$

we get the inhomogeneous space estimate,

$$\|\omega(t)\|_{F^s_{p,q}} \le \|\omega_0\|_{F^s_{p,q}} + C\int_0^t \|\nabla v(\tau)\|_{L^\infty}\|\omega(\tau)\|_{F^s_{p,q}} d\tau. \tag{2.4}$$

Using the embedding, $F^s_{p,q} \hookrightarrow L^\infty$, $s > n/p$, and translating the origin of time, we have

$$\|\omega(t_2)\|_{F^s_{p,q}} \le \|\omega(t_1)\|_{F^s_{p,q}} + C\int_{t_1}^{t_2} \|\omega(\tau)\|^2_{F^s_{p,q}} d\tau, \qquad \forall t_1 < t_2,$$

and

$$\frac{d}{dt}\|\omega(t)\|_{F^s_{p,q}} \le C\|\omega(t)\|^2_{F^s_{p,q}}.$$

Solving the differential inequality, we deduce

$$\|\omega(t)\|_{F^s_{p,q}} \le \frac{\|\omega_0\|_{F^s_{p,q}}}{1 - Ct\|\omega_0\|_{F^s_{p,q}}} \le 2\|\omega_0\|_{F^s_{p,q}} \qquad \forall t \in [0, \frac{1}{2C\|\omega_0\|_{F^s_{p,q}}}],$$

which is one of the key estimates for the construction of the local solution in $C([0,T]; F^s_{p,q})$. Using this estimate, and applying the successive iteration scheme, we could actually construct our desired local solution.

Next, we give outline of the proof of the following blow-up criterion for local solution belonging to $F^s_{p,q}$,

$$\limsup_{t \nearrow T_*} \|v(t)\|_{F^s_{p,q}} = \infty \text{ if and only if } \int_0^{T_*} \|\omega(t)\|_{\dot{B}^0_{\infty,\infty}} dt = \infty \tag{2.5}$$

for $s > n/p+1$. For this we use the following the BKM type of inequality.

$$\|f\|_{L^\infty} \le C(1 + \|f\|_{\dot{B}^0_{\infty,\infty}}(\log^+ \|f\|_{F^s_{p,q}} + 1)), \tag{2.6}$$

where $s > n/p$ (similarly for the Besov norms). The proof of (2.6) uses the Littlewood-Paley decomposition and the standard optimization of parameter argument(see [88, 67, 32, 9, 10, 15] for the similar and the related inequalities). The BKM type of inequality implies for $f = \nabla v$

$$\begin{aligned} \|\nabla v\|_{L^\infty} &\le C(1 + \|\nabla v\|_{\dot{B}^0_{\infty,\infty}}(\log^+ \|\nabla v\|_{F^s_{p,q}} + 1)) \\ &\le C(1 + \|\omega\|_{\dot{B}^0_{\infty,\infty}}(\log^+ \|\omega\|_{F^s_{p,q}} + 1)), \end{aligned}$$

where $s > n/p$, $1 < p, q < \infty$. We substitute this into (2.4) to obtain

$$\begin{aligned}\|\omega(t)\|_{F^s_{p,q}} \le\ & \|\omega_0\|_{F^s_{p,q}} \\ &+ C\int_0^t (\|\omega(s)\|_{\dot{B}^0_{\infty,\infty}} + 1)\left(\log^+ \|\omega(s)\|_{F^s_{p,q}} + 1\right) \|\omega(s)\|_{F^s_{p,q}} ds.\end{aligned}$$

By Gronwall's lemma we have

$$\|\omega(t)\|_{F^s_{p,q}} \le \|\omega_0\|_{F^s_{p,q}} \exp\left[C \exp[C\int_0^t (\|\omega(s)\|_{\dot{B}^0_{\infty,\infty}} + 1)ds]\right].$$

Hence

$$\lim\sup_{t\to T_*} \|\omega(t)\|_{F^s_{p,q}} = \infty \Rightarrow \int_0^{T_*} \|\omega(s)\|_{\dot{B}^0_{\infty,\infty}} ds = \infty.$$

The other direction of the criterion (2.5) follows from the estimates

$$\int_0^T \|\omega(s)\|_{\dot{B}^0_{\infty,\infty}} ds \le T \sup_{0\le t\le T} \|\omega(t)\|_{\dot{B}^0_{\infty,\infty}} \le CT \sup_{0\le t\le T} \|\omega(t)\|_{F^s_{p,q}}$$

for $s > n/p$.

§3. The Depletion of the Vortex Stretching Term

3.1. Regularity criterion for the generalized Navier-Stokes equations

Here we are concerned on the 'generalized' 3D Navier-Stokes equations in $\mathbb{R}^3$:

$$(NS)_\alpha \begin{cases} \dfrac{\partial v}{\partial t} + (v\cdot\nabla)v = -\nabla p - \nu\Lambda^\alpha v \\ \text{div } v = 0 \\ v(x,0) = v_0(x) \end{cases}$$

where $v = (v^1, v^2, v^3)$, $v^j = v^j(x,t)$, $j = 1, 2, 3$, $p = p(x,t)$, and $\nu > 0$ is a constant, and $\Lambda^\alpha = (-\Delta)^{\frac{\alpha}{2}}$. The case $\alpha = 2$ corresponds to the usual Navier-Stokes equations, which we denote by (NS). Heuristically, the number α represents the 'strength of dissipation', and in this lecture we are concerned on the case $0 < \alpha \le 2$. We note that the global regularity for $\alpha \ge \frac{5}{2}$ is well-known([73]). Leray([71]) and Hopf([57]) constructed global weak solution of (NS), which is called the Leray-Hopf weak solution(see e.g. [69, 89, 33] for more details). A different notion of weak solution, called the suitable weak solution is also constructed, and this

weak solution has regularity at least on 'large' subset of the space-time domain, the 'partial regularity'(see [5, 72, 84]). We are concerned here on the regularity conditions for weak solutions(continuation principle of local regular solution) to $(NS)_\alpha$. Regularity condition for weak solutions for (NS) was obtained by Prodi, Ohkyama and Serrin([83, 82, 84]), which states that $v \in L^r(0,T;L^p(\mathbb{R}^3))$ with $3/p+2/r \leq 1$, $3<p\leq\infty$ implies the regularity of the weak solution v of (NS) on $[0,T]$. Refinements of this result in various directions are derived by many authors([54, 86, 66, 52]). On the other hand, Beirão da Veiga obtained the regularity condition in terms of the vorticity([2]), which states:
Let $\omega = \text{curl } v$, $\frac{3}{2}<p\leq\infty$. If $\omega\in L^r(0,T;L^p(\mathbb{R}^3))$ with $3/p+2/r\leq 2$, then v becomes regular on $[0,T]$. We note the improvements of this result in [25, 67], imposing only on the two components of the vorticity. In this section we are concerned on the geometic type of regularity criterion initiated by Constantin-Fefferman([37]) and Constantin-Fefferman-Majda([38], see also [50] for a recent related result). We state the result in [37] in the following. Let $\xi(x,t)=\omega(x,t)/|\omega(x,t)|$ be the direction field of the vorticity, and let $\theta(x,y,t) =$ be the angle between $\xi(x,t)$ and $\xi(x+y,t)$. If $|\sin\theta(x,y,t)|\leq C|y|$ for all x,y in the domain(,or it is enough that this inequality holds in the region where vorticity $|\omega|$ is large), then v becomes regular on $[0,T]$. Later this result was improved by Beirão da Veiga and Berselli([3]) as follows. Let $s\in[1/2,1]$, $r\in[\frac{4}{2s-1},\infty]$. If $|\sin\theta(x,y,t)|\leq g(x,t)|y|^s$ for some function g satisfying $g\in L^r(0,T;L^p(\mathbb{R}^3))$ with $3/p+2/r=s-1/2$, then v becomes regular on $[0,T]$. We note here that for $p=r=\infty$, $s=1/2$ the above condition reduces to $|\sin\theta(x,y,t)|\leq C|y|^{\frac{1}{2}}$, which shows improvement of Constantin-Fefferman's result. For the case $s\in(0,1/2]$ there is a result due to Beirão da Veiga([4]) as follows. If $|\sin\theta(x,y,t)|\leq|y|^s$ and $\omega\in L^2(0,T;L^p(\mathbb{R}^3))$, $3/p=s+1$, then v becomes regular on $[0,T]$. One of our main results to be presented here unifies the above two separate results for $s\in[1/2,1]$ and $s\in(0,1/2]$, and generalize both of them. For the statement of our main theorem we first introduce function spaces. Given $\Omega\subset\mathbb{R}^n$, $0<s<1$, $1\leq p\leq\infty, 1\leq q\leq\infty$, the function space $\dot{\mathcal{F}}^s_{p,q}(\Omega)$ is defined by the seminorm,

$$\|f\|_{\dot{\mathcal{F}}^s_{p,q}} = \begin{cases} \left\| \left(\displaystyle\int_{\mathbb{R}^n} \frac{|f(x)-f(y)|^q}{|x-y|^{n+sq}} dy \right)^{\frac{1}{q}} \right\|_{L^p(\Omega,dx)} & \text{if } 1\leq p\leq\infty, 1\leq q<\infty \\ \left\| \displaystyle\sup_{y\in\Omega,|x-y|\neq 0} \frac{|f(x)-f(y)|}{|x-y|^s} \right\|_{L^p(\Omega,dx)} & \text{if } 1\leq p\leq\infty, q=\infty. \end{cases}$$

We compare our function spaces with other well-known classical function spaces. For $p = q = \infty$, $\dot{\mathcal{F}}^s_{\infty,\infty}(\Omega) \cong C^s(\Omega)$(Hölder space). Let us introduce the Banach space $\mathcal{F}^s_{p,q}(\Omega)$ by defining its norm, $\|f\|_{\mathcal{F}^s_{p,q}(\Omega)} = \|f\|_{L^p(\Omega)} + \|f\|_{\dot{\mathcal{F}}^s_{p,q}(\Omega)}$. For $2 \le p < \infty$ and $q = 2$, $\mathcal{F}^s_{p,2}(\mathbb{R}^n) \cong L^p_s(\mathbb{R}^n) := (1-\Delta)^{-\frac{s}{2}} L^p(\mathbb{R}^n)$(Bessel potential space) (see [85]). If $\frac{n}{\min\{p,q\}} < s < 1$, $n < p < \infty$ and $n < q \le \infty$, then $\mathcal{F}^s_{p,q}(\mathbb{R}^n) \cong F^s_{p,q}(\mathbb{R}^n)$(Triebel-Lizorkin space). The following is one of our main theorem in this section.

Theorem 3.1. *Let $v(x,t)$ be a solution to $(NS)_\alpha$, and $\omega(x,t)$ =curl $v(x,t)$. Let $\xi(x,t)$ be its direction field $\xi(x,t) = \omega(x)/|\omega(x)|$. Suppose there exists $s \in (0,1)$, $q \in (\frac{3}{3-s}, \infty]$, $p_1 \in (1,\infty]$, $p_2 \in (1, \frac{3}{s})$ satisfying $\frac{s}{3} < \frac{1}{p_1} + \frac{1}{p_2} < \frac{\alpha+s}{3}$, and $r_1, r_2 \in [1,\infty]$ such that the followings hold.*

$$\xi(x,t) \in L^{r_1}(0,T; \dot{\mathcal{F}}^s_{p_1,q}), \qquad |\omega(x,t)| \in L^{r_2}(0,T; L^{p_2}(\mathbb{R}^3))$$

with $3/p_1 + 3/p_2 + \alpha/r_1 + \alpha/r_2 \le \alpha + s$, Then, there is no singularity up to T.

For the proof we refer [24]. Intuitively, the above theorem says that assumption of higher regularity of the direction vector field compensates with the lower integrability of the amplitude of vorticity field, and vice versa. We consider the two special cases for (NS) below ($\alpha = 2$ case). First, let $p_2 = r_2 = 2$. Then, we know that the Leray-Hopf weak solution ω satisfies $\int_0^T \|\omega(t)\|^2_{L^2} dt < \infty$, and the condition of the above theorem becomes $\xi(x,t) \in L^{r_1}(0,T; \dot{\mathcal{F}}^s_{p_1,q})$, $3/p_1 + 2/r_1 \le s - 1/2$. Comparing this with Beirão da Veiga-Berselli's result in [3], we find the natural identification of the function $g(x,t)$ as the direction field $\xi(x,t)$. Moreover, since we allow any finite number for q in $(\frac{3}{3-s}, \infty]$, not necessarily infinity, the above is a generalization of Beirão da Veiga's result. Secondly, we observe that in the case $p_1 = r_1 = \infty$, and $s \in (0, 1/2]$ the condition of the above theorem reduces to $\xi \in \dot{\mathcal{F}}^s_{\infty,q}$, $\omega \in L^{r_2}(0,T; L^{p_2}(\mathbb{R}^3))$ with $3/p_2 + 2/r_2 \le s+2$. In the special case $q = \infty$, since $\dot{\mathcal{F}}^s_{\infty,\infty} \cong C^s$, and and by elementary geometry we have $|\sin\theta(x,y,t)| \le |\xi(x+y,t) - \xi(x,t)|$, it follows that $|\sin\theta(x,y,t)| \le C|y|^s$ if $\xi \in L^\infty(0,T; \dot{\mathcal{F}}^s_{\infty,\infty})$. Hence, we find that Beirão da Veiga's result in [4] is a special case for $q = \infty, r_2 = 2$ of Theorem 3.1. The proof of Theorem 3.1 uses the following:

Theorem 3.2. *Let ω =curl v satisfy $\omega \in L^r(0,T; L^p(\mathbb{R}^3))$ with $3/p + \alpha/r \le \alpha$, where $3/\alpha < p \le \infty$. Then, there is no singularity up to T.*

The proof is in [24]. The above theorem says quantitatively that for the regularity of solutions of weaker dissipative term we need higher integrability of the vorticity, and vice versa. For $\alpha = 2$ it reduces to Beirão da Veiga's criterion, while for $p = \infty$, $r = 1$ it reduces to the Beale-Kato-Majda's criterion.

3.2. The case of Euler's equations

Here we use the notion of particle trajectory $X(a,t)$, defined by

$$\frac{dX(a,t)}{dt} = v(X(a,t),t) \quad ; \quad X(a,0) = a \in \mathbb{R}^3,$$

where $v(x,t)$ is the classical solution of the Euler equations. We recall the fact that the mapping $a \mapsto X(a,t)$ is a volume preserving diffeomorphism. Let us denote

$$\Omega_0 = \{x \in \mathbb{R}^3 \,|\, \omega_0(x) \neq 0\}, \quad \Omega_t = X(\Omega_0, t).$$

Then we have the following theorem.

Theorem 3.3. *Let $v(x,t)$ be the local classical solution to the Euler system with initial data $v_0 \in H^m(\mathbb{R}^3)$, $m > 5/2$, and $\omega(x,t)$ =curl $v(x,t)$. We assume $\Omega_0 \neq \emptyset$. Then, the solution can be continued up to $T < \infty$ as the classical solution, namely $v(t) \in C([0,T]; H^m(\mathbb{R}^3))$, if there exists $p, p', q, q', s, r_1, r_2, r_3$ satisfying the following conditions,*

$$\frac{1}{p} + \frac{1}{p'} = 1, \qquad \frac{1}{q} + \frac{1}{q'} = 1,$$

and

$$\frac{1}{r_1} + \frac{p'}{r_2}\left(1 - \frac{sq'}{3}\right) + \frac{1}{r_3}\left\{1 - p'\left(1 - \frac{sq'}{3}\right)\right\} = 1$$

with

$$0 < s < 1, \qquad 1 \le \frac{3}{sq'} < p \le \infty, \qquad 1 \le q \le \infty,$$

and

$$r_1 \in [1,\infty],\ r_2 \in \left[p'\left(1 - \frac{sq'}{3}\right), \infty\right],\ r_3 \in \left[1 - p'\left(1 - \frac{sq'}{3}\right), \infty\right]$$

such that for direction field $\xi(x,t)$, and the magnitude of vorticity $|\omega(x,t)|$ the followings hold;

$$\int_0^T \|\xi(t)\|^{r_1}_{\dot{\mathcal{F}}^s_{\infty,q}(\Omega_t)} dt < \infty,$$

and

$$\int_0^T \|\omega(t)\|^{r_2}_{L^{pq'}(\Omega_t)} dt + \int_0^T \|\omega(t)\|^{r_3}_{L^{q'}(\Omega_t)} dt < \infty.$$

For the proof we refer [20]. Let us consider the special case of $p = \infty, q = 1$. In this case the conditions of the above theorem are satisfied if

$$\xi(x,t) \in L^{r_1}(0,T; C^s(\mathbb{R}^3)),$$

$$\omega(x,t) \in L^{r_2}(0,T; L^\infty(\mathbb{R}^3)) \cap L^{r_3}(0,T; L^\infty(\mathbb{R}^3)).$$

with

$$\frac{1}{r_1} + \frac{1}{r_2}\left(1 - \frac{s}{3}\right) + \frac{s}{3r_3} = 1.$$

In order to understand these conditions more intuitively we formally pass $s \to 0$, and choose $r_1 = \infty$ and $r_2 = r_3 = 1$, then we find that the above conditions reduce to the Beale-Kato-Majda's condition, since the condition $\xi(x,t) \in L^\infty(0,T; C^0(\mathbb{R}^3)) \cong L^\infty((0,T) \times \mathbb{R}^3)$ is obviously satisfied due to the fact that $|\xi(x,t)| \equiv 1$. The other case of interest is $q' = 3/s$, where the conditions are satisfied if

$$\xi(x,t) \in L^{r_1}(0,T; \dot{\mathcal{F}}^s_{\infty, \frac{3}{3-s}}(\mathbb{R}^3)), \qquad |\omega(x,t)| \in L^{r_2}(0,T; L^{\frac{3}{s}}(\mathbb{R}^3)).$$

with $1/r_1 + 1/r_2 = 1$. This shows explicitly the mutual compensation between the regularity of the direction field and the integrability of the vorticity magnitude in order to control regularity/singularity of solutions of the Euler equations.

3.3. Blow-up criterion in terms of reduced number of components of the vorticity

As another application of the depletion structure of the vortex stretching term of the Euler equations in the vorticity formulation we can have the blow-up criterion in terms of the reduced number of components of vorticity. The version of the Navier-Stokes equations is proved by Choe-Chae in [25](see also [67]). For the case of velocity component reduction instead of the vorticity there is a study by Neustupa-Penel in [79]. The following is the version for the Euler system.

Theorem 3.4. *Let $m > 5/2$. Suppose $v \in C([0,T_1); H^m(\mathbb{R}^3))$ is the local classical solution (1.1)-(1.3) for some $T_1 > 0$, corresponding to the initial data $v_0 \in H^m(\mathbb{R}^3)$, and $\omega = curl\ v$ is its vorticity. We decompose $\omega = \tilde{\omega} + \omega^3 e_3$, where $\tilde{\omega} = \omega^1 e_1 + \omega^2 e_2$, and $\{e_1, e_2, e_3\}$ is the canonical basis of $\mathbb{R}^3$. Then,*

$$\limsup_{t \nearrow T_*} \|v(t)\|_{H^m} = \infty \Leftrightarrow \int_0^{T_*} \|\tilde{\omega}(t)\|_{\dot{B}^0_{\infty,1}} dt = \infty. \tag{3.1}$$

For the proof we refer [15]. For plane flows(solution of the 2D Euler equations) $\tilde{\omega} \equiv 0$. Hence, as a trivial corollary of the above theorem we obtain the global regularity for the 2-D Euler equations. It would be interesting to improve the above result by replacing $\|\tilde{\omega}\|_{\dot{B}^0_{\infty,1}}$ by $\|\tilde{\omega}\|_{L^\infty}$, or even $\|\tilde{\omega}\|_{\dot{B}^0_{\infty,\infty}}$. For the 3D Navier-Stokes case it was possible to obtain the Serrin type of regularity criterion by $\|\tilde{\omega}\|_{L^{p,q}_T}$, where $L^{p,q}_T = L^p(0, T : L^q(\mathbb{R}^3))$, $2/p + 3/q = 1$ is the scale invariant space for the 3D Navier-Stokes equations([24]). Next, we consider the axisymmetric 3D Euler equations. By the axisymmetric flow we mean the velocity field $v(r, x_3, t)$ has the representation:

$$v(r, x_3, t) = v^r(r, x_3, t)e_r + v^\theta(r, x_3, t)e_\theta + v^3(r, x_3, t)e_3,$$

where $r = \sqrt{x_1^2 + x_2^2}$, and

$$e_r = (\frac{x_1}{r}, \frac{x_2}{r}, 0), \quad e_\theta = (-\frac{x_2}{r}, \frac{x_1}{r}, 0), \quad e_3 = (0, 0, 1).$$

The vorticity $\omega = \text{curl } v$ is computed, $\omega = \omega^r e_r + \omega^\theta e_\theta + \omega^3 e_3$, where $\omega^r = -\partial_{x_3} v^\theta, \omega^\theta = \partial_{x_3} v^r - \partial_r v^3, \omega^3 = \frac{1}{r}\partial_r(r v^\theta)$. We use the notations, $\tilde{v} = v^r e_r + v^3 e_3, \tilde{\omega} = \omega^r e_r + \omega^3 e_3$, and $\omega = \tilde{\omega} + \vec{\omega}_\theta$ with $\vec{\omega}_\theta = \omega^\theta e_\theta$. The Euler equations for the axisymmetric solution are

$$\begin{cases} \dfrac{\partial v^r}{\partial t} + (\tilde{v} \cdot \tilde{\nabla}) v^r = -\dfrac{\partial p}{\partial r}, \\ \dfrac{\partial v^\theta}{\partial t} + (\tilde{v} \cdot \tilde{\nabla}) v^\theta = -\dfrac{v^r v^\theta}{r}, \\ \dfrac{\partial v^3}{\partial t} + (\tilde{v} \cdot \tilde{\nabla}) v^3 = -\dfrac{\partial p}{\partial x_3} \end{cases}$$

with $\text{div } \tilde{v} = 0$, $v(r, x_3, 0) = v_0(r, x_3)$, where we set $\tilde{\nabla} = e_r \frac{\partial}{\partial r} + e_3 \frac{\partial}{\partial x_3}$. For the axisymmtric flow we have control of the blow-up by one component of vorticity.

Theorem 3.5. *Let v be the local classical axisymmetric solution of the 3-D Euler equations, corresponding to an axisymmetric initial data $v_0 \in H^m(\mathbb{R}^3)$. We decompose $\omega = \tilde{\omega} + \vec{\omega}_\theta$, where $\tilde{\omega} = \omega^r e_r + \omega^3 e_3$ and $\vec{\omega}_\theta = \omega^\theta e_\theta$. Then,*

$$\limsup_{t \nearrow T} \|v(t)\|_{H^m} = \infty \Leftrightarrow \int_0^T \|\vec{\omega}_\theta(t)\|_{\dot{B}^0_{\infty,1}} dt = \infty.$$

For the proof we refer [15]. We note that for the axisymmetric 3-D Navier-Stokes equations with swirl it is possible to control the regularity only by $\|\vec{\omega}_\theta\|_{L^{p,q}_T}$ with $2/p + 3/q = 1$([29]; see also[81]).

§4. Dynamics of the Deformation Tensor

Given velocity field v, we introduce the 3×3 matrices $V = (V_{ij})$, $P = (P_{ij})$ defined by

$$V_{ij} = \frac{\partial v_j}{\partial x_i}, \quad P_{ij} = \frac{\partial^2 p}{\partial x_i \partial x_j}.$$

Let us decompose the gradient matrix into the symmetric and the anti-symmetric part,

$$V_{ij} = \frac{1}{2}(V_{ij} + V_{ji}) + \frac{1}{2}(V_{ij} - V_{ji}) := S_{ij} + A_{ij}.$$

The symmetric part S_{ij} is the deformation tensor, while the antisymmetric part A_{ij} is related to the vorticity by (we use the summation convention)

$$A_{ij} = \frac{1}{2}\varepsilon_{ijk}\omega_k; \quad \omega_i = \varepsilon_{ijk}A_{jk}.$$

4.1. The spectral dynamics

The spectral dynamics approach in the fluid mechanics was initiated by Liu-Tadmor([74]). They investigated the pointwise dynamics of the eigenvalues of the matrix (V_{ij}) for a specific model system for the Euler equations, called the 'restricted Euler equation', where a non-local term arising from the pressure was replaced by an artificial local term. In this section we briefly review recent approach by the author of this article in [18], where the spectral dynamics of the deformation tensor S_{ij} was studied for the full 3D Euler system. We note the related approaches applied to the Navier-Stokes equations in [80]. In the Euler equations, taking partial derivatives $\partial/\partial x_k$ of (1.1) yields

$$\frac{DV}{Dt} = -V^2 - P, \quad \frac{D}{Dt} = \frac{\partial}{\partial t} + (v \cdot \nabla)v. \tag{4.1}$$

Using the decomposition $V = S + A$, the symmetric part of (4.1) can be written as

$$\frac{DS}{Dt} = -S^2 - A^2 - P,$$

from which, substituting the formula $A_{ij} = \frac{1}{2}\varepsilon_{ijk}\omega_k$, we derive

$$\frac{DS_{ij}}{Dt} = -S_{ik}S_{kj} + \frac{1}{4}(|\omega|^2\delta_{ij} - \omega_i\omega_j) - P_{ij}$$

Using this formula, we compute

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n} S_{ij}S_{ij}dx &= \int_{\mathbb{R}^n} S_{ij}\frac{DS_{ij}}{Dt}dx \\ &= -\int_{\mathbb{R}^n} S_{ik}S_{kj}S_{ij}dx - \frac{1}{4}\int_{\mathbb{R}^n}\omega_i S_{ij}\omega_j dx \\ &\qquad +\frac{1}{4}\int_{\mathbb{R}^n}|\omega|^2 S_{ii}dx + \int_{\mathbb{R}^n} P_{ij}S_{ij}dx \\ &= -\int_{\mathbb{R}^n} S_{ik}S_{kj}S_{ij}dx - \frac{1}{8}\frac{d}{dt}\int_{\mathbb{R}^n}|\omega|^2 dx,\end{aligned}$$

where we used the L^2-version of the vorticity equation,

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n}|\omega|^2 dx = \int_{\mathbb{R}^n}\omega_i S_{ij}\omega_j dx$$

We note

$$\begin{aligned}\int_{\mathbb{R}^n}|\omega|^2 dx &= \int_{\mathbb{R}^n}|\nabla v|^2 dx = \int_{\mathbb{R}^n} V_{ij}V_{ij}dx = \int_{\mathbb{R}^n}(S_{ij}+A_{ij})(S_{ij}+A_{ij})dx \\ &= \int_{\mathbb{R}^n} S_{ij}S_{ij}dx + \int_{\mathbb{R}^n} A_{ij}A_{ij}dx = \int_{\mathbb{R}^n} S_{ij}S_{ij}dx + \frac{1}{2}\int_{\mathbb{R}^n}|\omega|^2 dx.\end{aligned}$$

Hence,

$$\int_{\mathbb{R}^n} S_{ij}S_{ij}dx = \frac{1}{2}\int_{\mathbb{R}^n}|\omega|^2 dx$$

Substituting this into previous one, we obtain that

$$\frac{d}{dt}\int_{\mathbb{R}^n} S_{ij}S_{ij}dx = -\frac{4}{3}\int_{\mathbb{R}^n} S_{ik}S_{kj}S_{ij}dx$$

Let $\lambda_1, \lambda_2, \lambda_3$ be the eigenvalues of the deformation tensor S_{ij}, then we have just derived the following equation holds.

$$\frac{d}{dt}\int_{\mathbb{R}^n}(\lambda_1^2+\lambda_2^2+\lambda_3^2)dx = -\frac{4}{3}\int_{\mathbb{R}^n}(\lambda_1^3+\lambda_2^3+\lambda_3^3)dx.$$

Using the fact $\lambda_1+\lambda_2+\lambda_3=0$, we easily obtain $\lambda_1^3+\lambda_2^3+\lambda_3^3 = 3\lambda_1\lambda_2\lambda_3$. In conclusion we have just derived,

$$\frac{d}{dt}\int_{\mathbb{R}^n}(\lambda_1^2+\lambda_2^2+\lambda_3^2)dx = -4\int_{\mathbb{R}^n}\lambda_1\lambda_2\lambda_3 dx. \tag{4.2}$$

Using this formula, we can deduce the following theorem.

Theorem 4.1. *We set $\Omega = \mathbb{R}^3$ or $\mathbb{T}^3$. Let $v(x,t)$ be the classical solution of the 3D Euler equations in Ω, and $\lambda_1(x,t) \geq \lambda_2(x,t) \geq \lambda_3(x,t)$ be the eigenvalues of the deformation tensor S_{ij}. We denote $\lambda_2^+(x,t) = \max\{\lambda_2(x,t), 0\}$, and $\lambda_2^-(x,t) = \min\{\lambda_2(x,t), 0\}$. Then, the following (a priori) estimates hold.*

$$\exp\left[\int_0^t \left(\frac{1}{2}\inf_{x\in\Omega}\lambda_2^+(x,t) - \sup_{x\in\Omega}|\lambda_2^-(x,t)|\right)dt\right] \leq \frac{\|\omega(t)\|_{L^2}}{\|\omega_0\|_{L^2}}$$
$$\leq \exp\left[\int_0^t \left(\sup_{x\in\Omega}\lambda_2^+(x,t) - \frac{1}{2}\inf_{x\in\Omega}|\lambda_2^-(x,t)|\right)dt\right]$$

For the proof we refer [18]. From the previously known Constantin's equation([38]),

$$\frac{D|\omega|}{Dt} = \alpha|\omega|, \qquad \alpha(x,t) = \frac{\omega \cdot S\omega}{|\omega|^2},$$

we also have

$$\exp\left(\int_0^t \inf_{x\in\Omega}\alpha(x,s)ds\right) \leq \frac{\|\omega(t)\|_{L^2}}{\|\omega_0\|_{L^2}} \leq \exp\left(\int_0^t \sup_{x\in\Omega}\alpha(x,s)ds\right).$$

Combining this with the estimate for the Rayleigh quotient, $\lambda_3 \leq \alpha \leq \lambda_1$, we derive

$$\exp\left(\int_0^t \inf_{x\in\Omega}\lambda_3(x,s)ds\right) \leq \frac{\|\omega(t)\|_{L^2}}{\|\omega_0\|_{L^2}} \leq \exp\left(\int_0^t \sup_{x\in\Omega}\lambda_1(x,s)ds\right).$$

Notice that Theorem 4.1 contains only λ_2 to control the growth/decay, and could be regarded as a refinement.

Corollary 4.1. *Let $v_0 \in H^m$, $m > 5/2$ be given, and $\lambda_1(x,t)$,$\lambda_2(x,t)$, $\lambda_3(x,t)$ are as in Theorem 4.1. Then,*

$$\limsup_{t\to T_*}\|\omega(t)\|_{L^2} = \infty \Rightarrow \int_0^{T_*} \|\lambda_2^+(t)\|_{L^\infty}dt = \infty.$$

Proof. From the estimate of Theorem 4.1, we have immediately

$$\|\omega(t)\|_{L^2} \leq \|\omega_0\|_{L^2}\exp\left(\int_0^t \|\lambda_2^+(s)\|_{L^\infty}ds\right).$$

This implies the corollary. □

Let $f = (f_1, f_2, f_3)$, div $f = 0$. We denote by $\lambda_2(f)$ the second largest eigenvalue of the symmetric part of ∇f. We define admissible classes $\mathcal{A}_\pm$ defined by $\mathcal{A}_+ = \{f \in H^m(\Omega) \mid \inf_{x\in\Omega} \lambda_2(f)(x) > 0\}$, and $\mathcal{A}_- = \{f \in H^m(\Omega) \mid \sup_{x\in\Omega} \lambda_2(f)(x) < 0\}$. Given $v_0 \in H^m$, let $T_*(v_0)$ be the maximal time of unique existence of solution in H^m for the Euler system. Let $S_t : v_0 \mapsto v(t)$, from H^m into H^m, be the solution operator of the Euler system. Given $f \in \mathcal{A}_+$, we define the first zero touching time of $\lambda_2(f)$ as

$$T(f) = \inf\{t \in (0, T_*(v_0)) \mid \exists x \in \Omega \text{ such that } \lambda_2(S_t f)(x) < 0\}.$$

Similarly for $f \in \mathcal{A}_-$, we define

$$T(f) = \inf\{t \in (0, T_*(v_0)) \mid \exists x \in \Omega \text{ such that } \lambda_2(S_t f)(x) > 0\}.$$

Observing that $v_0 \in \mathcal{A}_+$(resp. $\mathcal{A}_-$) implies $\lambda_2^- = 0, \lambda_2^+ = \lambda_2$(resp. $\lambda_2^+ = 0, \lambda_2^- = \lambda_2$) on $\Omega \times (0, T(v_0))$, we have the following corollary of Theorem 4.1.

Corollary 4.2. *Let $v_0 \in \mathcal{A}_\pm$ be given. We set $\lambda_1(x,t) \geq \lambda_2(x,t) \geq \lambda_3(x,t)$ as the eigenvalues of the deformation tensor associated with $v(x,t) = (S_t v_0)(x)$ defined $t \in (0, T(v_0))$. Then, for all $t \in (0, T(v_0))$ we have the following estimates:*
(i) If $v_0 \in \mathcal{A}_+$, then

$$\exp\left(\frac{1}{2}\int_0^t \inf_{x\in\Omega} |\lambda_2(x,s)| ds\right) \leq \frac{\|\omega(t)\|_{L^2}}{\|\omega_0\|_{L^2}} \leq \exp\left(\int_0^t \sup_{x\in\Omega} |\lambda_2(x,s)| ds\right).$$

(ii) If $v_0 \in \mathcal{A}_-$, then

$$\exp\left(-\int_0^t \sup_{x\in\Omega} |\lambda_2(x,s)| ds\right) \leq \frac{\|\omega(t)\|_{L^2}}{\|\omega_0\|_{L^2}} \leq \exp\left(-\frac{1}{2}\int_0^t \inf_{x\in\Omega} |\lambda_2(x,s)| ds\right).$$

Using the spectral equation (4.2), we can also prove some decay in time estimates for some ratios of eigenvalues.

Theorem 4.2. *Here we consider the three dimensional periodic domain $\mathbb{T}^3$. Let $v_0 \in \mathcal{A}_\pm$ be given. We define*

$$\varepsilon(x,t) = \frac{|\lambda_2(x,t)|}{\lambda(x,t)} \quad \forall (x,t) \in \mathbb{T}^3 \times (0, T(v_0)),$$

where we set

$$\lambda(x,t) = \begin{cases} \lambda_1(x,t) & \text{if } \ v_0 \in \mathcal{A}_+ \\ -\lambda_3(x,t) & \text{if } \ v_0 \in \mathcal{A}_-. \end{cases}$$

Then, there exists a constant $C = C(v_0, |\mathbb{T}^3|)$, *with* $|\mathbb{T}^3|$ *denoting the volume of the box* $\mathbb{T}^3$, *such that*

$$\inf_{(x,s)\in\mathbb{T}^3\times(0,t)} \varepsilon(x,s) < \frac{C}{\sqrt{t}} \quad \forall t \in (0, T(v_0)).$$

For the proof we refer [18].

4.2. The points leading to possible singularities

The blow-up criteria introduced in the previous sections are independent on the initial data. Here we present recent results by the author of this article(see [19] for more details), where the blowing-up condition depends on the initial data and points on the domain. Let $\omega(x,t) \neq 0$ at (x,t), and $\xi(x,t) = \omega(x,t)/|\omega(x,t)|$ be the direction field of the vorticity. At such point (x,t) we define the scalar fields $\sigma(x,t) = \sum_{i,j=1}^3 \xi_i(x,t)S_{ij}(x,t)\xi_j(x,t)$, and $\rho(x,t) = \sum_{i,j=1}^3 \xi_i(x,t)P_{ij}(x,t)\xi_j(x,t)$, where $S(x,t)$ and $P(x,t)$ are the deformation tensor and the Hessian of the pressure respectively, associated with the flow. As will be seen below the quantities $\sigma(x,t)$ and $\rho(x,t)$ have prominent roles in leading to finite time singularity of solution to the Euler system (1.1)-(1.3). We denote $\sigma_0(x) = \sigma(x,0)$, $\rho_0(x) = \rho(x,0)$. The function $X(a,t)$, which is the particle trajectory introduced before. Then, we have

Theorem 4.3. *Given* $v_0 \in H^m(\Omega)$, $m > 5/2$ *with div* $v_0 = 0$, *let us define the set* $\mathcal{S}$ *by*

$$\mathcal{S} = \left\{ a \in \Omega \,\middle|\, \omega_0(a) \neq 0,\ \sigma_0(a) > 0,\ \text{and } \rho_0(a) < -\sigma_0^2(a) \right\}.$$

Suppose there exists $a \in \mathcal{S}$, *and* $\varepsilon \in (0,1]$ *satisfying,*

$$\sup_{0\leq t<\frac{1}{\varepsilon\sigma_0(a)}} \sqrt{(2\sigma^2+\rho)_+((X(a,t),t))} \leq (1-\varepsilon)\sigma_0(a),$$

where $(h)_+ = \max\{h, 0\}$. *Then the solution of (1.1)-(1.3) with the initial data* v_0 *blows up in finite time. Moreover, the blow-up time* T_* *is estimated from above by*

$$T_* \leq \frac{1}{\sup_{a\in\mathcal{S}} \varepsilon\sigma_0(a)}.$$

The set $\mathcal{S}$ of candidate points in Ω leading the finite time blow-up is specified by functions σ_0, ρ_0 which involve nonlocal operators. Recall that S contains singular integral operators acting on the vorticity ω(see e.g. [76]), and $P_{ij} = -R_iR_j\mathrm{Tr}[(\nabla v)^2]$, where R_i, $i = 1,2,3$, are the three dimensional Riesz transforms. We note that the similar type of specification of the corresponding set is given for the one dimensional model equation studied in [36], where the nonlocal operator is the Hilbert transform. In particular, as will be seen in the proof below, the condition $\sigma_0(x) > 0$ corresponds to the initial amplification of vorticity at the point x. We also note that $\sigma_0(-v_0) = -\sigma_0(v_0)$. Hence, by reversing the direction of initial velocity we can change the sign of σ_0. We can easily check that $\mathcal{S} = \emptyset$ for the 2D Euler equations. We note that for the well-known example of Taylor-Green vortex $\mathcal{S} \neq \emptyset$(see [19]). We note that $\sigma_0(\alpha) > \sqrt{[\rho_0(\alpha) + 2\sigma_0^2(\alpha)]_+}$ if and only if $\sigma_0(\alpha) > 0$ and $\rho_0(\alpha) < -\sigma_0^2(\alpha)$. Hence, as a special case of the above theorem we can choose

$$\varepsilon = 1 - \frac{\sqrt{[2\sigma_0^2(a) + \rho_0(a)]_+}}{\sigma_0(a)},$$

and we have the following corollary.

Corollary 4.3. *Let v_0, $\mathcal{S}$ be as in Theorem 2.1. Suppose there exists $a \in \mathcal{S}$ such that*

$$[(2\sigma^2 + \rho)(X(a,t),t)]_+ \leq [2\sigma_0^2(a) + \rho_0(a)]_+$$

for all $t \in [0, \frac{1}{\sigma_0(a) - h_0(a)})$, where $h_0(a) = \sqrt{[2\sigma_0^2(a) + \rho_0(a)]_+}$. Then the solution of (1.1)-(1.3) with the initial data v_0 blows up along the trajectory $\{X(a,t)\}$ in finite time, T_* with the upper estimate

$$T_* \leq \frac{1}{\sup_{a \in \mathcal{S}}[\sigma_0(a) - \sqrt{[2\sigma_0^2(a) + \rho_0(a)]_+}}$$

§5. Model Problems

5.1. The quasi-geostrophic equations

The 2D quasi-geostrophic equation(QG) models the dynamics of the mixture of cold and hot air and the fronts between them.

$$(QG) \begin{cases} \theta_t + (u \cdot \nabla)\theta = 0, \\ u = -\nabla^\perp(-\Delta)^{-\frac{1}{2}}\theta \\ \left(= \nabla^\perp \displaystyle\int_{\mathbb{R}^2} \frac{\theta(y,t)}{|x-y|} dy \right), \\ \theta(x,0) = \theta_0(x), \end{cases}$$

where $\nabla^\perp = (-\partial_2, \partial_1)$. Here, $\theta(x,t)$ represents the temperature of the air at $(x,t) \in \mathbb{R}^2 \times \mathbb{R}_+$. Taking $\nabla^\perp$ to the first equation of (QG), we obtain

$$\left(\frac{\partial}{\partial t} + u\cdot\nabla\right)\nabla^\perp\theta = (\nabla^\perp\theta\cdot\nabla)u,$$

where

$$u = \int_{\mathbb{R}^2} \frac{\nabla^\perp\theta(y,t)}{|x-y|}dy.$$

This is exactly the vorticity formulation of 3D Euler equation if we identify $\nabla^\perp\theta \Longleftrightarrow \omega$. The local existence can be proved easily following T. Kato's classical method. The blow-up criterion[40] is

$$\limsup_{t\nearrow T_*} \|\theta(t)\|_{H^s} = \infty \quad \text{if and only if} \quad \int_0^{T_*} \|\nabla^\perp\theta(s)\|_{L^\infty} ds = \infty.$$

This result has been refined, using the Triebel-Lizorkin spaces [14]. The question of finite time singularity/global regularity is still open, although there are some of interesting partial results in [42, 45, 46, 47, 48, 49](see also [41, 36, 43, 30, 95, 96, 97] for the studies of the viscous quasi-geostrophic equation).

5.2. One dimensional QG model

In this section we present the results in [26](see also [44] for the related result). We begin with

$$(QG)\begin{cases} \theta_t + (u\cdot\nabla)\theta = 0, \\ u = -\nabla^\perp(-\Delta)^{-\frac{1}{2}}\theta, \\ \theta(x,0) = \theta_0(x), \end{cases}$$

Note that

$$u = -R^\perp\theta = (-R_2\theta, R_1\theta),$$

where R_j, $j = 1,2$, for the two dimensional Riesz transform(see e.g. [85]) defined by

$$R_j(\theta)(x,t) = \frac{1}{2\pi}PV\int_{\mathbb{R}^2} \frac{(x_j-y_j)\theta(y,t)}{|x-y|^3}dy.$$

We can rewrite $\theta_t + \mathrm{div}\,[(\mathrm{R}^\perp\theta)\theta] = 0$, since $\mathrm{div}(\mathrm{R}^\perp\theta) = 0$. To construct the one dimensional model we replace $R^\perp(\cdot) \Rightarrow H(\cdot)$ and div $(\cdot) \Rightarrow \partial_x$ to obtain $\theta_t + (H(\theta)\theta)_x = 0$. Defining the complex valued function $z(x,t) = H\theta(x,t) + i\theta(x,t)$, and following the argument due to Constantin-Lax-Majda([39]), we find our equation is the imaginary part of complex Burgers' equation, $z_t + zz_x = 0$. Contrary to the real Burgers' equation The characteristics method does not work here for the proof of finite time singularity for smooth solutions. By different method we can prove the finite time blow-up for the generic initial data as follows.

Theorem 5.1. *Given a periodic non-constant initial data* $\theta_0 \in C^1([-\pi,\pi])$ *such that* $\int_{-\pi}^{\pi}\theta_0(x)dx = 0$, *there is no* $C^1([-\pi,\pi]\times[0,\infty))$ *periodic solution to the model equation.*

For the proof we refer [26]. Here we give an idea of construction of an explicit blowing up solution. We begin with the complex Burgers equation:

$$z_t + zz_x = 0, \quad z = u + i\theta$$

with $u(x,t) \equiv H\theta(x,t)$. Expanding it to real and imaginary parts, we obtain the system:

$$\begin{cases} u_t + uu_x - \theta\theta_x = 0, \\ \theta_t + u\theta_x + \theta u_x = 0 \end{cases}$$

In order to perform the *hodograph transform* we consider $x(u,\theta)$ and $t(u,\theta)$. We have,

$$\begin{aligned} u_x &= Jt_\theta\ , \quad \theta_x = -Jt_u\ , \\ u_t &= -Jx_\theta\ , \quad \theta_t = Jx_u\ , \end{aligned}$$

where $J = (x_u t_\theta - x_\theta t_u)^{-1}$. By direct substitution we obtain,

$$\begin{cases} -x_\theta + ut_\theta + \theta t_u = 0\ , \\ x_u - ut_u + \theta t_\theta = 0\ . \end{cases}$$

as far as $J^{-1} \neq 0$. This system can be written more compactly in the form:

$$\begin{cases} -(x-tu)_\theta + (t\theta)_u = 0 \\ (x-tu)_u + (t\theta)_\theta = 0, \end{cases}$$

which leads to the following Cauchy-Riemann system:

$$\xi_u = \eta_\theta, \quad \xi_\theta = -\eta_u$$

for

$$\begin{cases} \eta(u,\theta) \equiv x(u,\theta) - t(u,\theta)u \\ \xi(u,\theta) \equiv t(u,\theta)\theta \end{cases}.$$

Hence, $f(z) = \xi(u,\theta) + i\eta(u,\theta)$ with $z = u + i\theta$ is an analytic function. Choosing $f(z) = ln\, z$, we find,

$$\begin{cases} t\theta = \ln\sqrt{u^2+\theta^2} \\ x - tu = \arctan\dfrac{\theta}{u}, \end{cases} \tag{5.1}$$

which corresponds to the initial data,$z(x,0) = \cos x + i \sin x$. The relation 5.1 defines implicitly the real and imaginary parts $(u(x,t), \theta(x,t))$ of the solution. Removing θ from the system, we obtain $tu \tan(x - tu) = \ln |u/\cos(x - tu)|$, which defines $u(x,t)$ implicitly. By elementary computations we find both u_x and θ_x blow up at $t = e^{-1}$.

5.3. The Boussinesq equations

The system of (viscous) Boussinesq equations is

$$(B)\begin{cases} \dfrac{\partial v}{\partial t} + (v \cdot \nabla)v = -\nabla p + \nu \Delta v + \theta e_1, \\ \dfrac{\partial \theta}{\partial t} + (v \cdot \nabla)\theta = \kappa \Delta \theta \\ \text{div } v = 0, \\ v(x,0) = v_0(x), \ \theta(x,0) = \theta_0(x) \end{cases}$$

where $v = (v_1, v_2)$, $v_j = v_j(x,t), j = 1,2$, $e_1 = (1,0)$, $\nu, \kappa \geq 0$. The case $\nu = \kappa > 0 (= 0)$ is known to be essentially equivalent to the 3D axisymmetric Navier-Stokes(Euler) equations off the axis of symmetry. In the case $\nu = \kappa > 0$, the global regularity is well-known, while in the fully inviscid case of $\kappa = \nu = 0$ the problem of regularity/finite time singularity is completely open. In this case the local existence and blow-up criterion is obtained [51, 26, 29, 87]. Here we consider the case $\kappa = 0$, $\nu > 0$ or $\kappa > 0$, $\nu = 0$(the 'partial viscosity' case). We can view these cases as the system 'between the Navier-Stokes and the Euler equations'. In the article [77], H. K. Moffatt asked the question of existence of singularity in the case $\kappa = 0, \nu > 0$ and its possible development in the limit $\kappa \to 0$(vanishing diffusivity limit), and listed this as one of the 21th century problems. For this problem there is a partial result excluding the "squirt singularities" ([49]). The author of this article recently proved that actually there is no singularity at all in this case, and the strong convergence happens in the vanishing diffusivity limit. Actually we have:

- **Global regularity** for both of the cases($\kappa = 0$, $\nu > 0$, and $\kappa > 0$, $\nu = 0$)
- **Strong convergence** for vanishing viscosity($\nu \to 0$ with $\kappa > 0$ fixed), and vanishing diffusivity($\kappa \to 0$ with $\nu > 0$ fixed) limits.

For the proof we refer [16]. Our proof shows that both of the partial viscosity cases are the 'critical' ones in the sense that the classical method of a priori estimates 'barely' works. We note that the global regularity part of the above is also obtained independently in [58].

5.4. The modified Euler equations

We are first concerned on the system in a periodic domain $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$.

$$(\mathrm{P}_1)\begin{cases} \dfrac{\partial u}{\partial t} + (u\cdot\nabla)u = -\nabla q + (1+\varepsilon)\|\nabla u(t)\|_{L^\infty} u, \\ \quad \operatorname{div}\ u = 0, \\ u(x,0) = u_0(x), \end{cases}$$

where $u = (u_1, \cdots, u_n)$, $u_j = u_j(x,t)$, $j = 1, \cdots, n,$, $q(x,t)$ is a scalar, and $\varepsilon > 0$ is a constant. We note that the system (P_1) has the same scaling property as the Euler system.

Theorem 5.2. *Given sufficiently smooth u_0 with div $u_0 = 0$, the following statements hold true for (P_1).*

(i) (finite time blow-up) *If $\|\omega_0\|_{L^\infty} \neq 0$, then there exists time $t_* \leq \frac{1}{\varepsilon\|\omega_0\|_{L^\infty}}$ such that the solution actually blows up at t_*.*

(ii) (relation to the Euler equations) *The relation between the solution $u(x,t)$ of (P_1) and the solution $v(x,t)$ of the Euler system (1.1)-(1.3) is given by $u(x,t) = \varphi'(t)v(x,\varphi(t))$, where*

$$\varphi(t) = \lambda \int_0^t \exp\left[(1+\varepsilon)\int_0^\tau \|\nabla u(s)\|_{L^\infty} ds\right] d\tau.$$

(iii) (blow-up of the Euler system in terms of solutions of (P_1)) *The solution $v(x,t)$ of the Euler system (E) blows up at $T_* < \infty$ if and only if for $t_* := \varphi^{-1}(T_*) < \frac{1}{\varepsilon\|\omega_0\|_{L^\infty}}$ both of the followings hold true: $\int_0^{t_*} \exp\left[(1+\varepsilon)\int_0^\tau \|\nabla u(s)\|_{L^\infty} ds\right] d\tau < \infty$, and $\int_0^{t_*} \exp\left[(2+\varepsilon)\int_0^\tau \|\nabla u(s)\|_{L^\infty} ds\right] d\tau = \infty$.*

Here we present the proof of (i) below. For the other parts we refer [21]. Let $p \in (2,\infty)$. Taking L^2 inner product with $\omega|\omega|^{p-2}$, we obtain

$$\begin{aligned} \frac{1}{p}\frac{d}{dt}\|\omega\|_{L^p}^p &= \int_{\mathbb{R}^n} (\omega\cdot\nabla)u\cdot\omega|\omega|^{p-2}dx + (1+\varepsilon)\|\nabla u\|_{L^\infty}\|\omega\|_{L^p}^p \\ &\geq -\|\nabla u\|_{L^\infty}\|\omega\|_{L^p}^p + (1+\varepsilon)\|\nabla u\|_{L^\infty}\|\omega\|_{L^p}^p \\ &= \varepsilon\|\nabla u\|_{L^\infty}\|\omega\|_{L^p}^p \geq \varepsilon\|\omega\|_{L^\infty}\|\omega\|_{L^p}^p, \end{aligned}$$

where we used the pointwise estimate $|\omega(x,t)| \leq |\nabla u(x,t)|$. Hence $\frac{d}{dt}\|\omega\|_{L^p} \geq \varepsilon\|\omega\|_{L^\infty}\|\omega\|_{L^p}$. Integrating this over $(t, t+h)$, we have

$$\|\omega(t+h)\|_{L^p} - \|\omega(t)\|_{L^p} \geq \varepsilon\int_t^{t+h} \|\omega(s)\|_{L^\infty}\|\omega(s)\|_{L^p} ds.$$

Passing $p \to \infty$, we deduce $\|\omega(t+h)\|_{L^\infty} - \|\omega(t)\|_{L^\infty} \geq \varepsilon \int_t^{t+h} \|\omega(s)\|_{L^\infty}^2 ds$. Dividing both sides by $h > 0$, and passing to $h \to 0$, we obtain the differential inequality $\frac{d}{dt}\|\omega\|_{L^\infty} \geq \varepsilon \|\omega\|_{L^\infty}^2$. Solving the differential inequality, we have blow-up:

$$\|\omega(t)\|_{L^\infty} \geq \frac{\|\omega_0\|_{L^\infty}}{1 - \varepsilon\|\omega_0\|_{L^\infty} t} = \frac{1}{\varepsilon(t_* - t)}, \qquad t_* = \frac{1}{\varepsilon\|\omega_0\|_{L^\infty}}.$$

If, instead, we consider the following problem:

$$(\mathrm{P}_2) \begin{cases} \dfrac{\partial u}{\partial t} + (u \cdot \nabla) u = -\nabla q - (1+\varepsilon)\|\nabla u(t)\|_{L^\infty} u, \\ \quad \text{div } u = 0, \\ u(x,0) = u_0(x), \end{cases}$$

for $\varepsilon \geq 0$, then we can prove(see [21]):

Theorem 5.3. *Given $u_0 \in H^m(\Omega)$, $m > \frac{n}{2} + 2$, then the solution $u(x,t)$ of (P_2) belongs to $C([0,\infty) : H^m(\mathbb{R}^n))$. Moreover, we have the following decay estimate for the vorticity,*

$$\|\omega(t)\|_{L^\infty} \leq \frac{\|\omega_0\|_{L^\infty}}{1 + \varepsilon\|\omega_0\|_{L^\infty} t} \qquad \forall t \in [0, \infty).$$

§6. Nonexistence of the Self-Similar Blow-up

The Euler system (1.1)-(1.3) has scaling property that if (v,p) is a solution, then for any $\lambda > 0$ and $\alpha \in \mathbb{R}$ the functions

$$v^{\lambda,\alpha}(x,t) = \lambda^\alpha v(\lambda x, \lambda^{\alpha+1} t), \quad p^{\lambda,\alpha}(x,t) = \lambda^{2\alpha} p(\lambda x, \lambda^{\alpha+1} t)$$

are also solutions with the initial data $v_0^{\lambda,\alpha}(x) = \lambda^\alpha v_0(\lambda x)$.
In view of this it would be interesting to check if there exists any non-trivial solution $(v^\alpha(x,t), p^\alpha(x,t))$ of the form,

$$\begin{cases} v^\alpha(x,t) = \dfrac{1}{(T_* - t)^{\frac{\alpha}{\alpha+1}}} V\left(\dfrac{x}{(T_* - t)^{\frac{1}{\alpha+1}}}\right), \\ p^\alpha(x,t) = \dfrac{1}{(T_* - t)^{\frac{2\alpha}{\alpha+1}}} P\left(\dfrac{x}{(T_* - t)^{\frac{1}{\alpha+1}}}\right) \end{cases}$$

for $t \in [0, T_*)$ and $\alpha \neq -1$. Substituting this into the Euler equations, we find that (V,P) should be a solution of the system

$$\frac{\alpha}{\alpha+1} V + \frac{1}{\alpha+1}(x \cdot \nabla) V + (V \cdot \nabla) V = -\nabla P, \tag{6.1}$$

$$\text{div } V = 0, \tag{6.2}$$

which could be regarded as the Euler version of the Leray's equations([71]). Now the question of existence of self-similar blow-up is equivalent to that of solution to the system (6.1)-(6.2). For the solution of (6.1)-(6.2), we have the following earlier result in [12]:

Theorem 6.1. *If $V \in H^1(\mathbb{R}^3)$ is a nontrivial(nonzero) classical solution of (6.1)-(6.2) in $\mathbb{R}^3$, then the helicity of V is equal to zero, namely $\int_{\mathbb{R}^3} V \cdot \Omega dx = 0$, where $\Omega = curl V$.*

More recently the author ruled out completely the self-similar singularities assuming some integrability condition for the vorticity([22, 23]).

Theorem 6.2. *There exists no finite time blowing up self-similar solution to the 3D Euler equations of the form described above with $\alpha \neq -1$, if there exists $p_1 > 0$ such that the vorticity $\Omega = curl\ V \in L^p(\mathbb{R}^3)$ for all $p \in (0, p_1)$.*

We recall that the similar problem for the 3D Navier-Stokes equations is settled down in [78, 92].

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