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Estimates of maximal functions by Hausdorff contents in a metric space

Hisako Watanabe

Abstract.

Let M be the Hardy-Littlewood maximal operator in a quasi-metric space X. We give the estimates of Mf with weak type and strong type with respect to the α-Hausdorff content. To do these, we use the dyadic balls introduced by E. Sawyer and R. L. Wheeden.

§1. Introduction

In analysis many operators are dominated by constant multiples of the Hardy-Littlewood maximal operators. In $\mathbf{R}^n$ the maximal function Mf of f is defined by

$$Mf(x) = \sup \frac{1}{|B|} \int_B |f| dx,$$

where the supremum is taken over all balls B containing x and $|B|$ stands for the n-dimensional volume of B.

In 1988 D. R. Adams considered the estimates of the maximal functions with respect to the α-Hausdorff content H_∞^α and proved the following strong type inequality (cf. [1]).

Theorem A. *Let $0 < \alpha < n$. Then there is a constant c such that*

$$\int Mf dH_\infty^\alpha \leq c \int |f| dH_\infty^\alpha.$$

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In this theorem, the integral of a nonnegative function g with respect to H_∞^α is in the sense of Choquet and is defined by

$$\int g dH_\infty^\alpha := \int_0^\infty H_\infty^\alpha(\{x \in \mathbf{R}^n : g(x) > t\})dt.$$

In 1998 J. Orobitg and J. Verdera generalized Theorem A as follows (cf. [5]).

Theorem B. *Let $0 < \alpha < n$. Then, for some constant c depending only on α and n,*

$$\text{(i)}\quad \int (Mf)^p dH_\infty^\alpha \leq c \int |f|^p dH_\infty^\alpha, \quad \alpha/n < p,$$

$$\text{(ii)}\quad H_\infty^\alpha(\{x; Mf(x) > t\}) \leq ct^{-\alpha/n} \int |f|^{\alpha/n} dH_\infty^\alpha.$$

To prove Theorem A and Theorem B, the authors considered the maximal function and the α-Hausdorff content restricted to dyadic cubes. More precisely, let us define $\tilde{M}f$ and $\tilde{H}_\infty^\alpha$ in $\mathbf{R}^n$.

For each x

$$\tilde{M}f(x) := \sup \frac{1}{|Q|} \int_Q |f| dy,$$

where the supremum is taken over all dyadic cubes containing x and for a subset E of $\mathbf{R}^n$

$$\tilde{H}_\infty^\alpha(E) := \inf \sum_{j=1}^\infty l(Q_j)^\alpha,$$

where the infimum is taken over all coverings of E by countable families of dyadic cubes and $l(Q_j)$ stands for the side length of Q_j.

We see that Mf and $H_\infty^\alpha(E)$ are comparable to $\tilde{M}f$ and $\tilde{H}_\infty^\alpha(E)$, respectively. So they used $\tilde{M}$ and $\tilde{H}_\infty^\alpha$ instead of M and H_∞^α.

In [2] D. R. Adams defined a Choquet-Lorentz space $L^{q,p}(H_\infty^\delta)$ of the Lorentz type with respect to the Hausdorff capacity H_∞^δ in $\mathbf{R}^n$ and gave the estimates of the fractional maximal functions of order α in term of $L^{q,p}(H_\infty^\delta)$ (cf. Theorem 7 in [2]).

In this paper we estimate the Hardy-Littlewood maximal functions by Hausdorff contents in a quasi-metric space.

Recall that (X, ρ) is called a quasi-metric space if the mapping ρ from $X \times X$ to $[0, \infty)$ has the following three properties;

(i) $\rho(x, y) = 0$ if and only if $x = y$,
(ii) $\rho(x, y) = \rho(y, x)$ for all $x,\ y \in X$,
(iii) There is a constant $K \geq 1$ such that

$$\rho(x, y) \leq K(\rho(x, z) + \rho(z, y)) \quad \text{for all } x,\ y,\ z \in X. \tag{1.1}$$

In addition, we assume that the diameter of X is finite and set

$$\operatorname{diam} X = R.$$

Let M be the Hardy-Littlewood maximal operator and let H^α_∞ be the α-Hausdorff content. Furthermore we suppose that there are a non-negative Borel measure μ on X and a positive number d such that

$$b_1 r^d \le \mu(B(x,r)) \le b_2 r^d \tag{1.2}$$

for all positive $r \le R$, where

$$B(x,r) := \{y \in X : \rho(x,y) < r\}.$$

In a quasi-metric space there is no dyadic cube. Instead of dyadic cubes E. Sawyer and R. L. Wheeden [6] constructed a family of balls as follows:

Theorem C. *Put $\lambda = K + 2K^2$. Then, for each integer k, there exists a sequence $\{B_j^k\}_j$ $(B_j^k = B(x_{jk}, \lambda^k))$ of balls of radius λ^k having the following properties:*

(i) *Every ball of radius λ^{k-1} is contained in at least one of the balls B_j^k,*

(ii) *$\sum_j \chi_{B_j^k} \le M$ for all k in $\mathbf{Z}$,*

(iii) *$\hat{B}_i^k \cap \hat{B}_j^k = \emptyset$ for $i \ne j$, $k \in \mathbf{Z}$, where $\hat{B}_j^k = B(x_{jk}, \lambda^{k-1})$.*

They call these balls B_j^k dyadic balls. Denote by $\mathcal{B}_d$ the family of all dyadic balls. Using dyadic balls, we give the estimates of the maximal operator M in a quasi-metric space X by the integral with respect to H^α_∞, corresponding to the results of Orobitg–Verdera.

Theorem 1. *Let (X,ρ) be a quasi-metric space with diam $X < \infty$. Suppose that there are a positive number d and a Borel measure μ on X satisfying (1.2) for every ball $B(x,r) \subset X$. Furthermore, let $0 < \alpha < d$. Then*

$$H^\alpha_\infty(\{x : Mf(x) > t\}) \le ct^{-\alpha/d} \int |f|^{\alpha/d} dH^\alpha_\infty$$

for every f and $t > 0$.

Theorem 2. *Assume that X and μ satisfy the same conditions as Theorem 1. Let $\alpha/d < p$. Then*

$$\int (Mf)^p dH^\alpha_\infty \le c \int |f|^p dH^\alpha_\infty \quad \textit{for every } f.$$

We note that, for a nonnegative function g and a subset G of X,

$$\int_G g dH_\infty^\alpha := \int_0^\infty H_\infty^\alpha(\{x \in G : g(x) > t\})dt$$

and

$$\int_G g d\mu := \int_0^\infty \mu(\{x \in G : g(x) > t\})dt.$$

If $g \in L^1(\mu)$ and G is μ-measurable, then the integral with respect to the measure μ coincides with the usual one.

§2. Dyadic balls in a quasi-metric space

Throughout this paper let (X, ρ) be a quasi-metric space. The function ρ is called a quasi-metric. We assume that the diameter of X is finite and diam $X = R$. Furthermore we assume that there exists a positive Radon measure μ on X with $\mu(X) < \infty$ and satisfying (1.2) for some d. We note that, if (1.2) holds for all positive $r \leq R$, then (1.2) holds for all positive $r \leq 2(K + 2K^2)^2 R$ by changing the constants. So we may assume that (1.2) holds for all positive $r \leq 2(K + 2K^2)^2 R$. Consequently μ satisfies the doubling condition, i.e., there is a constant $c > 0$ such that

$$\mu(B(x, 2r)) \leq c\mu(B(x, r))$$

for $x \in X$ and $r \leq 2(K+2K^2)^2 R$. So X is a space of homogeneous type (See [3] on more precise properties on a space of homogeneous type).

For any quasi-metric ρ there exists an equivalent quasi-metric ρ' such that all balls with respect to ρ' are open (cf. [4]). Consequently we may assume that all balls $B(x, r)$ in X are open.

Let $B = B(x, r)$ be a ball and b be a positive real number. The notation bB stands for the ball of radius br centered at x and $r(B)$ stands for the radius of B. We often use the following value λ defined by

$$\lambda = 2K^2 + K,$$

where K is the constant in (1.1).

We begin with the following lemma.

Lemma 2.1. *Let B be a ball and $\{B_j\}$ be a sequence of disjoint balls. Put*

$$E = \{j : B \cap \lambda^{-1} B_j \neq \emptyset,\ r(B) \leq r(B_j)\}.$$

Then $\#E \leq N$, where N is a constant independent of B and $\{B_j\}$.

Proof. Case 1. We first consider the case where there exists $B_i \in \{B_j\}_j$ satisfying $B \cap \lambda^{-1}B_i \neq \emptyset$ and $r(B) \leq \lambda^{-1}r(B_i)$.

Let $w \in B$ and x_i be the center of B_i. Then, for $z \in B \cap \lambda^{-1}B_i$,

$$\begin{aligned} \rho(w, x_i) &\leq K(\rho(w,z) + \rho(z,x_i)) \\ &< 2K^2r(B) + K\lambda^{-1}r(B_i) \leq r(B_i). \end{aligned}$$

Hence $B \subset B_i$. Noting that $\{B_j\}$ are disjoint, we conclude that $\#E = 1$.

Case 2. We next consider the case where $r(B) > \lambda^{-1}r(B_j)$ for all $j \in E$. Let x be the center of B. Since $B_j \subset B(x, 2\lambda Kr(B))$, we have

$$\cup_{j\in E} B_j \subset B(x, 2\lambda Kr(B)).$$

Note that $\{B_j\}$ are disjoint and $r(B) \leq r(B_j)$ for all $j \in E$.

Let $\#E = n$. From (1.2), we deduce

$$\begin{aligned} &n\mu(B(x, 2K\lambda r(B)) \leq nb_2(2K\lambda r(B))^d \leq \frac{b_2}{b_1}(2K\lambda)^d \sum_{j\in E} \mu(B_j) \\ = \;& \frac{b_2}{b_1}(2K\lambda)^d \mu(\cup_{j\in E} B_j) \leq \frac{b_2}{b_1}(2K\lambda)^d \mu(B(x, 2K\lambda r(B))). \end{aligned}$$

Thus $n \leq \frac{b_2}{b_1}(2K\lambda)^d$. This leads to the conclusion. □

We have the following lemma for dyadic balls.

Lemma 2.2. *Let $\{B_j^k\} \subset \mathcal{B}_d$ and $B_j^k = B(x_{jk}, \lambda^k)$. Then there is a constant N_1, independent of j and k, such that*

$$\sum_j \chi_{\lambda B_j^k} \leq N_1.$$

Proof. Assume that $x \in \cap_{j=1}^n \lambda B_j^k$. Then $\hat{B}_j^k \subset B(x, 2K\lambda^{k+1})$. Similarly $B(x,\lambda^k) \subset B(x_{jk}, K\lambda^k(1+\lambda))$. Hence, by (1.2),

$$\begin{aligned} \mu(B(x, 2K\lambda^{k+1})) &\leq c_1\mu(B(x,\lambda^k)) \leq c_1\mu(B(x_{jk}, K\lambda^k(1+\lambda))) \\ &\leq c_2\mu(B(x_{jk}, \lambda^{k-1})) = c_2\mu(\hat{B}_j^k) \end{aligned}$$

for j. Noting that $\{\hat{B}_j^k\}$ are disjoint, we have

$$\frac{n}{c_2}\mu(B(x, 2K\lambda^{k+1})) \leq \sum_{j=1}^n \mu(\hat{B}_j^k) = \mu(\cup_{j=1}^n \hat{B}_j^k) \leq \mu(B(x, 2K\lambda^{k+1})),$$

whence $n \leq c_2$. Thus we have the conclusion. □

A sequence $\{B_j\}$ of balls is called maximal by inclusion if each B_j includes no B_i for $i \neq j$.

Lemma 2.3. *Let $\{B_j\} \subset \mathcal{B}_d$. If $\{\lambda^2 B_j\}$ is a maximal sequence by inclusion, then there is a constant N_1 such that*

$$\sum_j \chi_{\lambda B_j} \leq N_1.$$

Proof. Let $\{B^k_{j_l}\}_l$ be the subfamily of $\{B_j\}$ having radius λ^k. Lemma 2.2 yields that

$$\sum_l \chi_{\lambda B^k_{j_l}} \leq N_1.$$

We next consider two balls $B_j = B^k_j$ and $B_i = B^l_i$, $l < k$, in $\{B_j\}$. If $\lambda B^k_j \cap \lambda B^l_i \neq \emptyset$, then we pick $z \in \lambda B^k_j \cap \lambda B^l_i$. Let $w \in \lambda^2 B^l_i$. Writing $B^k_j = B(x_{jk}, \lambda^k)$ and $B^l_i = B(x_{il}, \lambda^l)$, we have

$$\begin{aligned} \rho(x_{jk}, w) &\leq K(\rho(x_{jk}, z) + K(\rho(z, x_{il}) + \rho(x_{il}, w))) \\ &< K\lambda^{k+1} + 2K^2\lambda^{l+2} \leq \lambda^{k+2}, \end{aligned}$$

whence $\lambda^2 B^l_i \subset \lambda^2 B^k_j$. This contradicts that $\{\lambda^2 B_j\}$ is maximal. Therefore we conclude that $\lambda B^k_j \cap \lambda B^l_i = \emptyset$. □

Using this lemma, we have

Lemma 2.4. *Let $\{B_j\} \subset \mathcal{B}_d$ such that $\{\lambda^2 B_j\}$ is a maximal sequence by inclusion. Furthermore let $B \in \mathcal{B}_d$. Put*

$$F = \{j : B \cap B_j \neq \emptyset,\ r(B) \leq r(B_j)\}.$$

Then $\#F \leq N_1$.

Proof. If $j \in F$, then $B \subset \lambda B_j$. Lemma 2.3 yields

$$\sum_j \chi_{\lambda B_j} \leq N_1.$$

Hence $\#F \leq N_1$.

□

Let $\{B_j\}$ be a (finite or infinite) sequence of subsets of X. Using it, we can construct a maximal sequence by inclusion. Indeed, we consider $\{B_1, B_2\}$ and, if $B_1 \subset B_2$ or $B_2 \subset B_1$, then we remove the less one from $\{B_1, B_2\}$ and denote by B'_1 the big one. Otherwise, put

$$B'_1 = B_1 \text{ and } B'_2 = B_2.$$

We next assume that $\{B_1', \cdots, B_m'\}$ has been constructed by using $\{B_1, \cdots, B_n\}$. Then we consider $\{B_1', \cdots, B_m', B_{n+1}\}$, remove all sets which are included by the other sets and make a new family $\{B_1', \cdots, B_l'\}$ of all balls which remain. Thus we inductively construct a subsequence $\{B_1, B_2, \cdots\}$ of $\{B_j\}$, which is a maximal sequence by inclusion, and call it the maximal sequence of $\{B_j\}$.

We are ready to prove our main lemma.

Lemma 2.5. *Let $\{B_j\} \subset \mathcal{B}_d$ and $\alpha > 0$. Then there exists a (finite or infinite) subsequence $\{B_{j_k}\}$ of $\{B_j\}$ having the following properties:*

(i)

$$\sum_{j_k \in S_B} r(B_{j_k})^\alpha \le 2r(B)^\alpha \quad \textit{for each } B \in \mathcal{B}_d,$$

where $S_B = \{j_k : B_{j_k} \cap B \neq \emptyset,\, r(B_{j_k}) \le r(B)\}$.

(ii) *For a positive number b there is a constant c such that*

$$H_\infty^\alpha(\cup_j bB_j) \le c \sum_k r(B_{j_k})^\alpha,$$

where c is independent of $\{B_j\}$.

Proof. We construct a subsequence $\{B_{j_k}\}$ of $\{B_j\}$ by induction. First, put $j_1 = 1$. The set $\{B_{j_1}\}$ has the property (i). Next, assume that $\{j_1, \cdots, j_m\}$ $(j_1 < \cdots < j_m)$ have been chosen so that (i) holds for $\{B_{j_1}, \cdots, B_{j_m}\}$. We set j_{m+1} the first number j such that $j_m < j$ and $\{B_{j_1}, \cdots, B_{j_m}, B_j\}$ satisfies (i). We note that, if $S_B = \emptyset$, then the left-hand side of the inequality in (i) is regarded as 0. Thus we construct $j_1, \cdots, j_n, \cdots$.

We next show that $\{B_{j_k}\}$ also satisfies (ii). Let j' be a number satisfying $j_m < j' < j_{m+1}$. Then there is a ball $C_{j'} \in \mathcal{B}_d$ such that $B_{j'} \cap C_{j'} \neq \emptyset$, $r(B_{j'}) \le r(C_{j'})$ and

$$\sum_{j_k \in S_{C_{j'}}} r(B_{j_k})^\alpha + r(B_{j'})^\alpha > 2r(C_{j'})^\alpha.$$

From this it follows that

$$\sum_{j_k \in S_{C_{j'}}} r(B_{j_k})^\alpha > r(C_{j'})^\alpha. \tag{2.1}$$

To prove (ii), we may suppose that $\sum_k r(B_{j_k})^\alpha < \infty$. We denote by $\{D_i\}$ the maximal sequence of $\{\lambda^2 C_{j'}\}$. Since $B_{j'} \cap C_{j'} \neq \emptyset$ and

$r(B_{j'}) \le r(C_{j'})$, we have $B_{j'} \subset \lambda C_{j'}$. Hence $B_{j'} \subset D_i$ for some i. Noting that

$$\cup_j bB_j \subset \cup_k bB_{j_k} \cup (\cup_{j'} bB_{j'}) \subset \cup_k bB_{j_k} \cup (\cup_i bD_i),$$

we have

$$H^\alpha_\infty(\cup_j bB_j) \le b^\alpha \sum_k r(B_{j_k})^\alpha + b^\alpha \lambda^{2\alpha} \sum_i r(\lambda^{-2} D_i)^\alpha.$$

The inequality (2.1) implies

$$\begin{aligned}\sum_i r(\lambda^{-2}D_i)^\alpha &\le \sum_i \sum_{j_k \in S_{\lambda^{-2}D_i}} r(B_{j_k})^\alpha \\ &= \sum_k \sum_{B_{j_k} \cap \lambda^{-2}D_i \neq \emptyset, r(B_{j_k}) \le \lambda^{-2} r(D_i)} r(B_{j_k})^\alpha.\end{aligned}$$

Fix a natural number k. We see by Lemma 2.4 that the number of $\lambda^{-2}D_i$ satisfying $B_{j_k} \cap \lambda^{-2}D_i \neq \emptyset$ and $r(B_{j_k}) \le \lambda^{-2} r(D_i)$ is at most N_1. Hence

$$\begin{aligned}H^\alpha_\infty(\cup_j bB_j) &\le b^\alpha \sum_k r(B_{j_k})^\alpha + b^\alpha \lambda^{2\alpha} N_1 \sum_k r(B_{j_k})^\alpha \\ &= b^\alpha (1 + N_1 \lambda^{2\alpha}) \sum_k r(B_{j_k})^\alpha.\end{aligned}$$

We may put $c = b^\alpha(1 + N_1\lambda^{2\alpha})$. Thus we have the assertion (ii). □

§3. Maximal functions and Hausdorff contents with respect to dyadic balls

In this section we introduce maximal functions and Hausdorff contents with respect to dyadic balls. We begin with maximal functions. For a function f we define

$$\tilde{M} f(x) = \sup \frac{1}{\mu(B)} \int_B |f| d\mu,$$

where the supremum is taken over all dyadic balls containing x. Here we note that, for a nonnegative function g,

$$\int g d\mu := \int_0^\infty \mu(\{x : g(x) > t\}) dt.$$

Using the properties in Theorem C, we can show the following lemma.

Lemma 3.1. *Let f be a function on X. Then there is a constant c independent of f such that*

$$\tilde{M}f(x) \le Mf(x) \le c\tilde{M}f(x)$$

for all $x \in X$.

Fix α satisfying $0 < \alpha < d$. Similarly we define, for $E \subset X$,

$$\tilde{H}^\alpha_\infty(E) = \inf \sum_j r(B_j)^\alpha,$$

where the infimum is taken over all coverings $\{B_j\}$ of E by dyadic balls B_j. Similarly we can show the following lemma.

Lemma 3.2. *Let $0 < \alpha < d$. Then there is a positive constant c such that*

$$c\tilde{H}^\alpha_\infty(E) \le H^\alpha_\infty(E) \le \tilde{H}^\alpha_\infty(E).$$

§4. Proofs of Theorem 1 and Theorem 2

In this section we will prove Theorem 1 and Theorem 2. To do these, we estimate the integral of a nonnegative function f with respect to the measure μ by the integral of f with respect to H^α_∞.

Lemma 4.1. *Let $0 < \alpha \le d$ and f be a nonnegative function on X. Then*

$$\int f d\mu \le c\left(\int f^{\alpha/d} dH^\alpha_\infty\right)^{d/\alpha},$$

where c is a positive constant independent of f.

Proof. Noting that μ satisfies (1.2), we can prove this lemma by the same method as in the proof of Lemma 3 in [5]. □

We note that $H^\alpha_\infty(\{x : f(x) > t\})$ is abbreviated to $H^\alpha_\infty(\{f > t\})$ in the proofs of Theorem 1 and Theorem 2.

Proof of Theorem 1. We may assume that $f \ge 0$. Put

$$E_t = \{x : \tilde{M}f(x) > t\}$$

for $t > 0$. For each $x \in E_t$ there is a ball $B_x \in \mathcal{B}_d$ such that

$$\frac{1}{\mu(B_x)} \int_{B_x} f d\mu > t. \tag{4.1}$$

Then $E_t \subset \cup_{x\in E_t} B_x \subset \cup_{x \in E_t} \lambda B_x$.

By Théorèm (1.2) on p.69 in [3] we can choose a countable family $\{\lambda B_j\} \subset \{\lambda B_x\}_{x \in E_t}$ such that $\{\lambda B_j\}$ ($B_j = B(x_j, r_j)$) are disjoint and $E_t \subset \cup_j B(x_j, h\lambda r_j)$ for some $h \geq 1$. Then, by Lemma 4.1 and (4.1),

$$r(B_j)^\alpha \leq \left(\frac{1}{b_1 t}\int_{B_j} f d\mu\right)^{\alpha/d} \leq c_1 t^{-\alpha/d}\int_{B_j} f^{\alpha/d} dH^\alpha_\infty. \tag{4.2}$$

Applying Lemma 2.5 to the sequence $\{B_j\}$, we choose a subsequence $\{B_{j_k}\}$ satisfying (i) and (ii) in Lemma 2.5 for $b = \lambda h$. Writing $B_{j_k} = B(x_k, r_k)$, we have, by (4.2),

$$H^\alpha_\infty(E_t) \leq H^\alpha_\infty(\cup_j B(x_j, \lambda h r_j)) \leq c_2 \sum_k r_k^\alpha \leq c_3 \sum_k t^{-\alpha/d}\int_{B_{j_k}} f^{\alpha/d} dH^\alpha_\infty.$$

We claim that

$$\sum_k \int_{B_{j_k}} f^{\alpha/d} dH^\alpha_\infty \leq c_4 \int f^{\alpha/d} dH^\alpha_\infty. \tag{4.3}$$

Indeed, if $\int f^{\alpha/d} dH^\alpha_\infty = +\infty$, then it is clear that (4.3) holds. Assume that $\int f^{\alpha/d} dH^\alpha_\infty < +\infty$. Since

$$\int_0^\infty H^\alpha_\infty(\{f^{\alpha/d} > \tau\}) d\tau < \infty,$$

we have

$$H^\alpha_\infty(\{f^{\alpha/d} > \tau\}) < \infty \quad \text{for a.e. } \tau$$

and hence, by Lemma 3.2,

$$\tilde{H}^\alpha_\infty(\{f^{\alpha/d} > \tau\}) < \infty \quad \text{for a.e. } \tau.$$

Fix τ satisfying $\tilde{H}^\alpha_\infty(\{f^{\alpha/d} > \tau\}) < \infty$. For $\epsilon > 0$ we take balls $Q_i \in \mathcal{B}_d$ such that

$$\{x : f(x)^{\alpha/d} > \tau\} \subset \cup_i Q_i$$

and

$$\sum_i r(Q_i)^\alpha < \tilde{H}^\alpha_\infty(\{f^{\alpha/d} > \tau\}) + \epsilon. \tag{4.4}$$

Since $\{\lambda B_{j_k}\}$ are disjoint, we see, by Lemma 2.1, that for each Q_i the number of B_{j_k} satisfying $Q_i \cap B_{j_k} \neq \emptyset$ and $r(Q_i) \leq \lambda r(B_{j_k})$ is at most

N. Hence

$$
\begin{aligned}
3\sum_i r(Q_i)^\alpha &= 2\sum_i r(Q_i)^\alpha + \sum_i r(Q_i)^\alpha \\
&\geq \sum_i \left(\sum_{\substack{Q_i\cap B_{j_k}\neq\emptyset \\ r(Q_i)>r(B_{j_k})}} r(B_{j_k})^\alpha + \frac{1}{N} N r(Q_i)^\alpha \right) \\
&\geq \sum_k \left(\sum_{\substack{Q_i\cap B_{j_k}\neq\emptyset \\ r(Q_i)>r(B_{j_k})}} r(B_{j_k})^\alpha + \frac{1}{N} \sum_{\substack{Q_i\cap B_{j_k}\neq\emptyset \\ r(Q_i)\leq r(B_{j_k})}} r(Q_i)^\alpha \right) \\
&\geq \frac{1}{N}\sum_k \tilde{H}^\alpha_\infty(B_{j_k}\cap(\cup_i Q_i)) \\
&\geq \frac{1}{N}\sum_k \tilde{H}^\alpha_\infty(B_{j_k}\cap\{f^{\alpha/d}>\tau\}).
\end{aligned}
$$

Hence, by (4.4),

$$
\tilde{H}^\alpha_\infty(\{f^{\alpha/d}>\tau\}) + \epsilon \geq \frac{1}{3N}\sum_k \tilde{H}^\alpha_\infty(B_{j_k}\cap\{f^{\alpha/d}>\tau\}).
$$

Thus, by Lemma 3.2, we have the claim (4.3). Therefore

$$
H^\alpha_\infty(E_t) \leq c_3 \sum_k t^{-\alpha/d}\int_{B_{j_k}} f^{\alpha/d}\,dH^\alpha_\infty \leq c_5 t^{-\alpha/d}\int f^{\alpha/d}\,dH^\alpha_\infty.
$$

This is the desired inequality. □

We next prove Theorem 2.

Proof of Theorem 2. Define

$$
f_1(x) = \begin{cases} f(x) & |f(x)| > \frac{t}{2}, \\ 0 & \text{otherwise.} \end{cases}
$$

Then

$$
|f(x)| \leq |f_1(x)| + t/2 \quad \text{and } Mf(x) \leq Mf_1(x) + t/2.
$$

Hence, by Theorem 1,

$$\begin{aligned} H_\infty^\alpha(\{x : Mf(x) > t\}) &\le H_\infty^\alpha(\{x : Mf_1(x) > t/2\}) \\ &\le c_1 t^{-\alpha/d} \int_{|f|>t/2} |f|^{\alpha/d} dH_\infty^\alpha. \end{aligned}$$

Therefore we write

$$\begin{aligned} \int (Mf)^p dH_\infty^\alpha &= \int_0^\infty H_\infty^\alpha(\{(Mf)^p > t\})dt \\ &= p \int_0^\infty H_\infty^\alpha(\{Mf > t\}) t^{p-1} dt \\ &\le c_1 p \int_0^\infty t^{p-1} t^{-\alpha/d} dt \int_{|f|>t/2} |f|^{\alpha/d} dH_\infty^\alpha \\ &\le I_1 + I_2, \end{aligned}$$

where

$$I_1 = c_1 p \int_0^\infty t^{p-1} t^{-\alpha/d} dt \int_0^\infty H_\infty^\alpha(\{|f| > s^{d/\alpha}\}) \chi_{\{s^{d/\alpha} \ge t/2\}} ds,$$

$$I_2 = c_1 p \int_0^\infty t^{p-1} t^{-\alpha/d} dt \int_0^\infty H_\infty^\alpha(\{|f| > t/2\}) \chi_{\{s^{d/\alpha} < t/2\}} ds.$$

Using Fubini's theorem, we have

$$\begin{aligned} I_1 &\le c_1 p \int_0^\infty H_\infty^\alpha(\{|f| > s^{d/\alpha}\}) ds \int_0^{2s^{d/\alpha}} t^{p-1-\alpha/d} dt \\ &= c_2 \int_0^\infty (s^{d/\alpha})^{p-\alpha/d} H_\infty^\alpha(\{|f| > s^{d/\alpha}\}) ds. \end{aligned}$$

Putting $t' = s^{dp/\alpha}$, we have

$$I_1 \le c_3 \int_0^\infty H_\infty^\alpha(\{|f|^p > t'\}) dt' = c_3 \int |f|^p dH_\infty^\alpha.$$

We next estimate I_2. Note

$$\begin{aligned} I_2 &\le c_1 p \int_0^\infty t^{p-1-\alpha/d} H_\infty^\alpha(\{|f| > t/2\}) dt \int_0^{(t/2)^{\alpha/d}} ds \\ &= c_1 p \int_0^\infty t^{p-1-\alpha/d} (t/2)^{\alpha/d} H_\infty^\alpha(\{|f| > t/2\}) dt. \end{aligned}$$

Put $t' = t/2$. Then

$$I_2 \le c_4 \int_0^\infty (t')^{p-1} H_\infty^\alpha(\{|f| > t'\}) dt' = c_5 \int |f|^p dH_\infty^\alpha.$$

Thus we have the conclusion. □

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Hisako Watanabe
Nishikosenba 2-13-2,
Kawagoeshi 350-0035, Japan
E-mail address: hisakowatanabe@nifty.com