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Partially Integrable Almost CR Manifolds of CR Dimension and Codimension Two

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Abstract.

We extend the results of [11] on embedded CR manifolds of CR dimension and codimension two to abstract partially integrable almost CR manifolds. We prove that points on such manifolds fall into three different classes, two of which (the hyperbolic and the elliptic points) always make up open sets. We prove that manifolds consisting entirely of hyperbolic (respectively elliptic) points admit canonical Cartan connections. More precisely, these structures are shown to be exactly the normal parabolic geometries of types $(PSU(2,1) \times PSU(2,1), B \times B)$, respectively $(PSL(3,\mathbb{C}), B)$, where B indicates a Borel subgroup. We then show how general tools for parabolic geometries can be used to obtain geometric interpretations of the torsion part of the harmonic components of the curvature of the Cartan connection in the elliptic case.

§1. Introduction

For non-degenerate real hypersurfaces in $\mathbb{C}^{n+1}$ (or more generally in complex manifolds) there is a nice geometric setup based on Cartan connections. The Levi form in any point of such a hypersurface is a non-degenerate Hermitian form, so up to isomorphism (in an obvious sense) it is determined by its signature. In particular, this signature is constant on connected components of M, so without loss of generality one may assume that it is constant on M. This can be interpreted as follows: The tangent spaces of M come with a canonical filtration provided by the maximal complex subspaces, and the constancy of the Levi form provides a reduction of the structure group of the associated graded bundle of the tangent bundle to a reductive subgroup of the group $PSU(p+1, q+1)$ of CR automorphisms of the homogeneous

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model. This observation is the basis for the construction of a canonical Cartan connection for non-degenerate CR manifolds of hypersurface type in [5] and [12], although in these two papers the associated graded vector bundle to the tangent bundle and the reduction of its structure group do not show up explicitly. In the paper [3], reductions of the above type are used as the basis of a general construction of normalized Cartan connections, which immediately shows that in the CR case, canonical Cartan connections do not only exist for (integrable) CR manifolds but more generally for partially integrable almost CR structures (still non-degenerate and of hypersurface type), which was also observed in [12]. Also, in the approach of [3] embeddability plays no role at all, it is only the filtration of the tangent bundle and the reduction of structure group of the associated graded vector bundle that is used.

For CR structures of higher codimension (i.e. real submanifolds of higher codimension in complex manifolds) the situation is much more complicated. The main problem is that the Levi form at a point now has values in a real vector space of dimension bigger than one, and the classification of such forms up to isomorphism is much more difficult. More drastically, one may have continuously varying isomorphism classes in general, so even the associated graded vector spaces to the tangent spaces are not locally isomorphic (under an isomorphism preserving the Levi form). In such cases, there seems to be no hope for geometric structures similar to Cartan connections.

There are, however, a few cases in which the set of isomorphism classes is still discrete, and which thus are more manageable. The simplest of these cases is the case of CR dimension two and codimension two, which is the one that will be treated in this paper. The basic examples of such structures are provided by non-degenerate codimension two submanifolds in $\mathbb{C}^4$, whose tangent spaces are not complex subspaces. In this case, there are three possible isomorphism classes of Levi-forms, referred to as hyperbolic, elliptic, and exceptional (to avoid confusion with parabolic geometries, we do not use the classical name "parabolic" for this class). Moreover, being of hyperbolic or elliptic type are open conditions, so one may study local properties around such points by restricting to manifolds all of whose points are of fixed type. In the paper [11], the authors used a simple normal form argument to show that embedded elliptic and hyperbolic CR manifolds of CR dimension and codimension two are examples of parabolic geometries, as discussed for example in [13], [3]. The general theory developed in the latter two papers as well as in [9] and [14] then shows that these manifolds carry a canonical Cartan connection. It is known in general that the harmonic part of the curvature of the Cartan connection is a complete obstruction against local flatness. This harmonic part splits into several irreducible components, and [11] contains geometric interpretations of those components which are of torsion type. It is also shown the latter paper, that some of these torsions vanish automatically in the embedded case, so [11] partly deals with the more general abstrac parabolic geometries of appropriate type, but without relating them to (almost) CR structures.

In this paper, we extend the approach of [11] to abstract CR structures. It turns out that, as in the hypersurface case, the integrability condition needed to obtain a parabolic geometry is exactly partial integrability, which is a quite simple and natural condition in the real picture. It is interesting that partial integrability is also exactly the condition under which the paper [8] develops analogs of Webster-Tanaka connections for higher codimension almost CR manifolds. We give an independent proof that for non-degenerate partially integrable almost CR manifolds of CR dimension two and codimension two (i.e. six dimensional manifolds endowed with a rank two complex subbundle in the tangent bundle which satisfying partial integrability and a non-degeneracy condition) the possible Levi-brackets (i.e. the bundle maps induced by the Lie bracket) fall into three different classes, called hyperbolic, elliptic and exceptional. Next, we show that the oriented manifolds all of whose points are hyperbolic (elliptic) are exactly the normal parabolic geometries of type $(PSU(2,1) \times PSU(2,1), B \times B)$ (respectively of type $(PSL(3,\mathbb{C}), B)$), where B indicates a Borel subgroup. Thus we prove that reducing to the connected component of the structure group used in [11] exactly corresponds to picking an orientation. While the passage from the CR structure to the parabolic geometry is rather straightforward in the hyperbolic case, there is a more tricky point in the elliptic case, which involves flipping of the almost complex structure on the subbundle. Next, we give a geometric interpretation of the harmonic curvature components in the elliptic case, which were discussed only briefly in [11]. We derive some improved interpretations of torsion components as well as complete proofs for results whose proofs where only sketched in [11]. For example we prove, that torsion-free elliptic manifolds are automatically real analytic and therefore embeddable. Throughout the presentation we stress the fact that these interpretations can be obtained by applying standard tools available for any parabolic geometry. We do not include a discussion of the hyperbolic case, since this was treated in detail in [11].

§2. Partially integrable almost CR manifolds of CR dimension and codimension two

2.1. almost CR manifolds

We start by considering an (abstract) almost CR manifold of CR dimension k and codimension l, i.e. a smooth manifold M of real dimension $2k+l$, together with a rank k complex subbundle $HM \subset TM$, the *CR subbundle* of M. Since we shall meet a different almost complex structure on HM which is more important in the sequel, we will denote the corresponding almost complex structure on H by $\tilde{J}$ or by $\tilde{J}^H$. A smooth map between two almost CR manifolds is called a *CR map*, if and only if its derivative in each point maps the CR subspace to the CR subspace and the restriction to the CR subspace is complex linear.

Let $QM := TM/HM$ be the quotient bundle, which is a real vector bundle of rank l over M and let $q\colon TM \to QM$ be the canonical quotient map. For two smooth sections ξ, η of HM the value of $q([\xi,\eta])$ in a point $x \in M$ depends only on the values of ξ and η in x, so this induces a bilinear, skew symmetric bundle map $\mathcal{L}\colon HM \times HM \to QM$, the *Levi-bracket* of M. Any CR map $f\colon M_1 \to M_2$ induces a homomorphism $QM_1 \to QM_2$ of vector bundles, which we also denote by Tf. A (local) *CR diffeomorphism* between manifolds of the same CR dimension and codimension is a (locally defined) CR map, which is a diffeomorphism (onto its image). If $f\colon M_1 \to M_2$ is a local CR diffeomorphism, then $f^*\mathcal{L}^{M_2} = \mathcal{L}^{M_1}$, i.e. $\mathcal{L}^{M_2}(T_xf\cdot\xi, T_xf\cdot\eta) = T_xf\cdot\mathcal{L}^{M_1}(\xi,\eta)$ for all elements ξ, $\eta \in HM_1$.

Definition. An almost CR manifold M is called *partially integrable* if the Levi bracket is totally real, i.e. $\mathcal{L}(\tilde{J}\xi, \tilde{J}\eta) = \mathcal{L}(\xi,\eta)$.

This partial integrability condition shows up as condition (A1) in [12, p. 170] (included into the definition of an almost CR manifold) and it plays an important role in [8]. To see why it is called partial integrability, one has to pass to the complexified tangent bundle $T_{\mathbb{C}}M = TM \otimes \mathbb{C}$ of M. Since $HM \subset TM$ is a complex subbundle, its complexification $H_{\mathbb{C}}M = HM \otimes \mathbb{C}$ splits into a holomorphic and an antiholomorphic part, $H_{\mathbb{C}}M = H_{1,0}M \oplus H_{0,1}M$. A typical smooth section of $H_{1,0}M$ is of the form $\xi - i\tilde{J}\xi$ for a smooth section ξ of HM. Taking two such sections and computing their bracket, we obtain

$$[\xi,\eta] - [\tilde{J}\xi, \tilde{J}\eta] - i([\tilde{J}\xi,\eta] + [\xi,\tilde{J}\eta]).$$

Now the condition of partial integrability as stated above is obviously equivalent to the fact that both the real and the imaginary part of this are again sections of HM, so it is equivalent to the bracket of two sections of $H_{1,0}M$ being a section of $H_{\mathbb{C}}M$. Recall that an almost CR manifold is called *integrable* or a *CR manifold* if and only if the bundle $H_{1,0}M$ is integrable, i.e. the bracket of two sections of $H_{1,0}M$ is again a section of $H_{1,0}M$. Assuming partial integrability, this condition can be nicely expressed without using complexifications as follows: By partial integrability, for two sections ξ, η of HM, both $[\xi,\eta]-[\tilde{J}\xi,\tilde{J}\eta]$ and $[\xi,\tilde{J}\eta]+[\tilde{J}\xi,\eta]$ are again sections of HM. Thus, we may define the *Nijenhuis-tensor* $\tilde{N}$ of M by

$$\tilde{N}(\xi,\eta)=[\xi,\eta]-[\tilde{J}\xi,\tilde{J}\eta]+\tilde{J}([\tilde{J}\xi,\eta]+[\xi,\tilde{J}\eta]).$$

The usual proof shows that this is bilinear over smooth functions, so it defines a smooth section of the bundle $H^*M\otimes H^*M\otimes HM$, where H^*M is the bundle dual to HM. Moreover, from the definition it follows immediately that $\tilde{N}$ is skew symmetric and conjugate linear in both variables. Clearly, integrability of $H_{1,0}M$ is equivalent to vanishing of the Nijenhuis tensor. Finally note that since $H_{0,1}M=\overline{H_{1,0}M}$, integrability of $H_{1,0}M$ and of $H_{0,1}M$ are equivalent.

2.2. Embedded almost CR manifolds

The typical examples of almost CR manifolds are certain submanifolds in smooth manifolds endowed with an almost complex structure. Suppose that $\mathcal{M}$ is a smooth manifold of real dimension $2k+2l$ endowed with an almost complex structure $J^{\mathcal{M}}$, and let $M\subset\mathcal{M}$ be a smooth submanifold of codimension l. For a point $x\in M$, consider the subspace $H_xM=T_xM\cap J^{\mathcal{M}}(T_xM)$. By construction, this is a complex subspace of T_xM (with the complex structure $\tilde{J}$ given by the restriction of $J^{\mathcal{M}}$). If the complex dimension of this spaces is equal to k for all $x\in M$, then the union of the H_xM defines a smooth subbundle $HM\subset TM$, which makes M into an almost CR manifold of CR dimension k and codimension l.

In fact, any almost CR manifold arises in that way. To see this, consider the quotient bundle $\pi\colon QM\to M$, which is a vector bundle of real rank l, into which M canonically embeds as the zero section. Choosing a linear connection on this vector bundle, we get an associated isomorphism $TQM\cong VQM\oplus\pi^*TM$, where VQM is the vertical tangent bundle and π^*TM is the pullback of the tangent bundle on M. Since the connection is linear, the zero section is covariantly constant, which implies that along the zero section, the horizontal complement to the vertical tangent space coincides with the image of the tangent map of the zero section. Let us further choose a splitting of the projection $q\colon TM\to QM$, which gives us an isomorphism $TM\cong HM\oplus QM$ of vector bundles. For a vector bundle, the vertical bundle is canonically isomorphic to the pullback of the original bundle, so putting all our choices and observations together, we see that we get an isomorphism from TQM to $\pi^*QM \oplus \pi^*HM \oplus \pi^*QM$, such that the first Q-factor corresponds to the vertical bundle. Now define an endomorphism J of TQM by $J(\xi,\eta,\zeta) = (-\zeta, \tilde{J}\eta, \xi)$ in the above decomposition, where $\tilde{J}$ comes from the almost complex structure on HM. Obviously, $J^2 = -\mathrm{id}$, so this defines an almost complex structure on the manifold QM. From the above observation on the horizontal subspaces in the points of the zero section we conclude that viewing M as a submanifold of QM, then in the above splitting the tangent space to M consists of all elements of the form $(0,\eta,\zeta)$ and hence the maximal complex subspace consists of all elements of the form $(0,\eta,0)$, so we exactly obtain the given subbundle HM.

If $\mathcal{M}$ is actually a complex manifold, then M is automatically a CR manifold. However, the converse question, whether any CR manifold can be embedded into a complex manifold is much more subtle, and it is well known that the answer is negative in general. There is however one case in which one gets a fairly simple general result. Although this is certainly well known to specialists, it seems that it is not easy to find a general proof in the literature, so we include the argument, which is based on the proof for the hypersurface case in [6].

Proposition. *Let $(M, HM, \tilde{J})$ be a real analytic CR manifold, i.e. M is a real analytic manifold, $HM \subset TM$ is a real analytic subbundle and the bundle map $\tilde{J}\colon HM \to HM$ is real analytic. Then M is locally embeddable, i.e. any point $x \in M$ has an open neighborhood U in M which embeds into $\mathbb{C}^{k+l}$ in such a way that HM becomes the subbundle of maximal complex subspaces.*

Proof. Choosing local coordinates on M in such a way that in each point the subspace generated by the last l coordinate vector fields is transversal to the subbundle HM, we may reduce to the case where M is an open subset $U \subset \mathbb{R}^{2k} \times \mathbb{R}^l$ such that for each $x \in U$ we have $H_xU \cap T_x\mathbb{R}^l = \{0\}$. We view $\mathbb{R}^{2k} \times \mathbb{R}^l$ as a subspace of $\mathbb{R}^{2k} \times \mathbb{C}^l$, and denote the coordinates on the latter space by $x_1, \dots, x_{2k}$, $x_{2k+1} + iy_{2k+1}, \dots, x_{2k+l} + iy_{2k+l}$. In particular, we may view the coordinate vector fields $\frac{\partial}{\partial x_j}$ as a $\mathbb{C}$-Basis of $TU \otimes \mathbb{C}$.

Since HM is a real analytic subbundle of TU and $\tilde{J}$ is real analytic, too, the subbundle $H_{0,1}U \subset TU \otimes \mathbb{C}$ is real analytic, and possibly shrinking U, we find a real analytic frame $\xi_1, \dots, \xi_k$ for this bundle, i.e. $\xi_j = \sum_{m=1}^{2k+l} a_{jm} \frac{\partial}{\partial x_m}$ for real analytic functions $a_{jm}\colon U \to \mathbb{C}$. Now there exists a neighborhood $\tilde{U}$ of U in $\mathbb{R}^{2k} \times \mathbb{C}^l$, such that all the functions a_{jm} extend to $\tilde{U}$ as functions holomorphic in the $\mathbb{C}^l$-factor. Next, we define sections $\tilde{\xi}_1, \dots, \tilde{\xi}_k$ of $T\tilde{U} \otimes \mathbb{C}$ by $\tilde{\xi}_j = \sum_{m=1}^{2k+l} \tilde{a}_{jm} \frac{\partial}{\partial x_m}$, and for $j = 1, \dots, l$ we put $\tilde{\xi}_{k+j} := \frac{\partial}{\partial x_{2k+j}} + i \frac{\partial}{\partial y_{2k+j}}$.

Since the values of $\xi_1, \dots, \xi_k$ in each point of U are linearly independent, we conclude that (possibly shrinking $\tilde{U}$) also the values of $\tilde{\xi}_1, \dots, \tilde{\xi}_k$ in each point of $\tilde{U}$ are linearly independent, which easily implies that $\tilde{\xi}_1, \dots, \tilde{\xi}_{k+l}$ span a rank $k + l$ complex subbundle V of $T\tilde{U} \otimes \mathbb{C}$. Next, we claim that $V \cap \bar{V} = \{0\}$: Suppose that $\lambda_1, \dots, \lambda_{k+l}, \mu_1, \dots, \mu_{k+l} \in \mathbb{C}$ are such that in a point of U we have

$$\lambda_1 \tilde{\xi}_1 + \cdots + \lambda_{k+l} \tilde{\xi}_{k+l} = \mu_1 \bar{\tilde{\xi}}_1 + \cdots + \mu_{k+l} \bar{\tilde{\xi}}_{k+l}.$$

Looking at the coefficients of the fields $\frac{\partial}{\partial y_j}$ one immediately concludes that $\mu_j = -\lambda_j$ for $j = k+1, \dots, k+l$. But then the equation reduces to

$$\mu_1 \bar{\tilde{\xi}}_1 + \cdots + \mu_k \bar{\tilde{\xi}}_k - \lambda_1 \tilde{\xi}_1 - \cdots - \lambda_k \tilde{\xi}_k = 2\lambda_{k+1} \frac{\partial}{\partial x_{k+1}} + \cdots + 2\lambda_{k+l} \frac{\partial}{\partial x_{k+l}}.$$

The left hand side of this equation lies in $HU \otimes \mathbb{C}$, while the right hand side lies in the complex span of the fields $\frac{\partial}{\partial x_j}$ for $j = 2k+1, \dots, 2k+l$. But by our assumptions, these two spaces are transversal, so the two sides have to vanish individually. Since $H_{1,0}M \cap \bar{H}_{1,0}M = \{0\}$, we thus conclude that $V \cap \bar{V} = \{0\}$ along U, so possibly shrinking $\tilde{U}$ again, we may assume that this holds on the whole of $\tilde{U}$.

Hence V defines an almost complex structure on $\tilde{U}$, and since U is just the subset on which $y_{2k+1} = \cdots = y_{2k+l} = 0$, one immediately concludes that for $x \in U$ the subspace $(T_xU \otimes \mathbb{C}) \cap V_x$ coincides with the fiber of $H_{0,1}U$ at x. To finish the proof, we only have to show that the distribution V is involutive, since then $\tilde{U}$ is a complex manifold and thus locally isomorphic to $\mathbb{C}^{k+l}$. But by assumption $H_{0,1}U$ is involutive, so any bracket $[\xi_i, \xi_j]$ is a linear combination (with real analytic coefficients) of ξ_m's. Since the $\tilde{\xi}$ are defined as partial holomorphic extensions of the ξ, we conclude that for $i, j \leq k$, any bracket $[\tilde{\xi}_i, \tilde{\xi}_j]$ can be written as a linear combination of $\tilde{\xi}_m$ with $m = 1, \dots, k$. On the other hand, we obviously have $[\tilde{\xi}_i, \tilde{\xi}_j] = 0$ if $i, j > k$, while for $i \leq k$ and $j > k$ we get the same result since the coefficients of $\tilde{\xi}_i$ are holomorphic in the last l factors, and the coefficients of $\tilde{\xi}_j$ are constant. Q.E.D.

2.3. Non-degeneracy

Let $(M, HM, \tilde{J})$ be a partially integrable almost CR manifold with Levi-bracket $\mathcal{L}$. Consider the dual Q^*M of the quotient bundle QM.

If $x \in M$ is a point and $\psi \in Q_x^*M$ is any element, we consider the totally real, skew symmetric bilinear map $\mathcal{L}^\psi \colon H_xM \times H_xM \to \mathbb{R}$ defined by $\mathcal{L}^\psi(\xi,\eta) = \psi(\mathcal{L}(\xi,\eta))$. A point $x \in M$ is called *non-degenerate* if $\mathcal{L}(\xi,\eta) = 0$ for all $\eta \in T_xM$ implies $\xi = 0$ and for each nonzero element $\psi \in Q_x^*M$ the map $\mathcal{L}^\psi$ is non-zero. The first condition just means that the Levi bracket at x is non-degenerate as a bilinear map, while the second condition is a coordinate-free version of linear independence of the components of $\mathcal{L}$. Obviously, non-degeneracy is an open condition, so if x is non-degenerate then there is an open neighborhood of x in M in which all points are non-degenerate. Since any CR diffeomorphism preserves the Levi brackets, it also preserves non-degeneracy, i.e. maps non-degenerate points to non-degenerate points. From now on, we will only consider non-degenerate partially integrable almost CR manifolds, i.e. manifolds all of whose points are non-degenerate.

Let us again compare this to the complexified picture. Here one usually considers the quotient bundle $Q_{\mathbb{C}}M = T_{\mathbb{C}}M/H_{\mathbb{C}}M$, which is a rank l complex bundle, and the *Levi-form* $\mathcal{H}\colon H_{1,0}M \times H_{1,0}M \to Q_{\mathbb{C}}M$ induced by $(\xi,\eta) \mapsto q_{\mathbb{C}}(\frac{i}{2}[\xi,\bar{\eta}])$. This turns out to be a hermitian form, and the usual non-degeneracy condition is just that $\mathcal{H}$ is non-degenerate and its components are linearly independent. A simple computation then shows that assuming partial integrability, the Levi bracket $\mathcal{L}$ corresponds exactly to the imaginary part of $\mathcal{H}$ under the identification $HM \to H_{1,0}M$ given by mapping ξ to $\xi - iJ\xi$. Now obviously non-degeneracy of $\mathcal{H}$ is equivalent to non-degeneracy of its imaginary part, while (complex) linear independence of the components of $\mathcal{H}$ is equivalent to (real) linear independence of the components of its imaginary part, so we recover the usual conditions.

2.4. The case of CR dimension and codimension two

For partially integrable almost CR manifolds of general CR dimension k and codimension l, the classification of possible Levi brackets up to the obvious notion of isomorphism is fairly complicated. In the special case $k = l = 2$ however, we shall see below that there are only three possible cases. This is quite well known, see e.g. [10] but since we will need detailed information about the classification in the further constructions we will reproduce it here in a form convenient for our purposes.

The key to this classification is to consider the bilinear maps $\mathcal{L}_x^\psi \colon H_xM \times H_xM \to \mathbb{R}$ for nonzero elements $\psi \in Q_x^*M$. By our non-degeneracy assumption, these maps are all nonzero, skew symmetric, and totally real, so since $k = 2$ they are either non-degenerate, or there is a complex subspace $H_x^\psi M$ of complex dimension one in H_xM

such that $\mathcal{L}^\psi(\xi,\eta)=0$ for all $\eta\in H_xM$ if and only if $\xi\in H_x^\psi M$. Note that clearly both the question whether $\mathcal{L}_x^\psi$ is degenerate and the space $H_x^\psi M$ in the degenerate case depend only on the class $[\psi]$ of ψ in the projectivization $P(Q_x^*M)\cong\mathbb{R}P^1$ of Q_x^*M. We write $\mathcal{L}_x^{[\psi]}$ for the class of forms corresponding to $[\psi]$ and in the case when $\mathcal{L}_x^{[\psi]}$ is degenerate, we write $H_x^{[\psi]}M$ for the corresponding subspace of H_xM.

Proposition. *Let M be a (non-degenerate) partially integrable almost CR manifold of CR dimension and codimension two and let $x\in M$ be a point. Then there are exactly three possibilities:*

1. *There are two points $[\psi_1]\neq[\psi_2]\in P(Q_x^*M)$ such that $\mathcal{L}^\psi: H_xM\times H_xM\to\mathbb{R}$ is degenerate if and only if $\psi\in[\psi_1]$ or $\psi\in[\psi_2]$. In this case, we have $H_xM=H_x^{[\psi_1]}M\oplus H_x^{[\psi_2]}M$ and the point x is called hyperbolic.*
2. *There is one point $[\psi_0]\in P(Q_x^*M)$ such that $\mathcal{L}^\psi$ is degenerate if and only if $\psi\in[\psi_0]$. In this case, the point x is called exceptional.*
3. *$\mathcal{L}^\psi$ is non-degenerate for all nonzero elements $\psi\in Q_x^*M$. In this case, the point x is called elliptic.*

Proof. Let us first assume that $[\psi_1]\neq[\psi_2]\in P(Q_x^*M)$ are such that $\mathcal{L}^{\psi_1}$ and $\mathcal{L}^{\psi_2}$ are degenerate. If $\xi\in H_x^{[\psi_1]}M\cap H_x^{[\psi_2]}M$ then $\mathcal{L}^{\psi_1}(\xi,\eta)=\mathcal{L}^{\psi_2}(\xi,\eta)=0$ for all $\eta\in H_xM$. But since $[\psi_1]\neq[\psi_2]$, the elements ψ_1 and ψ_2 form a basis of Q_x^*M, so $\mathcal{L}^\psi(\xi,\eta)=0$ for all $\psi\in Q_x^*M$. Thus, $\xi=0$ by non-degeneracy of $\mathcal{L}$, and consequently $H_xM=H_x^{[\psi_1]}M\oplus H_x^{[\psi_2]}M$.

If we assume $\psi\in Q_x^*M$ is another nonzero element such that $\mathcal{L}^\psi$ is degenerate, and such that $[\psi]$ is different from both $[\psi_1]$ and $[\psi_2]$, then on one hand, we may write $\mathcal{L}_x^\psi=a\mathcal{L}_x^{\psi_1}+b\mathcal{L}_x^{\psi_2}$ for nonzero real numbers a and b. On the other hand, $H_x^{[\psi]}M$ is complementary both to $H_x^{[\psi_1]}M$ and to $H_x^{[\psi_2]}M$ and thus for each element $\xi\in H_x^{[\psi_1]}M$, there is a unique element $\varphi(\xi)\in H_x^{[\psi_2]}M$, such that $\xi+\varphi(\xi)\in H_x^{[\psi]}M$, and φ is a linear isomorphism. Since $\mathcal{L}^\psi=a\mathcal{L}_x^{\psi_1}+b\mathcal{L}_x^{\psi_2}$, $\xi\in H_x^{[\psi_1]}M$ and $\varphi(\xi)\in H_x^{[\psi_2]}M$, we see that $\xi+\varphi(\xi)\in H_x^{[\psi]}M$ simply means that $0=a\mathcal{L}_x^{\psi_1}(\varphi(\xi),\eta)+b\mathcal{L}_x^{\psi_2}(\xi,\eta)$, for all η in H_xM. But inserting η from $H_x^{[\psi_1]}M$ we get $\mathcal{L}_x^{\psi_2}(\xi,\eta)=0$, for all ξ, $\eta\in H_x^{[\psi_1]}M$. Thus $\mathcal{L}_x^{\psi_2}$ would be identically zero, a contradiction. Q.E.D.

Remarks. (1) Clearly, the separation into these three classes is invariant under CR diffeomorphisms, so any CR diffeomorphism maps a point from one of the three classes to a point of the same class. Moreover, by definition, the properties of being hyperbolic or elliptic are stable, so any hyperbolic (elliptic) point has an open neighborhood in M which

consists entirely of hyperbolic (elliptic) points. Consequently, for local questions one may restrict to manifolds, all of whose points are either hyperbolic or elliptic.
(2) Traditionally, exceptional points are called parabolic points, but in view of the fact that parabolic geometries can be used to describe the hyperbolic and elliptic cases, we prefer to avoid this name. With these exceptional points, the situation is more complicated. On one hand, there are manifolds consisting entirely of exceptional points (for example an appropriate quadric), but it may also happen that any neighborhood of an exceptional point contains elliptic and/or hyperbolic points. Also, even the case in which all points are exceptional cannot be studied using parabolic geometries. It should also be mentioned that the exceptional points are more degenerate than the hyperbolic and elliptic ones. Namely, in our combination of dimensions, one can also consider the following (even more natural) non-degeneracy condition: The Levi bracket $\mathcal{L}$ may be considered as a smooth section of the bundle $\bigwedge^2_{\mathbb{R}} H^*M \otimes QM$. Hence, $\mathcal{L} \wedge \mathcal{L}$ is a smooth section of $\bigwedge^4_{\mathbb{R}} H^*M \otimes S^2QM$ and thus can be viewed as a quadratic polynomial on Q^*M defined up to scale. If one requires these polynomials to be non-degenerate, then only hyperbolic and elliptic points survive, see [2]. In the latter paper it is also shown that parts of the basic theory we develop here, carry over to the case where $HM \subset TM$ is only a real rank four vector bundle (i.e. without an almost complex structure).

2.5. The hyperbolic case

We next want to identify hyperbolic partially integrable almost CR manifolds of CR dimension and codimension two with a parabolic geometry. To avoid having to deal with non-connected groups, we restrict to the case when M is oriented, which definitely makes no problems locally. So let us assume that M is oriented and that all points of M are hyperbolic.

By proposition 2.4, for each point $x \in M$ we find the two distinguished classes $[\psi_1]$, $[\psi_2] \in P(Q^*_xM)$. Moreover, in local coordinates these are the solutions of a smooth equation, which implies that the classes $[\psi] \in P(Q^*M)$ which have the property that $\mathcal{L}^{[\psi]}$ is degenerate form a smooth submanifold of the bundle $P(Q^*M)$. Since M is hyperbolic, for any point $x \in M$ there are exactly two points in this submanifold lying over x, and since the projection $P(Q^*M) \to M$ is a surjective submersion, it restricts to a local diffeomorphism from the submanifold to M, so we get a two-sheeted covering of M. Hence, we can choose local smooth sections ψ_1 and ψ_2 of Q^*M which represent the distinguished classes in each point of their domain of definition. But then the null spaces of $\mathcal{L}^{\psi_1}$ and $\mathcal{L}^{\psi_2}$ form smooth subbundles H^1M and H^2M of HM, which are independent of the choice of the sections ψ_1 and ψ_2 (up to their numbering). So locally, we get a splitting $HM = H^1M \oplus H^2M$ of HM into a sum of complex line bundles. Now for $i = 1, 2$ let ξ_i be a local non vanishing section of H^iM and consider the local vector fields $\{\xi_1, J\xi_1, [\xi_1, J\xi_1], \xi_2, J\xi_2, [\xi_2, J\xi_2]\}$ on M. We claim that these form a local frame for TM. By construction, $\{\xi_i, J\xi_i\}$ is a local frame for H^iM, and since $HM = H^1M \oplus H^2M$ these four vector fields together form a local frame for HM. Applying $q\colon TM \to QM$ on the other hand, kills ξ_i and $J\xi_i$ for $i = 1, 2$ and maps the remaining vector fields to $\mathcal{L}(\xi_1, J\xi_1)$ and $\mathcal{L}(\xi_2, J\xi_2)$, respectively. Moreover, by definition $\psi_1(\mathcal{L}(\xi_1, J\xi_1)) = 0$. On the other hand, $\psi_1(\mathcal{L}(\xi_2, J\xi_2))$ must be nonzero, since $\psi_1 \circ \mathcal{L} = \mathcal{L}^{\psi_1}$ is nonzero, but has H^1M as its null space. Similarly, ψ_2 vanishes on $\mathcal{L}(\xi_2, J\xi_2)$ but is nonzero on $\mathcal{L}(\xi_1, J\xi_1)$, which implies our claim.

The question whether the above local frame is positively or negatively oriented is independent of the choice of ξ_1 and ξ_2. Indeed, one could even choose an arbitrary linearly independent section η_1 instead of $J\xi_1$, since exchanging the two sections changes the sign of the Lie bracket. Thus, we get a preferred order of the two subbundles, which we indicate by calling them H^+M and H^-M. Furthermore, choosing corresponding sections $\psi_\pm$ of Q^*M, they fit together globally, which shows that the two-fold covering constructed above is trivial. In particular, this means that we globally have $HM = H^+M \oplus H^-M$ and we find global smooth nonzero section $\psi_\pm$ of Q^*M representing the distinguished classes of functionals. Moreover, putting $Q^+M := \ker(\psi_-)$ and $Q^-M := \ker(\psi_+)$ we get a decomposition $QM = Q^+M \oplus Q^-M$ as a sum of real line bundles, which is independent of the choice of $\psi_\pm$, so it is intrinsic to M. In this language, we can rephrase the definition of the subbundles $H^\pm M$ as $\xi \in H^\pm M$ if and only if $\mathcal{L}(\xi, \eta) \in Q^\pm M$ for all $\eta \in HM$. Hence, $\mathcal{L}$ vanishes on $H^+M \times H^-M$, so it splits as $\mathcal{L} = \mathcal{L}^+ \oplus \mathcal{L}^-$, where $\mathcal{L}^\pm\colon H^\pm M \times H^\pm M \to Q^\pm M$.

2.6. Hyperbolic almost CR manifolds as parabolic geometries

Passing from the data $(M, H^+M, H^-M, QM = Q^+M \oplus Q^-M)$ to a parabolic geometry is now rather straightforward. Consider $\mathbb{C}^3$ with the hermitian inner product $\langle z, w\rangle = z_1\bar{w}_3 + z_3\bar{w}_1 + z_2\bar{w}_2$. This is clearly non-degenerate and has signature $(2, 1)$. Let $G := PSU(2, 1)$ be the quotient of the group of all complex automorphisms of $\mathbb{C}^3$ respecting this inner product by its center, and let $\mathfrak{su}(2,1)$ be the Lie algebra of this group. Then this is exactly the space of all complex 3×3-matrices, which are skew hermitian with respect to the above inner product. One easily computes directly that $\mathfrak{su}(2,1)$ is exactly the space of all matrices of the form $\begin{pmatrix} A & Z & iz \\ X & -2i\,\mathrm{im}(A) & -\bar{Z} \\ ix & -\bar{X} & -\bar{A} \end{pmatrix}$, where X, A, $Z \in \mathbb{C}$ and x, $z \in \mathbb{R}$. This Lie algebra gets a grading of the form $\mathfrak{g} = \mathfrak{g}_{-2} \oplus \cdots \oplus \mathfrak{g}_2$ by $\begin{pmatrix} \mathfrak{g}_0 & \mathfrak{g}_1 & \mathfrak{g}_2 \\ \mathfrak{g}_{-1} & \mathfrak{g}_0 & \mathfrak{g}_1 \\ \mathfrak{g}_{-2} & \mathfrak{g}_{-1} & \mathfrak{g}_0 \end{pmatrix}$. In particular, $\mathfrak{g}_{\pm 1} \cong \mathbb{C}$ and $\mathfrak{g}_{\pm 2} \cong \mathbb{R}$. Next, we define $G_0 \subset B \subset G$ as the subgroups of all elements whose adjoint action preserves the grading respectively the corresponding filtration of $\mathfrak{g}$. This means that g lies in G_0 (respectively B) if and only if $\mathrm{Ad}(g)(\mathfrak{g}_i) \subset \mathfrak{g}_i$ (respectively $\subset \mathfrak{g}_i \oplus \cdots \oplus \mathfrak{g}_2$) for all $i = -2, \dots, 2$. Note that B is actually a Borel subgroup of G.

Now one verifies directly, that any element of G_0 must be the class of a diagonal matrix, and a diagonal matrix lies in $SU(2,1)$ if and only if its entries on the diagonal are $(a, \bar{a}/a, 1/\bar{a})$ for $a \in \mathbb{C} \setminus \{0\}$. The Lie bracket $\mathfrak{g}_{-1} \times \mathfrak{g}_{-1} \to \mathfrak{g}_{-2}$ is just given by $(X, Y) \mapsto \bar{Y}X - \bar{X}Y$. A simple computation shows that the adjoint action of G induces an isomorphism from G_0 to the group of complex linear isomorphisms of $\mathfrak{g}_{-1}$ (which is isomorphic to $\mathbb{C} \setminus \{0\}$) and the action on $\mathfrak{g}_{-2}$ is chosen in such a way that it is compatible with the Lie bracket.

Now consider a product $G^+ \times G^-$ of two copies of G, with the corresponding Lie algebra $\mathfrak{g}^+ \oplus \mathfrak{g}^-$. Then the adjoint action is componentwise, we get a grading of $\mathfrak{g}^+ \oplus \mathfrak{g}^-$ and the subgroup of elements whose adjoint action preserves this grading (respectively the corresponding filtration) is exactly $G_0^+ \times G_0^-$ (respectively $B^+ \times B^-$). Now suppose that M is an oriented hyperbolic partially integrable almost CR manifold of CR dimension and codimension two. Let $\mathcal{G}_0$ be the complex frame bundle of the complex vector bundle $H^+M \oplus H^-M$ viewed as being modeled on $\mathfrak{g}_{-1}^+ \oplus \mathfrak{g}_{-1}^-$, i.e. the fiber of $\mathcal{G}_0$ over $x \in M$ is exactly the set of all pairs (u^+, u^-), where $u^\pm : \mathfrak{g}_{-1}^\pm \to H_x^\pm M$ is a complex linear isomorphism. For any $b \in G_0$, $\mathrm{Ad}(b) : \mathfrak{g}_{-1} \to \mathfrak{g}_{-1}$ is a complex linear isomorphism which respects the splitting, so $(u^+, u^-)\cdot b := (u^+, u^-) \circ \mathrm{Ad}(b)$ defines right action of G_0 on $\mathcal{G}_0$, which obviously is free and transitive on each fiber, thus making $p_0 : \mathcal{G}_0 \to M$ into a G_0-principal bundle.

Next, we define a filtration $V\mathcal{G}_0 \subset T^{-1}\mathcal{G}_0 \subset T\mathcal{G}_0$, where $V\mathcal{G}_0$ denotes the vertical bundle, by putting $T^{-1}\mathcal{G}_0 := (Tp_0)^{-1}(HM)$. We get a canonical one-form $\theta_{-2} \in \Omega^1(\mathcal{G}_0, \mathfrak{g}_{-2})$ as follows: Take a point

$u = (u^+, u^-) \in \mathcal{G}_0$. Then $u^+ \colon \mathfrak{g}_{-1}^+ \to H_x^+ M$ is a complex linear isomorphism, so there exist unique linear isomorphism $\tilde{u}^+ \colon \mathfrak{g}_{-2}^+ \to Q_x^+ M$ such that $\mathcal{L}_x(u^+(X), u^+(Y)) = \tilde{u}^+([X, Y])$, and similarly we get linear isomorphism $\tilde{u}^- \colon \mathfrak{g}_{-2}^- \to Q_x^- M$. For a tangent vector $\xi \in T_u\mathcal{G}_0$, consider $Tp_0 \cdot \xi \in T_x M$, and define

$$\theta_{-2}(\xi) := \big((\tilde{u}^+)^{-1}(q_+(Tp_0 \cdot \xi)), (\tilde{u}^-)^{-1}(q_-(Tp_0 \cdot \xi))\big),$$

where $q_\pm \colon TM \to Q^\pm M$ are the canonical projections. It is easy to see that this is a smooth one-form, and by construction its kernel in a point u is exactly the space $T_u^{-1}\mathcal{G}_0$. Moreover, since the action of G_0 on $\mathfrak{g}$ is compatible with the Lie bracket, it follows that for $(v^+, v^-) = (u^+, u^-) \cdot b$ we get $(\tilde{v}^+, \tilde{v}^-) = (\tilde{u}^+, \tilde{u}^-) \circ \mathrm{Ad}(b)$. Since $Tp_0 \circ Tr^b = Tp_0$, this implies that $(r^b)^* \theta_{-2} = \mathrm{Ad}(b^{-1}) \circ \theta_{-2}$, so θ_{-2} is equivariant.

Similarly, we get a canonical section $\theta_{-1} \in \Gamma(L(T^{-1}\mathcal{G}_0, \mathfrak{g}_{-1}))$. Namely, if $\xi \in T_u^{-1}\mathcal{G}_0$, then $Tp_0 \cdot \xi \in H_{p_0(u)} M$, so we uniquely decompose this as $\xi^+ + \xi^-$ with $\xi^\pm \in H_{p_0(u)}^\pm M$, and we define

$$\theta_{-1}(\xi) := \big((u^+)^{-1}(\xi^+), (u^-)^{-1}(\xi^-)\big).$$

Again, this is visibly smooth and its kernel in a point u is exactly the vertical tangent space $V_u\mathcal{G}_0$. Again since $Tp_0 \circ Tr^b = Tp_0$, the definition of the action of G_0 immediately implies that $(r^b)^* \theta_{-1} = \mathrm{Ad}(b^{-1}) \circ \theta_{-1}$. Consequently, $\theta = (\theta_{-2}, \theta_{-1})$ is a frame form of length one in the sense of [3, 3.2]. Moreover, by [3, proposition 4.2] this frame form satisfies the structure equation, so $(\mathcal{G}_0, \theta)$ is a P-frame bundle of degree one in the sense of [3, 3.6].

Theorem. *If M is an oriented hyperbolic partially integrable almost CR manifold of CR dimension and codimension two, then there is a canonical principal bundle $p \colon \mathcal{G} \to M$ with structure group $B \times B$, where B is the Borel subgroup in $PSU(2,1)$ endowed with a normal Cartan connection $\omega \in \Omega^1\big(\mathcal{G}, \mathfrak{su}(2,1) \times \mathfrak{su}(2,1)\big)$.*

Conversely, such a principal bundle and Cartan connection over a six dimensional smooth manifold M canonically make M into an oriented hyperbolic partially integrable almost CR manifolds of CR dimension and codimension two. These two constructions actually describe an equivalence of categories.

Proof. The first part is a special case of the main result of [3] or of the prolongation procedure of [9]. Moreover, the uniqueness part of this result implies that a local CR diffeomorphism $M_1 \to M_2$ lifts to a homomorphism $\Phi \colon \mathcal{G}_1 \to \mathcal{G}_2$ of principal bundles such that $\Phi^* \omega_2 = \omega_1$.

Conversely, let us assume that $p\colon \mathcal{G} \to M$ is a B-principal bundle over a smooth six dimensional manifold M endowed with a normal Cartan connection ω. For a point $x \in M$ choose a point $u \in \mathcal{G}$ with $p(u) = x$. Consider the component ω_{-2} of the Cartan connection which has values in $\mathfrak{g}_{-2}$. Then $\omega_{-2}(u)\colon T_u\mathcal{G} \to \mathfrak{g}^+_{-2} \oplus \mathfrak{g}^-_{-2}$ is a surjective linear map which vanishes on the vertical subbundle, so it induces a linear map $T_xM \to \mathfrak{g}^+_{-2} \oplus \mathfrak{g}^-_{-2}$. Let us denote by H_xM the kernel of this map and by Q_xM the quotient space T_xM/H_xM. Then we get a decomposition $Q_xM = Q^+_xM \oplus Q^-_xM$ by putting $Q^\pm_xM$ the subspaces mapped to $\mathfrak{g}^\pm_{-2}$. Further, the restriction of $\omega_{-1}(u)$ to the kernel of $\omega_{-2}(u)$ has as kernel exactly the vertical tangent space, so it descends to a linear isomorphism $H_xM \cong \mathfrak{g}^+_{-1} \oplus \mathfrak{g}^-_{-1}$, which gives us a decomposition $H_xM = H^+_xM \oplus H^-_xM$ into two one-dimensional complex spaces. If we choose a different point $\tilde{u}$ instead of u, then there is a unique element $b \in P$ such that $\tilde{u} = u{\cdot}b$ and equivariancy of the Cartan connection reads as $\omega(\tilde{u})(Tr^b{\cdot}\xi) = \mathrm{Ad}(b^{-1})(\omega(u)(\xi))$. Since $p \circ r^b = p$, this implies that the maps induced on tangent spaces of M change only by composition with $\mathrm{Ad}(b^{-1})$. Now B consists of elements respecting the filtration of $\mathfrak{g}$ so the adjoint action respects the set of elements with trivial $\mathfrak{g}_{-2}$-component, and thus the subspace H_xM remains unchanged. Moreover, on this subspace, the action of B factors through G_0 (see [3, 2.12]), which implies that also the complex structure and the decomposition of H_xM remain unchanged. Since the adjoint action of any element of G respects the decomposition $\mathfrak{g} = \mathfrak{g}^+ \oplus \mathfrak{g}^-$, also the decomposition of Q_xM remains unchanged.

Since the structures are independent of the choice of the point u, we can now use a local smooth section of $p\colon \mathcal{G} \to M$ to construct the data locally around a point and by uniqueness, they fit together globally, showing that we actually get a smooth subbundle HM, with a decomposition $H^+M \oplus H^-M$ into a sum of complex line bundles, as well as a decomposition $QM = Q^+M \oplus Q^-M$ of the quotient bundle $QM = TM/HM$. In particular, (M, HM) is an almost CR manifold of CR dimension and codimension two. The fact that the Cartan connection is normal implies that the underlying frame form of length one satisfies the structure equations, which in turn by [3, propositon 4.2] implies that the Levi bracket on M coincides with the algebraic bracket coming from the G_0-structure underlying the principal bundle $\mathcal{G}$. In particular, this implies that the Levi bracket is totally real, so (M, HM) is partially integrable. Moreover, we get that the bilinear map $\mathcal{L}$ is non-degenerate, it satisfies $\mathcal{L}(H^\pm M, HM) \subset Q^\pm M$ and for any point $x \in M$ the image of $\mathcal{L}$ contains Q^+_xM and Q^-_xM, and thus all of QM. But this immediately implies that for each nonzero $\psi \in Q^*_xM$ the map $\mathcal{L}^\psi$ is nonzero, so

(M, HM) is non-degenerate. Denoting by $\pi_\pm \colon QM \to Q^\pm M$ the canonical projections and choosing any local trivializations $\varphi_\pm$ of $Q_\pm$ we see that $\psi_\pm = \varphi_\pm \circ \pi_\pm$ are two linearly independent functionals such that $\mathcal{L}^{\psi_\pm}$ is degenerate (with null space $H^\pm$), so we see that (M, HM) is hyperbolic.

Finally, assume that $\mathcal{G}_1 \to M_1$ and $\mathcal{G}_2 \to M_2$ are two such principal bundles endowed with normal Cartan connections, and $\Phi \colon \mathcal{G}_1 \to \mathcal{G}_2$ is a homomorphism of principal bundles which covers a local diffeomorphism $f \colon M_1 \to M_2$. From the above construction of the CR structures on the M_i it is then immediate that $\Phi^*\omega_2 = \omega_1$ implies that $f \colon M_1 \to M_2$ is a CR map, and thus a local CR diffeomorphism, which establishes the equivalence of categories. Q.E.D.

2.7. The elliptic case

Next, we will consider the case of oriented elliptic partially integrable almost CR manifolds of CR dimension and codimension two. So we have $(M, HM, \tilde{J})$ such that for any point $x \in M$ and any nonzero element $\psi \in Q^*_x M$ the form $\mathcal{L}^\psi_x \colon H_x M \times H_x M \to \mathbb{R}$ introduced in 2.4 is non-degenerate. Note that since the complex subbundle $HM \subset TM$ is canonically oriented, choosing an orientation of M is equivalent to choosing an orientation of the quotient bundle QM.

Proposition. *Let $(M, HM, \tilde{J})$ be an oriented elliptic partially integrable almost CR manifold of CR dimension and codimension two with quotient bundle QM.*
(1) *There is a unique almost complex structure J^Q on the bundle QM which is compatible with the orientation of M and has the property that for each point $x \in M$ there is a nonzero element $\eta \in H_x M$ such that $\mathcal{L}_x(\tilde{J}\xi, \eta) = J^Q \mathcal{L}_x(\xi, \eta)$ for all $\xi \in H_x M$.*
(2) *If we define $H^\pm_x M$ to be the subspaces consisting of all $\eta \in H_x M$ such that $\mathcal{L}_x(\tilde{J}\xi, \eta) = \pm J^Q \mathcal{L}_x(\xi, \eta)$ for all $\xi \in H_x M$, then these subspaces fit together to form smooth subbundles $H^\pm M \subset HM$, which both are complex line bundles and have the property that $HM = H^+M \oplus H^-M$. Also, the Levi-bracket $\mathcal{L}$ vanishes on $H^+M \times H^+M$ and on $H^-M \times H^-M$.*
(3) *If we define a new almost complex structure J on HM by $J|_{H^+M} = -\tilde{J}$ and $J|_{H^-M} = \tilde{J}$, then with respect to the almost complex structures J and J^Q the Levi bracket $\mathcal{L} \colon HM \times HM \to QM$ is complex bilinear.*

Proof. Consider the complexification $Q_\mathbb{C} M = QM \otimes \mathbb{C}$ and the map $\mathcal{H} \colon HM \times HM \to Q_\mathbb{C} M$ defined by $\mathcal{H}(\xi, \eta) = \mathcal{L}(\tilde{J}\xi, \eta) + i\mathcal{L}(\xi, \eta)$. Using the fact that $\mathcal{L}$ is totally real, one immediately verifies that this is a $Q_\mathbb{C} M$-valued hermitian form on HM.

Next, for a point $x \in M$ consider the (complex) dual $(Q_{\mathbb{C}})^*_x M$ of the fiber of $Q_{\mathbb{C}}M$ at x. By definition, an element ψ of this space is a $\mathbb{C}$-linear map $(Q_{\mathbb{C}})_x M \to \mathbb{C}$. Similarly to 2.4 above, we can now consider $\mathcal{H}^\psi_x = \psi \circ \mathcal{H}_x \colon H_x M \times H_x M \to \mathbb{C}$. This is not a hermitian form any more, but it still is complex linear in the first and conjugate linear in the second variable, and we may still ask whether it is degenerate or non-degenerate. Moreover, as before the question whether $\mathcal{H}^\psi_x$ is degenerate or not, as well as the null space in the case where it is degenerate, depends only on the class of ψ in the (complex) projectivization $\mathcal{P}((Q_{\mathbb{C}})^*_x M) \cong \mathbb{C}P^1$. As before, we use square brackets to indicate the class in a projectivization. Choosing a real basis of $Q_x M$ (and considering the corresponding basis of the complexification) we can split $\mathcal{H}$ into two components $\mathcal{H}_1$, $\mathcal{H}_2$, and choosing further a complex basis ξ_1, ξ_2 of $H_x M$, we can consider the corresponding Hermitian matrices H_1, H_2 in this basis. Moreover, from the basis of $(Q_{\mathbb{C}})_x M$, we get homogeneous coordinates on $\mathcal{P}((Q_{\mathbb{C}})^*_x M)$ and in this picture the condition that $\mathcal{H}^\psi$ is degenerate exactly means that we have a solution $(\lambda : \mu)$ of the homogeneous polynomial of degree two given by $\det(\lambda H_1 + \mu H_2) = 0$. Since the matrices H_i are Hermitian, for any solution (λ, μ) of this equation, also $(\bar{\lambda}, \bar{\mu})$ is a solution. By definition, $\mathcal{H}^\psi_x(\xi, \eta) = 0$ if and only if $\psi(\mathcal{L}(\tilde{J}\xi, \eta)) = i\psi(\mathcal{L}(\xi, \eta))$. Now consider the image of $Q_x M \subset Q_x M \otimes \mathbb{C}$ under ψ, which is a real linear subspace of $\mathbb{C}$. If this is a proper subspace, then the above equation can only hold if the two sides both vanish. Since M is elliptic, we conclude that $\mathcal{H}^\psi_x$ is non-degenerate if $\psi(Q_x M) \neq \mathbb{C}$.

In particular, $\det(\lambda H_1 + \mu H_2)$ is not identically zero, so there are exactly two points $[\psi]$, $[\bar{\psi}] \in \mathcal{P}((Q_{\mathbb{C}})^*_x M)$ representing the solutions. From above, we know that the restrictions of ψ and $\bar{\psi}$ to $Q_x M$ are injective, so both maps define linear isomorphisms $Q_x M \to \mathbb{C}$. Clearly, exactly one of the maps ψ and $\bar{\psi}$ is orientation preserving as a real linear map from $Q_x M$ to $\mathbb{C}$ (with the canonical orientation), so we assume that ψ has this property. Thus, we get a complex structure J^Q_x on the vector space $Q_x M$, which clearly depends on the class $[\psi]$ only.

Using the formula from above, we see that the condition that $\mathcal{H}^\psi$ is degenerate exactly means that there is a nonzero element $\eta \in H_x M$ such that $\psi(\mathcal{L}(\tilde{J}\xi, \eta)) = i\psi(\mathcal{L}(\xi, \eta))$ for all $\xi \in H_x M$. By definition of J^Q, the last equation just reads as $\mathcal{L}(\tilde{J}\xi, \eta) = J^Q \mathcal{L}(\xi, \eta)$. Similarly, for an element η to lie in the null space of $\mathcal{H}^{\bar{\psi}}$ is equivalent to $\mathcal{L}_x(\tilde{J}\xi, \eta) = -J^Q \mathcal{L}_x(\xi, \eta)$, so the two null spaces are exactly $H^\pm_x M$ as defined in the theorem. By construction, they are both complex subspaces and nonzero, and their intersection is zero since $\mathcal{L}$ is non-degenerate, so the only possibility is that they both are of complex dimension one and $H_xM = H_x^+M \oplus H_x^-M$, so (1) follows.

To see that we get smooth subbundles $H^{\pm}M \subset HM$, one just has to note that the above constructions depend smoothly on the point x. Indeed, $\mathcal{H}$ is a smooth form $HM \times HM \to Q_{\mathbb{C}}M$. Choosing smooth local frames for HM and QM and the corresponding homogeneous coordinates on $\mathcal{P}((Q_{\mathbb{C}})^*M)$, we see that $(x, \lambda, \mu) \mapsto \det(\lambda H_1(x) + \mu H_2(x))$ is a smooth function, which is regular since the polynomial has different roots in each point, so the solutions form a smooth submanifold in $\mathcal{P}((Q_{\mathbb{C}})^*M)$, which by construction is a two-fold covering of M. The condition on the orientation distinguishes the two sheets of the covering, so we get a smooth section $M \to \mathcal{P}((Q_{\mathbb{C}})^*M)$ whose value in each point is exactly the class leading to the almost complex structure J_x^Q. Thus $J^Q \colon QM \to QM$ is smooth and hence an almost complex structure. Finally, choosing a local section ψ of $Q_{\mathbb{C}}^*M$ whose class in each point x is the distinguished element of $\mathcal{P}((Q_{\mathbb{C}})_x^*M)$, we get a smooth vector bundle map $HM \to L(HM, \mathbb{C})$ defined by $\eta_x \mapsto \mathcal{H}_x^{\psi}(\_, \eta)$, and H^+M is exactly the kernel of this bundle map, which we already know is of constant rank, so it is a smooth subbundle. Similarly, one deals with H^-M.

By construction, if one considers the restriction of the Levi-bracket $\mathcal{L}$ to $H^+M \times H^+M$, then this is complex bilinear and skew symmetric, so it vanishes since H^+M has complex rank one. Similarly, $\mathcal{L}$ vanishes on $H^-M \times H^-M$, so (2) is proved.

Thus, the only part of the Levi-bracket that has to be considered is its restriction to $H^+M \times H^-M$. By construction, this is conjugate linear in the first and complex linear in the second variable, so switching the almost complex structure on H^+M, it becomes complex bilinear.

Q.E.D.

2.8. An equivalence of categories

The structure (M, H^+M, H^-M, J^Q) obtained in proposition 2.7 is preserved under orientation preserving local CR diffeomorphisms. Suppose that $(M_k, HM_k, \tilde{J}_k)$ are oriented elliptic partially integrable almost CR manifolds and $f \colon M_1 \to M_2$ is an orientation preserving (local) CR diffeomorphism. For $x \in M_1$ consider $f(x) \in M_2$ and the induced linear isomorphism $T_xf \colon Q_xM_1 \to Q_xM_2$. Pulling back the complex structure on Q_xM_2 obtained from proposition 2.7(1) to Q_xM_1, we obtain a complex structure which is compatible with the orientation since f was assumed to be orientation preserving and the restriction of T_xf to the CR tangent spaces is complex linear and thus orientation preserving. Choosing $\eta \in H_xM_1$ such that $T_xf \cdot \eta \in H_{f(x)}^+M_2$, we conclude from $f^*\mathcal{L}^{M_2} = \mathcal{L}^{M_1}$ that $\mathcal{L}_x(\_,\eta)$ is complex linear for the pulled back structure, so by the uniqueness in proposition 2.7(1), we see that $T_xf\colon Q_xM_1 \to Q_xM_2$ is complex linear for the structures obtained from proposition 2.7(1). Moreover, the above argument also shows that $T_xf(H^\pm_x M_1) \subset H^\pm_{f(x)}M_2$. But then the fact that $T_xf\colon HM_1 \to HM_2$ is complex linear for the structures $\tilde{J}$ implies that it is also complex linear for the structures J obtained by proposition 2.7(3).

Conversely, assume that M is a smooth manifold of dimension 6 equipped with two complementary complex line bundles $H^\pm M \subset TM$, an almost complex structure J^Q on the quotient $QM = TM/HM$, where $HM = H^+M \oplus H^-M$, which is such that the bundle map $\mathcal{L}\colon HM \times HM \to QM$ induced by the Lie bracket is complex bilinear and non-degenerate. Then on HM consider the almost complex structure $\tilde{J}$ defined by flipping the complex structure on H^+M and keeping the complex structure on H^-M. The Levi bracket $\mathcal{L}\colon HM \times HM \to QM$ is by assumption non-degenerate. For ξ, $\eta \in HM$, we can split $\xi = \xi_+ + \xi_-$ and similarly for η and we compute

$$\begin{aligned}\mathcal{L}(\tilde{J}\xi, \tilde{J}\eta) &= \mathcal{L}(-J\xi_+ + J\xi_-, -J\eta_+ + J\eta_-)\\ &= -\mathcal{L}(J\xi_-, J\eta_+) - \mathcal{L}(J\xi_+, J\eta_-)\\ &= \mathcal{L}(\xi_-, \eta_+) + \mathcal{L}(\xi_+, \eta_-) = \mathcal{L}(\xi, \eta),\end{aligned}$$

where we have used that $\mathcal{L}$ is complex bilinear for J and $H^\pm M$ are isotropic for $\mathcal{L}$ since they have complex dimension one. Thus $(M, HM, \tilde{J})$ is a partially integrable almost CR manifold of CR dimension and codimension two. Finally, consider nonzero elements $\psi \in Q^*_xM$ and $\xi \in H_xM$. As before, we may split $\xi = \xi_+ + \xi_-$, and let us assume without loss of generality that $\xi_+ \neq 0$. Then $\mathcal{L}(\xi_+,\_)$ restricts to a nonzero complex linear (and thus surjective) map $H^-_xM \to Q_xM$, so we can find $\eta \in H^-_xM$ such that $\psi(\mathcal{L}(\xi_+, \eta)) \neq 0$. But since $\eta \in H^-_xM$, we have $\mathcal{L}(\xi, \eta) = \mathcal{L}(\xi_+, \eta)$, which implies that $\mathcal{L}^\psi$ is non-degenerate, and hence $(M, HM, \tilde{J})$ is elliptic. Finally, if $(M_j, H^+M_j, H^-M_j, J^Q_j)$ are such manifolds for $j = 1$, 2 and $f\colon M_1 \to M_2$ is a (local) diffeomorphism such that Tf restricts to complex linear isomorphism $H^\pm M_1 \to H^\pm M_2$ then f defines a (local) CR diffeomorphism from $(M_1, HM_1, \tilde{J}_1)$ to $(M_2, HM_2, \tilde{J}_2)$. Note that the condition on Tf preserving the subbundles $H^\pm$ implies that we get an induced map $Tf\colon QM_1 \to QM_2$, which is automatically complex linear since $Tf\cdot\mathcal{L}(\xi,\eta) = \mathcal{L}(Tf\cdot\xi, Tf\cdot\eta)$ because $\mathcal{L}$ is induced by the Lie bracket. Thus we get

Theorem. *The category of elliptic partially integrable almost CR manifolds of CR dimension and codimension two and local CR diffeomorphisms is equivalent to the category whose objects are six dimensional manifolds M endowed with two complementary complex line bundles $H^{\pm}M \subset TM$ and an almost complex structure on $QM = TM/HM$ such that the Levi bracket $\mathcal{L}\colon HM \times HM \to QM$ is non-degenerate and complex bilinear, and whose morphisms are the local diffeomorphisms whose derivative in each point restricts to complex linear isomorphisms between the subbundles $H^{\pm}M$.*

2.9. Real parabolic geometries of type $(PSL(3,\mathbb{C}),B)$

To show that the category from theorem 2.8 above is equivalent to a category of normal parabolic geometries is now strictly parallel to the hyperbolic case, so we are more brief on that. Consider first the Lie algebra $\mathfrak{g} = \mathfrak{sl}(3,\mathbb{C})$ (as a real Lie algebra), let $\mathfrak{g}_0$ be its Cartan subalgebra, i.e. the subalgebra of all trace free diagonal matrices and let $\mathfrak{b}$ be its Borel subalgebra, i.e. the subalgebra of all trace free upper triangular matrices. As a module over $\mathfrak{g}_0$, $\mathfrak{g}$ decomposes as a direct sum of $\mathfrak{g}_0$ and the root spaces, and we write this decomposition as $\mathfrak{g} = \mathfrak{g}_{-2} \oplus (\mathfrak{g}_{-1}^- \oplus \mathfrak{g}_{-1}^+) \oplus \mathfrak{g}_0 \oplus (\mathfrak{g}_1^- \oplus \mathfrak{g}_1^+) \oplus \mathfrak{g}_2$ defined by $\begin{pmatrix} \mathfrak{g}_0 & \mathfrak{g}_1^+ & \mathfrak{g}_2 \\ \mathfrak{g}_{-1}^+ & \mathfrak{g}_0 & \mathfrak{g}_1^- \\ \mathfrak{g}_{-2} & \mathfrak{g}_{-1}^- & \mathfrak{g}_0 \end{pmatrix}$. Clearly, this makes $\mathfrak{g}$ into a graded Lie algebra, i.e. $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$. The subalgebra $\mathfrak{b}$ is exactly the non-negative part in this grading, so the adjoint action of $\mathfrak{b}$ never moves down in the grading, which implies that the corresponding filtration is $\mathfrak{b}$-invariant. By the grading property, we have in particular the bracket $\mathfrak{g}_{-1} \times \mathfrak{g}_{-1} \to \mathfrak{g}_{-2}$. Like all brackets in $\mathfrak{g}$ this is complex bilinear, and hence $\mathfrak{g}_{-1}^+$ and $\mathfrak{g}_{-1}^-$ are isotropic, while the restriction of the bracket to $\mathfrak{g}_{-1}^+ \times \mathfrak{g}_{-1}^-$ is non-degenerate.

Next, consider the adjoint group $G = PSL(3,\mathbb{C})$ of $\mathfrak{g}$. We define subgroups $G_0 \subset B \subset G$ as the groups of those elements whose adjoint actions preserve the grading respectively the filtration on $\mathfrak{g}$. According to the general theory (see [3, proposition 2.9]), G_0 has Lie algebra $\mathfrak{g}_0$ and B has Lie algebra $\mathfrak{b}$, so it is a Borel subgroup in G. The group G is the quotient of $SL(3,\mathbb{C})$ by its center, which is just the third roots of unity times the identity matrix, so we will usually work in $SL(3,\mathbb{C})$ keeping in mind that we work modulo the center. Now it is easy to verify that for $g \in G_0$, any representative in $SL(3,\mathbb{C})$ must be diagonal. If a, b, c are the diagonal entries (and $abc = 1$), then one immediately verifies that the adjoint action on $\mathfrak{g}_{-1}^+$, $\mathfrak{g}_{-1}^-$ and $\mathfrak{g}_{-2}$ is given by multiplication by $a^{-1}b$, $b^{-1}c$ and $a^{-1}c$, respectively. Taking $0 \neq \lambda$, $\mu \in \mathbb{C}$ we see that putting $a = (\lambda^{-2}\mu^{-1})^{1/3}$, $b = (\lambda\mu^{-1})^{1/3}$ and $c = (\lambda\mu^2)^{1/3}$ we obtain an element that acts on $\mathfrak{g}^+_{-1}$ by multiplication with λ and on $\mathfrak{g}^-_{-1}$ by μ, while the action on $\mathfrak{g}_{-2}$ is fixed by compatibility with the Lie bracket. Moreover, one easily sees that by this condition the diagonal matrix with entries a, b, c is uniquely determined up to multiplication with a third root of unity times the identity matrix. Thus we see that the adjoint action identifies G_0 with the group of pairs φ_+, φ_-, where $\varphi_\pm$ is a complex linear isomorphism of $\mathfrak{g}^\pm_{-1}$, and the action on $\mathfrak{g}_{-2}$ is fixed by compatibility with the Lie bracket.

Now let M be a smooth manifold of dimension 6 equipped with two complementary complex line bundles $H^\pm M \subset TM$, an almost complex structure on $QM = TM/HM$ such that the Levi bracket $\mathcal{L}\colon HM \times HM \to QM$ is non-degenerate and complex bilinear. As in the hyperbolic case we consider the complex frame bundle $\mathcal{G}_0$ of $H^+M \oplus H^-M$ as modeled on $\mathfrak{g}_{-1}$, and via the adjoint action we can view this as a principal G_0-bundle. Denoting elements of $\mathcal{G}_0$ as (u^+, u^-), where $u^\pm\colon \mathfrak{g}^\pm_{-1} \to H^\pm_x M$ is a complex linear isomorphism, we now get a unique complex linear isomorphism $\tilde{u}\colon \mathfrak{g}_{-2} \to Q_xM$ such that $\mathcal{L}(u^+(X), u^-(Y)) = \tilde{u}([X,Y])$, which allows us to define a form $\theta_{-2} \in \Omega^1(\mathcal{G}_0, \mathfrak{g}_{-2})$ as in the hyperbolic case. As there one shows that the form is equivariant and its kernel is exactly $T^{-1}\mathcal{G}_0 = (Tp_0)^{-1}(HM)$. On the other hand, the definition of $\theta_{-1} \in \Gamma(L(T^{-1}\mathcal{G}_0, \mathfrak{g}_{-1}))$ is completely the same as in the hyperbolic case, and also the properties are verified in the same way. Hence we again get a frame form $(\theta_{-2}, \theta_{-1})$ of length one on $\mathcal{G}_0$, which satisfies the structure equations, since $\mathcal{L}$ is induced by the Lie bracket of vector fields, and we get:

Theorem. *For any elliptic partially integrable almost CR manifold M there exists a canonical principal bundle $p\colon \mathcal{G} \to M$ with group B equipped with a normal Cartan connection $\omega \in \Omega^1(\mathcal{G}, \mathfrak{sl}(3,\mathbb{C}))$. Conversely, a principal B-bundle $\mathcal{G}$ over a smooth 6-dimensional manifold M endowed with a normal Cartan connection makes M canonically into an oriented elliptic partially integrable almost CR manifold. These constructions actually give rise to an equivalence of categories.*

Proof. Existence of $\mathcal{G}$ and ω for an elliptic partially integrable almost CR manifold follows from the main result of [3] or the procedure of [9] or (with a reinterpretation of the underlying structure along the lines of [3, 4.4]) from the original procedure of Tanaka, see [13]. In view of theorem 2.8, the converse is completely analogous to the hyperbolic case. Explicitly, $HM = Tp(\ker(\omega_{-2}))$, the almost complex structure on QM is induced by ω_{-2} and the splitting of HM, as well as the almost complex structure J on HM are induced by ω_{-1}. (To get back to a

CR picture, one has to flip the almost complex structure on H^+M as in 2.8.)

Also, establishing the equivalence of categories is done exactly like in the hyperbolic case. Q.E.D.

§3. Interpretations of torsions in the elliptic case

One of the main advantages of having a canonical Cartan connection is that this offers a conceptual approach to obstructions against local flatness. It is well known in general (see e.g. [3, proposition 4.12] for a proof in the setting of parabolic geometries) that the curvature of a Cartan connection is a complete obstruction against local flatness. In our setting this means that a point $x \in M$ has a neighborhood which is CR diffeomorphic to an open subset of the flat model (which is a quadric in our case) if and only if the curvature of the Cartan connection vanishes identically locally around x. In the case of normal Cartan connection there is a further refinement of that. In this section we will show how general tools for parabolic geometries can be used to give geometric interpretations of these obstructions. Since this has been done in the hyperbolic case in some detail in [11], we restrict to the elliptic case here, in which we get several new results and improvements compared to the latter paper.

3.1. The curvature of the normal Cartan connection

As a start, we have to describe the normalization condition on our Cartan connections in a little more detail. For the Cartan connection $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$, we define the *curvature function* $\kappa\colon \mathcal{G} \to L(\bigwedge^2 \mathfrak{g}_-, \mathfrak{g})$, where $\mathfrak{g}_- = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$ and the exterior product is over $\mathbb{R}$, by $\kappa(u)(X,Y) = d\omega(\omega_u^{-1}(X), \omega_u^{-1}(Y)) + [X,Y]$. Recall that by definition, $\omega_u\colon T_u\mathcal{G} \to \mathfrak{g}$ is a linear isomorphism, so $\omega_u^{-1}(X)$ makes sense. This curvature function captures all the information about the curvature of the Cartan connection. Note that κ splits into homogeneous degrees with respect to the grading as $\kappa = \kappa^{(0)} + \dots + \kappa^{(6)}$, where $\kappa^{(0)}$ maps $\bigwedge^2 \mathfrak{g}_{-1}$ to $\mathfrak{g}_{-2}$, $\kappa^{(1)}$ maps $\bigwedge^2 \mathfrak{g}_{-1}$ to $\mathfrak{g}_{-1}$ and $\mathfrak{g}_{-2} \otimes \mathfrak{g}_{-1}$ to $\mathfrak{g}_{-2}$, and so on. The first normalization condition on ω is that $\kappa^{(0)}$ is identically zero.

To formulate the second normalization condition, observe that $L(\bigwedge^2 \mathfrak{g}_-, \mathfrak{g})$ is just the second chain group in the standard complex computing the (real!) Lie algebra cohomology $H^*_{\mathbb{R}}(\mathfrak{g}_-, \mathfrak{g})$. The spaces in this complex are just the spaces $L(\bigwedge^k_{\mathbb{R}} \mathfrak{g}_-, \mathfrak{g})$ of k-linear, alternating maps $\mathfrak{g}_-^k \to \mathfrak{g}$, and the differential $\partial\colon L(\bigwedge^k \mathfrak{g}_-, \mathfrak{g}) \to L(\bigwedge^{k+1} \mathfrak{g}_-, \mathfrak{g})$ is defined by the usual formula

$$\partial\varphi(X_0,\dots,X_k)=\sum_{i=0}^{k}(-1)^i[X_i,\varphi(X_0,\dots,\hat{X}_i,\dots,X_k)]$$
$$+\sum_{i<j}(-1)^{i+j}\varphi([X_i,X_j],X_0,\dots,\hat{X}_i,\dots,\hat{X}_j,\dots,X_k).$$

For our purpose the essential fact is that extending a construction of Kostant (see [7]) one can show that there is a natural adjoint $\partial^*\colon L(\bigwedge^k\mathfrak{g}_-,\mathfrak{g})\to L(\bigwedge^{k-1}\mathfrak{g}_-,\mathfrak{g})$ to the Lie algebra differential, see [3, 2.5, 2.6]. It turns out that both ∂ and ∂^* are differentials, so $\partial^2=(\partial^*)^2=0$, defining the Laplacian $\Box=\partial\partial^*+\partial^*\partial$ one gets a Hodge decomposition $L(\bigwedge^k\mathfrak{g}_-,\mathfrak{g})=\mathrm{Im}(\partial)\oplus\mathrm{Ker}(\Box)\oplus\mathrm{Im}(\partial^*)$ and $\mathrm{Ker}(\Box)$ is naturally isomorphic to the k-th cohomology group $H^k_{\mathbb{R}}(\mathfrak{g}_-,\mathfrak{g})$. A formula for ∂^* can be easily obtained from the fact that it is the dual map to another Lie algebra differential. For this, one has to note that the Killing form on $\mathfrak{g}$ induces a duality of $\mathfrak{g}_0$-modules between $\mathfrak{g}_-$ and $\mathfrak{g}_+$. We shall only need the formula for $\partial^*\colon L(\bigwedge^2\mathfrak{g}_-,\mathfrak{g})\to L(\mathfrak{g}_-,\mathfrak{g})$, which is given by

$$\partial^*\varphi(X)=\sum_\alpha\big([Z_\alpha,\varphi(X_\alpha,X)]+\tfrac{1}{2}\varphi([Z_\alpha,X]_-,X_\alpha)\big),$$

where $\{X_\alpha\}$ is a basis of $\mathfrak{g}_-$, $\{Z_\alpha\}$ is the dual basis of $\mathfrak{g}_+$ and $[\ ,\ ]_-$ denotes the $\mathfrak{g}_-$-component of the bracket, see [3, 2.5]. The second normalization condition on the Cartan connection is that the curvature function has values in $\mathrm{Ker}(\partial^*)=\mathrm{Ker}(\Box)\oplus\mathrm{Im}(\partial^*)$. Note that from the formulae it is obvious that both ∂ and ∂^* preserve homogeneous degrees, so this conditions can be applied to various homogeneous degrees of κ separately. Also, this implies that the cohomology groups split as direct sums $H^k(\mathfrak{g}_-,\mathfrak{g})=\sum_l H^k_{(l)}(\mathfrak{g}_-,\mathfrak{g})$ according to homogeneous degree.

For later use, we note one further refinement: Since $\mathfrak{g}$ is a complex representation of $\mathfrak{g}_-$, we can also view $L(\bigwedge^k_{\mathbb{R}}\mathfrak{g}_-,\mathfrak{g})\cong\bigwedge^k_{\mathbb{R}}\mathfrak{g}_-^*\otimes_{\mathbb{R}}\mathfrak{g}$ as $(\bigwedge^k_{\mathbb{R}}\mathfrak{g}_-^*\otimes_{\mathbb{R}}\mathbb{C})\otimes_{\mathbb{C}}\mathfrak{g}$. Now $\bigwedge^k_{\mathbb{R}}\mathfrak{g}_-^*\otimes_{\mathbb{R}}\mathbb{C}$ can be identified with $\bigwedge^k_{\mathbb{C}}(\mathfrak{g}_-^*\otimes_{\mathbb{R}}\mathbb{C})$ and the splitting of the complexification of $\mathfrak{g}_-^*$ into a holomorphic and an antiholomorphic part induces a splitting $L(\bigwedge^k_{\mathbb{R}}\mathfrak{g}_-,\mathbb{C})=\oplus_{p+q=k}L^{p,q}(\mathfrak{g}_-,\mathbb{C})$, which in turn induces a similar splitting for $L(\bigwedge^k_{\mathbb{R}}\mathfrak{g}_-,\mathfrak{g})$. Explicitly, $L^{1,0}(\mathfrak{g}_-,\mathfrak{g})$ and $L^{0,1}(\mathfrak{g}_-,\mathfrak{g})$ are the spaces of complex linear respectively conjugate linear maps. More generally, $L^{k,0}(\mathfrak{g}_-,\mathfrak{g})$ and $L^{0,k}(\mathfrak{g}_-,\mathfrak{g})$ are the spaces of those k-linear alternating maps which are complex linear respectively complex anti-linear in each variable. The last bit of information that we will need is that $L^{1,1}(\mathfrak{g}_-,\mathfrak{g})$ is exactly the set of those bilinear, alternating maps $\varphi\colon \mathfrak{g}_- \times \mathfrak{g}_- \to \mathfrak{g}$ which have the property that $\varphi(iX, iY) = \varphi(X,Y)$ for all X, $Y \in \mathfrak{g}_-$.

Thus, the final splitting of the spaces in the standard complex looks as $L(\bigwedge^k \mathfrak{g}_-,\mathfrak{g}) = \bigoplus_{p+q=k;l} L^{p,q}_{(l)}(\mathfrak{g}_-,\mathfrak{g})$, where l refers to the homogeneous degree. The fact that the brackets in $\mathfrak{g}$ all are complex bilinear together with the fact that ∂ preserves homogeneous degrees now implies that $\partial(L^{p,q}_{(l)}(\mathfrak{g}_-,\mathfrak{g})) \subset L^{p+1,q}_{(l)}(\mathfrak{g}_-,\mathfrak{g})$. For all fixed q and l we have a subcomplex $L^{*,q}_{(l)}(\mathfrak{g}_-,\mathfrak{g})$, and the cohomology groups finally split as $H^k(\mathfrak{g}_-,\mathfrak{g}) = \bigoplus_{p+q=k;l} H^{p,q}_{(l)}(\mathfrak{g}_-,\mathfrak{g})$. From the construction of the adjoint ∂^* one can verify that also ∂^* is compatible with bidegrees, so $\partial^*(L^{p,q}_{(l)}(\mathfrak{g}_-,\mathfrak{g})) \subset L^{p-1,q}_{(l)}(\mathfrak{g}_-,\mathfrak{g})$. In particular, ∂^* is identically zero on $L^{0,k}_{(l)}(\mathfrak{g}_-,\mathfrak{g})$ for all k, l.

The final point we have to observe is the Bianchi identity for the curvature of a Cartan connection, which can be used to compute $\partial \circ \kappa$. By [3, proposition 4.9] this reads as

$$(\partial \circ \kappa)(X,Y,Z) = \sum_{\text{cycl}} \big(\kappa(\kappa_-(X,Y),Z) + (\omega^{-1}(X)\cdot\kappa)(Y,Z)\big).$$

Here the sum is over all cyclic permutations of the arguments, κ_- is the component of κ in $\mathfrak{g}_-$, and in the last term we use the vector field $\omega^{-1}(X)$ to differentiate the function κ. The importance of this identity is that if we consider a fixed homogeneous degree on the left hand side, then only lower homogeneous degrees can enter on the right hand side. In particular, the lowest nonzero homogeneous degree of κ must have values in $\mathrm{Ker}(\partial) \cap \mathrm{Ker}(\partial^*) = \mathrm{Ker}(\Box)$ which is isomorphic to an appropriate cohomology group. This can be used to show (see [3, proposition 4.10]) that κ vanishes if and only if its harmonic part (i.e. its component in $\mathrm{Ker}(\Box)$) vanishes. Moreover, from the formula one can obviously get informations on possible bidegrees of $\partial \circ \kappa$ if the possible bidegrees of lower homogeneous components are known.

3.2. Harmonic curvature components

The relevant cohomology group $H^2(\mathfrak{g}_-,\mathfrak{g})$ for our case has been computed in [11]. Expressed in terms of bidegrees the results read as follows:

1. The only nonzero components in $H^2(\mathfrak{g}_-,\mathfrak{g})$ are $H^{0,2}_{(1)}$, $H^{1,1}_{(1)}$, and $H^{2,0}_{(4)}$.
2. $H^{0,2}_{(1)}$ splits as a B_0-module into two one-dimensional components, which are represented by maps $\mathfrak{g}_{-2} \times \mathfrak{g}^{\pm}_{-1} \to \mathfrak{g}_{-2}$.

3. $H^{1,1}_{(1)}$ splits into four one-dimensional components which are represented by totally real maps $\bigwedge^2 \mathfrak{g}^+_{-1} \to \mathfrak{g}^-_{-1}$, and $\bigwedge^2 \mathfrak{g}^-_{-1} \to \mathfrak{g}^+_{-1}$ respectively by maps $\mathfrak{g}^+_{-1} \times \mathfrak{g}^-_{-1} \to \mathfrak{g}^-_{-1}$ and $\mathfrak{g}^-_{-1} \times \mathfrak{g}^+_{-1} \to \mathfrak{g}^+_{-1}$, which are complex linear in the first and conjugate linear in the second variable (this last point needs a slightly closer look on the description of cohomologies in [11]).
4. $H^{2,0}_{(4)}$ splits into two one dimensional components represented by complex bilinear maps $\mathfrak{g}_{-2} \times \mathfrak{g}^+_{-1} \to \mathfrak{g}^+_1$ respectively $\mathfrak{g}_{-2} \times \mathfrak{g}^-_{-1} \to \mathfrak{g}^-_1$.

In particular, we have six irreducible torsion-type components (the components homogeneous of degree one) and two curvature-type components. Note moreover, that the homogeneous component $\kappa^{(1)}$ of the curvature function is harmonic by the Bianchi identity, so only the components indicated in 2 and 3 of the above list may actually be nonzero.

Our next task is to give geometric interpretations of the harmonic components of the curvature which are of torsion-type. Most of this has already been done in [11], but we partly have different interpretations and partly simpler and more complete proofs. The main tool for deducing these interpretations is the following general result on parabolic geometries which is a variant of [11, lemma 2.10]:

Lemma. *Let $(p\colon \mathcal{G} \to M, \omega)$ be a parabolic geometry, i.e. a principal P-bundle endowed with a $\mathfrak{g}$-valued Cartan connection, where $P \subset G$ is a parabolic subgroup, and let κ be the curvature function of ω. Suppose that $x \in M$ is a point and $u \in \mathcal{G}$ is such that $p(u) = x$. Then there is a neighborhood U of $x \in M$ and an extension operator $\xi_x \mapsto \xi$ from the tangent space T_xM to the set of local vector fields defined on U, which is compatible with all structures on TM carried over from $\mathfrak{g}_-$ using ω, and which has the following property: For ξ_x, $\eta_x \in T_xM$ let X, $Y \in \mathfrak{g}_-$ be the unique elements such that $\xi_x = T_up\cdot\omega_u^{-1}(X)$ and $\eta_x = T_up\cdot\omega_u^{-1}(Y)$. Then*

$$[\xi,\eta](x) = T_up\cdot\omega_u^{-1}\big([X,Y] - \kappa(u)(X,Y)\big).$$

Proof. (see [11, lemma 2.10]) Note first that X and Y are well defined since the difference of two lifts of a tangent vector on M lies in the vertical bundle, which is mapped by ω to $\mathfrak{b}$. Denoting by Fl^ξ the flow of a vector field ξ, we consider the map $\varphi(X) := \mathrm{Fl}_1^{\omega^{-1}(X)}(u)$, which is defined on a neighborhood V of 0 in $\mathfrak{g}_-$. The tangent map in zero of $p \circ \varphi\colon V \to M$ is a linear isomorphism, so possibly shrinking V we may assume that φ and $p \circ \varphi$ are both diffeomorphisms onto their images. Denoting $U := p(\varphi(V))$, we get a smooth local section $\sigma\colon U \to \varphi(V)$, since p has to restrict to a diffeomorphism $\varphi(V) \to U$.

Now given $\xi_x \in T_xM$ we define the extension $\xi \in \mathfrak{X}(U)$ by $\xi(y) := Tp\cdot\big(\omega(\sigma(y))^{-1}(X)\big)$. Clearly, this extension operator is compatible with all structures on TM carried over from $\mathfrak{g}_-$ via ω. To verify the condition on the brackets, let us start with two tangent vectors ξ_x and η_x and the corresponding elements X, $Y \in \mathfrak{g}_-$. Any point in $p^{-1}(U)$ can be uniquely written as $\sigma(y)b$ for $y \in U$ and $b \in P$. Defining a local vector field $\tilde{\xi}$ on $p^{-1}(U)$ by $\tilde{\xi}(\sigma(y)\cdot b) := Tr^b\cdot\big(\omega(\sigma(y))^{-1}(X)\big)$, we obviously get a projectable vector field, which projects onto ξ and similarly we get a field $\tilde{\eta}$ projecting onto η. Hence, $[\xi,\eta](x) = T_up\cdot[\tilde{\xi},\tilde{\eta}](u)$. Moreover, $u = \sigma(p)$, so the flow lines of $\tilde{\xi}$ and $\tilde{\eta}$ through u stay in the image of σ, and thus $\omega(\tilde{\xi})$ is constant along the flow line of $\tilde{\eta}$ through u and vice versa. But then the definition of the exterior derivative implies that $\omega([\tilde{\xi},\tilde{\eta}])(u) = -d\omega(u)(\tilde{\xi}(u),\tilde{\eta}(u))$, and since $\tilde{\xi}(u) = \omega(u)^{-1}(X)$ and similarly for η, the result follows from the definition of κ. Q.E.D.

The main upshot of this result is that whenever one has a tensorial quantity which can be expressed as a combination of Lie brackets of vector fields (i.e. whenever one has an expression in Lie brackets which becomes linear over smooth functions in all arguments), then one can compute the value of the corresponding tensor on tangent vectors using the extensions described in the proposition, and thus immediately get an interpretation of the tensor in terms of the curvature function κ.

3.3. The Nijenhuis tensor

Let $(M, HM, \tilde{J})$ be an elliptic partially integrable almost CR manifold. Then there is an obvious first candidate for a torsion type object, namely the Nijenhuis tensor $\tilde{N}\colon HM \times HM \to HM$ introduced in 2.1. As we have seen, $\tilde{N}$ is skew symmetric and conjugate linear (with respect to $\tilde{J}$) in both arguments, so its restriction to $H^+M \times H^+M$ and to $H^-M \times H^-M$ vanishes since both these subbundles are of complex rank one, so what remains is $\tilde{N}\colon H^+M \times H^-M \to HM$, which we can split as $\tilde{N}^+ + \tilde{N}^-$ according to the splitting of the values. Then both components are conjugate linear with respect to $\tilde{J}$ in both arguments, so with respect to J they both become sesquilinear, which fits well to two of the harmonic curvature components described above. In fact, vanishing of the parts of the Nijenhuis tensor is equivalent to vanishing of the corresponding irreducible components of the curvature function:

Proposition. *Let κ be the curvature function of the canonical Cartan connection. Then the restriction of $\kappa^{(1)}$ to $\mathfrak{g}^+_{-1} \times \mathfrak{g}^-_{-1}$ has values in $\mathfrak{g}^+_{-1}$ (respectively $\mathfrak{g}^-_{-1}$) if and only if $\tilde{N}^- = 0$ (respectively $\tilde{N}^+ = 0$). In particular, M is integrable and thus a CR manifold if and only if $\kappa^{(1)}$ vanishes on $\mathfrak{g}^+_{-1} \times \mathfrak{g}^-_{-1}$.*

Proof. For tangent vectors $\xi_x \in H^+_x M$ and $\eta_x \in H^-_x M$, the Nijenhuis tensor can be computed as

$$\tilde{N}(\xi_x,\eta_x) = [\xi,\eta](x) - [\tilde{J}\xi,\tilde{J}\eta](x) + \tilde{J}\big([\tilde{J}\xi,\eta](x) + [\xi,\tilde{J}\eta](x)\big),$$

for any extensions ξ, η to local smooth vector fields. Splitting the result into $\tilde{N}^+ + \tilde{N}^-$, we may replace the $\tilde{J}'s$ by plus or minus J. Denoting by $\pi_\pm \colon H_x M \to H^\pm_x M$ the canonical projections, we obtain the following expressions for $\tilde{N}^+(\xi_x,\eta_x)$ and $\tilde{N}^-(\xi_x,\eta_x)$

$$\pi_+\big([\xi,\eta](x) + [J\xi,J\eta](x)\big) - J\pi_+\big(-[J\xi,\eta](x) + [\xi,J\eta](x)\big)$$
$$\pi_-\big([\xi,\eta](x) + [J\xi,J\eta](x)\big) + J\pi_-\big(-[J\xi,\eta](x) + [\xi,\tilde{J}\eta](x)\big).$$

Let us start by discussing $\tilde{N}^+$. Choose a point $u \in \mathcal{G}$ with $p(u) = x$, and let X, $Y \in \mathfrak{g}_-$ be the elements corresponding to ξ_x and η_x as in lemma 3.2. Then $X \in \mathfrak{g}^+_{-1}$ and $Y \in \mathfrak{g}^-_{-1}$. By lemma 3.2 and complex bilinearity of the bracket on $\mathfrak{g}$, the element $[\xi,\eta](x) + [J\xi,J\eta](x)$ is simply given by $-T_u p\cdot\big(\omega_u^{-1}(\kappa(X,Y) + \kappa(iX,iY))\big)$. Moreover, since $\omega_u^{-1}(\mathfrak{b})$ is killed by $T_u p$, we may in this equation as well replace κ by $\kappa^{(1)}$, and since we know from 3.2 that the restriction of $\kappa^{(1)}$ to $\mathfrak{g}^+_{-1} \times \mathfrak{g}^-_{-1}$ is sesquilinear, we may replace $\kappa^{(1)}(X,Y) + \kappa^{(1)}(iX,iY)$ by $2\kappa^{(1)}(X,Y)$. Thus, $\pi_+\big([\xi,\eta](x) + [J\xi,J\eta](x)\big)$ is obtained by projecting down ω_u^{-1} of the $\mathfrak{g}^+_{-1}$-component of $-2\kappa^{(1)}(X,Y)$. Similarly, one sees that $-J\pi_+\big(-[J\xi,\eta](x) + [\xi,J\eta](x)\big)$ is obtained by projecting down ω_u^{-1} of the $\mathfrak{g}^+_{-1}$-component of $-2i\kappa^{(1)}(iX,Y)$. But from 3.2 we know that this component is conjugate linear in the first variable, so we see that $\frac{1}{4}\tilde{N}^+(\xi_x,\eta_x)$ is obtained by projecting down the $\mathfrak{g}^+_{-1}$-component of $-\kappa^{(1)}(X,Y)$, so we see that vanishing of $\tilde{N}^+$ is equivalent to the fact that $\kappa^{(1)}(X,Y) \in \mathfrak{g}^-_{-1}$ for all $X \in \mathfrak{g}^+_{-1}$ and $Y \in \mathfrak{g}^-_{-1}$. Similarly, one deals with $\tilde{N}^-$. Q.E.D.

Remark. Note that embedded partially integrable almost CR structures (i.e. manifolds for which the almost CR structure comes from an embedding into a *complex* manifold of appropriate dimension) are automatically CR, so the restriction of $\kappa^{(1)}$ to $\mathfrak{g}^+_{-1} \times \mathfrak{g}^-_{-1}$ vanishes automatically in the embedded case.

3.4. The other torsions of type $(1,1)$

The other two $(1,1)$-components in the homogeneous part of degree one of the curvature are even easier to interpret. Recall that by construction both H^+M and H^-M are isotropic with respect to the (complex bilinear) Levi-bracket. Consequently, for two sections ξ, $\eta \in \Gamma(H^+M)$, the Lie bracket $[\xi, \eta]$ is a section of HM, so we can project it to H^-M. Obviously, the result is bilinear over smooth functions, so there is a well defined tensorial map $T^+\colon \bigwedge^2_{\mathbb{R}} H^+M \to H^-M$ defined by $T^+(\xi, \eta) = \pi_-([\xi, \eta])$ for smooth sections ξ, η as above. Note that since H^+M is of complex rank one and T^+ is skew symmetric, it must be automatically totally real. Clearly T^+ vanishes identically if and only if the Lie bracket of two sections of H^+M is again a section of H^+M, i.e. if and only if H^+M is integrable. Similarly, we obtain a bundle map $T^-\colon \bigwedge^2 H^-M \to H^+M$ whose vanishing is equivalent to integrability of the bundle H^-M.

Proposition. *Let κ be the curvature function of the canonical Cartan connection. Then the restriction of $\kappa^{(1)}$ to $\bigwedge^2 \mathfrak{g}^+_{-1}$ (respectively to $\bigwedge^2 \mathfrak{g}^-_{-1}$) vanishes if and only if the subbundle $H^+M \subset TM$ (respectively $H^-M \subset TM$) is integrable.*

Proof. Consider tangent vectors ξ_x, $\eta_x \in H^+_xM$ and a point $u \in \mathcal{G}$ with $p(u) = x$. Then by definition the corresponding elements X, $Y \in \mathfrak{g}_-$ from lemma 3.2 lie in $\mathfrak{g}^+_{-1}$. Moreover, the extensions ξ, η provided by lemma 3.2 are sections of H^+M, and we have $[\xi, \eta](x) = -T_up{\cdot}\omega_u^{-1}(\kappa(X, Y))$, since $[X, Y] = 0$. Again, since vertical elements are killed by Tp, we may replace κ by $\kappa^{(1)}$ in this expression. Moreover, from 3.2 we know that $\kappa^{(1)}(X, Y) \in \mathfrak{g}^-_{-1}$ for X, $Y \in \mathfrak{g}^+_{-1}$, so the projection coincides with $\pi_-([\xi, \eta](x)) = T^+(\xi_x, \eta_x)$. Consequently, vanishing of the restriction of $\kappa^{(1)}$ to $\bigwedge^2 \mathfrak{g}^+_{-1}$ is equivalent to vanishing of T^+ and thus to integrability of H^+M. The other component is treated similarly. Q.E.D.

3.5. Torsions of type $(0, 2)$

To interpret the remaining two components of $\kappa^{(1)}$ it is convenient (although not formally necessary) to construct first an almost complex structure J on M, which combines the almost complex structures J on HM and J^Q on QM. Using the canonical Cartan connection ω it is clear how to get such an extension. In fact, for each point $u \in \mathcal{G}$, we get an isomorphism $\mathfrak{g}_- \to T_{p(u)}M$ defined by $X \mapsto T_up{\cdot}\omega_u^{-1}(X)$ and thus a complex structure on T_xM, where $x = p(u)$. Moreover, changing the point u to $u{\cdot}g$ for $g \in B$, equivariancy of ω implies that the new isomorphism is given by composing the old one with $\mathrm{Ad}(g)$, which is complex linear, so the complex structure on T_xM is canonical. Clearly, this defines a smooth almost complex structure J on M. Moreover, this has the property that the bundle maps $H^{\pm}M \to TM$ and $TM \to QM$ are complex linear, since the structures on the other spaces are also induced by ω.

Now we can easily characterize this almost complex structure: To do this, consider an arbitrary almost complex structure $\hat{J}$ on M such that $H^{\pm}M \to TM$ and $q\colon TM \to QM$ are complex linear. For a vector field ξ on M and a section η of HM consider the expression $q([\hat{J}\xi,\eta]) - J^Q q([\xi,\eta])$. Since $q(\eta)=0$ this is linear over smooth functions in η and since $q(\hat{J}\xi) = J^Q q(\xi)$ is also linear over smooth functions in ξ, so it defines a tensor $TM \times HM \to QM$. Moreover, for $\xi \in HM$, the tensor is given by $\mathcal{L}(J\xi,\eta) - J^Q(\mathcal{L}(\xi,\eta)) = 0$ by complex bilinearity of the Levi bracket. Hence we can factor to QM in the first variable and splitting $HM = H^+M \oplus H^-M$ we obtain two tensors $S^{\pm}\colon QM \times H^{\pm}M \to QM$. Note that these tensors by construction are conjugate linear in the first variable.

Proposition. *The almost complex structure J on M induced by the canonical Cartan connection ω is the unique almost complex structure on M such that the bundle maps $H^{\pm}M \to TM$ and $TM \to QM$ are complex linear and such that the tensors $S^{\pm}\colon QM \times H^{\pm}M \to QM$ induced by $(\xi,\eta) \mapsto q([J\xi,\eta]) - J^Q([\xi,\eta])$ are both conjugate linear in the second variable. Moreover, the restriction of $\kappa^{(1)}$ to $\mathfrak{g}_{-2} \times \mathfrak{g}_{-1}^{+}$ (respectively $\mathfrak{g}_{-2} \times \mathfrak{g}_{-1}^{-}$) vanishes if and only if S^+ (respectively S^-) is identically zero.*

Finally, vanishing of both S^+ and S^- is equivalent to integrability of the almost complex structure J on M.

Proof. Let us first verify that the almost complex structure induced by ω has the stated property. Take $\xi_x \in T_xM$ and $\eta_x \in H_xM$ and a point $u \in \mathcal{G}$ with $p(u) = X$. Let $X \in \mathfrak{g}_-$ and $Y \in \mathfrak{g}_{-1}$ be the corresponding elements from lemma 3.2, and ξ, η the extensions provided by lemma 3.2. Then $J\xi$ is the extension of $J_x\xi_x$ provided by lemma 3.2, so $[J\xi,\eta](x) = -T_up\cdot\omega_u^{-1}\big(\kappa(iX,Y)+[iX,Y]\big)$ and $[\xi,\eta](x) = -T_up\cdot\omega_u^{-1}\big(\kappa(X,Y)+[X,Y]\big)$. For the classes provided by q, only the $\mathfrak{g}_{-2}$-component of the result is significant, so we can again replace κ by $\kappa^{(1)}$, and we see that our tensor is given by taking the class of the image under T_up of

$$-\omega_u^{-1}\big(\kappa^{(1)}(iX_{-2},Y) + [iX,Y]_{-2} - i\kappa^{(1)}(X_{-2},Y) - i[X,Y]_{-2}\big),$$

where the subscripts -2 indicate the component in $\mathfrak{g}_{-2}$. Since the bracket in $\mathfrak{g}$ is complex bilinear, the bracket terms cancel. Moreover, from 3.2 we know that the restriction of $\kappa^{(1)}$ to $\mathfrak{g}_{-2} \times \mathfrak{g}_{-1}$ is conjugate linear in both variables. Now conjugate linearity in the first variable implies that the tensor is actually given by taking the class of the projection of $\omega_u^{-1}(2\kappa^{(1)}(X_{-2},Y))$, and conjugate linearity in the second variable then implies our claim. Moreover, the equivalent conditions for vanishing of components of $\kappa^{(1)}$ are obvious from this computation.

To prove the uniqueness of J, assume that $\hat{J}$ is another almost complex structure on M such that $H^{\pm}M \to TM$ and $TM \to QM$ are complex linear. For a vector field ξ on M we have by assumption $q(J\xi) = q(\hat{J}\xi)$, so there is a smooth section ξ' of HM such that $\hat{J}\xi = J\xi + \xi'$. But then $q([\hat{J}\xi, \eta]) = q([J\xi, \eta]) + \mathcal{L}(\xi', \eta)$ for all η, and the second term is complex linear in η. Since $\xi' \mapsto \mathcal{L}(\xi', \_)$ induces an isomorphism between HM and the bundle of complex linear maps $HM \to QM$ by non-degeneracy and complex bilinearity of the Levi bracket, the uniqueness follows.

The final statement is a little more subtle to prove. If both S^+ and S^- vanish, then from above we know that the restriction of $\kappa^{(1)}$ to $\mathfrak{g}_{-2} \times \mathfrak{g}_{-1}$ vanishes, so from 3.2 we conclude that $\kappa^{(1)}$ is totally real and has values in $\mathfrak{g}_{-1}$ only. Now take two elements ξ_x, $\eta_x \in T_xM$ and a point $u \in \mathcal{G}$ with $p(u) = x$, and let X, $Y \in \mathfrak{g}_-$ be the corresponding elements from lemma 3.2. Applying lemma 3.2 one sees that the value $N(\xi_x, \eta_x)$ of the Nijenhuis tensor of J on ξ_x and η_x is given by

$$-T_u p \cdot \omega_u^{-1}\big(\kappa(X,Y) - \kappa(iX, iY) + i\big(\kappa(iX,Y) + \kappa(X, iY)\big)\big),$$

which obviously equals $-4T_u p \cdot \omega_u^{-1}(\kappa_{0,2}(X,Y))$, where $\kappa_{0,2}$ denotes the component of κ which is conjugate linear in both arguments, and the terms involving brackets in $\mathfrak{g}$ do not show up since this bracket is complex bilinear. From above, we know that in this expression only components of κ of homogeneity ≥ 2 may enter. Moreover, since vertical elements are killed by Tp, we only have to consider components of $\kappa_{0,2}(X,Y)$ which lie in $\mathfrak{g}_-$. Now the homogeneous component of degree l of κ maps $\bigwedge^2_{\mathbb{R}} \mathfrak{g}_{-1}$ to $\mathfrak{g}_{l-2}$, $\mathfrak{g}_{-2} \otimes \mathfrak{g}_{-1}$ to $\mathfrak{g}_{l-3}$ and $\bigwedge^2_{\mathbb{R}} \mathfrak{g}_{-2}$ to $\mathfrak{g}_{l-4}$. Moreover, maps which are conjugate linear in both arguments automatically vanish on $\bigwedge^2 \mathfrak{g}_{-2}$ since $\mathfrak{g}_{-2}$ is of complex dimension one. Thus we see that the only relevant contribution to the above expression could come from the part of $\kappa^{(2)}_{0,2}$ which maps $\mathfrak{g}_{-2} \otimes \mathfrak{g}_{-1}$ to $\mathfrak{g}_{-1}$. The only other possible component of $\kappa^{(2)}_{0,2}$ maps $\bigwedge^2 \mathfrak{g}_{-1}$ to $\mathfrak{g}_0$.

Specialized to homogeneity two, the Bianchi identity from 3.1 tells us that $(\partial \circ \kappa^{(2)})(X,Y,Z)$ can be computed as

$$\sum_{\text{cycl}} \big(\kappa^{(1)}(\kappa^{(1)}(X,Y), Z) + (\omega^{-1}(Z) \cdot \kappa^{(1)})(X,Y)\big).$$

Since $\kappa^{(1)}$ has values in $\mathfrak{g}_{-1}$, the same must hold for $\partial \circ \kappa^{(2)}$ by this formula. Consequently, $\partial \circ \kappa^{(2)}$ has to vanish on $\mathfrak{g}_{-2} \otimes \bigwedge^2 \mathfrak{g}_{-1}$ since

∂ preserves homogeneities. Moreover, since we have observed in 3.1 that $\partial(L^{p,q}(\mathfrak{g}_-,\mathfrak{g}))\subset L^{p+1,q}(\mathfrak{g}_-,\mathfrak{g})$, we conclude that also $\partial\circ\kappa^{(2)}_{0,2}$ must vanish on $\mathfrak{g}_{-2}\otimes\bigwedge^2\mathfrak{g}_{-1}$. For $X\in\mathfrak{g}_{-2}$ and $Y,\ Z\in\mathfrak{g}_{-1}$ we have $[X,Y]=[X,Z]=0$ and $\kappa^{(2)}_{0,2}([Y,Z],X)=0$ since $\kappa^{(2)}_{0,2}$ must vanish on $\bigwedge^2\mathfrak{g}_{-2}$, and inserting the definition of ∂, we see that $(\partial\circ\kappa^{(2)}_{0,2})(X,Y,Z)=0$ is equivalent to

$$[Y,\kappa^{(2)}_{0,2}(X,Z)]=[X,\kappa^{(2)}_{0,2}(Y,Z)]+[Z,\kappa^{(2)}_{0,2}(X,Y)].$$

Replacing X by iX the same equation must hold. On the other hand, doing that multiplies the left hand side by $-i$, the first term in the right hand side by i and the second term in the right hand side by $-i$, so we conclude that $[X,\kappa^{(2)}_{0,2}(Y,Z)]=0$ and thus $[Y,\kappa^{(2)}_{0,2}(X,Z)]=[Z,\kappa^{(2)}_{0,2}(X,Y)]$ for all $X\in\mathfrak{g}_{-2}$ and $Y,\ Z\in\mathfrak{g}_{-1}$. But replacing Y by iY in this equation, the left hand side gets multiplied by i and the right hand side by $-i$, so we must have $[Y,\kappa^{(2)}_{0,2}(X,Z)]=0$ for all X, Y, Z, and thus the restriction of $\kappa^{(2)}_{0,2}$ to $\mathfrak{g}_{-2}\otimes\mathfrak{g}_{-1}$ vanishes by non-degeneracy of the bracket and the result follows. Q.E.D.

3.6. Torsion free elliptic CR manifolds

We conclude the discussion of geometric interpretations of the torsion-type components of the curvature of the canonical Cartan connection by discussing the case where all torsion type components vanish simultaneously. In this case, since by 3.1 the lowest nonzero homogeneous component of κ must be harmonic, we immediately see that there are no nonzero components of homogeneity less than four, and moreover $\kappa^{(4)}$ is complex bilinear and its only nonzero components are $\mathfrak{g}_{-2}\otimes\mathfrak{g}^+_{-1}\to\mathfrak{g}^+_1$ and $\mathfrak{g}_{-2}\otimes\mathfrak{g}^-_{-1}\to\mathfrak{g}^-_1$. But indeed, much more can be said in this case:

Theorem. *Suppose that M is an elliptic partially integrable almost CR manifold of CR dimension and codimension two such that $\tilde{N}^\pm=T^\pm=S^\pm=0$. Then the almost complex structure on $\mathcal{G}$ induced by ω is integrable, and the projection $p\colon\mathcal{G}\to M$ is holomorphic, so $\mathcal{G}$ is a holomorphic principal B-bundle over M. Moreover, the Cartan connection $\omega\in\Omega^1(\mathcal{G},\mathfrak{g})$ is a holomorphic $(1,0)$-form, so $(p\colon\mathcal{G}\to M,\omega)$ is a complex parabolic geometry of type $(PSL(3,\mathbb{C}),B)$. Conversely, any complex parabolic geometry of that type is torsion free when viewed as a real parabolic geometry.*

Proof. The Cartan connection ω defines a trivialization $T\mathcal{G}\cong\mathcal{G}\times\mathfrak{g}$ of the tangent bundle of $\mathcal{G}$, so since $\mathfrak{g}$ is a complex vector space, it induces an almost complex structure $J^{\mathcal{G}}$ on $\mathcal{G}$. The almost complex structure J on M was defined via ω, so it follows immediately that $p\colon \mathcal{G} \to M$ has complex linear derivative. Moreover, since $S^{\pm} = 0$, the almost complex structure J is integrable by proposition 3.4. To prove the first statement, we only have to show that torsion freeness implies integrability of $J^{\mathcal{G}}$. For $X \in \mathfrak{g}$, let us denote by $\tilde{X} \in \mathfrak{X}(\mathcal{G})$ the vector field $\omega^{-1}(X)$. The definition of the curvature function together with the fact that the curvature of ω is horizontal easily implies that $\omega([\tilde{X}, \tilde{Y}]) = -\kappa(X_-, Y_-) - [X, Y]$, where the subscript $-$ denotes the $\mathfrak{g}_-$-component. Using this and the fact that the bracket in $\mathfrak{g}$ is complex bilinear, one now concludes that the Nijenhuis-tensor on $\mathcal{G}$ evaluated on $\tilde{X}$ and $\tilde{Y}$ is just $-\frac{1}{4}\kappa_{0,2}(X_-, Y_-)$, where $\kappa_{0,2}$ denotes the component of the curvature which is conjugate linear in both arguments.

Once we have shown integrability of $J^{\mathcal{G}}$, we know that ω is a smooth $(1,0)$-form, and holomorphicity of this form is equivalent to $\bar{\partial}\omega = 0$, i.e. to $d\omega$ being a $(2,0)$-form. (Mistakenly, it was claimed in [11] that holomorphicity of ω is trivially satisfied.) But since the bracket on $\mathfrak{g}$ is complex bilinear, the fact that $d\omega$ is of type $(2,0)$ is equivalent to $\kappa(u)$ being complex bilinear for any $u \in G$. Hence proving complex bilinearity of κ suffices to prove the theorem. One possibility to prove this is to eliminate first the possibilities of having a nontrivial component of type $(0,2)$ in κ and then eliminating possible $(1,1)$-components using a pretty involved analysis of the Bianchi identity, similar to the proof of proposition 3.5. Following an idea of [1] on strengthening the Bianchi identity, there is a neat way around all that using fairly heavy tools:

Since the curvature function κ is an equivariant map $\mathcal{G} \to \bigwedge^2 \mathfrak{g}_- \otimes \mathfrak{g}$, it can be viewed as an element of $\Omega^2(M, \mathcal{A})$, where $\mathcal{A} = \mathcal{G} \times_B \mathfrak{g}$ is the adjoint tractor bundle. In [4], the twisted exterior derivative $d_{\mathfrak{g}}\colon \Omega^i(M, \mathcal{A}) \to \Omega^{i+1}(\mathcal{A})$ is constructed. On the other hand, the Cartan connection induces a principal connection $\tilde{\omega}$ on $\mathcal{G} \times_B G$ and since $\mathcal{A}$ can be viewed as associated to that bundle, we get an induced covariant exterior derivative $d^{\tilde{\omega}}$ between the same spaces. In [4, section 2] it is shown that in the torsion free case these two operators coincide. Moreover, the curvature of $\tilde{\omega}$ is also given by κ, so the Bianchi identity for principal connections implies $d^{\tilde{\omega}}(\kappa) = 0$. Now since ω is normal, the curvature κ has ∂^*-closed values and the harmonic part κ_0 may simply be viewed as the image of κ under the bundle map π_H corresponding to the projection $\ker(\partial^*) \to \ker(\partial^*)/\operatorname{im}(\partial^*)$, see [4, section 2]. This projection is splitted by an invariant differential operator L, whose construction is one of the main achievements of the paper [4]. In [4, Lemma 2.7] it is shown that this operator is characterized by $\pi_H(L(s)) = s$ and $\partial^* \circ d_{\mathfrak{g}} \circ L = 0$, which implies that $\kappa = L(\kappa_0)$ in the torsion free case.

But then [4, Theorem 2.5] shows that κ has values in the B-submodule generated by the values of κ_0, and since κ_0 has only complex bilinear values, the generated B-module consists of complex bilinear maps only. Q.E.D.

Corollary. *Let $(M, HM, \tilde{J})$ be a torsion free elliptic CR manifold of CR dimension and codimension two. Then M is a real analytic manifold and the subbundle $HM \subset TM$ and the endomorphism $\tilde{J}\colon HM \to HM$ are real analytic. In particular, M is automatically locally embeddable.*

Proof. By proposition 3.5, M is a complex manifold, and thus in particular real analytic. Moreover, by the last theorem, the Cartan bundle $p\colon \mathcal{G} \to M$ is a holomorphic principal bundle and the Cartan connection $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$ is holomorphic. Hence, $\omega^{-1}(\mathfrak{g}_{-1} \oplus \cdots \oplus \mathfrak{g}_2)$ is a real analytic subbundle of $T\mathcal{G}$. By construction, this projects onto the subbundle $HM \subset TM$, which thus also is real analytic. Similarly, the subbundles $H^{\pm}M$ and the almost complex structure J is analytic, since they all are induced from ω. Since $\tilde{J}$ is obtained by keeping J on H^-M and flipping it on H^+M, it is real analytic, too. Embeddability then follows from proposition 2.2. Q.E.D.

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