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Bases of Chambers of Linear Coxeter Groups

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§1. Introduction

Let V be a vector space over the real numbers $\mathbb{R}$. The subgroups of $\mathrm{GL}(V)$ that are generated by reflections are called *reflection groups.* We study in this paper those reflection groups from which a polyhedral cone may be constructed and which lead to a chamber system in V. Using a result of J. Tits [5], it follows that these groups are obtained from representations of Coxeter groups. So they are called *linear Coxeter groups.* From this point of view, these groups were also extensively studied by E.B. Vinberg [6] in the case where they have a finite number of canonical generators. We extend this theory in order to investigate the reflection subgroups of a linear Coxeter group. We make no restriction on the number of generators or on the dimension of V. Our object is to present this subject using the concrete geometric methods that are associated with the chamber systems in a real vector space.

We apply these results to give a proof that a reflection subgroup of a linear Coxeter group is again a linear Coxeter group. This generalizes the result that asserts that a reflection subgroup of a Coxeter group is a Coxeter group which was independently proved by M. Dyer [3] and V.V. Deodhar [2]. Our results also characterize a base for the reflection subgroup, which will be useful in a sequel to this paper.

§2. Linear Coxeter Groups

2.1. Polyhedral Cones

Let V be a vector space over $\mathbb{R}$, and denote its dual by $V^\vee$. Let T be a subset of V. We are interested in reflection groups that act on T. Commonly the choice for T will be V itself, but in dealing with reflection subgroups, it is useful to choose T to be the convex set that

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is left invariant by the associated linear Coxeter group, namely, its Tits cone.

Let $\Lambda^\vee$ be a subset of $V^\vee$ and set

$$C(\Lambda^\vee) = \{v \in T \mid \lambda^\vee(v) \geq 0 \text{ for all } \lambda^\vee \in \Lambda^\vee\}, \tag{1}$$

$$C(\Lambda^\vee)^\circ = \{v \in T \mid \lambda^\vee(v) > 0 \text{ for all } \lambda^\vee \in \Lambda^\vee\}. \tag{2}$$

For $\lambda^\vee \in V^\vee$, respectively set $D_{\lambda^\vee}$ and $D^\circ_{\lambda^\vee}$ to be the half-spaces $C(\{\lambda^\vee\})$ and $C(\{\lambda^\vee\})^\circ$. Then

$$C(\Lambda^\vee) = \bigcap_{\lambda^\vee \in \Lambda^\vee} D_{\lambda^\vee} \text{ and } C(\Lambda^\vee)^\circ = \bigcap_{\lambda^\vee \in \Lambda^\vee} D^\circ_{\lambda^\vee}. \tag{3}$$

Likewise set $H_{\lambda^\vee} = \lambda^{\vee -1}(0)$ for $\lambda^\vee \in V^\vee$. Then $H_{\lambda^\vee}$ is the hyperplane in V which is the envelope for $D_{\lambda^\vee}$. A convex subset $C(\Lambda^\vee)$ of V given in (3) is said to be a *polyhedral cone* in T if $C(\Lambda^\vee)^\circ \neq \emptyset$. If $|\Lambda^\vee| = 2$, it is sometimes called a *dihedral cone.*

Definition 2.1. Let $\Pi^\vee \subseteq V^\vee$. For $\alpha^\vee \in \Pi^\vee$, set $F_{\alpha^\vee}(\Pi^\vee) = H_{\alpha^\vee} \cap C(\Pi^\vee) = H_{\alpha^\vee} \cap C(\Pi^\vee \setminus \{\alpha^\vee\})$ and $F^\circ_{\alpha^\vee}(\Pi^\vee) = H_{\alpha^\vee} \cap C(\Pi^\vee \setminus \{\alpha^\vee\})^\circ$. Given $\Lambda^\vee \subseteq V^\vee$, a subset $\Pi^\vee$ is said to be a *base* for $C(\Lambda^\vee)$ if $C(\Pi^\vee) = C(\Lambda^\vee)$, and $F^\circ_{\alpha^\vee}(\Pi^\vee) \neq \emptyset$ for all $\alpha^\vee \in \Pi^\vee$. In this case, $F_{\alpha^\vee}(\Pi^\vee)$ is said to be a *face* of $C(\Pi^\vee)$. We say that $\Pi^\vee$ is a *base* if it is a base for $C(\Pi^\vee)$.

Clearly if $\Pi^\vee$ is a base, it is a base for $C(\Lambda^\vee)$ for any $\Lambda^\vee \supseteq \Pi^\vee$ such that $C(\Lambda^\vee) \supseteq C(\Pi^\vee)$. If $\Pi^\vee$ is a base for $C(\Lambda^\vee)$, then the hyperplanes $H_{\alpha^\vee}$ with $\alpha^\vee \in \Pi^\vee$ are called the *walls* of $C(\Lambda^\vee)$. Note that having $F^\circ_{\alpha^\vee}(\Pi^\vee) \neq \emptyset$ is equivalent to having $C(\Pi^\vee) \supset C(\Pi^\vee \setminus \{\alpha^\vee\})$. Thus if $\Pi^\vee$ is a minimal subset of $\Lambda^\vee$ such that $C(\Pi^\vee) = C(\Lambda^\vee)$, it is a base for $C(\Lambda^\vee)$.

2.2. Reflection Groups

Denote the pairing $V^\vee \times V \to \mathbb{R}$ given by $(\lambda^\vee, x) \longmapsto \langle \lambda^\vee, x\rangle = \lambda^\vee(x)$. A reflection $r \in \mathrm{GL}(V)$ is determined by two elements $\alpha_r \in V$ and $\alpha_r^\vee \in V^\vee$ with

$$\langle \alpha_r^\vee, \alpha_r \rangle = 2 \tag{4}$$

so that

$$r : x \to x - \langle \alpha_r^\vee, x\rangle \alpha_r. \tag{5}$$

The vectors $\alpha_r^\vee$ and α_r respectively are said to be a *coroot* and *root* of r. Hence $H_r = \alpha_r^{\vee -1}(0)$ is the fixed hyperplane of r and $R\alpha_r$ is its complementary eigenspace. When $\alpha_r^\vee$ and α_r satisfy (4), they are said to be paired to r. Thus $(c\alpha_r^\vee, c^{-1}\alpha_r)$, $c \neq 0$, are the coroots and roots that are paired to r.

Given a set S of reflections, $W(S)$ will designate the reflection group given by $W(S) = \langle s \mid s \in S \rangle$. Designate by $w^\vee$ the transformation of V which is contragredient to $w \in \mathrm{GL}(V)$. Associated with $W(S)$ is the contragredient group $W(S)^\vee = \{w^\vee \mid w \in W(S)\}$, which acts on $V^\vee$. If r is given by (5), then $r^\vee : x^\vee \to x^\vee - \langle x^\vee, \alpha_r \rangle \alpha_r^\vee$. Because $\langle \alpha_r^\vee, x \rangle = 0$ implies $\langle w^\vee \alpha_r^\vee, wx \rangle = 0$, it follows that $wH_{\alpha_r^\vee} = H_{w^\vee \alpha_r^\vee}$.

Set $\mathcal{H}(W(S)) = \{H_r \mid r \text{ is a reflection in } W(S)\}$.

Definition 2.2. Let T be a subset of V, and $\Pi^\vee = \{\alpha_i^\vee \in V^\vee \mid i \in I\}$. Take $C(\Pi^\vee)$ to be a polyhedral cone in T. Let $S = S(\Pi^\vee)$ be a set of reflections s_i, $i \in I$, where for each $i \in I$, $\alpha_i^\vee$ is a coroot of s_i. Assume that T is $W(S(\Pi^\vee))$-invariant. Then $C(\Pi^\vee)$ is said to be a *chamber* of $W(S(\Pi^\vee))$ for the action of $W(S(\Pi^\vee))$ on T if

$$wH_{\alpha_i^\vee} \cap C(\Pi^\vee)^\circ = \emptyset \tag{6}$$

for all $w \in W(S(\Pi^\vee))$ and $\alpha_i^\vee \in \Pi^\vee$.

Set $\mathcal{H}(W(S);\Pi^\vee) = \{H_{\beta^\vee} \mid \beta^\vee \in W(S)^\vee \Pi^\vee\}$. As $wH_{\alpha_i^\vee} = H_{w^\vee \alpha_i^\vee}$, (6) is equivalent to having $H_{\beta^\vee} \cap C(\Pi)^\circ = \emptyset$ for all $H_{\beta^\vee} \in \mathcal{H}(W(S), \Pi^\vee)$.

Definition 2.3. If $C(\Pi^\vee)$ is a chamber such that $wC(\Pi^\vee) = C(\Pi^\vee)$ implies $w = 1$, then $C(\Pi^\vee)$ is said to be a *regular chamber* for the action of $W(S(\Pi^\vee))$ on T and $W(S(\Pi^\vee))$ is said to be a *linear Coxeter group*[1].

The translates $wC(\Pi^\vee)$ of $C(\Pi^\vee)$, $w \in W(S)$, will also be called *chambers* of $W(S(\Pi))$, and we set $\mathcal{C}(W(S))$ to be the set of chambers of $W(S)$. When considering a given reflection group $W(S(\Pi^\vee))$ acting on a set T, it will be understood that the chambers in $\mathcal{C}(W(S))$ are chambers for the action on T. The set $\mathcal{C}(W(S))$ is sometimes called the *chamber system* for $W(S)$. When $C(\Pi^\vee)$ is a regular chamber, then

$$wC(\Pi^\vee)^\circ \cap C(\Pi^\vee)^\circ = \emptyset \tag{7}$$

for every $w \in W(S(\Pi^\vee)) \setminus \{1\}$, in which case $C(\Pi^\vee)^\circ$ is a fundamental domain for the action of $W(S(\Pi^\vee))$ on the subset $T(W(S(\Pi^\vee))) = \bigcup_{w \in W(S)} wC(\Pi^\vee)$.

Proposition 2.1. *Let $\Pi^\vee = \{\alpha_i^\vee \mid i \in I\} \subseteq V^\vee$. Take $S(\Pi^\vee)$ to be a set of reflections s_i with coroots $\alpha_i^\vee$, $i \in I$, and let T be a $W(S(\Pi^\vee))$-invariant subset of V. A polyhedral cone $C(\Pi^\vee)$ is a chamber for the*

[1] Linear Coxeter groups were defined as such by E.B.Vinberg [6].

action of $W(S(\Pi^\vee))$ on T if and only if for all $w \in W(S(\Pi^\vee))$, either $wC(\Pi^\vee)^\circ \cap C(\Pi^\vee)^\circ = \emptyset$ or $wC(\Pi^\vee)^\circ = C(\Pi^\vee)^\circ$. If it is a regular chamber, then $H_r \cap wC(\Pi)^\circ = \emptyset$ for all $H_r \in \mathcal{H}(W(S))$ and $w \in W(S(\Pi^\vee))$.

Proof. Assume that $C(\Pi^\vee)$ is a chamber so that $wH_{\alpha_i^\vee} \cap C(\Pi^\vee)^\circ = \emptyset$ for all $i \in I$ and $w \in W(S(\Pi^\vee))$. So either $w^\vee\alpha_i^\vee(C(\Pi^\vee)^\circ) > 0$ or $w^\vee\alpha_i^\vee(C(\Pi^\vee)^\circ) < 0$ for $i \in I$. If $w^\vee\alpha_i^\vee(C(\Pi^\vee)^\circ) > 0$ for all $i \in I$, then $wC(\Pi^\vee)^\circ = C(w^\vee\Pi^\vee)^\circ \supseteq C(\Pi^\vee)^\circ$. But also $w^\vee\alpha_i^\vee(x) = \alpha_i^\vee(wx)$ for $x \in V$; then $\alpha_i^\vee(wC(\Pi^\vee)^\circ) > 0$ for all $i \in I$. Hence $C(\Pi^\vee)^\circ \supseteq C(w^\vee\Pi^\vee)^\circ = wC(\Pi^\vee)^\circ$. Thus $C(\Pi^\vee)^\circ = wC(\Pi^\vee)^\circ$. On the other hand, if $w^\vee\alpha_i^\vee(C(\Pi^\vee)^\circ) < 0$ for some $i \in I$, then $wC(\Pi^\vee)^\circ \cap C(\Pi^\vee)^\circ \subseteq -D^\circ_{\alpha_i^\vee} \cap D^\circ_{\alpha_i^\vee} = \emptyset$.

Conversely, assume that $wC(\Pi^\vee)^\circ \cap C(\Pi^\vee)^\circ = \emptyset$ or $wC(\Pi^\vee)^\circ = C(\Pi^\vee)^\circ$. Then in first instance, $wH_{\alpha_i^\vee} \cap C(\Pi^\vee)^\circ = \emptyset$ for $i \in I$. In the second instance, $wH_{\alpha_i^\vee}$ intersects only the envelope $C(\Pi^\vee) \setminus C(\Pi^\vee)^\circ$ of $C(\Pi^\vee)$, and again $wH_{\alpha_i^\vee} \cap C(\Pi^\vee)^\circ = \emptyset$ for $i \in I$.

Finally consider that $C(\Pi^\vee)$ is a regular chamber. Suppose that $H_r \cap wC(\Pi^\vee)^\circ \neq \emptyset$ for some reflection $r \in W(S(\Pi^\vee))$ and $w \in W(S(\Pi^\vee))$. Then $rwC(\Pi^\vee) = wC(\Pi^\vee)$. But then the regularity of $C(\Pi^\vee)$ implies that $w^{-1}rw = 1$ and so $r = 1$. Hence $H_r \cap C(\Pi^\vee)^\circ = \emptyset$. Q.E.D.

Take $\Pi^\vee = \{\alpha_i^\vee \mid i \in I\} \subseteq V^\vee$, and let $S(\Pi^\vee)$ be a set of reflections s_i, $i \in I$, in $\mathrm{GL}(V)$ each with coroot $\alpha_i^\vee$ in $\Pi^\vee$. Suppose that $C(\Pi^\vee)$ is a polyhedral cone. Let $\Sigma^\vee(W(S(\Pi^\vee)))$ be the set of coroots of the reflections in $W(S(\Pi^\vee))$. To each $\alpha^\vee \in \Sigma^\vee(W(S(\Pi^\vee)))$ such that $H_{\alpha^\vee} \cap C(\Pi^\vee)^\circ = \emptyset$, either $\alpha^\vee(C(\Pi^\vee)^\circ) > 0$ or $\alpha^\vee(C(\Pi^\vee)^\circ) < 0$. Let

$$\Sigma^{\vee+}(W(S(\Pi^\vee))) = \{\alpha^\vee \in \Sigma^\vee(W(S(\Pi^\vee))) \mid \alpha^\vee(C(\Pi^\vee)^\circ) > 0\}. \tag{8}$$

The elements of $\Sigma^{\vee+}(W(S(\Pi^\vee)))$ will be said to be *positive* with respect to $C(\Pi^\vee)$. Because $\Pi^\vee \subseteq \Sigma^{\vee+}(W(S(\Pi^\vee)))$, the following proposition follows from Proposition 2.1.

Proposition 2.2. *A polyhedral cone $C(\Pi^\vee)$ with base $\Pi^\vee$ is a regular chamber if and only if*

$$C(\Pi^\vee) = \bigcap_{\alpha^\vee \in \Sigma^{\vee+}(W(S(\Pi^\vee)))} D_{\alpha^\vee}. \tag{9}$$

To each $\beta^\vee \in W(S(\Pi^\vee))^\vee\Pi^\vee$, $s_{\beta^\vee} = s_{w^\vee\alpha^\vee} = ws_{\alpha^\vee}w^{-1}$ is in $W(\Pi^\vee))$. So for all $H_{\beta^\vee}$ such that $\beta^\vee \in W(S(\Pi^\vee))^\vee\Pi^\vee$ and $H_{\beta^\vee} \cap C(\Pi^\vee)^\circ = \emptyset$, either $\beta^\vee(C(\Pi^\vee)^\circ) > 0$ or $s_{\beta^\vee}\beta^\vee(C(\Pi^\vee)) > 0$. Set $\Sigma^\vee(\Pi^\vee) = \{\beta^\vee \in \Sigma^\vee(W(S(\Pi^\vee))) \mid H_{\beta^\vee} \cap C(\Pi^\vee)^\circ = \emptyset\}$ and set $\Sigma^{\vee+}(\Pi^\vee) = \Sigma^\vee(\Pi^\vee) \cap \Sigma^{\vee+}(W(S(\Pi^\vee)))$. Then $C(\Pi^\vee)$ is a chamber of

$W(S(\Pi^\vee))$ if and only if $\Sigma^\vee(\Pi^\vee) = W(S(\Pi^\vee))\Pi^\vee$. This is equivalent to having $D_{\beta^\vee} \supseteq C(\Pi^\vee)$ for $\beta^\vee \in \Sigma^{\vee+}(\Pi^\vee)$. But $\Pi^\vee \subseteq \Sigma^\vee(\Pi^\vee)$; so the following proposition follows.

Proposition 2.3. *A polyhedral cone $C(\Pi^\vee)$ with base $\Pi^\vee$ is a chamber for $W(S(\Pi^\vee))$ if and only if*

$$C(\Pi^\vee) = \bigcap_{\beta^\vee \in \Sigma^{\vee+}(\Pi^\vee)} D_{\beta^\vee}. \tag{10}$$

2.3. Dihedral Groups

The argument which we present is directed towards the utilization of Theorem 3.1 which establishes that $(W(S(\Pi^\vee)), S(\Pi^\vee))$ is a Coxeter system if each $C(\Pi^\vee_{ij})$ is a regular chamber, $\Pi^\vee_{ij}$ being any pair contained in $\Pi^\vee$. Thus the case where $W(S(\Pi^\vee))$ is a dihedral group requires special attention.[2]

Theorem 2.4. *Let $S = \{r, s\}$ where r and s are reflections in $\mathrm{GL}(V)$. Respectively, let $\alpha^\vee, \alpha$ and $\beta^\vee, \beta$ be coroot and root pairs for r and s. Let $\Pi^\vee = \{\alpha^\vee, \beta^\vee\}$, and let $C(\Pi^\vee)$ be the dihedral cone given by $C(\Pi^\vee) = D_{\alpha^\vee} \cap D_{\beta^\vee} \cap T$ where T is a $W(S)$-invariant subset of V and $S = S(\Pi^\vee)$. The following conditions on the roots and coroots of r and s are necessary and sufficient for $C(\Pi^\vee)$ to be a chamber for the action of $W(S)$ on T.*

$$\langle \alpha^\vee, \beta\rangle \le 0 \text{ and } \langle \beta^\vee, \alpha\rangle \le 0, \tag{11}$$

$$\langle \alpha^\vee, \beta\rangle = 0 \text{ if and only if } \langle \beta^\vee, \alpha\rangle = 0, \tag{12}$$

$$\langle \alpha^\vee, \beta\rangle\langle \beta^\vee, \alpha\rangle = 4\cos^2\frac{\pi}{n}, \tag{13}$$

${}^{\vee}_{\alpha}n \in \mathbb{Z} \setminus \{0\}$, *when* $\langle \alpha^\vee, \beta\rangle\langle \beta^\vee, \alpha\rangle \le 4$. *Furthermore, $W(S(\Pi^\vee))$ is finite if and only if (13) holds. If $C(\Pi^\vee)$ is a chamber, then it is a regular chamber.*

Proof. Since $D_{\alpha^\vee} \cap D_{\beta^\vee}$ is a chamber for the action of $W(S(\Pi^\vee))$ on V if and only if $D_{\alpha^\vee} \cap D_{\beta^\vee} \cap T$ is also a chamber for the action of $W(S(\Pi^\vee))$ on T, we take $T = V$. Thus $C(\Pi^\vee) = D_{\alpha^\vee} \cap D_{\beta^\vee}$. Let $V_0 = H_{\alpha^\vee} \cap H_{\beta^\vee}$. Then V_0 is the fixed subspace for the action of $W(S)$ on V, and $V_o \subseteq C(\Pi^\vee)$. Clearly $W(S)$ acts faithfully on V/V_0 and $C(\Pi^\vee)/V_0$ is a chamber of $W(S)$ on V/V_0 if and only if $C(\Pi^\vee)$ is a chamber on V. Without loss of generality, we may assume that $V_0 = 0$. Then $\dim V = 2$, and $C(\Pi^\vee)$ is bounded by the half lines $K_{\alpha^\vee} = H_{\alpha^\vee} \cap$

[2]This result clarifies a result stated by Vinberg [6].

$C(\Pi^\vee)$ and $K_{\beta^\vee} = H_{\beta^\vee} \cap C(\Pi^\vee)$. Set $C_s(\Pi^\vee) = C(\Pi^\vee) \cup sC(\Pi^\vee)$. Since $C(\Pi^\vee) \cap sC(\Pi^\vee) = K_{\beta^\vee}$, $C_s(\Pi^\vee)$ is the sector in V that is bounded by $K_{\alpha^\vee}$ and $sK_{\alpha^\vee}$.

Consider first that $C(\Pi^\vee)$ is a chamber and that $\langle \alpha^\vee, \beta \rangle \geq 0$. Let $\mathbb{R}^+$ be the set of positive real numbers. Then (4) implies that $\mathbb{R}^+\beta \subseteq C(\Pi^\vee)$. Hence $-\mathbb{R}^+\beta = s\mathbb{R}^+\beta \subseteq sC(\Pi^\vee)$. Because $C_s(\Pi^\vee)$ contains $\mathbb{R}\beta = \mathbb{R}^+\beta \cup -\mathbb{R}^+\beta$, the angle θ_s from $K_{\alpha^\vee}$ to $sK_{\alpha^\vee}$ satisfies $\theta_s \geq \pi$. But $sH_{\alpha^\vee} \cap C(\Pi^\vee)^\circ = \emptyset$; so $H_{\alpha^\vee} \cap sC(\Pi^\vee)^\circ = \emptyset$. Therefore $\theta_s = \pi$. Hence $H_{\alpha^\vee} \supseteq \mathbb{R}\beta$, which is equivalent to $\langle \alpha^\vee, \beta \rangle = 0$. Because $H_{\alpha^\vee}$ is a wall of $C_s(\Pi^\vee)$, $V = C_s(\Pi^\vee) \cup sC_s(\Pi^\vee) = C(\Pi^\vee) \cup sC(\Pi^\vee) \cup rC(\Pi^\vee) \cup rsC(\Pi^\vee)$. Consequently $W(S)$ is a fours group; so $rs = sr$. This implies $\mathbb{R}\alpha \subseteq H_{\beta^\vee}$; thus $\langle \beta^\vee, \alpha \rangle = 0$. Likewise $\langle \alpha^\vee, \beta \rangle = 0$ is a consequence of $\langle \beta^\vee, \alpha \rangle \geq 0$. This establishes (11) and (12). The condition (13) is established at the end of this argument.

Now consider that (11), (12) and (13) hold. If $\langle \alpha^\vee, \beta \rangle = \langle \beta^\vee, \alpha \rangle = 0$, then $W(S)$ must be a fours group, in which case, $C(\Pi^\vee)$ is a regular chamber. So consider that $\langle \alpha^\vee, \beta \rangle < 0$ and $\langle \beta^\vee, \alpha \rangle < 0$. Replace the pair $\beta^\vee, \beta$ by the pair $c\beta^\vee, c^{-1}\beta$ where $c^2 = \frac{\langle \alpha^\vee, \beta \rangle}{\langle \beta^\vee, \alpha \rangle}$. Then $\langle \alpha^\vee, \beta \rangle = \langle \beta^\vee, \alpha \rangle$, and $C(\Pi^\vee)$ remains unchanged along with $\langle \alpha^\vee, \beta \rangle \langle \beta^\vee, \alpha \rangle$. Let $\phi : V^\vee \to V$ be the correlation that is defined by $\phi : \alpha^\vee \mapsto \alpha$and $\phi : \beta^\vee \mapsto \beta$. Let $f : V \times V \to \mathbb{R}$ be the bilinear form that is given by setting $f(x, y) = \langle \phi^{-1}(x), y \rangle$. Then f is $W(S)$-invariant and symmetric. Also $\langle \alpha^\vee, \beta \rangle = f(\alpha, \beta)$. By (4), $f(\alpha, \alpha) = f(\beta, \beta) = 2$; set $a = f(\alpha, \beta)$. The discriminant of f is $4 - a^2 = 4 - \langle \alpha^\vee, \beta \rangle \langle \beta^\vee, \alpha \rangle$. So f is indefinite, degenerate or positive definite according as $a^2 > 4$, $a^2 = 4$, or $a^2 < 4$. Let $u = sr$, and set $U = \langle u \rangle$. Since $|W(S)| > 4$, $u^2 \neq 1$. The discriminant of the characteristic polynomial of u is $a^2(4 - a^2)$. So u has 2, 1, or 0 eigenspaces according as f is indefinite, degenerate or positive definite. In the first two cases, u has real eigenvalues; so $|u| = \infty$. Then u and u^2 have the same eigenspaces. These must be the isotropic lines of f.

When $\langle \alpha^\vee, \beta \rangle \langle \beta^\vee, \alpha \rangle > 4$, f is indefinite, its isotropic lines divide V into four sectors V_1, V_2, V_3, V_4, which are permuted by the group $W(S)/U$. These lines are interchanged by r and s; hence they are the eigenspaces for u. As $C_s(\Pi^\vee) \cap sC_s(\Pi^\vee) = uK_{\alpha^\vee} = tK_{\alpha^\vee}$, $C_s(\Pi^\vee)$ is contained in one of these sectors, say, V_1. It follows then that $V_1 = \bigcup_{n=-\infty}^{n=\infty} u^n C_s(\Pi^\vee)$ and that U acts regularly on $\{u^n C_s(\Pi^\vee) \mid n \in \mathbb{Z}\}$. From this, it follows that $W(S)$ acts regularly on $\{wC(\Pi^\vee) \mid w \in W(S)\}$. Therefore $C(\Pi^\vee)$ is a regular chamber.

The situation is similar when $\langle \alpha^\vee, \beta \rangle \langle \beta^\vee, \alpha \rangle = 4$and f is degenerate. The difference is that in this case there two sectors V_1 and V_2 which are separated by the unique isotropic line. This forces $\mathbb{R}\alpha = \mathbb{R}\beta$.

Next suppose that $\langle \alpha^\vee, \beta\rangle \langle \beta^\vee, \alpha\rangle < 4$, in which case f is positive definite and $W(S)$ is finite. Then f gives rise to a scalar product[3] where $a = \alpha \cdot \beta = 2\cos\theta$ and θ is the angle between the half lines $\mathbb{R}^+\alpha$ and $\mathbb{R}^+\beta$. The difference $\theta_0 = \Pi - \theta$ is the angle between the half lines $K_{\alpha^\vee}$ and $K_{\beta^\vee}$ and hence θ_0 is the angle of the sector $C(\Pi^\vee)$. So $2\theta_0$ is the angle of the sector $C_s(\Pi^\vee)$, which is also the angle of the rotation u. Let n be the least positive integer such that $C_s(\Pi^\vee) \cap u^n C_s(\Pi^\vee) \neq \emptyset$. Then $C_s(\Pi^\vee)$ is a chamber for U if and only if $2\theta n = 2\pi$. This is equivalent to having $\langle \alpha^\vee, \beta\rangle\langle \beta^\vee, \alpha\rangle = f(\alpha,\beta)^2 = a^2 = 4\cos^2\frac{\pi}{n}$ where $n \in \mathbb{Z}\setminus\{0\}$. Clearly $C_s(\Pi^\vee)$ is a chamber for U if and only if $C(\Pi^\vee)$ is a chamber for $W(S)$. This proves that $C(\Pi^\vee)$ is a chamber as well as showing that (13) is a consequence of $C(\Pi^\vee)$ being a chamber. Since $|\mathcal{C}(W(S))| = |W(S)|$, $C(\Pi^\vee)$ is also regular.

Finally, note that is finite if and only if u has no real eigenvalues, which is equivalent to (13) Also we have shown that (12), (11) and (13) imply that $C(\Pi^\vee)$ is regular and that these conditions are implied when $C(\Pi^\vee)$ is a chamber, in which case it must be regular. Q.E.D.

§3. Characterizations

3.1. Characterization of Linear Coxeter groups

The next result is due to J. Tits [5]. This argument was developed from his result which establishes the contragredient representation of a Coxeter group (*cf.* Bourbaki [1, V, §4.4] or Humphreys [4, p. 126]).

Theorem 3.1. *Let S be a set of reflections s_i, $i \in I$, in $\mathrm{GL}(V)$ and let $\alpha_i^\vee, \alpha_i$ be a paired coroot and root of s_i. Set $\Pi^\vee = \{\alpha_i^\vee \mid i \in I\}$ and $\Pi = \{\alpha_i \mid i \in I\}$. Let T be a $W(S)$-invariant subset of V. Suppose that $C(\Pi^\vee)$ is a chamber for the action of $W(S(\Pi^\vee))$ on T such that $C(\Pi_{ij}^\vee)$ is a regular chamber for $W(S(\Pi_{ij}))$ for each pair $\Pi_{ij}^\vee = \{\alpha_i^\vee, \alpha_j^\vee\} \subseteq \Pi^\vee$. Then $(W(S), S)$ is a Coxeter system, and $W(S)$ is a linear Coxeter group acting on T.*

Proof. The proof of Theorem 3.1 as we have stated it is obtained from Tits [5, Lemme 1]. Tits' argument is centered about the proof of the following statement[4]:

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[3] *cf.* Bourbaki [1, V, §2.3].

[4] Actually by replacing w by sw,the second statement becomes a consequence of the first; so the argument is directed to proving the first statement. Also Tits' statement does not require that s be a reflection.

(P) *Let* $w \in W(S)$. *Then, given* $s \in S$ *with coroot* $\alpha_s^\vee$, *either* $wC(\Pi^\vee) \subseteq sD_{\alpha_s^\vee}$ *and* $\ell(sw) = \ell(w) - 1$ *or* $wC(\Pi^\vee) \subseteq D_{\alpha_s^\vee}$ *and* $\ell(sw) = \ell(w) + 1$.

where $\ell(w)$ is the number of factors from S in a shortest expression of w as a product of elements of S. The argument is by induction on $\ell(w)$. Assuming that *(P)* holds for each dihedral group $W(S(\Pi_{ij}^\vee))$, $i, j \in I$, Tits argues by induction on $\ell(w)$ that *(P)* holds for $W(S)$. Either Lemma 1 of [1, V, §4.5] or the description of the action of $W(S(\Pi_{ij}^\vee))$ on its chambers given in Theorem 2.4 can be used to establish *(P)* for the subgroups $W(S(\Pi_{ij}^\vee))$. The condition *(P)* for the group $W(S)$ immediately implies the regularity of its chambers in the following way. Suppose that $w(C(\Pi^\vee) = C(\Pi^\vee)$ for some $w \in W(S)$. Then $wC(\Pi^\vee) \subseteq D_{\alpha_i^\vee}$ for all $i \in I$. So by *(P)*, $\ell(s_{\alpha_j^\vee} w) = \ell(w) + 1$ for all $s_{\alpha_j^\vee} \in S$. But this fails when $w \neq 1$ since there exist $\alpha_j^\vee \in \Pi^\vee$ such that $\ell(s_{\alpha_j^\vee} w) < \ell(w)$. Because $W(S)$ can be regarded as a Coxeter group acting on the chamber system $\mathcal{C}(W(S))$ the above argument also shows that this action is effective. Hence $(W(S), S)$ is a Coxeter system. Q.E.D.

Let $S = \{s_i \mid i \in I\}$ be a set of reflections of a reflection group W. Let $\alpha_i^\vee$ and α_i respectively be paired coroots and roots for s_i, $i \in I$. Set $\Pi^\vee = \{\alpha_i^\vee \mid i \in I\}$ and $\Pi = \{\alpha_i \mid i \in I\}$. We say that the sets $\Pi^\vee$ and Π have the *Cartan property* if every pair $(\alpha_i^\vee, \alpha_j)$, $i, j \in I$, $i \neq j$, satisfies the conditions (11), (12) and (13) of Theorem 2.4. A direct application of Theorem 3.1 and Theorem 2.4 gives the following corollary.

Corollary 3.2. *Let* S *be a set of reflections* s_i, $i \in I$, *in* $\mathrm{GL}(V)$ *and let* $\alpha_i^\vee, \alpha_i$ *be a coroot and root of* s_i. *Set* $\Pi^\vee = \{\alpha_i^\vee \mid i \in I\}$ *and* $\Pi = \{\alpha_i \mid i \in I\}$. *Suppose that* $C(\Pi^\vee)$ *is a polyhedral cone in a* $W(S)$-*invariant subset* T *of* V. *If* $C(\Pi^\vee)$ *is a chamber for the action of* $W(S)$ *on* T *and if* $\Pi^\vee$ *and* Π *have the Cartan property, then* $W(S)$ *is a linear Coxeter group.*

Theorem 3.3. *Let* S *be a set of reflections* s_i, $i \in I$, *in* $\mathrm{GL}(V)$ *and let* $\alpha_i^\vee$ *and* α_i, *respectively, be a paired coroot and root of* r_i. *Set* $\Pi^\vee = \{\alpha_i^\vee \mid i \in I\}$ *and* $\Pi = \{\alpha_i \mid i \in I\}$ *so that* $S = S(\Pi^\vee)$. *Let* $C(\Pi^\vee)$ *be a chamber for the action of* $W(S)$ *on a* $W(S)$-*invariant subset* T, *and let* $\Pi^\vee$ *be a base for* $C(\Pi^\vee)$. *Then the sets* $\Pi^\vee$ *and* Π *have the Cartan property, and* $W(S(\Pi^\vee))$ *is a linear Coxeter group acting on* T.

Proof. For each pair $\Pi_{ij}^\vee = \{\alpha_i^\vee, \alpha_j^\vee\} \subseteq \Pi^\vee$, we argue that $C(\Pi_{ij}^\vee)$ is a chamber for $W(S(\Pi_{ij}^\vee))$. By (10), $C(\Pi^\vee) = \bigcap \{D_{\alpha^\vee} \mid \alpha^\vee \in \Sigma^{\vee+}(\Pi^\vee)\}$. It is required to show that $H_{\alpha^\vee} \cap C(\Pi_{ij}^\vee)^\circ = \emptyset$ for $\alpha^\vee \in \Sigma^{\vee+}(\Pi_{ij}^\vee)$. So suppose that for some $\alpha^\vee \in \Sigma^{\vee+}(\Pi_{ij}^\vee)$, $H_{\alpha^\vee} \cap C(\Pi_{ij}^\vee)^\circ \neq \emptyset$. Now

$C(\Pi^\vee) \subseteq C(\{\alpha_k^\vee, \alpha\})$ where $k = i$ or j. For definiteness, suppose $k = i$. Let $V_0 = H_{\alpha_i^\vee} \cap H_{\alpha_j^\vee}$; then $H_{\alpha^\vee} \supseteq V_0$ and $H_{\alpha_j^\vee} \cap C(\{\alpha^\vee, \alpha_i^\vee\}) = V_0$. Hence $H_{\alpha_j^\vee} \cap C(\Pi^\vee)^\circ = \emptyset$ inasmuch as $C(\Pi^\vee)^\circ \subseteq C(\{\alpha^\vee, \alpha_k^\vee\})$. In particular, this implies that $F^\circ_{\alpha_j^\vee}(\Pi^\vee) \cap C(\Pi^\vee) = \emptyset$. But $\Pi^\vee$ is a base; so we have a contradiction. Therefore, $C(\Pi_{ij}^\vee)$ is a chamber for $W(S(\Pi_{ij}^\vee))$. By Theorem 2.4, it is a regular chamber and $\Pi_{ij}^\vee$ and $\Pi_{ij} = \{\alpha_i, \alpha_j\}$ have the Cartan property. Thus also $\Pi^\vee$ and Π have the Cartan property. Corollary 3.2 implies that $W(S(\Pi^\vee))$ is a linear Coxeter group. Q.E.D.

3.2. The Tits Cone

In this section, all linear Coxeter groups will be regarded as acting on V. We consider a linear Coxeter group $W(S)$ where S is the set of reflections $S = \{s_i \mid i \in I\}$, and $C(\Pi^\vee)$ is a regular chamber such that $\Pi^\vee = \{\alpha_i^\vee \mid i \in I\}$, $\alpha_i^\vee$ being a coroot of s_i. For $\emptyset \subset J \subseteq I$, set $V_J = \bigcap_{j \in J} H_{\alpha_j^\vee}$; then $V_\emptyset = V$ and $\Pi_\emptyset^\vee = \emptyset$. Set $F_J = C(\Pi^\vee) \cap V_J$ and

$$F_J^\circ(\Pi^\vee) = C(\Pi \setminus \Pi_J^\vee)^\circ \cap V_J \tag{14}$$

where $\emptyset \subseteq J \subseteq I$. The subset $F_J^\circ(\Pi^\vee)$ is called a *facet* of $C(\Pi^\vee)$ provided that it is nonempty. Then $C(\Pi^\vee) = \bigcup_{\emptyset \subseteq J \subseteq I} F_J^\circ(\Pi^\vee)$. The subspace V_j is said to be the *support* of $F_J(\Pi^\vee)$ and $F_J^\circ(\Pi^\vee)$. The subgroup $W_J = \langle s_j \mid j \in J\rangle$ is called a *parabolic* subgroup of $W(S)$. Set $\Pi_J^\vee = \{\alpha_j^\vee \in \Pi^\vee \mid j \in J\}$. Theorem 3.1 implies that W_J is a linear Coxeter group for which $C(\Pi_J^\vee)$ is a chamber. Since W_J leaves fixed V_J, it also leaves fixed $F_J(\Pi^\vee)$. If $\emptyset \subseteq J \subset K \subseteq I$, then $V_J \supseteq V_K$. Let J^* be the subset of I such that $H_{\alpha_j^\vee} \supseteq V_J$ for $j \in J^*$. Then J^* is the maximal subset of I such that $V_{J^*} = V_J$. Hence $\alpha_j^\vee(F_J^\circ(\Pi^\vee)) = 0$ for all $j \in J^*$. So $F_J^\circ(\Pi^\vee) = V_J \cap C(\Pi^\vee \setminus \Pi_J^\vee)^\circ = V_{J^*} \cap C(\Pi^\vee \setminus \Pi_{J^*}^\vee)^\circ = F_{J^*}^\circ(\Pi^\vee)$. Let $\mathcal{M}(\Pi^\vee)$ be the set of such maximal subsets J^* of I. Thus the set $\mathcal{F}(C(\Pi^\vee))$ of facets contained in $C(\Pi^\vee)$ is given by $\mathcal{F}(C(\Pi^\vee)) = \{F_J^\circ(\Pi^\vee) \mid J \in \mathcal{M}(\Pi^\vee)\}$. It is clear that the facets in $\mathcal{F}(C(\Pi^\vee))$ are mutually disjoint and that $C(\Pi^\vee) = \bigcup\{F_J^\circ(\Pi^\vee) \mid F_j(\Pi^\vee) \in \mathcal{F}(C(\Pi^\vee))\}$.

Set

$$T(W(S)) = \bigcup_{w \in W(S)} wC(\Pi^\vee). \tag{15}$$

Denote the complement of the envelope of the convex hull of $T(W(S))$ by $T(W(S))^\circ$. The set $T(W(S))$ is convex. Consequently $T(W(S))$ is called a *Tits cone*. For $w \in W(S)$ and $\emptyset \subset J \subseteq I$, $wF_J^\circ(\Pi^\vee)$ is a facet of $wC(\Pi^\vee)$ with support wV_J. The corresponding parabolic subgroup is wW_Jw^{-1}. Designate $\mathcal{F}(W(S))$ to be the set of facets of the chambers of $W(S)$. By (15),

$$T(W(S)) = \bigcup \mathcal{F}(W(S)) . \tag{16}$$

Standard arguments[5] give the next two propositions and, together with (16), show that two chambers in $\mathcal{C}(W(S))$ can intersect only in a common facet and that the decomposition (16) is a partition of $T(W(S))$.

Proposition 3.4. *Let $F_J^\circ(\Pi^\vee)$, $F_K^\circ(\Pi) \in \mathcal{F}(C(\Pi^\vee))$ and take $w \in W(S)$. Then if $F_J^\circ(\Pi^\vee) \cap wF_K^\circ(\Pi^\vee) \neq \emptyset$, $J = K$ and $w \in W_J$. In particular, for $wF_J^\circ(\Pi^\vee) \in \mathcal{F}(W(S))$,*

$$wW_Jw^{-1} = \{u \in W(S) \mid uwF_J^\circ(\Pi^\vee) = wF_J^\circ(\Pi^\vee)\} .$$

Proposition 3.5. *$T(W(S))$ is convex.*

Proposition 3.6. *Let $W(S)$ be any linear Coxeter group acting on V. Let $C(\Pi^\vee)$ be a chamber for $W(S)$ with a base $\Pi^\vee$. Then $W(S)$ is finite if and only if $-C(\Pi^\vee) \subseteq T(W(S))$ and thus if and only if $T(W(S)) = V$.*

Proof. The convex hull of $C(\Pi^\vee) \cup -C(\Pi^\vee)$ is V. So $T(W(S)) = V$ if and only if $-C(\Pi^\vee) \subseteq T(W(S))$. Let $\Pi^\vee = \{\alpha_i^\vee \mid i \in I\}$ where $S = \{s_i \mid i \in I\}$ and $\alpha_i^\vee$ is a coroot of s_i. It is well-known[6] that a linear Coxeter group $W(S)$ with a finite set S of generating reflections is finite if and only if $-C(\Pi^\vee) \subseteq T(W(S))$. Of course, if $W(S)$ is finite, then S is finite. Therefore, it remains to show that if $-C(\Pi^\vee)$ is a chamber for $W(S)$, then S is finite. So assume that there exists $w_0 \in W(S)$ such that $w_0C(\Pi^\vee) = -C(\Pi^\vee)$. Then $w_0C(\Pi^\vee) \subseteq s_iD_{\alpha_i^\vee} = -D_{\alpha_i^\vee}$ for all $i \in I$. However, by *(P)* of §3.1, this occurs only if s_i appears in a reduced expression of w_0 as a product of reflections in S. Since $\ell(w) < \infty$, this implies that S is finite. Q.E.D.

If W_J is finite, then $F_J^\circ(\Pi^\vee)$ is said to have *finite type.* Set $\mathcal{E}(W(S))$ to be the subset of $\mathcal{F}(W(S))$ consisting of facets of finite type. Clearly $\mathcal{E}(W(S))$ is $W(S)$-invariant. Finally set

$$E(W(S)) = \bigcup \mathcal{E}(W(S)) \tag{17}$$

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[5] *cf.* [1, V, §4.6]. This argument is an induction based on the mutual disjointness of the facets in $\mathcal{F}(C(\Pi^\vee))$.

[6] *cf.* [1, Ex. 2, p.130]. This exercise pertains to the present situation since *(P) of* §3.1 is available.

Define the *star* of a facet $wF_J^\circ(\Pi^\vee)$ to be the subset

$$\operatorname{st} wF_J^\circ(\Pi^\vee) = \{uF_K^\circ(\Pi^\vee) \in \mathcal{F}(W(S)) \mid uF_K(\Pi^\vee) \supseteq wF_J^\circ(\Pi^\vee)\}$$

of $\mathcal{F}(W(S))$. It is clear that $uC(\Pi^\vee) = uF_\emptyset^\circ(\Pi^\vee) \supseteq uF_K(\Pi^\vee)$. Therefore the facets in $\operatorname{st} wF_J^\circ(\Pi^\vee)$ are those that are contained in a chamber $uC(\Pi^\vee) = C(u^\vee\Pi^\vee)$ for which $u^\vee\Pi^\vee \supseteq w^\vee\Pi_J^\vee$. Set

$$\text{(18)} \qquad \mathcal{C}(wF_J^\circ(\Pi^\vee)) = \{C \in \mathcal{C}(W(S)) \mid C^\circ \in \operatorname{st} wF_J^\circ(\Pi^\vee)\}.$$

Each chamber $C \in \mathcal{C}(wF_J^\circ(\Pi^\vee))$ has the form $C = uwC(\Pi^\vee)$ for some $u \in W(S)$. In particular, we may take $u = 1$ since clearly $wC(\Pi^\vee)$ is in $\mathcal{C}(wF_J^\circ(\Pi^\vee))$. Any two chambers $v_1wC(\Pi^\vee)$ and $v_2wC(\Pi^\vee)$ in $\mathcal{C}(wF_J^\circ(\Pi^\vee))$ intersect in $F_{12} = v_1wC(\Pi^\vee) \cap v_2wC(\Pi^\vee)$ where $F_{12}^\circ \in \operatorname{st} wF_J^\circ(\Pi^\vee)$. For $wF_J^\circ(\Pi^\vee) \in \mathcal{F}(W(S))$, set

$$\text{(19)} \qquad N(wF_J^\circ(\Pi^\vee)) = \bigcup \mathcal{C}(wF_J^\circ(\Pi^\vee)).$$

Theorem 3.7. *Let $F^\circ = F_J^\circ(\Pi^\vee) \in \mathcal{F}(W(S))$ where $\Pi^\vee = \{\alpha_i^\vee \mid i \in I\}$, and let $uC(\Pi^\vee) = C(u^\vee\Pi^\vee) \in \mathcal{C}(F^\circ)$ where $u \in W(S)$, Set $J_u = \{j \in I \mid u^\vee\alpha_j^\vee(F^\circ) = 0\}$ and $K_u = I \setminus J_u = \{j \in I \mid u^\vee\alpha_k^\vee(F^\circ) > 0\}$. Set $\Gamma^\vee = \bigcup\{u^\vee\Pi_{K_u}^\vee \mid C(u^\vee\Pi^\vee) \in \mathcal{C}(F^\circ)\}$. Then the following holds.*

(i) *$uC(\Pi_{J_u}^\vee) \in \mathcal{C}(W_J)$. When $C' \in \mathcal{C}(W_J)$, then $C' \cap C(\Gamma^\vee) \in \mathcal{C}(F^\circ)$, and $\mathcal{C}(\mathcal{F}^\circ) = \{uC(\Pi^\vee) \mid u \in W_J\}$.*

(ii) *$N(F^\circ) = C(\Gamma^\vee) \subseteq T(W_J)$, and $\Gamma^\vee$ is a base for the polyhedral cone $C(\Gamma^\vee)$ where $\Gamma^\vee = \bigcup\{(u^\vee\Pi_{K_u}^\vee) \mid u \in W_J\}$.*

(iii) *$F^\circ \subseteq N(F^\circ)^\circ$ if and only if $F^\circ \in \mathcal{E}(W(S))$.*

Proof. (i) The hyperplanes $H_{u^\vee\alpha_j^\vee}$, $\alpha_j^\vee \in \Pi_{J_u}^\vee$ are hyperplanes of reflections $r_{u^\vee\alpha_j^\vee} \in W_J$. Clearly $uC(\Pi_{J_u}^\vee)$ is a polyhedral cone. It is a chamber for W_J, for otherwise there would exist $H_r \in \mathcal{H}(W_J)$ such that $H_r \cap uC(\Pi_{J_u}^\vee)^\circ \neq \emptyset$ in contradiction to $H_r \cap uC(\Pi^\vee)^\circ = \emptyset$. On the other hand, if $C' \in \mathcal{C}(W_J)$. Then $C' \supseteq V_J \supseteq F^\circ$; hence it contains a chamber $C_1 \in \mathcal{C}(F^\circ)$. Since the chambers in $\mathcal{C}(F^\circ)$ belong to distinct chambers of the stabilizer of F°, which is W_J, C' contains only one chamber of $W(S)$. This forces $C' \cap C(\Gamma^\vee) \in \mathcal{C}(F^\circ)$. Clearly $C(\Pi^\vee) \in \mathcal{C}(F^\circ)$; so $\Pi^\vee = \Pi_{J_1}^\vee \cup \Pi_{K_1}^\vee$ and $\mathcal{C}(\mathcal{F}^\circ) = \{uC(\Pi^\vee) \mid u \in W_J\}$ inasmuch as $\mathcal{C}(W_J) = \{uC(\Pi_{J_1}^\vee) \mid u \in W_J\}$.

(ii) Now $\Gamma^\vee = \bigcup\{(u^\vee\Pi_{K_u}^\vee) \mid uC(\Pi^\vee) \supseteq F^\circ\}$. But because $F^\circ \not\subseteq H_{u^\vee\alpha_k^\vee}$ for $u^\vee\alpha_{^\vee} \in \Gamma^\vee$, $F^\circ \subseteq D_{u^\vee\alpha^\vee}^\circ$. But then $uC(\Pi^\vee) \subseteq D_{u^\vee\alpha^\vee}^\circ$ for $uC(\Pi^\vee) \in \mathcal{C}(F^\circ)$. Thus $N(F^\circ) \subseteq C(\Gamma^\circ)$. However, $T(W(S)) \subseteq T(W_J)$ and the set $\mathcal{C}(W_J)$ partitions $T(W_J)$. Then the set $\{C' \cap C(\Gamma^\circ) \mid C' \in \mathcal{C}(W_J)\}$ partitions $C(\Gamma^\circ)$ into the set $\mathcal{C}(F^\circ)$. Thus $N(F^\circ) = C(\Gamma^\circ)$.

Now $\bigcup_{u^\vee\alpha_k^\vee\in\Gamma^\vee} F^{\circ}_{u^\vee\alpha_k^\vee}(\Gamma^\vee)$ is the envelope $B(\Gamma^\vee)$ of $C(\Gamma^\vee)$, and each $F^{\circ}_{u^\vee\alpha_k^\vee}(\Gamma^\vee)$ contains the face $F^{\circ}_{u^\vee\alpha_k^\vee}(uC(\Pi^\vee)$ of $uC(\Pi^\vee)\in \operatorname{st} F^{\circ}$. Consequently $F^{\circ}_{u^\vee\alpha_k^\vee}(\Gamma^\vee)\neq\emptyset$; so for each $u^\vee\alpha_k^\vee\in\Gamma^\vee$, $F^{\circ}_{u^\vee\alpha_k^\vee}(\Gamma^\vee)$ is a face of $C(\Gamma^\vee)$, and thus $\Gamma^\vee$ is a base.

(iii) Because $F^{\circ}\subseteq V_J$, it is contained in the Tits cone $T(W_J)$ of W_J. But $F^{\circ}\subseteq T(W_J)^{\circ}$ if and only if $T(W_J)=V$. It follows from Proposition 3.6 that $T(W_J)=V$ if and only if W_J is finite, in which case $F^{\circ}\in\mathcal{E}(W(S))$. So it remains to show that $F^{\circ}\subseteq N(F^{\circ})^{\circ}$ if and only if $T(W_J)=V$. Now $T(W_J)\supseteq T(W(S))$. So $F^{\circ}\not\subseteq T(W_J)^{\circ}$ is equivalent to both $T(W_J)\neq V$ and $F^{\circ}\not\subseteq N(F^{\circ})^{\circ}$. On the other hand, $F^{\circ}\subseteq T(W_J)^{\circ}$ is equivalent to $T(W_J)=V$ and thus to having the envelope of $C(\Gamma^\vee)$ being contained in the walls $H_{\gamma^\vee}$, $\gamma\in\Gamma^\vee$, of $C(\Gamma^{\circ})$. But $u^\vee\alpha_k^\vee(F^{\circ})>0$ for all $u^\vee\alpha_k^\vee\in\Gamma^\vee$. This means that $F^{\circ}\not\subseteq C(\Gamma^\vee)^{\circ}=N(F^{\circ})^{\circ}$. Q.E.D.

Corollary 3.8. *Let $W(S)$ be a linear Coxeter group acting on V, and let $T(W(S))^{\circ}$ be the interior of its Tits cone $T(W(S))$. Then $E(W(S))=T(W(S))^{\circ}$.*

Proof. By virtue of Theorem 3.7, it follows that $F^{\circ}\subseteq N(F^{\circ})^{\circ}=C(\Gamma^\vee)^{\circ}$ if and only $F^{\circ}\in\mathcal{E}(W(S))$ where $\Gamma^\vee$ is defined by (3.7). But $C(\Gamma^\vee)^{\circ}\subseteq T(W(S))$. So $F^{\circ}\subseteq T(W(S))^{\circ}$ if $F^{\circ}\in\mathcal{E}(W(S))$. On the other hand, if $F^{\circ}\not\subseteq T(W(S))^{\circ}$, then $F^{\circ}\not\supseteq E(W(S))$, and it follows that $F^{\circ}\not\subseteq T(wW_Jw^{-1})^{\circ}$ where wW_Jw^{-1} is the subgroup of $W(S)$ that fixes F°. As we argued in Theorem 3.7, this implies that W_J is infinite and so $F^{\circ}\notin\mathcal{E}(W(S))$. Then $F^{\circ}\not\subseteq E(W(S))$, and by (16), $T(W(S))^{\circ}=E(W(S))$. Q.E.D.

3.3. Reflection Subgroups

A Coxeter group is given by a Coxeter system $(W(S),S)$, which specifies its presentation, and $W(S)$ may always be represented as a linear Coxeter group[7] by means of the contragredient representation. The involutions in $W(S)$ that correspond to the reflections in this representation are those that belong to the set R of the conjugates of the elements of S. Independently by M. Dyer [3] and V.V. Deodhar [2] showed that a subgroup of a Coxeter group that is generated by these involutions is again a Coxeter group. Also J. Tits has noted that Theorem 3.1 is applicable to this problem. Here we offer a direct proof that a reflection subgroup of a linear Coxeter group is a linear Coxeter group. This immediately implies that it is a Coxeter group. The importance of a direct proof lies in the geometrical insight which it provides, which is

[7] *cf.* Bourbaki [1].

useful when investigating particular reflection subgroups which can be identified by an explicit construction of the base of a chamber.

In this section, we work with a given linear Coxeter group $W = W(S)$ and a reflection subgroup W_0. We take $T = E(W)$ to be the underlying set T that is used to define the chambers of W and W_0 by means of (1).

Theorem 3.9. *Let W be a linear Coxeter group acting on V with a regular chamber $C(\Pi^\vee)$ that has a base $\Pi^\vee$. Let W_0 be a reflection subgroup of W that is generated by a set of reflections. Then W_0 is a linear Coxeter group with a chamber $C(\Pi_0^\vee)$ that contains $C(\Pi^\vee)$.*

Proof. Set

$$C_0 = \left(\bigcap_{\gamma^\vee \in \Sigma_0^{\vee +}} D_{\gamma^\vee} \right) \cap E(W). \tag{20}$$

It follows from Corollary 3.8 that $E(W) = T(W)^\circ$; so $C(\Pi^\vee)^\circ \neq \emptyset$. Because $C_0 \supseteq C(\Pi^\vee)^\circ$, it follows from (10) that C_0 is a chamber for W_0. By virtue of Theorem 3.3, it remains to show that C_0 has a base. Let $\Pi_0^\vee$ be the subset of $\Sigma_0^{\vee +}$ consisting of those coroots $\gamma^\vee$ such that $H_{\gamma^\vee}$ is a wall of a chamber $w_{\gamma^\vee} C(\Pi^\vee)$ of W that is contained in C_0; then $\gamma^\vee = w_{\gamma^\vee}\alpha_i^\vee$ for some $\alpha_i^\vee \in \Pi^\vee$ and

$$F_{\gamma^\vee}^\circ(\Pi_0^\vee) = H_{\gamma^\vee} \cap C(\Pi_0^\vee \setminus \{\gamma^\vee\})^\circ \supseteq H_{\gamma^\vee} \cap C(w_{\gamma^\vee}\Pi^\vee \setminus \{w_{\gamma^\vee}\alpha_i^\vee\})^\circ \neq \emptyset.$$

Therefore $\Pi_0^\vee$ is a base. By virtue of (20), $C(\Pi_0^\vee) \supseteq C_0$.

So it suffices to show that $C_0 \supseteq C(\Pi_0^\vee)$. Let B_0 be the envelope $C_0 \setminus C_0^\circ$ of C_0. By virtue of Theorem 3.4, $B_0 = \bigcup\{wF_J^\circ(\Pi) \in \mathcal{F}(W) \mid wF_J^\circ(\Pi) \subseteq B_0\}$. Take $wF_J^\circ(\Pi^\vee) \in \mathcal{F}(W)$ where $wF_J^\circ(\Pi^\vee) \subseteq B_0$. Then by Theorem 3.7, $N(wF_J^\circ(\Pi^\vee))$ is polyhedral cone, and as $T(W) = E(W)$, $wF^\circ(\Pi^\vee) \subseteq N(wF^\circ(\Pi^\vee))^\circ$. So as $wF_J^\circ(\Pi^\vee) \subseteq B_0$, $C_0 \cap N(wF_J^\circ(\Pi^\vee))^\circ \neq \emptyset$. Hence $C_0 \cap N(wF_J^\circ(\Pi^\vee))$ is a polyhedral cone $C(\Lambda_0^\vee)$ where $\Lambda_0^\vee \subseteq \Sigma^\vee$.

As $wF_J^\circ(\Pi^\vee) \subseteq B_0$, it follows from Theorem 3.7 that $wF_J^\circ(\Pi^\vee) \not\subseteq C(\Lambda_0^\vee)^\circ$. This means that $wF_J^\circ(\Pi^\vee)$ is contained in the envelope of $C(\Lambda_0^\vee)$. Because $wF_J^\circ(\Pi^\vee) \in \mathcal{E}(W(S))$, the parabolic subgroup wW_Jw^{-1} is finite. Therefore $\Lambda_0^\vee$ is finite and $C(\Lambda_0^\vee)$ has a base $\Pi^\vee(wF_J^\circ(\Pi^\vee))$. Let $\Pi_0^\vee(wF_J^\circ(\Pi^\vee))$ denote the subset of $\Pi^\vee(wF_J^\circ(\Pi^\vee))$ which consists of those $\gamma^\vee \in \Sigma_0^{\vee +}$ such that $H_{\gamma^\vee} \supseteq wF_J^\circ(\Pi^\vee)$. Then $\Pi_0^\vee(wF_J^\circ(\Pi^\vee)) \subseteq \Sigma_0^\vee$, and $C_0 \cap N(wF_J^\circ(\Pi^\vee)) = C(\Pi_0^\vee(wF_J^\circ(\Pi^\vee))) \cap N(wF_J^\circ(\Pi^\vee))$. Since $H_{\gamma^\vee} \cap N(wF_J^\circ(\Pi^\vee))^\circ \neq \emptyset$ for $\gamma^\vee \in \Pi_0^\vee(wF_J^\circ(\Pi^\vee))$, it follows that $\Pi_0^\vee(wF_J^\circ(\Pi^\vee)) \subseteq \Pi_0^\vee$. Set $\Pi_1^\vee = \bigcup_{wF_J^\circ(\Pi^\vee) \subseteq B_0} \Pi_0^\vee(wF_J^\circ(\Pi^\vee))$. Then $\Pi_1^\vee \subseteq$

$\Pi_0^\vee$ and $C(\Pi_1^\vee) \supseteq C(\Pi_0^\vee)$. By virtue of Corollary 3.2 and Theorem 3.3, $\Pi_1^\vee$ inherits the Cartan property from $\Pi_0^\vee$. Therefore $\Pi_1^\vee$ is a base for the polyhedral cone $C(\Pi_1^\vee)$, and $C(\Pi_1^\vee) = \bigcap_{wF_J^\circ(\Pi^\vee)\subseteq B_0} C(\Pi^\vee(wF_J^\circ(\Pi^\vee)))$. Since $B_0 = \bigcup\{wF_J^\circ(\Pi) \mid wF_J^\circ(\Pi) \subseteq B_0\}$, B_0 is contained in the envelope of $C(\Pi_1^\vee)$. Since C_0 is the convex hull of B_0, we now have $C_0 \supseteq C(\Pi_1^\vee) \supseteq C(\Pi_0^\vee)$. Q.E.D.

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