Advanced Studies in Pure Mathematics 29, 2000
Singularities - Sapporo 1998
pp. 203–220

The Milnor fiber as a virtual motive

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In this text, which correponds to our talk at the Conference "Singularities in Geometry and Topology" held in Sapporo in July 1998, we present our results, obtained in collaboration with Jan Denef, on the virtual motive associated to the Milnor fiber.

§1. Introduction

1.1. Let X be a smooth and connected complex algebraic variety and consider $f : X \to \mathbf{C}$ a non constant morphism. For any singular point x of $f^{-1}(0)$, the Milnor fiber at x is defined as

$$F_x := B(x, \varepsilon) \cap f^{-1}(t),$$

for $0 < |t| \ll \varepsilon \ll 1$, with $B(x, \varepsilon)$ the open ball of radius ε centered at x. There is some abuse of notation here, since, strictly speaking, F_x depends on the choice of ε and t, but all the invariants we shall consider will not.

Maybe the most natural invariants of the Milnor fiber to look at first are the Betti numbers

$$b_i(F_x) := \mathrm{rk} H^i(F_x, \mathbf{C}).$$

In fact, these numbers are in general very difficult to compute as soon as the singularity of $f = 0$ at x is not isolated. Much more easy to determine is the Euler characteristic

$$\chi(F_x) := \sum_i (-1)^i b_i(F_x).$$

When X is of dimension n and the singularity of $f = 0$ at x is isolated, $\chi(F_x) = 1 + (-1)^{n-1} b_{n-1}(F_x)$, and $b_{n-1}(F_x)$ is nothing else but the Milnor number.

Received September 23, 1998
Revised March 2, 1999

1.2. Of course, the information given by this Euler characteristic is quite weak. An already better invariant may be obtained by taking in account the natural monodromy action on the cohomology of F_x. The action of the monodromy operator M gives a canonical decomposition

$$H^i(F_x, \mathbf{C}) = \bigoplus_{\lambda \in \mathbf{C}^\times} H^i(F_x, \mathbf{C})_\lambda, \tag{1.2.1}$$

with $H^i(F_x, \mathbf{C})_\lambda$ the part where the eigenvalues of M are equal to λ. Hence one can refine the invariant $\chi(F_x)$ by defining

$$\chi(F_x)_\lambda := \sum_i (-1)^i \mathrm{rk} H^i(F_x, \mathbf{C})_\lambda.$$

By A'Campo's formula [1] (in fact a direct consequence of the commutation of the nearby cycle functor with the direct image with proper support functor [22]), the following simple formula for $\chi(F_x)_\lambda$ in terms of a resolution of $f = 0$ holds:

$$\chi(F_x)_\lambda = \sum_{\lambda^m = 1} \chi(S_m \cap \pi^{-1}(x)).$$

Here the notation is the following: we are given a resolution $\pi : \tilde{X} \to X$of $f = 0$, $\tilde{X}$ is smooth, π is proper and birational, the preimage E of the singular locus of $f^{-1}(0)$ is a divisor with (strict) normal crossings, and π is an isomorphism onto its image outside E, and S_m denotes the open subvariety of E where $\pi^{-1}(f^{-1}(0))$ is locally given by $z^m = 0$, z being a local coordinate.

Since the cohomology groups $H^i(F_x, \mathbf{C})$ carry a natural mixed Hodge structure [19] [21] [12] [13] [14], one can consider generalized Hodge numbers

$$e^{p,q} := \sum (-1)^i h^{p,q} H^i(F_x, \mathbf{C})$$

and

$$e_\lambda^{p,q} := \sum (-1)^i h^{p,q} H^i(F_x, \mathbf{C})_\lambda.$$

In fact the data of the $e_\lambda^{p,q}$'s is equivalent to that of the Hodge spectrum defined in [19] [21] [20] [14]. (For an analogue of A'Campo's formula for the Hodge spectrum see Remark 4.2.2.)

The commun feature for all these invariants is that they all may be defined as some kind of Euler characteristics. The main object of this paper is to provide, in some sense, universal invariants of Euler characteristic type for the Milnor fiber.

§2. Universal Euler characteristics and motives

2.1. Universal Euler characteristics

Invariants of Euler characteristic type take usually their values in a ring and satisfy relations of the type $F(A \cup B) = F(A) + F(B) - F(A \cap B)$ and $F(A \times B) = F(A)F(B)$. Consider now Sch, the category of reduced and separated schemes of finite type over $\mathbf{C}$ (i.e. varieties) and define the abelian group $K_0(\mathrm{Sch})$ as the quotient of the free abelian group generated by symbols $[S]$, S in Sch, by the relations

$$[S] = [S'],$$

for S' isomorphic to S and

$$[S] = [S'] + [S \setminus S'],$$

for S' closed in S. There is a natural product on $K_0(\mathrm{Sch})$ such that

$$[S][S'] = [S \times S'],$$

which provides $K_0(\mathrm{Sch})$ with a ring structure. To any constructible subset W of a variety S one can naturally associate an element $[W]$ in $K_0(\mathrm{Sch})$ such that

$$[W \cup W'] = [W] + [W'] - [W \cap W']$$

(just write W as the disjoint union of a finite family of varieties S_i and set $[W] = \sum [S_i]$; this is independent of the choice of the S_i's). Clearly $S \mapsto [S]$ is the "universal Euler characteristic" of algebraic varieties.

2.2. Motives

In our situation we are interested in keeping track of the monodromy action, in particular we want to have some analogue of the eigenvalue decomposition (1.2.1). This is in fact one of the reasons why motives enter in the picture: if a finite group G acts on a smooth projective variety X, there is a direct sum decomposition $h(X) = \bigoplus h(X)_\alpha$ of the motive $h(X)$ associated to X, with α running over the set of irreducible characters of G. The notion of motives being maybe not so familiar to people in singularity theory (though they are in fact easy to define, natural, and, we hope to convince the reader, useful), we shall give now some basic definitions (a good recent reference is [18]). Let $\mathcal{V}$ denote the category of smooth and projective $\mathbf{C}$-schemes. For an object X in $\mathcal{V}$ and an integer d, $\mathcal{Z}^d(X)$ denotes the free abelian group generated by irreducible subvarieties of X of codimension d. We define the rational Chow group $A^d(X)$

as the quotient of $\mathcal{Z}^d(X)\otimes\mathbf{Q}$ modulo rational equivalence. For X and Y in $\mathcal{V}$, we denote by $\mathrm{Corr}^r(X,Y)$ the group of correspondences of degree r from X to Y. If X is purely d-dimensional, $\mathrm{Corr}^r(X,Y) = A^{d+r}(X\times Y)$, and if $X = \coprod X_i$, $\mathrm{Corr}^r(X,Y) = \oplus\,\mathrm{Corr}^r(X_i,Y)$. The category Mot of $\mathbf{C}$-motives may be defined as follows (cf. [18]). Objects of Mot are triples (X,p,n) where X is in $\mathcal{V}$, p is an idempotent (i.e. $p^2 = p$) in $\mathrm{Corr}^0(X,X)$, and n is an integer in $\mathbf{Z}$. If (X,p,n) and (Y,q,m) are motives, then

$$\mathrm{Hom}_{\mathrm{Mot}}((X,p,n),(Y,q,m)) = q\,\mathrm{Corr}^{m-n}(X,Y)\,p.$$

Composition of morphisms is given by composition of correspondences. The category Mot is additive, $\mathbf{Q}$-linear, and pseudo-abelian. There is a natural tensor product on Mot, defined on objects by

$$(X,p,n)\otimes(Y,q,m) = (X\times Y, p\otimes q, n+m).$$

We denote by h the functor $h : \mathcal{V}^\circ \to \mathrm{Mot}$ which sends an object X to $h(X) = (X,\mathrm{id},0)$ and a morphism $f : Y \to X$ to its graph in $\mathrm{Corr}^0(X,Y)$. This functor is compatible with the tensor product and the unit motive $1 = h(\mathrm{Spec}\,\mathbf{C})$ is the identity for the product. We denote by $\mathbf{L}$ the Lefschetz motive $\mathbf{L} = (\mathrm{Spec}\,\mathbf{C},\mathrm{id},-1)$. One can prove there is a canonical isomorphism

$$h(\mathbf{P}^1)\simeq 1\oplus\mathbf{L},$$

so, in some sense, $\mathbf{L}$ corresponds to $H^2(\mathbf{P}^1)$. We denote by $^\vee$ the involution $^\vee : \mathrm{Mot}^\circ \to \mathrm{Mot}$, defined on objects by $(X,p,n)^\vee = (X,{}^tp,d-n)$ if X is purely d-dimensional, and as the transpose of correspondences on morphisms. For X in $\mathcal{V}$ purely of dimension d, $h(X)^\vee = h(X)\otimes\mathbf{L}^{-d}$ (Poincaré duality). For any field E containing $\mathbf{Q}$ one defines similarly the category $\mathrm{Mot}\otimes E$ of motives with coefficients in E, by replacing the Chow groups $A^\cdot$ by $A^\cdot\otimes_{\mathbf{Q}} E$.

Since algebraic correspondences naturally act on cohomology, any cohomology theory on the category $\mathcal{V}$ factors through Mot and $\mathrm{Mot}\otimes E$, for E an extension of $\mathbf{Q}$, hence motives have canonical Betti and Hodge realizations.

Consider $K_0(\mathrm{Mot})$, the Grothendieck group of the pseudo-abelian category Mot. It is the abelian group associated to the monoid of isomorphism classes of motives with respect to $\oplus$. The tensor product on Mot induces a natural ring structure on $K_0(\mathrm{Mot})$. One defines similarly the ring $K_0(\mathrm{Mot}\otimes E)$ for E an extension of $\mathbf{Q}$. Of particular interest to us will be the case when E is the extension $\mathbf{Q}(\mu_\infty)$ of $\mathbf{Q}$ generated by all roots of unity in $\mathbf{C}$. To simplify notation we set $A := K_0(\mathrm{Mot}\otimes\mathbf{Q}(\mu_\infty))$.

Realization functors on Mot induce realization morphisms on the level on Grothendieck groups. In particular, we shall consider the Hodge realization morphism

$$H : A \longrightarrow K_0(\mathrm{MHS}_{\mathbf{C}}),$$

with $K_0(\mathrm{MHS}_{\mathbf{C}})$ the Grothendieck group of the abelian category of complex mixed Hodge structures.

By the following result of Gillet and Soulé [9] and Guillén and Navarro Aznar [10] one can assign to any algebraic variety a natural Euler characteristic (with proper suppports) with value into the ring $K_0(\mathrm{Mot})$ of virtual motives.

Theorem 2.2.1. *There exists a unique morphism of rings*

$$\chi_c : K_0(\mathrm{Sch}) \longrightarrow K_0(\mathrm{Mot})$$

such that $\chi_c([X]) = [h(X)]$ *for* X *projective and smooth.*

Remark that $\chi_c([\mathbf{A}^1]) = \mathbf{L}$. From now on we shall also denote by $\mathbf{L}$ the element $[\mathbf{A}^1]$ in $K_0(\mathrm{Sch})$.

Let G be a abelian finite group (in fact the assumption that G is abelian is irrelevant). Let X be an algebraic variety over $\mathbf{C}$ endowed with a G-action. We say X is a G-variety if the G-orbit of any closed point in X is contained in an affine open scheme (this condition is always satisfied when X is quasi-projective). One defines in the usual way isomorphisms and closed immersions of G-varieties and so one may define a ring $K_0(\mathrm{Sch}, G)$, the Grothendieck ring of G-varieties over k, similarly as we defined $K_0(\mathrm{Sch})$.

For any character α of G, let us denote by p_α the corresponding idempotent in $\mathbf{Q}(\mu_\infty)[G]$. Let X be a smooth projective variety on which G acts. There is a natural ring morphism μ from $\mathbf{Q}(\mu_\infty)[G]$ to the ring of correspondences on X with coefficients in $\mathbf{Q}(\mu_\infty)$ sending a group element g onto the graph of multiplication by g. Let us denote by $h(X, \alpha)$ the motive $(X, \mu(p_\alpha), 0)$ in $\mathrm{Mot} \otimes \mathbf{Q}(\mu_\infty)$.

The following equivariant analogue of Theorem 2.2.1 is proved in [6].

Theorem 2.2.2. *For any character* α *of* G*, there exists a unique morphism of rings*

$$\chi_c(\_, \alpha) : K_0(\mathrm{Sch}, G) \longrightarrow A$$

such that $\chi_c([X], \alpha) = [h(X, \alpha)]$ *for* X *projective and smooth with* G*-action.*

2.3. An example: Fermat Hypersurfaces and Jacobi motives

An important and classical example of varieties with group action giving rise to interesting motives are Fermat hypersurfaces. These motives will also occur naturally in our motivic analogue of the Thom-Sebastiani formula (Theorem 5.4.2). For $n \geq 1$, we consider the affine Fermat variety F_d^n defined by the equation $x_1^d + \cdots + x_n^d = 1$ in $\mathbf{A}^n$. The action of μ_d, the group of d-th roots of unity, on each coordinate induces a natural action of the group μ_d^n on F_d^n. Hence, for $\alpha_1, \ldots, \alpha_n$ characters of μ_d, one defines the Jacobi motive $J(\alpha_1, \ldots, \alpha_n)$ as the element

$$J(\alpha_1, \ldots, \alpha_n) := \chi_c(F_d^n, (\alpha_1, \ldots, \alpha_n))$$

in A. It is clear that $J(\alpha_1, \ldots, \alpha_n)$ is symmetric in the α_i's. In fact, as is quite clasical, one can recover from $J(\alpha_1, \ldots, \alpha_n)$ the usual Jacobi sums (via étale realization using the Galois action) and the Beta function (via the period pairing for the Hodge realization) (cf., *e.g.*, [2]).

The following identities which are analogues of classical identities for Jacobi sums and Beta functions are proved in [8].

Proposition 2.3.1. (1) *We have* $J(1,1) = \mathbf{L}$.
(2) *We have* $J(1, \alpha) = 0$ *if* $\alpha \neq 1$.
(3) *If* $\alpha \neq 1$, $J(\alpha, \alpha^{-1}) = -1$.
(4) *We have*

$$J(\alpha_1, \alpha_2)[J(\alpha_1\alpha_2, \alpha_3) - \varepsilon] = J(\alpha_1, \alpha_2, \alpha_3) - \delta,$$

with $\varepsilon = \delta = 0$ *if* $\alpha_1\alpha_2 \neq 1$, $\varepsilon = 1$, $\delta = (\mathbf{L} - 1)$, *if* $\alpha_1\alpha_2 = 1$ *and* $\alpha_1 \neq 1$, *and* $\varepsilon = 1$, $\delta = \mathbf{L}$, *if* $\alpha_1 = \alpha_2 = 1$.

§3. An interlude: Motivic Igusa Zeta functions

3.1. Let p be a prime number and let K be a finite extension of $\mathbf{Q}_p$. Let R be the valuation ring of K, P the maximal ideal of R, and $\bar{K} = R/P$ the residue field of K. Let q denote the cardinality of $\bar{K}$, so $\bar{K} \simeq \mathbf{F}_q$. For z in K, let $\operatorname{ord} z$ denote the valuation of z, and set $|z| = q^{-\operatorname{ord} z}$. Let f be a non constant element of $K[x_1, \ldots, x_m]$. The p-adic Igusa local zeta function $Z(s)$ associated to f (relative to the trivial multiplicative character) is defined as the p-adic integral

$$Z(s) = \int_{R^m} |f(x)|^s |dx|, \tag{3.1.1}$$

for $s \in \mathbf{C}$, $\mathrm{Re}(s) > 0$, where $|dx|$ denotes the Haar measure on K^m normalized in such of way that R^m is of volume 1. For n in $\mathbf{N}$, set $Z_n = \{x \in R^m \mid \operatorname{ord} f(x) = n\}$. We may express $Z(s)$ as a series

$$Z(s) = \sum_{n \geq 0} \operatorname{vol}(Z_n)\, q^{-ns}. \tag{3.1.2}$$

Now, if we denote by X_n the image of Z_n in $(R/P^{n+1})^m$, we may rewrite the series as

$$Z(s) = \sum_{n \geq 0} \operatorname{card}(X_n)\, q^{-ns-(n+1)m} \tag{3.1.3}$$

since $\operatorname{vol}(Z_n) = \operatorname{card}(X_n)\, q^{-(n+1)m}$.

3.2. Now let X be a smooth and connected complex algebraic variety and consider $f : X \to \mathbf{C}$ a non constant morphism. We denote by $\mathcal{L}(X)$ the space of formal arcs on X: there is a natural bijection between the space of $\mathbf{C}$-points of $\mathcal{L}(X)$, $\mathcal{L}(X)(\mathbf{C})$, and $X(\mathbf{C}[[t]])$. There is a natural structure of $\mathbf{C}$-scheme on $\mathcal{L}(X)$, but we shall always consider $\mathcal{L}(X)$ with its reduced structure. Similarly, for $n \geq 0$, we can consider the space $\mathcal{L}_n(X)$ of arcs modulo t^{n+1}: a $\mathbf{C}$-point of $\mathcal{L}_n(X)$ corresponds to a $\mathbf{C}[t]/t^{n+1}\mathbf{C}[t]$-point on X. The space $\mathcal{L}_n(X)$ may be endowed with a natural structure of $\mathbf{C}$-scheme of finite type, and there is a natural morphism

$$\pi_n : \mathcal{L}(X) \longrightarrow \mathcal{L}_n(X)$$

given by truncation. In this setting $\mathcal{L}(\mathbf{A}^m)$ and $\mathcal{L}_n(\mathbf{A}^m)$ may be considered as analogues of R^m and $(R/P^{n+1})^m$. Pursuing this analogy further, one considers the reduced subscheme Z_n of $\mathcal{L}(\mathbf{A}_k^m)$ whose points are the series φ such that $\operatorname{ord}_t f(\varphi) = n$ and the image X_n of Z_n in $\mathcal{L}_n(\mathbf{A}_k^m)$, which has a natural structure of variety over $\mathbf{C}$. More generally, for W closed in X, we shall denote by $Z_{W,n}$ the closed subscheme of Z_n whose points are arcs φ with $\varphi(0)$ in W and by $X_{W,n}$ its image in $\mathcal{L}_n(\mathbf{A}_k^m)$.

A natural analogue of the right-hand side of (3.1.3), which is a series in $\mathbf{Z}[p^{-1}][[p^{-s}]]$, is the following series in $K_0(\mathrm{Sch}_k)[\mathbf{L}^{-1}][[\mathbf{L}^{-s}]]$

$$Z_{\mathrm{geom}}(s) = \sum_{n \geq 0} [X_n]\, \mathbf{L}^{-ns-(n+1)m}. \tag{3.2.1}$$

Here $\mathbf{L}^{-s}$ is just the name for a formal variable which could as well be written $T = \mathbf{L}^{-s}$.

3.3. More generally, p-adic Igusa local zeta functions involve multiplicative characters. Let π be a fixed uniformizing parameter of R and set $\mathrm{ac}(z) = z\pi^{-\operatorname{ord} z}$ for z in K. For any character $\alpha : R^\times \to \mathbf{C}^\times$ (i.e.

a group morphism with finite image), one defines the p-adic Igusa local zeta function $Z(s,\alpha)$ as the integral

$$Z(s,\alpha)=\int_{R^m}\alpha(\mathrm{ac}(f(x)))|f(x)|^s|dx|, \tag{3.3.1}$$

for $s\in\mathbf{C}$, $\mathrm{Re}(s)>0$ (see [11], [4]). To extend definition (3.2.1) to the more general situation involving characters, we shall use motives in the following way.

We fix an integer $d\geq 1$. Let $g: W\to\mathbf{C}^\times$ be a morphism of $\mathbf{C}$-varieties. For any character α of μ_d, one may define an element $[W]_{g,\alpha}$ of $K_0(\mathrm{Mot}_k\otimes\mathbf{Q})$ as follows.

The morphism $[d]:\mathbf{C}^\times\to\mathbf{C}^\times$ given by $x\mapsto x^d$ is a Galois covering with Galois group μ_d. We consider the fiber product

$$\begin{array}{ccc}\widetilde{W}_{g,d} & \longrightarrow & W\\ \downarrow & & \downarrow g\\ \mathbf{C}^\times & \xrightarrow{[d]} & \mathbf{C}^\times.\end{array}$$

The scheme $\widetilde{W}_{g,d}$ is endowed with an action of μ_d, so we can define

$$[W]_{g,\alpha}:=\chi_c(\widetilde{W}_{g,d},\alpha).$$

In our setting we can consider the morphism $f_n: X_n\to\mathbf{C}^\times$ whose value at a series φ is the coefficient of order n of $f(\varphi)$. When d divides d' we have a canonical surjective morphism of groups $\mu_{d'}\to\mu_d$ given by $x\mapsto x^{d'/d}$ which dualizes to a injective morphism of character groups $\widehat{\mu}_d\to\widehat{\mu}_{d'}$. We set $\widehat{\mu}:=\varinjlim\widehat{\mu}_d$. We shall identify $\widehat{\mu}_d$ with the subgroup of elements of order dividing d in $\widehat{\mu}$.

Now let α be in $\widehat{\mu}$ an element of order d. Viewing α as a character of μ_d, we may now define the series

$$Z_{\mathrm{mot}}(s,\alpha)=\sum_{n\geq 0}[X_n]_{f_n,\alpha}\,\mathbf{L}^{-ns-(n+1)m} \tag{3.3.2}$$

in $K_0(\mathrm{Mot}_k\otimes\mathbf{Q})[[\mathbf{L}^{-s}]]$. More generally, for W a closed subvariety of X, one defines similarly a series $Z_{\mathrm{mot},W}(s,\alpha)$ by replacing in the previous definition X_n by the variety $X_{W,n}$.

3.4. Rationality and formula on a resolution

Let D be the divisor defined by $f=0$ in X. Let (Y,h) be a resolution of f. By this, we mean that Y is a smooth and connected k-scheme of finite type, $h: Y\to X$ is proper, that the restriction

$h : Y \setminus h^{-1}(D) \to X \setminus D$ is an isomorphism, and that $(h^{-1}(D))_{\mathrm{red}}$ has only normal crossings as a subscheme of Y. Let E_i, $i \in J$, be the irreducible (smooth) components of $(h^{-1}(D))_{\mathrm{red}}$. For each $i \in J$, denote by N_i the multiplicity of E_i in the divisor of $f \circ h$ on Y, and by $\nu_i - 1$ the multiplicity of E_i in the divisor of h^*dx, where dx is a local non vanishing volume form, i.e. a local generator of the sheaf of differential forms of maximal degree. For $i \in J$ and $I \subset J$, we consider the schemes $E_i^\circ := E_i \setminus \cup_{j \neq i} E_j$, $E_I := \cap_{i \in I} E_i$, and $E_I^\circ := E_I \setminus \cup_{j \in J \setminus I} E_j$. When $I = \emptyset$, we have $E_\emptyset = Y$.

Now denote by J_d the set of $I \subset J$ such that $d \,|\, N_i$ for all i in I and by U_d the union of the E_I°'s, with I in J_d. Let Z be locally closed in U_d. For any character α of $\mu_d(k)$ of order d, we will construct an element $[Z]_{f,\alpha}$ in $K_0(\mathrm{Mot}_k \otimes \mathbf{Q})$ as follows. If on Z we may write $f \circ h = uv^d$ with u non vanishing on Z, we set $[Z_{f,\alpha}] = [Z]_{u,\alpha}$. In general, one covers Z by a finite set of Z_r's for which the previous condition holds, and we set

$$[Z_{f,\alpha}] = \sum_r [(Z_r)_{f,\alpha}] - \sum_{r_1 \neq r_2} [(Z_{r_1} \cap Z_{r_2})_{f,\alpha}] + \cdots .$$

One can check this definition does not depend of any choice.

We can now state the following result which is proved in [6]:

Theorem 3.4.1. *For any element α of $\hat{\mu}$ of order d,*

$$Z_{\mathrm{mot},W}(s,\alpha) = \mathbf{L}^{-m} \sum_{I \in J_d} [(E_I^\circ \cap h^{-1}(W))_{f,\alpha}] \cdot \prod_{i \in I} \frac{(\mathbf{L}-1)\,\mathbf{L}^{-N_i s - \nu_i}}{1 - \mathbf{L}^{-N_i s - \nu_i}} \tag{3.4.1}$$

in $A[[\mathbf{L}^{-s}]]$.

In particular it follows that $Z_{\mathrm{mot},W}(s,\alpha)$ is a rational series in $\mathbf{L}^{-s}$. It also follows that if the order of the character α does not divide any of the N_i's, then $Z_{\mathrm{mot},W}(s,\alpha)$ is identically zero (hence only of finite number of the functions $Z_{\mathrm{mot},W}(s,\alpha)$ are not identically zero).

The proof of Theorem 3.4.1 is based on the following geometric lemma which is a special case of Lemma 3.4 in [7].

Let X, Y and F be algebraic varieties over $\mathbf{C}$, and let A, *resp.* B, be a constructible subset of X, *resp.* Y. We say that a map $\pi : A \to B$ is *piecewise trivial fibration with fiber F*, if there exists a finite partition of B in subsets S which are locally closed in Y such that $\pi^{-1}(S)$ is locally closed in X and isomorphic to $S \times F$, with π corresponding under the isomorphism to the projection $S \times F \to S$. We say that the map π is a *piecewise trivial fibration over* some constructible subset C of B, if the restriction of π to $\pi^{-1}(C)$ is a piecewise trivial fibration.

Lemma 3.4.2. *Let X and Y be connected smooth schemes over $\mathbf{C}$ and let $h : Y \to X$ be a birational morphism. For e in $\mathbf{N}$, let Δ_e be the reduced subscheme of $\mathcal{L}(Y)$ defined by*

$$\Delta_e := \{\varphi \in Y(\mathbf{C}[[t]]) \mid \mathrm{ord}_t \det \mathcal{J}_\varphi = e\},$$

where $\mathcal{J}_\varphi$ is the jacobian of h at φ. For n in $\mathbf{N}$, let $h_{n*} : \mathcal{L}_n(Y) \to \mathcal{L}_n(X)$ be the morphism induced by h, and let $\Delta_{e,n}$ be the image of Δ_e in $\mathcal{L}_n(Y)$. If $n \geq 2e$, the following holds.

a) *The set $\Delta_{e,n}$ is a union of fibers of h_{n*}.*
b) *The restriction of h_{n*} to $\Delta_{e,n}$ is a piecewise trivial fibration with fiber $\mathbf{A}^e$ onto its image.*

Remark 3.4.3. These motivic Igusa functions specialize, by considering the trace of the Frobenius on their étale realization, in the p-adic case with good reduction, to the usual p-adic Igusa local zeta functions. They also specialize, by considering Euler characteristic of their Betti realization, to the topological zeta functions $Z_{\mathrm{top}}(s)$ introduced in [5], which were, heuristically, obtained as a limit as q goes to 1 of p-adic Igusa local zeta functions. We refer to [6] for details.

§4. The virtual motive attached to the Milnor fiber

4.1. Since $Z_{\mathrm{mot},W}(s,\alpha)$ is an A-linear combination of rational series of the form $\mathbf{L}^{-Ns-n}/(1-\mathbf{L}^{-Ns-n})$, with N and n in $\mathbf{N} \setminus \{0\}$, one can consider its limit as $s \to -\infty$, by defining

$$\lim_{s \to -\infty} \frac{\mathbf{L}^{-Ns-n}}{1-\mathbf{L}^{-Ns-n}} = -1.$$

One easily checks that one obtains in this way a well defined element

$$\lim_{s \to -\infty} Z_{\mathrm{mot},W}(s,\alpha)$$

in A. It follows from Theorem 3.4.1 that we have the following expression for $\lim_{s \to -\infty} Z_{\mathrm{mot},W}(s,\alpha)$ in terms of a resolution of $f = 0$:

$$\lim_{s \to -\infty} Z_{\mathrm{mot},W}(s,\alpha) = \mathbf{L}^{-m} \sum_{I \in J_d} [(E_I^\circ \cap h^{-1}(W))_{f,\alpha}](1-\mathbf{L})^{|I|}. \tag{4.1.1}$$

Note that it is not a priori clear that the right hand side of (4.1.1) is independent of the resolution, but it follows from the fact that the left hand side is canonical.

4.2. We assume from now on that W is contained in $f^{-1}(0)$. In that case, it follows from (4.1.1) that $\lim_{s \to -\infty} Z_{\mathrm{mot},W}(s,\alpha)$ is divisible by $1-\mathbf{L}$, so we may define

$$S^{\psi}_{\alpha,W,f} := \frac{\mathbf{L}^m}{1-\mathbf{L}} \lim_{s \to -\infty} Z_{\mathrm{mot},W}(s,\alpha).$$

Strictly speaking $S^{\psi}_{\alpha,W,f}$ is only defined up to $(\mathbf{L}-1)$-torsion in A, but this is not a serious problem, since $(\mathbf{L}-1)$-torsion is killed by realization functors. (In fact we do not know whether there exists or not any non trivial $(\mathbf{L}-1)$-torsion element in A.)

By the following result, proved in [6], the Hodge realization of $S^{\psi}_{\alpha,\{x\},f}$ is equal to the virtual Hodge structure defined by the Milnor fiber at x for the eigenvalue $\alpha(e^{2\pi i/d})$. Hence it is very natural to consider $S^{\psi}_{\alpha,\{x\},f}$ as the virtual motive associated to the Milnor fiber at x for the eigenvalue $\alpha(e^{2\pi i/d})$.

Theorem 4.2.1. *Let x be a point of $f^{-1}(0)$. Denote by $[H^i(F_x,\mathbf{C})_{\alpha(e^{2\pi i/d})}]$ the class in $K_0(\mathrm{MHS}_{\mathbf{C}})$ of $H^i(F_x,\mathbf{C})_{\alpha(e^{2\pi i/d})}$ with its canonical Hodge structure. The equality*

$$H(S^{\psi}_{\alpha,\{x\},f}) = \sum_i (-1)^i [H^i(F_x,\mathbf{C})_{\alpha(e^{2\pi i/d})}]$$

holds in $K_0(\mathrm{MHS}_{\mathbf{C}})$.

Remark 4.2.2. As a consequence of (4.1.1) and Theorem 4.2.1, one deduces an analogue of A'Campo's formula for the Hodge spectrum.

§5. Exponential integrals and a motivic Thom-Sebastiani Theorem

5.1. We begin by reviewing exponential integrals in the p-adic case, so we use again the notations of 3.1.

Let $f \in R[x_1,\ldots,x_m]$ be a non constant polynomial. Let $\Phi : R^m \to \mathbf{C}$ be a locally constant function with compact support. Let α be a character of $R^\times$. For i in $\mathbf{N}$, we set

$$Z_{\Phi,f,i}(\alpha) := \int_{\{x \in R^m \,|\, \mathrm{ord} f(x) = i\}} \Phi(x)\alpha(\mathrm{ac} f(x))|dx|.$$

We denote by Ψ the standard additive character on K, defined by

$$z \longmapsto \Psi(z) = \exp(2i\pi \mathrm{Tr}_{K/\mathbf{Q}_p} z).$$

For i in $\mathbf{N}$, we consider the exponential integral

$$E_{\Phi,f,i} := \int_{R^m} \Phi(x)\Psi(\pi^{-(i+1)}f(x))|dx|. \tag{5.1.1}$$

For α a character of $R^\times$, the conductor of α, $c(\alpha)$, is defined as the smallest $c \geq 1$ such that α is trivial on $1 + P^c$, and one associates to α the Gauss sum

$$g(\alpha) = q^{1-c(\alpha)} \sum_{v \in (R/P^{c(\alpha)})^\times} \alpha(v)\Psi(v/\pi^{c(\alpha)}).$$

The following result is a consequence of § 1 of [4].

Proposition 5.1.1. *For any i in $\mathbf{N}$,*

$$\begin{aligned} E_{\Phi,f,i} &= \int_{\{x \in R^m \,|\, \mathrm{ord} f(x) > i\}} \Phi(x)|dx| \\ &\quad + (q-1)^{-1} \sum_{\alpha} g(\alpha^{-1}) Z_{\Phi,f,i-c(\alpha)+1}(\alpha). \end{aligned} \tag{5.1.2}$$

Here $i - c(\alpha) + 1 \geq 0$. If moreover the critical locus of f in Supp Φ *is contained in $f^{-1}(0)$, then, for all except a finite number of characters α, the integrals $Z_{\Phi,f,j}(\alpha)$ are zero for all j.*

Using Theorem 3.3 of [4], one deduces from Proposition 5.1.1 that, assuming that Φ is residual, i.e. that Supp Φ is contained in R^m and that $\Phi(x)$ depends only on x modulo P, that the critical locus of f in Supp Φ is contained in $f^{-1}(0)$ and that the divisor $f = 0$ has good reduction (in the sense that the conditions in Theorem 3.3 of [4] are satisfied), then

$$\begin{aligned} E_{\Phi,f,i} &= \int_{\{x \in R^m \,|\, \mathrm{ord} f(x) > i\}} \Phi(x)|dx| \\ &\quad + (q-1)^{-1} \sum_{\substack{\alpha \\ c(\alpha)=1}} g(\alpha^{-1}) Z_{\Phi,f,i}(\alpha). \end{aligned} \tag{5.1.3}$$

5.2. Exponential integrals

Let X be a smooth connected variety over $\mathbf{C}$ of dimension m and let $f : X \to \mathbf{C}$ be a morphism. If one is looking for a motivic analogue of p-adic exponential integrals, a hint is given by formula (5.1.3) which expresses p-adic exponential integrals as linear combinations of p-adic integrals involving multiplicative characters with Gauss sums as coefficients. Though Gauss sums are not motivic themeselves, they are related to Jacobi sums by the familiar relation

$$g(\alpha)g(\beta) = g(\alpha\beta)j(\alpha, \beta), \tag{5.2.1}$$

when α, β and $\alpha\beta$ are not equal to 1 and have conductor 1, with

$$j(\alpha,\beta) = \sum_{x \in \bar{K}\setminus\{0,1\}} \alpha(x)\beta(1-x).$$

But the Jacobi sums $j(\alpha,\beta)$ are motivic, being equal to the trace of the Frobenius on the étale realization of a Jacobi motive, hence we may follow the idea, introduced by Greg Anderson in [2], of enlarging the world of motives by adding Gauss sums motives related to Jacobi motives by a relation similar to 5.2.1. More precisely, one considers the free A-module U with basis G_α, α in $\widehat{\mu}(k)$. We define an A-algebra structure on U by putting the following relations:

$$(5.2.2) \qquad G_1 = -1$$

$$(5.2.3) \qquad G_\alpha G_{\alpha^{-1}} = \mathbf{L} \qquad \text{for} \quad \alpha \neq 1$$

$$(5.2.4) \qquad G_{\alpha_1} G_{\alpha_2} = J(\alpha_1,\alpha_2)\, G_{\alpha_1\alpha_2} \qquad \text{for} \quad \alpha_1, \alpha_2, \alpha_1\alpha_2 \neq 1.$$

It follows from Proposition 2.3.1 that U is a commutative and associative algebra.

For m in $\mathbf{Z}$, let $F^m A$ denote the subgroup of A generated by $h(S,f,i)$, with $i - \dim S \geq m$. This gives a filtration on the ring A; we denote by $\widehat{A}$ the completion of A with respect to this filtration and we set $\widehat{U} := U \otimes_A \widehat{A}$. We shall also consider the subring A_{loc} of $\widehat{A}$ generated by the image of A in $\widehat{A}$ and the series $(1-\mathbf{L}^{-n})^{-1}$, $n \in \mathbf{N}\setminus\{0\}$. We denote by U_{loc} the tensor product $U \otimes_A A_{\mathrm{loc}}$, which is naturally a subring of $\widehat{U}$.

Let W be a closed subvariety of $f^{-1}(0)$. We define, for $i \geq 0$, the motivic analogue $E_{i,W,f,\mathrm{mot}}$ of $E_{\Phi,f,i}$ as the series

$$(5.2.5) \qquad E_{i,W,f,\mathrm{mot}} := \sum_{n>i} \frac{\chi_c([X_{W,n}])}{\mathbf{L}^{(n+1)m}} + \sum_{\alpha \in \widehat{\mu}(k)} \frac{1}{\mathbf{L}-1} G_{\alpha^{-1}} \frac{[X_{W,i}]_{f_i,\alpha}}{\mathbf{L}^{(i+1)m}}$$

in $\widehat{U}$. Remark that, since only a finite number of the functions $Z_{\mathrm{mot},W}(s,\alpha)$ are non zero, the second sum in (5.2.5) is finite. Furthermore, one can deduce from Theorem 3.4.1 that $E_{i,W,f,\mathrm{mot}}$ belongs in fact to the ring U_{loc}. In [8] the definition of $E_{i,W,f,\mathrm{mot}}$ is extended to the case where X is no longer smooth.

Now the standard multiplicativity property of exponential integrals is no longer trivial. In fact the following result is one of the main results in [8]:

Theorem 5.2.1. *Let X and X' be irreducible complex algebraic varieties over $\mathbf{C}$, let $f : X \to \mathbf{C}$ and $f' : X' \to \mathbf{C}$ be morphisms of $\mathbf{C}$-varieties. Denote by $f \oplus f' : X \times X' \to \mathbf{C}$ the morphism $(x,x') \mapsto$*

$f(x)+f'(x')$. Let W (resp. W') be a reduced subvariety of $f^{-1}(0)$ (resp. $f'^{-1}(0)$). For every $i \geq 0$,

$$E_{i,f\oplus f',W\times W',\mathrm{mot}} = E_{i,f,W,\mathrm{mot}} \cdot E_{i,f',W',\mathrm{mot}}. \tag{5.2.6}$$

5.3. An algebraic lemma on power expansions of rational functions

Denote by B the ring $\mathcal{U}_{\mathrm{loc}}$. We consider the ring of Laurent polynomials $B[T,T^{-1}]$ and its localisation $B[T,T^{-1}]_{\mathrm{rat}}$ obtained by inverting the multiplicative family generated by the polynomials $1-\mathbf{L}^a T^b$, a, b in $\mathbf{Z}$, $b \neq 0$. Remark that, in this definition, we could restrict to $b>0$ or to $b<0$. Hence, by expanding denominators into formal series, there are canonical embeddings of rings

$$\exp_T : B[T,T^{-1}]_{\mathrm{rat}} \hookrightarrow B[T^{-1},T]]$$

and

$$\exp_{T^{-1}} : B[T,T^{-1}]_{\mathrm{rat}} \hookrightarrow B[[T^{-1},T].$$

Here $B[T^{-1},T]]$ (resp. $B[[T^{-1},T]$) denotes the ring of series $\sum_{i\in\mathbf{Z}} a_i T^i$ with $a_i = 0$ for $i \ll 0$ (resp. $i \gg 0$). By taking the difference $\exp_T - \exp_{T^{-1}}$ of the two expansions one obtains an embedding

$$\tau : B[T,T^{-1}]_{\mathrm{rat}}/B[T,T^{-1}] \hookrightarrow B[[T^{-1},T]],$$

where $B[[T^{-1},T]]$ is the group of formal Laurent series with coefficients in B.

Let $\varphi = \sum_{i\in\mathbf{Z}} a_i T^i$ and $\psi = \sum_{i\in\mathbf{Z}} b_i T^i$ be series in $B[[T^{-1},T]]$. We define their Hadamard product as the series

$$\varphi * \psi := \sum_{i\in\mathbf{Z}} a_i b_i \, T^i.$$

A basic elementary result (see Proposition 5.1.1 of [8] for a proof) states that if two series φ and ψ in $B[[T^{-1},T]]$ belong to the image of τ, then their Hadamard product $\varphi*\psi$ is also in the image of τ. It follows in particular that the intersection of $B[[T]]$ with the image of $\exp_T$, which we shall denote by $B[[T]]_{\mathrm{rat}}$, is stable under Hadamard product.

Let $\varphi = \exp_T(P)$ be in $B[[T]]_{\mathrm{rat}}$. We denote by $\lambda(\varphi)$ the constant term in the expansion of $\exp_{T^{-1}}(P)$.

We shall need the following lemma, whose proof is completely elementary (see Proposition 5.1.2 of [8] for the proof).

Funny Lemma 5.3.1. *Let φ and ψ be series in $TB[[T]]_{\mathrm{rat}}$. Then*

$$\lambda(\varphi * \psi) = -\lambda(\varphi)\cdot\lambda(\psi).$$

5.4. A motivic stationary phase formula

Now we consider the Poincaré series

$$E_{W,f}(T) := \sum_{i>0} E_{i,W,f,\mathrm{mot}} T^i.$$

Note that $E_{W,f}(T)$ has no constant term. One may deduce from Theorem 3.4.1 that the series $E_{W,f}(T)$ belongs in fact to $\mathcal{U}_{\mathrm{loc}}[[T]]_{\mathrm{rat}}$.

We shall now consider $\mathcal{S}^{\psi}_{\alpha,W,f}$ as an element of $\mathcal{A}_{\mathrm{loc}}$ (note there is no $(\mathbf{L}-1)$-torsion in $\mathcal{A}_{\mathrm{loc}}$), and we define $\mathcal{S}^{\phi}_{\alpha,W,f} = \mathcal{S}^{\psi}_{\alpha,W,f}$ for $\alpha \neq 1$, and $\mathcal{S}^{\phi}_{\alpha,W,f} = \mathcal{S}^{\psi}_{\alpha,W,f} - \chi_c([W])$, for $\alpha = 1$, in $\mathcal{A}_{\mathrm{loc}}$. Remark that, since $\mathcal{S}^{\psi}_{\alpha,W,f}$ corresponds to motivic Euler characteristic of nearby cycles, $\mathcal{S}^{\phi}_{\alpha,W,f}$ corresponds to motivic Euler characteristic of vanishing cycles.

One easily gets the following formula, which may be viewed as a motivic analogue of the stationary phase formula:

Motivic stationary phase formula 5.4.1. *The following relation holds in $\mathcal{A}_{\mathrm{loc}}$:*

$$\lambda(E_{W,f}(T)) = -\mathbf{L}^{-m} \sum_{\alpha\in\widehat{\mu}(k)} G_{\alpha^{-1}} \mathcal{S}^{\phi}_{\alpha,W,f}.$$

The following Motivic Thom-Sebastiani Theorem follows directly from the motivic analogue stationary phase formula and the Funny Lemma 5.3.1.

Theorem 5.4.2. *Let X and X' be smooth and connected algebraic varieties over $\mathbf{C}$ of pure dimension m and m'. Let $f : X \to \mathbf{A}^1_k$ and $f' : X' \to \mathbf{A}^1_k$ be morphisms of k-varieties. Let W (resp. W') be a reduced subscheme of $f^{-1}(0)$ (resp. $f'^{-1}(0)$). Then*

$$\begin{aligned}(5.4.1)\qquad & \sum_{\alpha} G_{\alpha^{-1}} \mathcal{S}^{\phi}_{\alpha,W\times W',f\oplus f'} \\ &= \Big(\sum_{\alpha} G_{\alpha^{-1}} \mathcal{S}^{\phi}_{\alpha,W,f}\Big) \cdot \Big(\sum_{\alpha} G_{\alpha^{-1}} \mathcal{S}^{\phi}_{\alpha,W',f'}\Big).\end{aligned}$$

One can observe that the appearance in the Thom-Sebastiani formula of vanishing cycles instead of nearby cycles is explained here by the Funny Lemma 5.3.1 which is only valid for series without constant terms!

5.5. We now explain how one can deduce from Theorem 5.4.2 a Thom-Sebastiani Theorem for the Hodge spectrum.

Since a $\mathbf{C}$-Hodge structure of weight n is just a finite dimensional bigraded vector space $V = \bigoplus_{p+q=n} V^{p,q}$, or, equivalently, a finite dimensional vector space V with decreasing filtrations $F^{\cdot}$ and $\overline{F}^{\cdot}$ such that $V = F^p \oplus \overline{F}^q$ when $p+q = n+1$, one can define similarly a *rational* $\mathbf{C}$-Hodge structure of weight n, by allowing p and q to belong to $\mathbf{Q}$ but still requiring $p+q \in \mathbf{Z}$.

We denote by $K_0(\mathrm{RMHS}_{\mathbf{C}})$ the Grothendieck group of the abelian category of rational $\mathbf{C}$-Hodge structures. For $d \geq 1$, there is an embedding of $\widehat{\mu}_d(\mathbf{C})$ in $\mathbf{Q}/\mathbf{Z}$ given by $\alpha \mapsto a$ with $\alpha(e^{2\pi i/d}) = e^{2\pi i a}$. This gives an isomorphism $\widehat{\mu}(\mathbf{C}) \simeq \mathbf{Q}/\mathbf{Z}$. We denote by γ the section $\mathbf{Q}/\mathbf{Z} \to [0,1)$.

The morphism $H : A \to K_0(\mathrm{MHS}_{\mathbf{C}})$ may be extended to a morphism $H : U \to K_0(\mathrm{RMHS}_{\mathbf{C}})$ as follows. For p and q in $\mathbf{Q}$ with $p+q$ in $\mathbf{Z}$, we denote by $H^{p,q}$ the class of the rank 1 vector space with bigrading (p,q). We set $H(G_1) = -1$ and $H(G_\alpha) = -H^{1-\gamma(\alpha),\gamma(\alpha)}$ for $\alpha \neq 1$. This is compatible with the relations 5.2.2–5.2.4 since, by a standard calculation,

$$H(J_{\alpha_1,\alpha_2}) = -H^{1-(\gamma(\alpha_1)+\gamma(\alpha_2)-\gamma(\alpha_1+\alpha_2)),\gamma(\alpha_1)+\gamma(\alpha_2)-\gamma(\alpha_1+\alpha_2)},$$

when $\alpha_1 \neq 1$, $\alpha_2 \neq 1$ and $\alpha_1\alpha_2 \neq 1$.

By using a weight argument one can prove that $H : A \to K_0(\mathrm{MHS}_{\mathbf{C}})$ is zero on the kernel of the morphism $A \to \widehat{A}$. Hence H vanishes also on the kernel of the morphism $U \to \widehat{U}$, and we can extend it to the image of this morphism.

Assume now X is smooth and let x be a closed point of $f^{-1}(0)$. We shall denote by $\mathrm{Sp}(f,x)$ the Hodge spectrum as defined in [19] and [14] (which differs from that of [20] by multiplication by t).

By applying H to both sides of (5.4.1), when X and X' are smooth and W and W' are points one obtains the following Thom-Sebastiani Theorem for the Hodge spectrum, which was first proved by A. Varchenko in [21] when f and f' have isolated singularities (see also [17]), the general case being due to M. Saito [20], [15], [16]).

Theorem 5.5.1. *Let X and X' be smooth and connected complex algebraic varieties. Let $f : X \to \mathbf{A}^1_{\mathbf{C}}$ and $f' : X' \to \mathbf{A}^1_{\mathbf{C}}$ be morphisms of algebraic varieties. Let x and x' be closed points in $f^{-1}(0)$ and $f'^{-1}(0)$. Then*

$$\mathrm{Sp}(f \oplus f', (x,x')) = \mathrm{Sp}(f,x) \cdot \mathrm{Sp}(f',x').$$

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