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Infinitesimal CR Automorphisms

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To Masatake Kuranishi on his seventieth birthday

Let M be a real hypersurface through the origin in $\mathbf{C}^n$ or, more generally, an integrable CR manifold of hypersurface type. A smooth vector field X on M is called an *infinitesimal CR automorphism* of M if the local one-parameter group it generates is a local group of CR automorphisms of M. Fix $p \in M$ and let $\operatorname{aut}(M,p)$ denote the space of infinitesimal CR automorphisms of M which are defined in a neighborhood of p.

Throughout this paper, M will denote a connected analytic real hypersurface in $\mathbf{C}^n$. For $p \in M$, there is a distinguished subspace $\operatorname{hol}(M,p) \subset \operatorname{aut}(M,p)$ defined as follows. If Z is a holomorphic vector field defined in a neighborhood of $p \in \mathbf{C}^n$ and $X = \operatorname{Re} Z$, then the local one-parameter group of X is a group of biholomorphic transformations [KN, remarks preceding Proposition IX.2.10]. Here, by holomorphic vector field, I mean a vector field of type $(1,0)$ with holomorphic coefficients. Hence, if X is tangent to M, then $X \in \operatorname{aut}(M,p)$. Let $\operatorname{hol}(M,p)$ denote the space of all infinitesimal CR automorphisms X of M defined in some neighborhood of p which are of the form $X = \operatorname{Re} Z$ for some holomorphic vector field Z, $\operatorname{hol}(M,p) \subset \operatorname{aut}(M,p)$. Let $\operatorname{hol}(M) = \operatorname{hol}(M,0)$ and $\operatorname{aut}(M) = \operatorname{aut}(M,0)$.

Infinitesimal CR automorphisms are useful in the study of hypersurfaces with degenerate Levi form. I will survey some recent results about $\operatorname{hol}(M)$ and $\operatorname{aut}(M)$ and their applications. In Section 1, I use infinitesimal CR automorphisms to characterize homogeneous hypersurfaces. Section 2 describes applications of holomorphic nondegeneracy to finite dimensionality of $\operatorname{hol}(M)$ and to mappings of algebraic hypersurfaces. I will discuss some conditions for equality of $\operatorname{hol}(M)$ and $\operatorname{aut}(M)$ in Section 3.

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1. Homogeneous hypersurfaces Following the terminology of Baouendi, Rothschild and Trèves ([BRT]), a real hypersurface in $\mathbf{C}^n$ is called *rigid* if there are coordinates $(z_1, \dots, z_{n-1}, w = u + iv)$ such that M is given by an equation of the form

$$v = F(z, \overline{z}),$$

a *rigid equation.* Tanaka [T] called these *regular* and D'Angelo [DA] called them *T-regular.*

Among rigid hypersurfaces, the simplest ones are the homogeneous hypersurfaces. A rigid hypersurface is *homogeneous* if it is locally biholomorphically equivalent to

$$v = p(z, \overline{z}) \tag{1.1}$$

with p a homogeneous polynomial. This terminology comes from the fact that (1.1) is invariant under the nonisotropic dilations

$$(z, w) \to (tz, t^m w) = \delta_t(z, w) \tag{1.2}$$

where m is the degree of the polynomial p.

How can you tell if a rigid hypersurface is homogeneous? This problem was first posed by Linda Rothschild. The problem is local, so I will assume that $0 \in M$ and will work locally in a neighborhood of 0. Equivalences will preserve the origin. I can make a biholomorphic change of coordinates so that either M is the hyperplane $v = 0$ or M is given by an equation of the form

$$v = p(z, \overline{z}) + O(m + 1)$$

where p is a nontrivial homogeneous polynomial of degree m with no pure terms in z or $\overline{z}$. In this case, m is an invariant, the *type* of M at the origin, and M is of *finite type* at the origin. Suppose that the origin is a point of type m. A vector field Y is *homogeneous* of weight j if

$$Y(f \circ \delta_t) = t^{-j}(Yf) \circ \delta_t$$

where δ_t is the nonisotropic dilation (1.2).

If M is homogeneous, given by

$$v = p(z, \overline{z})$$

with p homogeneous of degree m, then

$$Y_0 = 2\operatorname{Re}\left(\sum_{j=1}^{n-1} z_j \frac{\partial}{\partial z_j} + mw\frac{\partial}{\partial w}\right)$$

is in $\mathrm{hol}(M)$ and is homogeneous of weight 0. It is the infinitesimal generator of the dilations δ_{e^t}. Call a vector field $Y \in \mathrm{hol}(M)$ an *approximate infinitesimal dilation* if

$$Y = Y_0 + \text{terms of weight} \geq 1.$$

Theorem 1.3 ([S5, Theorem 4.1]). *Let M be a rigid analytic real hypersurface through the origin in $\mathbf{C}^n$. Suppose M is given by a rigid equation of the form*

$$v = p(z, \overline{z}) + O(m+1)$$

with p a nontrivial polynomial homogeneous of degree m having no pure terms. Then M is homogeneous if and only if M has an approximate infinitesimal dilation.

This theorem was first proved in $\mathbf{C}^2$ ([S1], [S2], [S3]), then in $\mathbf{C}^n$ under the additional hypothesis that $\dim \mathrm{hol}(M) < \infty$ ([S4]).

Theorem 1.3 can be generalized to characterize weighted homogeneous hypersurfaces. Fix positive integers $m_1, \ldots, m_n$. Now I will use $(z_1, \ldots, z_n)$ as coordinates. The non-isotropic group of dilations determined by $(m_1, \ldots, m_n)$ is the group $\{\delta_t : t > 0\}$ where

$$\delta_t(z) = (t^{m_1} z_1, \ldots, t^{m_n} z_n).$$

A function h is *homogeneous of weight j* if $h \circ \delta_t = t^j h$. A vector field Y is *homogeneous of weight j* if

$$Y(f \circ \delta_t) = t^{-j}(Yf) \circ \delta_t.$$

Let

$$Y_0 = 2\,\mathrm{Re} \sum_{j=1}^{n} m_j z_j \frac{\partial}{\partial z_j}.$$

The one-parameter group generated by Y_0 is the group of non-isotropic dilations $\{\delta_{e^t} : t \in \mathbf{R}\}$. An analytic real hypersurface M is *weighted homogeneous* (with respect to the non-isotropic group of dilations) if it is locally equivalent, via a biholomorphic map which preserves the origin, to a hypersurface given by an equation of the form

$$P(z, \overline{z}) = 0$$

where P a polynomial which is homogeneous with respect to the non-isotropic group of dilations.

As before, call a vector field $Y \in \mathrm{hol}(M)$ an *approximate infinitesimal dilation* if

$$Y = Y_0 + \text{terms of weight} \geq 1.$$

Theorem 1.4 ([S5, Theorem 4.1]). *Let M be an analytic real hypersurface through the origin in $\mathbf{C}^n$ and suppose there is an approximate infinitesimal dilation $Y \in \mathrm{hol}(M)$. Then M is weighted homogeneous.*

This theorem does not require the hypothesis that M be rigid and there is no nondegeneracy hypothesis or finite type hypothesis on M.

The theorem can be proved by a technique used by Poincaré in his thesis [P] and generalized by Dulac [Du]. One linearizes Y, that is, one finds a change of coordinates so that in the new coordinates $\widetilde{z}$,

$$Y = 2\,\mathrm{Re} \sum_{j=1}^{n} m_j \widetilde{z}_j \frac{\partial}{\partial \widetilde{z}_j}.$$

To do this, one first finds a formal change of variables, then one applies Poincaré's by now standard domination argument to prove that the formal change converges.

Now, after reordering the coordinates and multiplying $\widetilde{z}_n$ by i if necessary, I can assume M is given by an equation of the form

$$\mathrm{Im}\, \widetilde{z}_n = \widetilde{F}(\widetilde{z}\,', \overline{\widetilde{z}\,'}, \mathrm{Re}\, \widetilde{z}_n) \tag{1.5}$$

where $\widetilde{z}\,' = (\widetilde{z}_1, \dots, \widetilde{z}_{n-1})$. Applying Y to this equation shows that the right side of this equation is a weighted homogeneous polynomial and hence M is homogeneous.

By replacing $\widetilde{z}_n$ with $a\widetilde{z}_n$ for an appropriate $a \in \mathbf{C}$, one may assume that (1.5) is a rigid equation. This yields the following proposition.

Proposition 1.6 ([S5, Proposition 4.3]). *If M is weighted homogeneous then M is rigid.*

2. Holomorphic nondegeneracy How can one tell whether $\mathrm{hol}(M)$ is finite dimensional? In $\mathbf{C}^2$ it is for any hypersurface M of finite type. The example

$$v = |z_1|^2$$

in $\mathbf{C}^n$, $n \geq 3$, shows that some stronger nondegeneracy hypothesis is required in higher dimensions. In this example, $\mathrm{Re}\, f(z,w)\frac{\partial}{\partial z_2} \in \mathrm{hol}(M)$ for any holomorphic function f.

Definition. *Let M be an analytic real hypersurface in $\mathbf{C}^n$. A nontrivial holomorphic vector field W is called a holomorphic tangent to M at p if W is defined in a neighborhood of p and $W\big|_M$ is tangent to M. The hypersurface M is holomorphically nondegenerate at p if M*

has no holomorphic tangent at p. If M has a holomorphic tangent at p, M is holomorphically degenerate at p.

Theorem 2.1 ([S4, Theorem 4.3]). *Let M be an analytic real hypersurface through the origin in $\mathbf{C}^2$. The following are equivalent.*

(1) $\operatorname{hol}(M)$ *is finite dimensional;*
(2) *M is not flat;*
(3) *the Levi form of M is somewhere nondegenerate;*
(4) *M is holomorphically nondegenerate at the origin.*

In higher dimensions holomorphic nondegeneracy is not the same as nonflat, finite type, essentially finite or somewhere Levi nondegenerate. (See [BJT] for the definition of essentially finite.)

Theorem 2.2 ([BR2, Theorem 2, Proposition 4.2], [S6, Corollaries 3.3, 3.4]). *Let M be an analytic real hypersurface through the origin in $\mathbf{C}^n$. The following are equivalent.*

(1) *M is holomorphically nondegenerate at the origin.*
(2) *M is everywhere holomorphically nondegenerate.*
(3) *M is essentially finite on an open dense set.*

In general, and even for many simple examples of hypersurfaces with polynomial defining equations, it is very difficult to compute $\operatorname{hol}(M)$. If M is rigid with a rigid defining equation which is a polynomial, in principle—and often in fact—it is easy to check whether M is holomorphically nondegenerate at the origin.

Holomorphic nondegeneracy is a natural condition to introduce in connection with finite dimensionality of $\operatorname{hol}(M)$. Suppose M is a holomorphically degenerate real hypersurface, with holomorphic tangent Z. Then for all multi-indices α, $X_\alpha = \operatorname{Re} z^\alpha Z \in \operatorname{hol}(M)$ so $\dim \operatorname{hol}(M) = \infty$. This gives one direction of the following theorem.

Theorem 2.3 ([S4, Theorem 4.16], [S6, Theorem 1.7]). *Let M be an analytic real hypersurface through the origin in $\mathbf{C}^n$. Then the space* $\operatorname{hol}(M)$ *is finite dimensional if and only if M is holomorphically nondegenerate.*

In $\mathbf{C}^2$ the theorem follows easily from Theorem 2.1. Theorem 2.3 was first proved in the case of rigid hypersurfaces [S4]. In the rigid case the proof is long and technical; much of the work goes into proving an approximate version of the theorem, which requires a polynomial hypersurface to approximate M and an approximate version of $\operatorname{hol}(M)$. In dimensions greater than 2, the approximating hypersurface must include some higher order terms; the homogeneous part may not give a good approximation. The proof gives a bound on $\dim \mathrm{hol}(M)$ which depends on the type at the origin and the defining equation. To prove the theorem in the general case, one shows that if M is holomorphically nondegenerate and $\dim \mathrm{hol}(M) \geq 1$, then there is an open dense set $U \subset M$ and an integer ℓ (computable in terms of an appropriate defining function for M) such that if $p \in U$, then M is rigid, essentially finite and of type 2 at p, and $\dim \mathrm{hol}(M,p) \leq \ell$.

The following theorem of Baouendi and Rothschild gives an application of holomorphic nondegeneracy to mappings of algebraic hypersurfaces. A real hypersurface is *algebraic* if it is contained in the zero set of a nontrivial real valued polynomial. A holomorphic map is *algebraic* if its components satisfy polynomial equations with polynomial coefficients.

Theorem 2.4 ([BR2, Theorem 1]). *Let M be a holomorphically nondegenerate algebraic real hypersurface in $\mathbf{C}^n$ and let M' be an algebraic real hypersurface in $\mathbf{C}^n$. If f is a biholomorphic map taking M to M' then f is algebraic. Conversely, if M is a holomorphically degenerate algebraic real hypersurface which contains the origin, then there is a nonalgebraic biholomorphic map f defined in a neighborhood of the origin, with $f(0) = 0$, which takes M to itself.*

3. Analyticity of infinitesimal CR automorphisms For any analytic real hypersurface M and any $p \in M$, $\mathrm{hol}(M,p) \subset \mathrm{aut}(M,p)$. The two spaces are not always equal.

Example 3.1 ([S4, Example 7.11]). Let $M = \{v = 0\} \subset \mathbf{C}^2$. Then

$$X = e^{-1/u^2} \frac{\partial}{\partial u} \in \mathrm{aut}(M).$$

However, $X \notin \mathrm{hol}(M)$ so $\mathrm{hol}(M) \subsetneq \mathrm{aut}(M)$.

There is a sufficient condition for equality of $\mathrm{hol}(M)$ and $\mathrm{aut}(M)$.

Proposition 3.2 ([S3, Remark 2.5]). *Let M be an analytic real hypersurface through the origin in $\mathbf{C}^n$. Suppose every CR diffeomorphism on M is analytic. Then* $\mathrm{hol}(M) = \mathrm{aut}(M)$.

The next theorem summarizes what is known about equality of $\mathrm{hol}(M)$ and $\mathrm{aut}(M)$ in the case that $\mathrm{hol}(M)$ is finite dimensional.

Theorem 3.3. *Let M be an analytic real hypersurface through the origin in $\mathbf{C}^n$. Suppose that one of the following holds.*

(1) *M is essentially finite;*

(2) *M is rigid and every neighborhood U of 0 contains a point $p \in M$ such that the Levi form of M is nondegenerate at p;*
(3) *M is algebraic and holomorphically nondegenerate.*

Then $\operatorname{aut}(M)$ *is finite dimensional and* $\operatorname{aut}(M) = \operatorname{hol}(M)$.

Theorem 3.3 was proved for hypersurfaces satisfying (1) and (2) in [S4, Theorem 6.1]. For hypersurfaces satisfying (3) it follows from Proposition 3.2 and the following theorem of Baouendi, Huang and Rothschild.

Theorem 3.4 [BHR, Theorem 1]. *Let M and M' be algebraic real hypersurfaces in $\mathbf{C}^n$ and suppose that M is holomorphically nondegenerate. If H is a smooth CR map from M to M' and the Jacobian determinant of H is not everywhere 0, then H extends holomorphically to a neighborhood of M.*

To describe additional results on the question of when $\operatorname{hol}(M) = \operatorname{aut}(M)$, I need a characterization of infinitesimal CR automorphisms analogous to the definition of $\operatorname{hol}(M)$.

Proposition 3.5. *Let M be a real hypersurface through the origin in $\mathbf{C}^n$ and let X be a smooth tangent vector field defined in a neighborhood of the origin on M. Then $X \in \operatorname{aut}(M)$ if and only if*

$$X = \operatorname{Re} \sum_{j=1}^{n} f_j \frac{\partial}{\partial z_j} \tag{3.6}$$

where each f_j is a CR function on a neighborhood of the origin in M.

Proof. Let X be a $\mathcal{C}^\infty$ real vector field tangent to M. By Theorem 1 of [BR1], it suffices to show that X is of the form (3.6) if and only if for every smooth section Y of $T^{0,1}(M)$ on a neighborhood of the origin,

$$[X, Y] \in T^{0,1}(M). \tag{3.7}$$

Now $X = (Z + \overline{Z})\big|_M$ for some smooth vector field $Z = \sum_{j=1}^n f_j \frac{\partial}{\partial z_j}$ defined in a neighborhood of the origin. Let $Y = \sum_{j=1}^n g_j \frac{\partial}{\partial \overline{z}_j} \in \mathcal{C}^\infty(T^{0,1}(M))$. Then Y extends to a $\mathcal{C}^\infty$ vector field $\widetilde{Y}$ of type $(0,1)$ defined in a neighborhood of the origin. Now

$$\begin{aligned} [X, Y] &= ([Z, \widetilde{Y}] + [\overline{Z}, \widetilde{Y}])\big|_M \\ &= \Big(\sum_{j=1}^{n} (Z g_j) \frac{\partial}{\partial \overline{z}_j} - \sum_{j=1}^{n} (Y f_j) \frac{\partial}{\partial z_j} + [\overline{Z}, \widetilde{Y}]\Big)\Big|_M . \end{aligned}$$

The first and last terms are of type $(0,1)$. Hence (3.7) holds for all Y if and only if $Yf_j \equiv 0$ for all smooth sections Y of $T^{0,1}(M)$, so if and only if f_j is a CR function for each j.

Baouendi, Huang and Rothschild proved the following theorem about failure of analyticity of CR diffeomorphisms for holomorphically degenerate hypersurfaces.

Theorem 3.8 ([BHR, Theorem 4]). *Let M be an analytic holomorphically degenerate real hypersurface through the origin in $\mathbf{C}^n$. If there is a germ at 0 of a smooth CR function on M which does not extend to be holomorphic in any neighborhood of 0, then there is a germ of a smooth CR diffeomorphism from M to itself, fixing 0, which does not extend holomorphically to any neighborhood of 0.*

This result is closely related to the question of when $\mathrm{hol}(M) = \mathrm{aut}(M)$ in the holomorphically degenerate case.

Theorem 3.9. *Let M be a holomorphically degenerate analytic real hypersurface through the origin in $\mathbf{C}^n$. Then $\mathrm{hol}(M) = \mathrm{aut}(M)$ if and only if every CR function defined on a neighborhood of the origin in M extends to be holomorphic on a neighborhood of the origin in $\mathbf{C}^n$.*

Proof. Suppose every CR function on a neighborhood of the origin in M extends to be holomorphic. Let $X \in \mathrm{aut}(M)$. Then X is given by (3.6) for some CR functions f_j. There is a neighborhood U of the origin in $\mathbf{C}^n$ such that f_j, $j = 1, \dots, n$, extends to a holomorphic function F_j on U. Hence, $X = \mathrm{Re}\, Z\big|_M$ where $Z = \sum F_j \frac{\partial}{\partial z_j}$, and $X \in \mathrm{hol}(M)$.

Suppose $\mathrm{hol}(M) = \mathrm{aut}(M)$. Let Z be a holomorphic tangent to M at the origin, $Z = \sum f_j \frac{\partial}{\partial z_j}$, for some holomorphic functions f_j. Let f be a CR function defined on a neighborhood of the origin in M. Then, by Proposition 3.5,

$$X = \mathrm{Re} \sum_{j=1}^{n} f f_j \frac{\partial}{\partial z_j}$$

is in $\mathrm{aut}(M)$, so $X \in \mathrm{hol}(M)$. Because $X \in \mathrm{hol}(M)$, the proof of Theorem 3.8 shows that f extends to be holomorphic in a neighborhood of the origin, so every CR function extends.

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