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# Cohomology mod $p$ of the 4-connected Cover of the Classifying Space of Simple Lie Groups

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## §0. Introduction

Let $G$ be a compact, connected, simply connected, simple Lie group and $BG$ its classifying space. A prime $p$ is called good (for $G$) (resp. exceptional (for $G$)) if $H_*(G; Z)$ is $p$-torsion free (resp. not $p$-torsion free). As is well known $BG$ is 3-connected and $\pi_4(BG)=H_4(GB; Z)=H^4(BG; Z)=Z$ (cf. [3]). Represent a generator of $H^4(BG; Z)$ by a map $Q''\colon BG\to K(Z, 4)$ and denote its homotopy fibre by $B\tilde{G}$. The purpose of this paper is to determine $H^*(B\tilde{G}; F_p)$ for any odd prime $p$.

Consider the following pull back diagram:

$$\begin{tikzcd}
K(Z,3) \arrow[r, "\pi'"] \arrow[d, equal] & B\tilde{T} \arrow[r, "Q'"] \arrow[d, "i"] & BT \arrow[d, "\bar{i}"] \\
K(Z,3) \arrow[r, "\pi"'] & B\tilde{G} \arrow[r, "Q''"'] & BG
\end{tikzcd}$$

where $T$ is a maximal torus, $i$ and $\bar{i}$ are the maps induced by the inclusion. Note that $\bar{i}^*\colon H^4(BG; Z)\to H^4(BT; Z)$ is a monomorphism and $\operatorname{Im}\bar{i}^*=H^4(BT; Z)^{W(G)}$ where $W(G)$ is the Weyl group of $G$. Therefore $Q'=\bar{i}^*Q''$ is a generator of $H^4(BT\colon Z)^{W(G)}$. Denote the mod $p$ reduction of $Q'$ by $Q$. Since $H^*(BT; F_p)\cong S(H_2(BT, F_p)^*)$, where $S$ denotes the symmetric algebra, we may consider that $Q$ is a quadratic form. Let $h=h(G,p)$ be the codimension of a $Q$-isotropic subspace of maximum dimension.

As is well known that

$$H^*(K(Z,3); F_p)\cong S(\beta P_k u_3;\ k\geq 1)\otimes E(P_k u_3;\ k\geq 0)$$

where $E$ denotes the exterior algebra, $P_k=\mathscr{P}^{p^{k-1}}\cdots\mathscr{P}^1$ and $u_3$ is a generator of $H^3(K(Z, 3); F_p)$ $(=Z/p)$. Denote the subalgebra generated by $\{\beta P_k u_3;\ k\geq 1\}\cup\{P_k u_3;\ k\geq j\}$ by $R_j$. Then the main results of this paper are the following:

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**Theorem 2.2.** *As an algebra* $H^*(B\tilde{T}; F_p)$ *is isomorphic to* $H^*(BT; F_p)/J \otimes R_h$ *where* $J$ *is the ideal generated by* $Q, P_1Q, \cdots, P_{h-1}Q$.

**Theorem 2.3.** *For a good prime* $p$, $H^*(B\tilde{G}; F_p)$ *is isomorphic to* $H^*(BG; F_p)/J' \otimes R_h$ *as an algebra where* $J'$ *is the ideal generated by* $x_4, P_1x_4, \cdots, P_{h-1}x_4$ $(x_4 = i^*Q)$.

**Theorem 4.1.** *The Serre spectral sequence for the fibering* $G/T \to B\tilde{T} \to B\tilde{G}$ *collapses for any* $G$ *and any odd prime* $p$.

The paper is organized as follows: In Section 1 we prove certain algebraic results which are used in Section 2. In Section 2 we determine $H^*(B\tilde{G}; F_p)$ for a good prime $p$. In Section 3 we determine $h = h(G, p)$. For an exceptional prime $p$, the module structure and the algebra structure of $H^*(B\tilde{G}; F_p)$ are determined in Section 4 and Section 5 respectively.

For a classical type $G$ the result was announced in [5].

Throughout the paper $p$ is an odd prime.

## § 1. A note on a quadratic form over $F_p$

In this section we prepare some algebraic results. Let $V$ be an $n$-dimensional vector space over $F_p$. Let $S(V^*)$ be the symmetric algebra over $V^*$, the dual of $V$. Consider a quadratic form $Q$ on $V$ and define its associated bilinear form by $B(x, y) = \frac{1}{2}(Q(x+y) - Q(x) - Q(y))$. We consider the following sequence of homogeneous elements in $S(V^*)$:

$$(1.1) \qquad Q(x), B(x, x^p), \cdots, B(x, x^{p^{h-1}})$$

where $h$ is the codimension of a $Q$-isotropic subspace of maximum dimension. Firtst we should do is to prove the following:

**Theorem 1.2.** *The sequence* (1.1) *is a regular sequence and all maximal* $Q$*-isotropic subspaces of* $V$ *are of same dimension* $n-h$.

*Proof.* Let $J$ be an ideal of $S(V^*)$ generated by (1.1) and Var $J$ the common zeros of (1.1) in $V_\Omega = V \otimes \Omega$, where $\Omega$ is a universal field of $F_p$. It is well known that (1.1) is a regular sequence if and only if dim Var $J = n-h$ (see Theorem 2 of p. 397 of [15]). Therefore Theorem 1.2 is an easy consequence of the following Lemma 1.3.

**Lemma 1.3.** Var $J = \bigcup W_\Omega$, *where* $W$ *ranges over maximal* $Q$*-isotropic subspaces.*

*Proof of Lemma* 1.3. Using the identity

$$Q(\sum t_i x^{p^i}) = \sum t_i^2 Q(x)^{p^i} + 2 \sum_{i<j} t_i t_j B(x, x^{p^{j-i}})^{p^i}$$

we see easily that $x \in \operatorname{Var} J$ if and only if the $\Omega$-subspace

$$M_x = \Omega x + \cdots + \Omega x^{p^{h-1}}$$

is $Q$-isotropic in $V_\Omega$ ($Q$ is extended to $V_\Omega$ naturally). It is also seen that $\operatorname{Var} J = \bigcup_{x \in \operatorname{Var} J} M_x$. Clearly $W_\Omega \subset \operatorname{Var} J$. We need only show $M_x \subset W_\Omega$ for some maximal $Q$-isotropic subspace $W$. Since a space which is stable under the Frobenius map $F$ should have a form $W'_\Omega$ for some subspace $W'$ in $V$, $W' \subset W$ and so $M_x = W'_\Omega$ is a subspace of $W_\Omega$ for some maximal $Q$-isotropic subspace $W$.

Therefore we will show that $M_x$ is stable under $F$. Recall the classification of quadratic forms over $\boldsymbol{F}_p$. First $V = V' \perp V_0$, where $\perp$ denotes the orthogonal decomposition, $B$ is nondegenerate on $V'$ and $V_0$ is the radical of $V$. $V'$ can be decomposed as follows:

$$V' = P_1 \perp P_2 \perp \cdots \perp P_m \perp S$$

where $P_i$ is a hyperbolic plane ($\dim P_i = 2$ and $Q = x_1 x_2$ on $P_i$) and $S$ is one of the following four types:

$$(1.4)\qquad \begin{array}{lll} \text{type } 0: & \dim S = 0 & \\ \text{type } \mathrm{I}_+: & \dim S = 1 & Q = x^2 \text{ on } S \\ \text{type } \mathrm{I}_-: & \dim S = 1 & Q = gx^2 \text{ on } S \\ \text{type } \mathrm{II}: & \dim S = 2 & Q = x_1^2 - gx_2^2 \text{ on } S \end{array}$$

where $g$ is one of non-square elements of $\boldsymbol{F}_p$, fixed once for all (see for example Ch. IV. 3 of [1]). We check that $M_x$ is stable under $F$ in each form of four types in (1.4). It is enough to prove the lemma when $V_0 = 0$.

If $\dim M_x \leq h-1$, there is a linear relation

$$x^{p^{h-1}} = \sum_{i=0}^{h-2} \lambda_i x^{p^i} \quad (\lambda_i \in \Omega).$$

Hence $M_x$ is stable under $F$. Now we explain in each form of four types:

type $\mathrm{I}_\pm$: $\dim V = 2m+1$ and $h = m+1$. $M_x$ is $Q$-isotropic. Therefore $\dim M_x = \dim V - h = m = h-1$ and so Lemma 1.3 holds.

type II: $\dim V = 2m+2$ and $h = m+2$. $S_\Omega$ is a hyperbolic plane. Therefore $\dim M_x = \frac{1}{2} \dim V = m+1 = h-1$ and so Lemma 1.3 holds.

type 0: $\dim V = 2m$ and $h = m$. If $\dim M_x \leq m-1 = h-1$, Lemma 1.3 holds as above. Assume $\dim M_x = m$. In this case we can write $V = U \oplus U^*$ with $Q(u+v) = \langle u, v \rangle$, where $U$ is a subspace of dimension

$m$, $U^*$ its dual and $\langle\ ,\ \rangle$ is the pairing of $U$ and $U^*$. By the assumption $\pi_1\colon M_x \to U_\Omega$ may be surjective. Then there is a unique linear transformation $T\colon U \to U^*$ such that

$$M_x = \{z + Tz;\ z \in U_\Omega\}.$$

The fact that $M_x$ is $Q$-isotropic can be rewritten as

$$\langle z, Tz\rangle = 0 \quad \text{and} \quad \langle z, Tz'\rangle = \langle z', Tz\rangle$$

for any $z, z' \in U_\Omega$. Let $x = u + v$ for $u \in U$ and $v \in U^*$. As $x^{p^i} \in M_x$ for $0 \le i < m$, we have $T(u^{p^i}) = v^{p^i}$ for $0 \le i < m$. So for $1 \le i < m$

$$\begin{aligned}\langle u^{p^i}, Tu^{p^m}\rangle &= \langle u^{p^m}, Tu^{p^i}\rangle = \langle u^{p^m}, v^{p^i}\rangle = \langle u^{p^{m-1}}, v^{p^{i-1}}\rangle^p \\ &= \langle u^{p^{m-1}}, Tu^{p^{i-1}}\rangle^p = \langle u^{p^{i-1}}, Tu^{p^{m-1}}\rangle^p = \langle u^{p^{i-1}}, v^{p^{m-1}}\rangle^p \\ &= \langle u^{p^i}, v^{p^m}\rangle.\end{aligned}$$

And also $\langle u^{p^m}, Tu^{p^m}\rangle = 0 = \langle u, v\rangle^{p^m} = \langle u^{p^m}, v^{p^m}\rangle$. In $U_\Omega$, $u, \cdots, u^{p^{m-1}}$ form a basis and also $u^p = F(u), \cdots, u^{p^m} = F(u^{p^{m-1}})$ form a basis since $F$ is a semi-linear automorphism. Therefore $Tu^{p^m} = v^{p^m}$ in $U^*$ and so $M_x$ is stable under $F$. This completes the proof of Lemma 1.3 and so Theorem 1.2 is proved.

Next we examine primary components of the ideal $J$. $J$ has a primary decomposition

$$J = \bigcap q_i \tag{1.5}$$

where $q_i$ is a primary ideal associated to a prime ideal $p_i$. The irreducible components of Var $J$ are in 1-1 correspondence with minimal primes $p_i$ (cf. p. 163 of [15]). The theorem of Macauley says that there is no embedded component in $J$ for it is generated by the regular sequence (1.1) (cf. p. 203 of [15]). Therefore all $p_i$'s are distinct and minimal. Now we have the following:

**Proposition 1.6.** *$J = \bigcap_W q_W$ where $W$ ranges all maximal $Q$-isotropic subspaces of $V$ and $q_W$ is a primary ideal whose associated prime ideal $p_W$ is functions in $S(V^*)$ vanishing on $W_\Omega$.*

We now assume that $V_0 = 0$ until Remark 1.12 for some technical reasons.

We determine $e(q_W)$, multiplicity of $q_W$. Because Var $p_W = W_\Omega$, deg $p_W = 1$. Thus the generalized Bezout's theorem implies (cf. § 27 of [9])

$$\prod_{j=0}^{h-1} (1 + p^j) = \sum_W e(q_W). \tag{1.7}$$

To determine $e(q_W)$ and $q_W$, we need to count the number of maximal $Q$-isotropic subspaces of $V$ in each form of four types in (1.4).

type 0: Let $W_m \supset \cdots \supset W_1 \supset 0$ be a $Q$-isotropic flag where $W_m$ is a maximal $Q$-isotropic subspace. The number of isotropic vectors is $(p^m-1)(p^{m-1}+1)$ (see p. 146 of [1]). Once $W_1$ is chosen the rest of the flag are the same as an isotropic flag in the space $W_1^\perp/W_1$ which has dimension $2m-2$ and still type 0. Therefore the number of maximal $Q$-isotropic subspaces is

(1.8) $$\prod_{j=1}^{m}(p^{j-1}+1)$$

(see [11]).

type $\mathrm{I}_\pm$: Using the method as above. Note that the number of isotropic vectors is $p^{2m}-1$ (see p. 146 of [1]). Hence the number of maximal $Q$-isotropic subspaces is

(1.9) $$\prod_{j=1}^{m}(p^{j}+1).$$

type II: In this case the number of isotropic vectors is $(p^m-1)(p^{m+1}+1)$ (see p. 146 of [1]). As before the number of maximal $Q$-isotropic subspaces is

(1.10) $$\prod_{j=1}^{m}(p^{j+1}+1).$$

Using the above computation we can now determine $e(q_W)$ and the ideal $q_W$.

**Theorem 1.11.** *If $W$ is a maximal $Q$-isotropic subspace of $V$, then*

$$q_W=\mathrm{Ker}\,\{r_W\colon S(V^*)\to S((W^\perp)^*)/J(W^\perp)\}$$

*where $W^\perp$ is the annihilator subspace of $W$, $J(W^\perp)$ is the ideal generated by $Q'(x)$, $B'(x, x^p), \cdots, B'(x, x^{p^{h'-1}})$ ($Q'$ or $B'$ is the restriction of $Q$ or $B$ to $W^\perp$ and $h'=\dim W^\perp-\dim W$.) and $r_W$ is the natural map induced by the inclusion. Moreover $e(q_W)=1$, $2$, or $2(p+1)$ if $Q$ is of* type 0, $\mathrm{I}_\pm$, *or* II *respectively.*

**Remark 1.12.** We have assumed that $V_0=0$ (i.e. $B$ is nondegenerate) since Proposition 1.6. But it is obvious that Theorem 1.11 holds for all non-degenerate cases, it is also valid in degenerate cases. Therefore we still assume in the proof that $V_0=0$. Theorem 1.11 holds unless $V_0=0$.

*Proof of Theorem* 1.11. We prove in each form of four types in (1.4).

type 0: Compare (1.7) and (1.8). These are equal and so $e(q_W)=1$ for all $W$. $W^{\perp}=W$ and $J(W^{\perp})=0$ and so the theorem holds.

type $\mathrm{I}_{\pm}$: $W^{\perp}=W\oplus S$ then

$$S((W^{\perp})^*)/J(W^{\perp})\cong S(W^*)\otimes \boldsymbol{F}_p[x]/(x^2) \tag{1.13}$$

and the zero ideal of this ring is a primary ideal of multiplicity 2. Let $q'_W=\mathrm{Ker}\,\{S(V^*)\to S((W^{\perp})^*)/J(W^{\perp})\}$. Then $q'_W$ is a primary ideal of multiplicity 2 associated with $p_W=\mathrm{Ker}\,\{S(V^*)\to S(W^*)\}$. Because $J\subset q'_W$, $q_W\subset q'_W$ and so $e(q_W)\geq e(q'_W)=2$. Compare now (1.7) and (1.9), $e(q_W)\geq 2$ implies $e(q_W)=2$ and so $q_W=q'_W$.

type II: $W^{\perp}=W\oplus S$ then

$$S((W^{\perp})^*)/J(W^{\perp})=S(W^*)\otimes S(S^*)/J(S)\cong S(W^*)\otimes \boldsymbol{F}_p[x_1, x_2]/J(S) \tag{1.14}$$

where $J(S)$ is the ideal generated by $x_1^2-gx_2^2$ and $x_1^{p+1}-gx_2^{p+1}$. The multiplicity of $J(S)$ in $\boldsymbol{F}_p[x_1, x_2]$ is a special case of this theorem. Set $W=0$ and $V=S$. By (1.7) $e(J(S))=2(p+1)$. As before we can prove the theorem by (1.7) and (1.10).

Finally we show the following:

**Theorem 1.15.** *For all $m$, $B(x, x^{p^m})\in J$.*

*Proof.* From the proof of Theorem 1.11, only type II case is non trivial. Here $\dim S=2$ and $Q=x_1^2-gx_2^2$ and so $J=J(S)$ in the proof of Theorem 1.11. $x_1^2\equiv gx_2^2 \bmod J$ and so $g^{(p+1)/2}x_2^{p+1}\equiv x_1^{p+1}\equiv gx_2^{p+1} \bmod J$. Thus we have

$$\begin{aligned} x_1^{p^{2n+1}+1} &\equiv g^{(p^{2n+1}+1)/2}x_2^{p^{2n+1}+1}\equiv g^{(p+1)p(1)/2}x_2^{(p+1)p(1)} \\ &\equiv g^{p(1)}x_2^{p^{2n+1}+1}\equiv gx_2^{p^{2n+1}+1} \end{aligned}$$

and

$$\begin{aligned} x_1^{p^{2n}+1} &\equiv x_1^{p^{2n}-1}x_2^2\equiv g^{(p^{2n}-1)/2}x_2^{p^{2n}-1}\times gx_2^2\equiv g^{(p+1)p(2)}x_2^{(p+1)p(2)}\times gx_2^2 \\ &\equiv g^{p(2)}\cdot gx_2^{p^{2n}+1}\equiv gx_2^{p^{2n}+1} \end{aligned}$$

mod $J$ where $(p+1)p(1)=p^{2n+1}+1$ and $(p+1)p(2)=p^{2n}-1$. Thus $B(x, x^{p^m})\in J$ for $m\geq 2$.

## § 2. $H^*(B\tilde{G}; F_p)$ for a good prime $p$

In this section we determine the algebra structure of $H^*(B\tilde{G}; F_p)$ for a good prime $p$. Note that an odd prime $p$ is exceptional if and only if $(G, p)$ is one of the following:

$$(E_6, 3),\ (E_7, 3),\ (E_8, 3),\ (F_4, 3),\ (E_8, 5). \tag{2.1}$$

First we determine $H^*(B\tilde{T}; F_p)$. Consider the Serre spectral sequence for the fibering $K(Z, 3)\xrightarrow{j'} B\tilde{T}\xrightarrow{\pi'} BT$ with $F_p$ coefficient

$$E_2 = H^*(BT; F_p)\otimes H^*(K(Z, 3); F_p) \Longrightarrow E_\infty = \mathrm{Gr}\,(H^*(B\tilde{T}; F_p)).$$

The element $u_3$ is transgressive with $\tau(u_3)=Q$. Therefore $P_k u_3$ and $\beta P_k u_3$ are transgressive with $\tau(P_k u_3)=P_k Q=2^k B(x, x^k)$ and $\tau(\beta P_k u_3)=\beta P_k Q=0$. Theorem 1.2 says $\tau(u_3)$, $\tau(P_1 u_3)$, $\cdots$, $\tau(P_{k-1}u_3)$ is a regular sequence. On the other hand $\tau(P_h u_3)\in J=(\tau(u_3), \cdots, \tau(P_{h-1}u_3))$ by Theorem 1.15 and so $P_h u_3 \in \mathrm{Im}\, j'^*$. Thus we have $E_\infty = H^*(BT; F_p)/J\otimes R_h$. Since $H^*(BT; F_p)/J$ is $\mathrm{Im}\,\pi'^*$ and $R_h$ is a free commutative algebra we have the following:

**Theorem 2.2.** *As an algebra $H^*(B\tilde{T}; F_p)$ is isomorphic to $H^*(BT; F_p)/J\otimes R_h$ where $J$ is the ideal generated by $Q$, $P_1Q$, $\cdots$, $P_{h-1}Q$.*

From now on we assume that $p$ is good for $G$. In this case $H^{2j-1}(BG; F_p)$ and $H^{2j-1}(G/T; F_p)=0$ for any $j$ (see Borel [2] and Bott [3]), and the Serre spectral sequence for the fibering $G/T\xrightarrow{\bar{\lambda}} BT\xrightarrow{\bar{i}} BG$ with $F_p$ coefficient collapses. Hence $H^*(BT; F_p)$ is a free module over $H^*(BG; F_p)$ and so $i^*$ is faithfully flat. Put $x_4 = i^*Q$, then in the Serre spectral sequence for the fibering $u_3$ is transgressive with $\tau(u_3)=x_4$, $\tau(P_k u_3)=P_k x_4 = i^* 2^k(x, x^{p^k})$ and $\tau(\beta P_k u_3)=0$. Since $Q$, $B(x, x^p)$, $\cdots$, $B(x, x^{p^{h-1}})$ is a regular sequence, $B(x, x^{p^h})\in (Q, B(x, x^p), \cdots, B(x, x^{p^{h-1}}))$ and $i^*$ is faithfully flat, we have $x_4$, $P_1x_4$, $\cdots$, $P_{h-1}x_4$ is a regular sequence and $P_h x_4 \in J' = (x_4, \cdots, P_{h-1}x_4)$. Thus we have

**Theorem 2.3.** *If $p$ is good for $G$, then as an algebra, $H^*(B\tilde{G}; F_p)$ is isomorphic to $H^*(BG; F_p)/J'\otimes R_h$ where $J'$ is the ideal generated by $x_4$, $P_1x_4$, $\cdots$, $P_{h-1}x_4$.*

**Remark 2.4.** If $p$ is good for $G$, then $\bar{\lambda}^*$ is surjective and so $\lambda^*$ is also surjective, where $\lambda$: $G/T\to B\tilde{T}$. Therefore the Serre spectral sequence for the fibering $G/T\to B\tilde{T}\xrightarrow{i} B\tilde{G}$ with $F_p$ coefficient collapses if $p$ is good for $G$.

## § 3. The number $h(G, p)$

In this section we determine the numbers $h(G, p)$. First of all it is well known that two non degenerate quadratic forms over $F_p$ ($p$ is an odd prime) are equivalent if and only if they have same rank and same discriminant (see for example Serre [12]). If $G$ is of classical type then $Q$ is given by the following:

**Proposition 3.1.** (1) *If $G=A_\ell$, then there exists $x_0, \cdots, x_\ell$ such that $Q(x)=\sum_{i<j} x_i x_i \mid V$ where $V$ is the hyperplane defined by $x_0+\cdots+x_\ell=0$.*

(2) *If $G=B_\ell$, $C_\ell$ or $D_\ell$, then there exists $x_1, \cdots, x_\ell$ such that $Q(x)=x_1^2+\cdots+x_\ell^2$.*

(3) *In $F_p^\times/(F_p^\times)^2$,*

$$\operatorname{disc}(Q(x))=\begin{cases}(-\frac{1}{2})(\ell+1) & \text{if } G=A_\ell, \\ 1 & \text{if } G=B_\ell,\ C_\ell \text{ or } D_\ell.\end{cases}$$

*Proof.* (1) and (2) are well known since $H^*(BG; Z_{(p)})$ is generated by $c_2$ (the second Chern class) or $P_1$ (the first Pontrjagin class). Therefore we need only show (3). For $G=A_\ell$, define $a_j$ by $x_i(a_j)=\delta_{ij}$ and put $v_j=a_j-a_0$ for $h=1, 2, \cdots, \ell$. Then $v_1, \cdots, v_\ell$ is a basis of $V$. Note that $B(v_i, v_j)=-\frac{1}{2}$ if $i\neq j$ and $Q(v_i)=-1$. Therefore disc $(Q(x))=(-\frac{1}{2})^\ell$ det $(E-B'(\ell))$ where $B'(\ell)=(b_{ij})$ is defined by $b_{ij}=-1$ for any $i, j$ and $E$ is the identity matrix. Since $B'(\ell)^2=-\ell B'(\ell)$ and rank $B'(\ell)=1$, $\det(tE-B'(\ell))=t^\ell+\ell t^{\ell-1}$ and therefore $\det(E-B'(\ell))=\ell+1$. For $G=B_\ell$, $C_\ell$ or $D_\ell$ the proof is easy.

**Remark 3.2.** If $\ell+1\equiv 0 \bmod p$, then $Q(x)$ for $G=A_\ell$ is degenerate. But $Q\mid V'$ is non degenerate where $V'$ is the hyperplane (of $V$) defined by $x_\ell=x_0$. Moreover disc $(Q/V')=(-\frac{1}{2})^{\ell-1}$ in $F_p^\times/(F_p^\times)^2$.

Now we can prove the following:

**Theorem 3.3.** (1) *If $\left(\frac{\ell+1}{p}\right)=0$, then $h(A_\ell, p)=h(A_{\ell-1}, p)$ and if $\left(\frac{\ell+1}{p}\right)=\pm 1$, then $h(A_\ell, p)$ is given by the following table:*

| | $\ell\equiv 1$ | $\ell=0$ | $\ell\equiv 2$ | |
|---|---|---|---|---|
| | | | $p\equiv 1$ | $p\equiv -1$ |
| $\left(\frac{\ell+1}{p}\right)=1$ | $\frac{\ell+1}{2}$ | $\frac{\ell}{2}$ | $\frac{\ell}{2}$ | $\frac{\ell}{2}+1$ |
| $\left(\frac{\ell+1}{p}\right)=-1$ | $\frac{\ell+1}{2}$ | $\frac{\ell}{2}+1$ | $\frac{\ell}{2}+1$ | $\frac{\ell}{2}$ |

(ii) *$h(G,p)$ for $G=B_\ell$, $C_\ell$ or $D_\ell$ is given by the following table:*

| | $\ell\equiv 1$ | $\ell\equiv 0$ | $\ell\equiv 2$ |
|---|---|---|---|
| $p\equiv 1$ | $\frac{\ell+1}{2}$ | $\frac{\ell}{2}$ | $\frac{\ell}{2}$ |
| $p\equiv -1$ | $\frac{\ell+1}{2}$ | $\frac{\ell}{2}$ | $\frac{\ell}{2}+1$ |

*where* $\equiv$ *means congruence modulo* 4 *and* $\left(\frac{\ell+1}{p}\right)$ *is the Legendre symbol.*

The following is Theorem 2.4 and Remark 2.5 of [7]

**Theorem 3.4.** (1) $h(G_2, p)=h(A_2, p)$, $h(F_4, p)=h(B_4, p)$,

(2) $$h(E_6, p)=\begin{cases} 3 & \text{if } \left(\frac{-3}{p}\right)\neq -1 \\ 4 & \text{if } \left(\frac{-3}{p}\right)=-1, \end{cases}$$

(3) $h(E_7, p)=h(E_8, p)=4$.

## § 4. The module structure of $H^*(B\tilde{G}; F_p)$ for an exceptional prime $p$

In this section we will prove the following:

**Theorem 4.1.** *For any $G$ and any odd prime $p$, the Serre spectral sequence for the fibering $G/T \xrightarrow{\lambda} B\tilde{T} \xrightarrow{i} B\tilde{G}$ with $F_p$ coefficient collapses.*

For a good prime $p$, Theorem 4.1 was proved in Section 2 (see Remark 2.4). Therefore we assume that $(G, p)$ is one of the five pairs in (2.1).

First recall the following facts on the Poincaré polynomials:

**Lemma 4.2** (Bott [3]). *$P(H^*(G/T; F_p))=(1-t^2)^{-\ell}\prod_{i=1}^{\ell}(1-t^{2m(i)+2})$, where $P(H^*(X; F_p))=\sum_{k=0}^{\infty}\dim H^k(X; F_p)t^k$, $\ell=\mathrm{rank}\, G$ and $m(1)<m(2)\leq\cdots\leq m(\ell)$ is the exponent of $W(G)$.*

**Lemma 4.3** (Toda). *If $p$ is exceptional for $G$, then as an algebra $H^*(G/T; F_p)$ is generated by $H^k(G/T; F_p)$ for $k\leq 2g(G,p)$, where*

$$g(G,p)=\begin{cases} 4 & \text{if } (G,p)=(E_6, 3), (E_7, 3) \text{ or } (F_4, 3), \\ 10 & \text{if } (G,p)=(E_8, 3), \\ 5 & \text{if } (G,p)=(E_8, 5). \end{cases}$$

See Theorem 3.2 of [14].

On the other hand we can easily show the following:

**Lemma 4.4.** *If $p$ is exceptional for $G$, then $H^{2k+1}(B\tilde{G}; F_p)=0$ for $k\leq g(G,p)$.*

*Proof.* See p. 140 of [10] for $(F_4, 3)$, Theorem V of [4] for $(E_6, 3)$, $(E_7, 3)$ or $(E_8, 5)$ and Proposition 4.4 of [8] for $(E_8, 3)$.

*Proof of Theorem* 4.1. We need only show $\lambda^*$ is surjective. By Lemma 4.2 and Lemma 4.4 $\lambda^*$ is surjective for $\deg\leq 2g(G,p)$. Since $H^*(G/T; F_p)$ is generated by $H^k(G/T; F_p)$ for $k\leq 2g(G,p)$ as an algebra by Lemma 4.3, and $\lambda^*$ is an algebra homomorphism, $\lambda^*$ is surjective.

**Corollary 4.5.** $P(H^*(B\tilde{G}; F_p))=\left(\prod_{i=2}^{\ell}(1-t^{2m(i)+2})\right)^{-1}\prod_{j=h}^{\infty}\frac{(1+t^{2p^j+1})}{(1-t^{2p^j+2})}$

*Proof.* By Theorem 2.2

$$P(H^*(B\tilde{T}; F_p))=(1-t^2)^{-\ell}(1-t^4)\prod_{j=h}^{\ell}\frac{(1+t^{2p^j+1})}{(1-t^{2p^j+2})}$$

On the other hand $P(H^*(B\tilde{T}; F_p))=P(H^*(G/T; F_p))P(H^*(B\tilde{G}; F_p))$ by Theorem 4.1. Note that $m(1)=1$ for any $G$.

**Remark 4.6.** The Serre spectral sequence for the fibering $E_8/T\to B\tilde{T}\to B\tilde{E}_8$ with $F_2$ coefficient does not collapse.

## § 5. The algebra structure of $H^*(B\tilde{G}; F_p)$ for an exceptional prime $p$

In this section we concern mainly the case $(G,p)=(E_8, 5)$. We will say results of other pairs in (2.1) only because these are similar. So $H^*(\ )$ means $H^*(\ ; F_5)$.

Put $R=F_5[T_1, \cdots, T_8, X_{12}]$ where $\deg T_j=2$ and $\deg X_{12}=12$. Recall that $h(E_8, 5)=4$. Denote the subalgebra of $H^*(K(Z,3))$ generated by $\{P_k u_3, \beta P_k u_3; k\geq j\}$ by $S_j$. Theorem 2.2 implies

**Lemma 5.1.** *There is a surjective homomorphism*

$$e\colon R\otimes F_5[X_{52}, X_{252}]\otimes S_4 \longrightarrow H^*(B\tilde{T})$$

*such that* Ker $e=(r_4, r_{12}, r_{52}, r_{252})$, *where* $\deg r_j=j$ *and* $r_4, r_{12}, r_{52}, r_{252}$ *is a regular sequence in* $F_5[T_1, \cdots, T_8]$ (*and so in* $R$).

Put $J(0)=\{16, 24, 28, 36, 40, 48\}$, $J(1)=J(0)\cup\{4, 12, 60\}$ and $J(2)=J(0)\cup\{52, 60, 252\}$. The following is Theorem 3.2 of [14]:

**Proposition 5.2.** *There exist* $\rho_j \in R$ $(j \in J(1))$ *such that* $H^*(G/T)$ *is isomorphic to* $R/(\rho_j; j \in J(1))$, *where* $\deg \rho_j = j$ *and* $\rho_j; j \in J(1)$ *is a regular sequence.*

Put $\rho'_j = e(\rho_j)$. First it is easy to show that as an algebra

$$H^*(BG) \cong F_5[y_j; j \in J(0)] \qquad \text{for } * \leq 51 \tag{5.3}$$

where $\deg y_j = j$ (see [6]). Since the Serre spectral sequence for $G/T \xrightarrow{\lambda} B\tilde{T} \xrightarrow{i} B\tilde{G}$ collapses, $\rho_4 = r_4$, $\rho_{12} \equiv r_{12} \mod (\rho_4)$ and for $j \in J(0)$

$$i^*(y_j) \equiv \rho'_j \mod (\rho'_k; k < j). \tag{5.4}$$

Since $H^{52}(G/T)$ is decomposable, we have

**Lemma 5.5.** *There exists* $y_{52} \in H^{52}(B\tilde{G})$ *such that* $i^*(y_{52}) \equiv x_{52} \mod$ *decomposables where* $x_{52} = e(X_{52})$. *Moreover* $i^*(y_{52}) - x_{52} \in (\rho'_j; j \in J(0))$.

**Lemma 5.6.** *There exists a (weighted) homogeneous polynomial* $f_{52} \neq 0$ *of degree* 52 *such that* $f_{52}(y_j; j \in J(0)) = 0$ *in* $H^*(B\tilde{G})$.

*Proof.* There are no relations in degree less than 52 by (5.3) and there is an indecomposable element in degree 52 by Lemma 5.5. Therefore there must be a relation in degree 52 by Corollary 4.5.

Also we have

**Lemma 5.7.** *There is* $y_{60} \in H^{60}(B\tilde{G})$ *such that* $i^*(y_{60}) \equiv \rho'_{60} \mod (\rho'_k; k \in J(0), x_{52})$.

Summing up these results we can say that

**Proposition 5.8.** *There is an algebra homomorphism* $I$: $F_5[Y_j; j \in J(0) \cup \{52, 60\}] \to R \otimes F_5[X_{52}]$ *such that the following diagram commutes*:

$$\begin{array}{ccc} F_5[Y_j; j \in J(0) \cup \{52, 60\}] & \xrightarrow{I} & R \otimes F_5[X_{52}] \\ \downarrow{\scriptstyle e''_1} & & \downarrow{\scriptstyle e'_1} \\ H^*(B\tilde{G}) & \xrightarrow{i^*} & H^*(B\tilde{T}) \end{array}$$

*where* $e''_1(Y_j) = y_j$, *and* $e'_1(X_{52}) = x_{52}$.

*Proof.* From 5.4, for $j \in J(0)$, there exist $f_{ij} \in H^*(B\tilde{T})$ $(i<j)$ such that $i^*(y_j) = \rho'_j + \sum_{i<j} f_{ij}\rho'_i$. Similarly $i^*(y_{52}) = x_{52} + \sum_{i \in J(0)} g_i \rho'_i$ and $i^*(y_{60}) = \rho'_{60} + \sum_{i \in J(0)} h_i \rho'_i + h_{52} x_{52}$ for $g_i$, $h_i \in H^*(B\tilde{T})$. Choose $F_{ij}$, $G_i$,

$H_i \in R$ such that $e'_1(F_{ij})=f_{ij}$, $e'_k(G_i)=g_i$ and $e'_1(H_i)=h_i$. Define $I$ by $I(Y_j)=\rho_j+\sum_{i<j} F_{ij}\rho_i$, $I(Y_{52})=X_{52}+\sum_{i\in J(0)} G_i\rho_i$ and $I(Y_{60})=\rho_{60}+\sum_{i\in J} H_i\rho_i+H_{52}X_{52}$. It is easy that $I$ satisfies the above commutativity.

**Lemma 5.9.** *Let $k$ be a field and $a_1, \cdots, a_n \in k[b_1, \cdots, b_m]$ be a sequence of homogeneous elements. Then $a_1, \cdots, a_n$ is a regular sequence if and only if $a_1, \cdots, a_n$ generates a polynomial subalgebra over which $k[b_1, \cdots, b_n]$ is free.*

See [11].

Note that $I(f_{52}) \in (r_4, r_{12}, r_{52})$ and $I(f_{52}) \notin (r_4, r_{12})=(\rho_4, \rho_{12})$. Therefore

**Lemma 5.10.** $I(f_{52})\equiv r_{52} \bmod (r_4, r_{12})$.

On the other hand the induced map $F_5[Y_j; j\in J(0)\cup\{52, 60\}]/(f_{52})\to H^*(B\tilde{T})$ is injective for $\deg\leq 251$ and so we have

**Lemma 5.11.** $\bar{e}''_1$; $F_5[Y; j\in J(0)\cup\{52, 60\}]/(f_{52})\to H^*(B\tilde{G})$ *is an isomorphism for* $\deg\leq 251$.

Quite similarly we have

**Lemma 5.12.** (1) *There is an element $Y_{252}\in H^{252}(B\tilde{G})$ such that* $i^*(y_{252})-x_{252}\in(\rho'_j; j\in J(0)\cup\{60\}, x_{52})$.

(2) *There is a homogeneous element $f_{252}$ of degree* 252 *such that*

$$f_{252}(y_j; j\in J(0)\cup\{52, 60\})=0 \qquad \textit{in } H^*(B\tilde{G}).$$

Moreover we have

**Proposition 5.13.** (1) *There is an algebra homomorphism $I'$: $F_5[Y_j; j\in J(2)]\to R\otimes F_5[X_{52}, X_{252}]$ such that the following diagram commutes*:

$$\begin{array}{ccc} F_5[Y_j; j\in J(2)] & \xrightarrow{I'} & R\otimes F_5[X_{52}, X_{252}] \\ \downarrow{\scriptstyle e''_2} & & \downarrow{\scriptstyle e'_2} \\ H^*(B\tilde{G}) & \xrightarrow{i^*} & H^*(B\tilde{T}), \end{array}$$

*where $e''_2(Y_j)=y_j$, $e'_2(X_{52})=x_{52}$ and $e'_2(X_{252})=x_{252}$.*

(2) $I'(Y_j)=I(Y_j)$ *for $j\neq 252$ and* $I'(f_{252})\equiv r_{252} \bmod (r_4, r_{12}, r_{52})$.

Now applying Lemma 5.9, we have

**Lemma 5.14.** *$f_{52}, f_{252}$ is a regular sequence.*

On the other hand there are $y_{2\cdot5^k+1}$, $y_{2\cdot5^k+2}$ $k\geq4$ such that $\pi^*(y_{2\cdot5^k+1})\equiv P_k u_3$ and $\pi^*(y_{2\cdot5^k+2})=\beta P_k u_3$ mod decomp. We have thus an algebra homomorphism $\lambda$: $\boldsymbol{F}_5[Y_j; j\in J(2)]/(f_{52}, f_{252})\otimes S_4\to H^*(B\tilde{G})$.

Obviously $i^*\circ\lambda$ is a monomorphism. By the observation of the Poincaré series in Corollary 4.5, we can easily prove $\lambda$ is an isomorphism. Remaining cases can be determined by a similar method.

**Theorem 5.15.** *Let $S_j$ be the subalgebra of $H^*(K(Z, 3); \boldsymbol{F}_p)$ generated by $\{P_k u_3, \beta P_k u_3; k\geq j\}$, then as an algebra.*

(1) $H^*(B\tilde{E}_8; \boldsymbol{F}_5)\cong \boldsymbol{F}_5[y_j; j=16, 24, 28, 36, 40, 48, 52, 60, 252]/(f_{52}, f_{252})\otimes S_4$,

(2) $H^*(B\tilde{E}_8; \boldsymbol{F}_3)\cong \boldsymbol{F}_3[y_j; j=16, 24, 28, 36, 40, 48, 56, 60]/(f_{56})\otimes S_4$,

(3) $H^*(B\tilde{E}_7; \boldsymbol{F}_3)\cong \boldsymbol{F}_3[y_j; j=12, 16, 20, 24, 28, 36, 56]/(f_{56})\otimes S_4$,

(4) $H^*(B\tilde{E}_6; \boldsymbol{F}_3)\cong \boldsymbol{F}_3[y_j; j=10, 12, 16, 18, 20, 24]/(f_{20})\otimes S_3$,

(5) $H^*(B\tilde{F}_4; \boldsymbol{F}_3)\cong \boldsymbol{F}_3[y_j; j=12, 16, 24]\otimes S_2$.

*where* $\deg y_j=j$, $\deg f_j=j$ *and in* (1) $f_{52}$, $f_{252}$ *is a regular sequence.*

**Remark 5.16.** $H^*(B\tilde{G}; \boldsymbol{F}_2)$ is known for $G=A_\ell$, $C_\ell$, $E_6$, $E_7$, $E_8$, $F_4$ or $G_2$ (See [5], [13]). But it seems to be difficult to determine $H^*(B\tilde{G}; \boldsymbol{F}_2)$ for $G=B_\ell$ or $D_\ell$.

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