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On the Maximal Connected Algebraic Subgroups of the Cremona Group II

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In preceding papers [U1], [U2], we classified certain types of maximal connected algebraic subgroups of the Cremona group of 3 variables; primitive groups and imprimitive groups which are not of de Jonquières type. The aim of this paper is to accomplish the classification of the maximal connected algebraic subgroups of the Cremona group of 3 variables as promised in [U2]. For this purpose we have to classify de Jonquières type subgroups (Theorem (2.1)). We explained our method of approach in [U3] for the Cremona group of 2 variables. Therefore it is preferable to understand [U3] before reading this paper. The main theorem (2.1) is essentially due to Enriques and Fano [E.F] and Fano [F] written almost 100 years ago but it does not seem that their proofs are considered to be rigorous. We believe that despite of its group theoretic form, Theorem (2.1) is a quite substantial result on the rational threefolds and should have applications. An analysis of the Theorem from the view point of minimal models is done in [M.U] and to be completed in [U4]. As in the preceding papers *we work over the complex number field* $k=\mathbf{C}$.

§ 1. Review of preceding papers, [U1], [U2], [U3]

(1.1) Cremona group Cr_n of n variables is the automorphism group of the rational function field of n variables. It seems that the most natural interpretation is to regard Cr_n as the group functor Autbirat $\mathbf{P}^n$. Let G be an algebraic group. Then *a morphism* $G \to \mathrm{Cr}_n$ *is, by definition, a morphism of group functors. The following are in* 1:1 *correspondence*:

(1) *Morphisms* $G \to \mathrm{Cr}_n$,

(2) *Pseudo-operations* $(G, \mathbf{P}^n)$.

Since any pseudo-operation is equivalent to an algebraic operation, the following is in 1:1 correspondence with (1) and (2):

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(3) $\{(G, X)$ *and* $f \mid (G, X)$ *is an algebraic operation,* f *is a birational map* $f\colon X \to \mathbf{P}^n\}$.

We want to study algebraic subgroups of Cr_n or effective pseudo-operations $(G, \mathbf{P}^n)$. In (3) if we ignore f, then a subgroup G is determined up to a birational automorphism of $\mathbf{P}^n$. Namely an effective operation (G, X) with X rational and n-dimensional determines a conjugacy class in Cr_n. The converse also holds. Hence, *to give a conjugacy class of an algebraic subgroup G of* Cr_n *is equivalent to giving an effective algebraic operation* (G, X) *with X rational and n-dimensional. We say that* (G, X) *is a birational realization of the conjugacy class of G. If* $(\varphi, f)\colon (G', X') \to (G, X)$ *is a morphism of algebraic operations with f birational and φ finite, then we say* (G', X') *is an almost effective realization of the conjugacy class of G. Let* G_1, G_2 *be algebraic subgroups of* Cr_n. *We say that the conjugacy class of* G_1 *is contained in that of* G_2 *(or* G_1 *is contained in* G_2 *up to conjugacy) if there exists an algebraic subgroup* G_1' *conjugate to* G_1 *in* Cr_n *such that* $G_1' \subset G_2$. *The conjugacy class of* G_1 *is contained in* G_2 *if and only if there exist effective realizations* (G_1, X_1) *and* (G_2, X_2) *of* G_1 *and* G_2 *and a morphism* $(\varphi, f)\colon (G_1, X_1) \to (G_2, X_2)$ *with f birational.* We refer, for the definitions of primitive, imprimitive or de Jonquières type operations, to [U1], [U2]. An algebraic subgroup of Cr_n is said to be *primitive, imprimitive* or de Jonquières type if it is realized by an algebraic operation with the respective property.

(1.2.1) Let X be a non-singular variety. We denote by $\mathbf{T}$ the tangent bundle or the sheaf of its sections.

(1.2.2) *For an algebraic group G, G^0 denotes its connected component of the unity* 1. *If U (resp. R) is the unipotent radical (resp. radical) of a connected algebraic group G, by the reductive (resp. semi-simple) part of G we mean the quotient group G/U (resp. G/R). But by abuse of language, any group isogeneous to G/U (resp. G/R) and also its Lie algebra are called a reductive (resp. semi-simple) part of G.*

(1.2.3) *When we treat* SL_2, *B denotes the Borel subgroup of upper (or lower) triangular matrices of* SL_2.

(1.2.4) *An irreducible* SL_2*-module of dimension n is denoted by U_n or V_n without any explanation but the notation U_n is not reserved for irreducible* SL_2*-module and when we use U_n for other purposes, its meaning is explained so that there is no danger of confusion.*

(1.2.5) *Let* $f\colon X \to Y$ *be a surjective morphism of algebraic varieties. We denote by* $\mathrm{Aut}_Y X$ *the Y-automorphism group of X and* $\mathrm{Aut}(X; Y)$ *denotes the set of the automorphisms* $g\colon X \to X$ *such that there exists an automorphism* $g'\colon Y \to Y$ *depending on g with* $f \circ g = g' \circ f$.

(1.2.6) *We denote by* $h^i(X, \mathscr{F})$ *the dimension of the cohomology* *group* $H^i(X, \mathscr{F})$.

(1.2.7) For an integer $m \geqq 0$, we define F'_m to be

$$\operatorname{Spec}\left(\bigoplus_{k=0}^{\infty} \mathcal{O}_{\mathbf{P}^1}(-km)\right),$$

which is the total space of the line bundle of degree m over $\mathbf{P}^1$. Let

$$E = \mathcal{O}_{\mathbf{P}^1} \oplus \mathcal{O}_{\mathbf{P}^1}(-m) \quad \text{and} \quad \mathrm{F}_m = \operatorname{Proj}\left(\bigoplus_{k=0}^{\infty} S^k(E)\right).$$

Then $\mathrm{F}'_m = \mathrm{F}_m - $(a section of $\mathrm{F}_m/\mathbf{P}$). $\operatorname{Pic} \mathrm{F}'_m \simeq \operatorname{Pic} \mathbf{P}^1$ by projection $p: \mathrm{F}'_m \to \mathbf{P}^1$. $p^*\mathcal{O}_{\mathbf{P}^1}(l)$ is sometimes denoted by $\mathcal{O}_{\mathrm{F}'_m}(l)$ or simply $\mathcal{O}(l)$.

(1.3) Let $X \to Y$ be an $\mathbf{A}^1$-bundle and G an algebraic group operating effectively on X/Y. Then the rank of G is at most 1 by Lemma (1.2.1) [U3], G is solvable and the unipotent radical of G is abelian because $\operatorname{Aut} \mathbf{A}^1 \simeq \mathbf{G}_a \cdot \mathbf{G}_m$.

§ 2. Main Theorem

We state the main theorem. The varieties appearing in the theorem such as $\mathrm{F}'_{m,l}$, J'_m etc. will be defined in Section 3 after the statement of the theorem. The theorem consists of two parts. The first part says that, in the Cremona group Cr_3 of three variables, any algebraic subgroup is contained, up to inner automorphisms of Cr_3, in one of the algebraic subgroups found in the list. The second part asserts that the algebraic subgroups in the list are maximal.

Theorem (2.1). (I) *Let G be a connected algebraic group in* Cr_3. *Then G is contained in the conjugacy class of one of the following algebraic operations*:

(P1) $(\mathrm{PGL}_4, \mathbf{P}_3)$,

(P2) $(\mathrm{PSO}_5$, *quadric* $\subset \mathbf{P}_4)$.

(E1) $(\mathrm{PGL}_2, \mathrm{PGL}_2/\Gamma)$, *$\Gamma$ is an octahedral subgroup of* PGL_2.

(E2) $(\mathrm{PGL}_2, \mathrm{PGL}_2/\Gamma)$, *$\Gamma$ is an icosahedral subgroup of* PGL_2.

(J1) $(\mathrm{PGL}_3 \times \mathrm{PGL}_2, \mathbf{P}^2 \times \mathbf{P}^1)$.

(J2) $(\mathrm{PGL}_2 \times \mathrm{PGL}_2 \times \mathrm{PGL}_2, \mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1)$.

(J3) $(\mathrm{PGL}_2 \times \mathrm{Aut}^0 \mathrm{F}'_m, \mathbf{P}^1 \times \mathrm{F}'_m)$ *where m is an integer* $\geqq 2$.

(J4) $(\mathrm{PGL}_3, \mathrm{PGL}_3/B)$ *where B is a Borel subgroup of* PGL_3.

(J5) $(\mathrm{PGL}_2, \mathrm{PGL}_2/\mathrm{D}_{2n})$ *where n is an integer* $\geqq 4$.

(J6) $(G, G/H_{m,n})$ *where* $G = \mathbf{G}_m \times \mathrm{SL}_2 \times \mathrm{SL}_2$,

$$H_{m,n} = \left\{ \left(t_1^m t_2^n, \begin{pmatrix} t_1 & x \\ 0 & t_1^{-1} \end{pmatrix}, \begin{pmatrix} t_2 & y \\ 0 & t_2^{-1} \end{pmatrix} \right) \in \mathbf{G}_m \times \mathrm{SL}_2 \times \mathrm{SL}_2 \,|\, t_1, t_2 \in k^*, x, y \in k \right\},$$

and m, n *are integers with* $m > 2$, $-2 > n$.

(J7) $(\mathrm{Aut}^0 \mathrm{J}'_m, \mathrm{J}'_m)$ *where* m *is an integer* $m \geqq 2$.

(J8) $(\mathrm{Aut}^0 \mathrm{L}'_{m,n}, \mathrm{L}'_{m,n})$ *where* m, n *are integers with* $m \geqq n \geqq 1$.

(J9) $(\mathrm{Aut}^0 \mathrm{F}'_{m,n}, \mathrm{F}'_{m,n})$ *where* m, n *are integers with* $m > n \geqq 2$.

(J10) $(\mathrm{Aut}^0 \mathrm{F}'_{m,m}, \mathrm{F}'_{m,m})$ *where* m *is an integer* $\geqq 2$.

(J11) $(\mathrm{Aut}^0 \mathrm{E}'^l_m, \mathrm{E}'^l_m)$ *where* l, m *are integers with* $m \geqq 2$, $l \geqq 2$ *or* $m = 1$, $l \geqq 3$.

(J12) *Generically intransitive operation* (PGL_2, X_π) *with general orbits isomorphic to* $(\mathrm{PGL}_2, \mathrm{PGL}_2/\mathbf{G}_m)$, *where* $\pi\colon C_1 \to C_2$ *is an étale* 2-*covering of a rational curve* C_2 *with genus* $(C_1) \geqq 1$. *We shall see that these operations are effectively parametrized by the moduli space of nonsingular elliptic or hyperelliptic curves of genus* $\geqq 1$ (*Corollary* (4.27)).

(II) *The* (*conjugacy classes of*) *algebraic subgroups of* Cr_3 *determined by the above operations* (P1), (P2), (E1), (E2), (J1), $\cdots$, (J12) *are maximal* (*conjugacy classes of*) *algebraic subgroups of* Cr_3.

We prove in fact a seemingly stronger assertion.

Theorem (2.2). *If we replace in Theorem* (2.1), *the operations in* (J11) by (J′11) $(\mathrm{Aut}^0(\mathrm{E}'^l_m; \mathrm{F}'_m), (\mathrm{E}'^l_m))$ *the conclusions of Theorem* (2.1) *hold.*

Corollary (2.3). $\mathrm{Aut}^0 \mathrm{E}'^l_m = \mathrm{Aut}^0(\mathrm{E}'^l_m; \mathrm{F}'_m)$.

§ 3. Definitions of the algebraic operations

(3.1) The algebraic operations (P1), (P2), (E1), (E2) are studied in our preceding papers [U1], [U2]. The algebraic operations (J1), (J2), (J4) and (J6) are lucidly understood.

(3.2) As in [U3], F'_m *is the total space of the line bundle over* $\mathbf{P}^1$ *of degree* m. More precisely $\mathrm{F}'_m = \mathrm{Spec}\,(\bigoplus_{k=0}^{\infty} \mathcal{O}_{\mathbf{P}^1}(-km))$.

(3.3) The algebraic group $\mathrm{Aut}^0 X$ is the group of the connected component of 1 of the automorphism group of X. Since in our list the variety X is open, $\mathrm{Aut}^0 X$ is not a representable functor but still we can speak of $\mathrm{Aut}^0 X$ because the dimension of $H^0(X, \mathrm{T}_X)$ is finite (see Section 4).

(3.4) D_{2n} denotes the *dihedral group* of order 2n.

(3.5) J'_m *is the total space of the line bundle over* $\mathbf{P}^2$ *of degree* m. Namely $J'_m = \mathrm{Spec}\,(\bigoplus_{k=0}^{\infty} \mathcal{O}_{\mathbf{P}^2}(-km))$.

(3.6) $L'_{m,n}$ *is the total space of the line bundle over* $\mathbf{P}^1 \times \mathbf{P}^1$ *of bidegree* (m, n). Precisely speaking, $L'_{m,n} = \mathrm{Spec}\,(\bigoplus_{k=0}^{\infty} \mathcal{O}_{\mathbf{P}^1 \times \mathbf{P}^1}(-km, -kn))$ where $\mathcal{O}_{\mathbf{P}^1 \times \mathbf{P}^1}(a, b) = p_1^* \mathcal{O}_{\mathbf{P}^1}(a) \otimes p_2^* \mathcal{O}_{\mathbf{P}^1}(b)$, p_i being the projection of $\mathbf{P}^1 \times \mathbf{P}^1$ onto the i-th factor.

(3.7) $F'_{m,n}$ *is the total space of the vector bundle* $\mathcal{O}_{\mathbf{P}^1}(m) \oplus \mathcal{O}_{\mathbf{P}^1}(n)$ *over* $\mathbf{P}^1$. Therefore, $F'_{m,n} = \mathrm{Spec}\,(S(\mathcal{O}_{\mathbf{P}^1}(-m) \oplus \mathcal{O}_{\mathbf{P}^1}(-n)))$.

(3.8) Let m, l be integers with $m \geqq 1$, $lm \geqq 2$ and $k = 2 - lm$. Consider an extension of coherent sheaves on F'_m,

$$0 \longrightarrow \mathcal{O}_{F'_m} \longrightarrow E \longrightarrow \mathcal{O}_{F'_m}(k) \longrightarrow 0. \tag{3.8.1}$$

It is easy to show Pic F'_m is isomorphic to $\mathbf{Z}$ and $\mathcal{O}_{F'_m}(1) = \pi^* \mathcal{O}_{\mathbf{P}^1}(1)$ is a generator, where $\pi\colon F'_m \to \mathbf{P}^1$ is the projection. $\mathcal{O}_{F'_m}(k)$ is by definition $\pi^* \mathcal{O}_{\mathbf{P}^1}(k)$. F'_m is a homogeneous space (Umemura [U3]). Let us look for a condition for the extension (3.8.1) to be homogeneous. The extension (3.8.1) is parametrized by $H^1(F'_m, \mathcal{O}_{F'_m}(-k))$ which is $\mathrm{Aut}^0 F'_m$-module (to be precise, we have to replace $\mathrm{Aut}^0 F'_m$ by its 2-covering if m is even so that the operation is linearized (cf. [D], [U3])). Since the projection π is a morphism of homogeneous spaces and π is affine, we have $\mathrm{Aut}^0 F'_m$-isomorphism,

$$\begin{aligned} H^1(F'_m, \mathcal{O}_{F'_m}(-k)) &\simeq H^1(\mathbf{P}^1, (\pi_* \mathcal{O}_{F'_m}) \otimes \mathcal{O}_{\mathbf{P}^1}(-k)) \\ &= H^1(\mathbf{P}^1, (\bigoplus_{j \geqq 0} \mathcal{O}_{\mathbf{P}^1}(-jm)) \otimes \mathcal{O}(-k)) \\ &= H^1(\mathbf{P}^1, \bigoplus_{j \geqq 0} \mathcal{O}_{\mathbf{P}^1}(-k-jm)) = \bigoplus_{j \geqq 0} H^1(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(-k-jm)). \end{aligned}$$

For $j \geqq 0$, $H^1(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(-k-jm))$ is an irreducible SL_2 hence $\mathrm{Aut}^0 F'_m$-module. *The extension E in* (3.8.1) *is homogeneous if and only if it corresponds to an* $\mathrm{Aut}^0 F'_m$*-eigen-vector of* $H^1(F'_m, \mathcal{O}_{F'_m}(-k))$. *Therefore, there is only one homogeneous E corresponding to* $H^1(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(-2))$. *The homogeneous extension determines a homogeneous* $\mathbf{A}^1$*-bundle over* F'_m *which is denoted by* E'^l_m. $\mathrm{Aut}^0(E'^l_m, F'_m)$ is the connected component of 1 of the closed subgroup of $\mathrm{Aut}^0 E'^l_m$ respecting the fibration $E'^l_m \to F'_m$. *We recall the reader not to confuse* $\mathrm{Aut}^0(E'^l_m, F'_m)$ *with* $\mathrm{Aut}^0_{F'_m} E'_m$ *which is the* F'_m*-automorphism group of* E'_m

(3.9) Let $\pi\colon C_1 \to C_2$ be an étale 2-covering of a non-singular rational curve C_2. When we speak of a covering, we always assume C_1 to be irreducible. Let ι be the involution of C_1 so that $C_1/\langle\iota\rangle \simeq C_2$. Let T be a Cartan subgroup of PGL_2 consisting of diagonal matrices. Let us determine the automorphisms of PGL_2/T compatible with the operation of PGL_2.

Lemma (3.9.1). *The* PGL_2*-automorphism group of* PGL_2/T *is a cyclic group of order* 2.

Proof. Let $f\colon \mathrm{PGL}_2/T \to \mathrm{PGL}_2/T$ be a PGL_2-automorphism. Let $f(T)=aT$, $a \in \mathrm{PGL}_2$. Then as for any $b \in \mathrm{PGL}_2$, $baT=bf(T)=f(bT)$, f is determined by $f(aT) \in \mathrm{PGL}_2/T$. But for any $t \in T$, $aT=f(T)=f(tT)=taT$. Namely $TaT=aT$ or $Ta=aT$. a is in the normalizer of T. Therefore $a=\mathrm{I}_2$ or $\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$ mod T.

Consider now the product operation $(1\times \mathrm{PGL}_2,\ C_1\times \mathrm{PGL}_2/T)$. Let f be the non-trivial PGL_2-automorphism of PGL_2/T, $f(aT)=a\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}T$. An automorphism $\iota\times f$ defines an involution on $C_1\times \mathrm{PGL}_2/T$ hence a descent datum on $C_1\times \mathrm{PGL}_2/T$ and an operation (PGL_2, X_π). It follows from Lemma (3.9.1), (PGL_2, X_π) is not isomorphic to the product $(1\times \mathrm{PGL}_2,\ C_2\times \mathrm{PGL}_2/T)$ as algebraic operations.

Lemma (3.9.2). *The variety* X_π *is rational.*

Proof. $(\mathrm{PGL}_2, \mathrm{PGL}_2/T)$ is a suboperation of $(\mathrm{PGL}_2\times \mathrm{PGL}_2,\ \mathrm{PGL}_2/B \times \mathrm{PGL}_2/B)$ by $a\mapsto(a, a)$, at $aT\mapsto\left(aB,\ a\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}B\right)$, where B denotes the upper triangular Borel subgroup of PGL_2. Then the involution $f(aT)=a\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}T$ on PGL_2/T is induced by the automorphism of interchanging factors of $\mathrm{PGL}_2/B\times \mathrm{PGL}_2/B$. Let y, z denote the coordinate on the product $\mathrm{SL}_2/B\times \mathrm{SL}_2/B$ hence they are independent variables over k. Let $k(C_1)$, $k(C_2)$ be the fields of rational functions on C_1 and C_2. Then there exist a variable x over k, $\alpha \in k(C_1)$ and an $a(x) \in k(x)$ such that $\alpha^2-a(x)=0$. The descent datum operates on the rational function field

$$k(C_1\times \mathrm{PGL}_2/T)\simeq k(C_1\times \mathrm{PGL}_2/B\times \mathrm{PGL}_2/B)=k(x, y, z)$$

by $y\mapsto z$, $z\mapsto y$, $\alpha\mapsto -\alpha$, $x\mapsto x$. Let us denote this involution by h. It is sufficient to show that the invariant field $k(C_1)(y, z)^h=K$ is rational. In fact $k(C_2)(y+z, yz)\subset K$ and $[k(C_1)(y, z): K]=2$. On the other hand $[k(C_1)(y, z): k(C_2)(y+z, yz)]=4$. Thus $[k(C_1)(y, z)^h: k(Y)(y+z, yz)]=2$. We notice $\alpha(y-z) \notin k(x, y+z, yz)$ since the elements of $k(x, y+z, yz)$ is invariant under $\alpha\mapsto -\alpha$, $z\mapsto y$, $y\mapsto z$. But $\alpha(y-z) \in k(\alpha, x, y, z)^h$. Therefore $k(x, y+z, yz, \alpha(y-z))=k(\alpha, x, y, z)^h$. Now we show $k(x, y+z, yz,$

$\alpha(y-z))=k(x, y+z, \alpha(y-z))$ which implies $k(\alpha, x, y, z)^h=k(x, y+z, \alpha(y-z))$ is rational since $\mathrm{tr}(k(\alpha, x, y, z)^h : h)=3$. In fact, $\alpha^2(y-z)^2=a(x)((y+z)^2-4yz)$ hence $yz=(\alpha^2(y-z)^2-a(x)(y+z)^2)/4a(x) \in k(x, y+z, \alpha(y-z))$.

§ 4. Automorphism groups

It is known for a complete variety X the automorphism group functor $\mathrm{Aut}\, X$ is representable (Murre [M], Matsumura-Oort [M. O]). Let Now X be a non-singular variety. We denote by $\mathbf{T}_X$ the tangent bundle of X. If X is not complete, the group functor $\mathrm{Aut}\, X$ is not always representable. But if the dimension of $H^0(X, \mathbf{T}_X)$ is finite, the group functor $\mathrm{Aut}\, X$ is close to be representable. We recall a result of Matsumura-Oort [M. O]. Let (Sch/k) be the category of schemes over k and $(\mathrm{Sch}/k)_{\mathrm{red}}$ be the full subcategory of (Sch/k) consisting of reduced schemes.

Theorem (4.1) (Matsumura-Oort [M. O]). *Let X be an algebraic variety over k such that $h^0(X, \mathbf{T}_X)$ is finite. Then the restriction* $\mathrm{Aut}\, X | (\mathrm{Sch}/k)_{\mathrm{red}}$ *to the category of reduced schemes is representable by a reduced group scheme which is locally of finite type over k.*

Notice in the original theorem in [M. O], k is assumed to be *perfect* and this assumption is satisfied as we work over $\mathbf{C}$.

If we recall that a group scheme over a field of characteristic 0 is reduced, we get

Corollary (4.2). *Let X be as in Theorem* (4.1) *and G a connected algebraic group. If G operates on X, then we get a morphism of algebraic operations* $(\varphi, \mathrm{Id}) : (G, X) \to (\mathrm{Aut}^0 X, X)$.

This corollary is quite useful for our purpose. To be able to apply this theorem, we want to show that the dimension of $H^0(X, \mathbf{T}_X)$ is finite for the open varieties X in (J3), (J7), (J8), (J9), (J10), (J11) of Theorem (2.1).

Lemma (4.3). *Using the notation of Section* 2, *we have the following estimations.*

(1) $h^0(\mathbf{P}^1 \times \mathrm{F}'_m, \mathbf{T}) \leq \begin{cases} m+8 & \text{if } m \geq 3, \\ 11 & \text{if } m=2, \\ 12 & \text{if } m=1. \end{cases}$

(2) $h^0(\mathrm{J}'_m, \mathbf{T}) \leq \dfrac{(m+2)(m+1)}{2}+9 \quad \text{for } m \geq 1.$

(3) $h^0(\mathrm{L}'_m, \mathbf{T}) \leq (m+1)(n+1)+7$ *for* $m, n \geq 0$.

(4) $h^0(\mathrm{F}'_{m,n}, \mathbf{T})$ *is finite if* $m, n \geq 1$ *and* $\leq \dfrac{(m+r+2)(q+1)}{2} + n + 6$

if $m > n \geq 2$, *where* $m = qn + r$, $q, r \in \mathbf{Z}$, $0 \leq r < n$.

(5) $h^0(\mathrm{F}'_{m,m}, \mathbf{T}) \leq 2m+9$ *for* $m \geq 1$.

(6) $h^0(\mathrm{E}'^{l}_m, \mathbf{T})$ *finite if* $m \geq 1$, $lm \geq 2$ *and*

$$\leq \begin{cases} \dfrac{(ml+m-2)l}{2} + m + 6 & \text{if } m \geq 2,\ l \geq 1, \\ \dfrac{(ml-2)(ml-1)}{2} + m + 6 & \text{if } m = 1,\ l \geq 3. \end{cases}$$

Proof. To prove (1) for $m \geq 3$ it is sufficient to show $h^0(\mathrm{F}'_m, \mathbf{T}) = m+5$ since $h^0(\mathbf{P}^1, \mathbf{T}) = 3$ and $h^0(\mathbf{P}^1 \times \mathrm{F}'_m, \mathbf{T}) = h^0(\mathbf{P}^1, \mathbf{T}) + h^0(\mathrm{F}'_m, \mathbf{T})$. F'_m is the line bundle over $\mathbf{P}^1$ of degree m hence obtained by gluing two $\mathbf{A}^2$'s; $(x, y) \in \mathbf{A}^2$ and $(x', y') \in \mathbf{A}^2$ are identified if $x \neq 0$, $\dfrac{1}{x} = x'$, $y' = \left(\dfrac{1}{x}\right)^m y$. Therefore we have an exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathrm{F}'_m}(2) \longrightarrow \mathbf{T}_{\mathrm{F}'_m} \longrightarrow \mathcal{O}_{\mathrm{F}'_m}(m) \longrightarrow 0,$$

where π is the projection $\pi: \mathrm{F}'_m \rightarrow \mathbf{P}^1$, $\mathcal{O}_{\mathrm{F}'_m}(l) = \pi^* \mathcal{O}_{\mathbf{P}^1}(l)$. It follows from the exact sequence

$$h^0(\mathrm{F}'_m, \mathbf{T}) \leq h^0(\mathrm{F}'_m, \mathcal{O}_{\mathrm{F}'_m}(2)) + h^0(\mathcal{O}_{\mathrm{F}'_m}(m)).$$

Since the spectral sequence for π degenerates, the inequality (1) follows from

$$H^0(\mathrm{F}'_m, \mathcal{O}_{\mathrm{F}'_m}(l)) = \bigoplus_{k=0}^{\infty} H^0(\mathbf{P}^1, \mathcal{O}(l-km)).$$

The other inequalities are proved similarly and hence we omit their proofs.

Corollary (4.3.1). *For a variety* X *in Lemma* (4.3), $\mathrm{Aut}^0 X$ *is an algebraic group.*

Proof. Corollary follows from Theorem (4.1) and Lemma (4.3).

Lemma (4.4). *Let* X be *a variety and* L *a line bundle over* X. *We denote by* $\boldsymbol{L}$ *the total space* $\mathrm{Spec}\,(\bigoplus_{i=0}^{\infty} L^{-i})$ *of* L. *If the automorphism group* $\mathrm{Aut}^0_X \boldsymbol{L}$ *is representable in the category of the reduced schemes,* $\mathrm{Aut}^0_X \boldsymbol{L}$ *is solvable, of rank* 1 *and the unipotent radical of* $\mathrm{Aut}^0_X \boldsymbol{L}$ *coincides with* $H^0(X, L)$.

Proof. Since no semi-simple group operates on the fibre $\mathbf{A}^1$ of L/X, $\mathrm{Aut}_X^0 L$ is solvable. By Demazure [D] or by Lemma (1.21), [U3], $\mathrm{Aut}_X^0 L$ is of rank ≤ 1 but it is of rank 1 because $\mathbf{G}_m$ operates on the line bundle through scalar multiplication. Let U be a unipotent group operating effectively on L/X. The operation of U on each fibre lies in Aut $\mathbf{A}^1$ hence in the translations. Therefore the operation of U on each fibre is abelian. Thus U itself is abelian. It is easy to see, writing the operation in terms of local coordinates, that to give an effective operation of $\mathbf{G}_a$ on L/X is equivalent to giving a non-zero element of $H^0(X, L)$.

We want to give some remarks on the automorphism groups of the transformation spaces in the main theorem.

Lemma (4.5). *Let (G, X) be an effective algebraic operation. If X is rational, 3-dimensional and open, then the Lie algebra of G is not isomorphic to any of the following Lie algebras;* $\mathfrak{so}_5$, $\mathfrak{sl}_4$, $\mathfrak{sl}_2 \times \mathfrak{sl}_3$, $\mathfrak{sl}_2 \times \mathfrak{sl}_2 \times \mathfrak{sl}_2$.

Proof. Assume that the Lie algebra of G is isomorphic to $\mathfrak{so}_5$, $\mathfrak{sl}_4$, $\mathfrak{sl}_2 \times \mathfrak{sl}_3$ or $\mathfrak{sl}_2 \times \mathfrak{sl}_2 \times \mathfrak{sl}_2$. If G does not operate transitively even generically, then by a Theorem of Rosenlicht [R] there exists a non-empty G-invariant open set U such that the quotient $U \rightarrow G\backslash U$ exists. The fibres of $U \rightarrow G\backslash U$ are G-orbits hence unirational. By a Theorem of Zariski [Z] or Lüroth's Theorem, the fibres are rational since their dimension ≤ 2. Now by Enriques' Theorem, Umemura [U3], the Lie algebra of a semi-simple algebraic subgroup of the 2 variable Cremona group is $\mathfrak{sl}_2 \times \mathfrak{sl}_2$, $\mathfrak{sl}_3$ or $\mathfrak{sl}_2$. In particular G does not operate effectively on each fibre of $U \rightarrow G\backslash U$. Hence by Lemma (1.21) Umemura [U3], the operation of G on U is not effective. Thus the operation of G on X should have an open orbit V. Let H be the stabilizer group at a point of V so that $G/H \simeq V$. Then H is a closed subgroup of codimension 3 in G and H does not contain a proper normal subgroup of G since the operation of G on X is faithful.

We show that H is parabolic hence $V \simeq G/H$ is projective which contradicts the openness of X. This would be a consequence of a direct calculation but here we argue differently. If the Lie algebra of G is isomorphic to $\mathfrak{sl}_2 \times \mathfrak{sl}_3$ or $\mathfrak{sl}_2 \times \mathfrak{sl}_2 \times \mathfrak{sl}_2$, we show in (5.6) and (5.12) H is parabolic. If the Lie algebra of G is isomorphic to $\mathfrak{sl}_4$ or $\mathfrak{so}_5$, then the operation (G, V) is primitive. For otherwise by Umemura [U3], (G, V) is of de Jonquières type and G operates transitively on a variety W of dimension ≤ 2. Since W is unirational, it is rational. But by a Theorem of Enriques an algebraic group whose Lie algebra is isomorphic to $\mathfrak{sl}_4$ or $\mathfrak{so}_5$ is neither contained in Cr_2 nor Cr_1 (See [U3]). By Umemura [U1], we conclude H is parabolic in this case.

Remark (4.6). In Lemma (4.5), the assumption X to be rational is unnecessary. But we proved the lemma under the rationality assumption because for our purpose this weak assertion is sufficient.

Lemma (4.7). *Let (G, X) be an effective algebraic operation. If G is semi-simple and if X is rational, 3-dimensional and open, then the Lie algebra of G is isomorphic to one of the following*; $\mathfrak{sl}_2$, $\mathfrak{sl}_2\times\mathfrak{sl}_2$, $\mathfrak{sl}_3$.

Proof. If a torus T operates effectively on X, then $\dim T\leq\dim X$ by 1.6. Corollaire 1, Demazure [D]. Therefore the rank $G\leq 3$ and G must have a closed proper subgroup of codimension ≤ 3. The following are all the simple Lie algebras of rank ≤ 3: (1) rank 1 $\mathfrak{sl}_2$: (2) rank 2 $\mathfrak{sl}_3$, $\mathfrak{so}_5$, G_2; (3) rank 3 $\mathfrak{sl}_4$, $\mathfrak{so}_7$, $\mathfrak{sp}_6$. Since G_2, $\mathfrak{so}_7$, $\mathfrak{sp}_6$ have no Lie subalgebra of codimension ≤ 3, an algebraic group with one of those Lie algebra can not effectively act on X. We can also exclude $\mathfrak{sl}_2\times\mathfrak{sl}_3$, $\mathfrak{sl}_2\times\mathfrak{sl}_2\times\mathfrak{sl}_2$, $\mathfrak{sl}_4$ and $\mathfrak{so}_5$ by Lemma (4.5). Now if we consider all the possible products $\mathfrak{g}$ of simple Lie algebras such that rank $\mathfrak{g}\leq 3$, the Lemma follows.

Proposition (4.8). *For $m\geq 1$, $\mathrm{Aut}^0\,\mathrm{J}'_m$ respects the fibration of the line bundle J'_m over $\mathbf{P}^2$, i.e. there exists an exact sequence*;

$$\mathbf{1}\to\mathrm{Aut}^0_{\mathbf{P}^2}\mathrm{J}'_m\to\mathrm{Aut}^0\mathrm{J}'_m\to\mathrm{Aut}^0\mathbf{P}^2\to\mathbf{1}. \tag{4.9}$$

The semi-simple part of $\mathrm{Aut}^0\,\mathrm{J}'_m$ *is* $\mathfrak{sl}_3$.

Proof. Let V be the natural irreducible representation of degree 3 of SL_3. As in Section 2, Umemura [U3], the semi-direct product $S^m(V)_{\bf .}\mathrm{SL}_3$ operates transitively on J'_m. By Lemma (4.7), the semi-simple part of $\mathrm{Aut}^0\,\mathrm{J}'_m$ is $\mathfrak{sl}_3$ and $S^m(V)$ is in the unipotent radical U of $\mathrm{Aut}^0\,\mathrm{J}'_m$. Let U_Z be the center of U. $U_Z{}_{\bf .}S^m(V)$ is a vector group and SL_3-invariant. J'_m is a homogeneous space under the operation of $U_ZS^m(V)_{\bf .}\mathrm{SL}_3$. $U_Z{}_{\bf .}S^m(V)$ has not an open orbit on J'_m. For, otherwise by Lemma (1.12) Umemura [U3], J'_m would be isomorphic to $\mathbf{A}^3$ in particular affine but this is absurd since the line bundle J'_m over $\mathbf{P}^2$ contains $\mathbf{P}^2$ as the 0-section. $U_Z{}_{\bf .}S^m(V)$ has not a 2-dimensional orbit. In fact, otherwise since $U_ZS^m(V)$ is normal in $U_Z{}_{\bf .}S^m(V)\mathrm{SL}_3$ which operates transitively on J'_m, all the $U_ZS^m(V)$-orbits on J'_m would be 2-dimensional. Let H be an stabilizer of $U_ZS^m(V)_{\bf .}\mathrm{SL}_3$ at a point of J'_m so that $U_ZS^m(V)_{\bf .}\mathrm{SL}_3/H\simeq\mathrm{J}'_m$. We proved $U_ZS^m(V)_{\bf .}\mathrm{SL}_3/U_Z{}_{\bf .}S^m(V)_{\bf .}H$ would be 1-dimensional hence $U_ZS^m(V)_{\bf .}\mathrm{SL}_3$ therefore SL_3 would operate transitively on 1-dimensional variety

$$U_ZS^m(V)_{\bf .}\mathrm{SL}_3/U_ZS^m(V)_{\bf .}H$$

which is a contradiction. We have thus proved the $U_Z{}_{\bf .}S^m(V)$ has only 1-

dimensional orbits on J'_m, Namely U_Z-orbits coincide with $S^m(V)$-orbits. Hence the quotient by U_Z is the quotients by $S^m(V)$ which is the fibration $J'_m \to \mathbf{P}^2$. Since U_Z is normal in $\mathrm{Aut}^0 J'_m$, $\mathrm{Aut}^0 J'_m$ respects this fibration.

Corollary (4.10). *For* $m \geq 1$, $\mathrm{Aut}^0 J'_m$ *consists of the following birational automorphisms;*

$$x \mapsto \frac{ax+by+c}{gx+hy+i}, \quad y \mapsto \frac{dx+ey+f}{gx+hy+i}, \quad z \mapsto \frac{\lambda z + f(x, y)}{(gx+hy+i)^m},$$

where $\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \in \mathrm{SL}_3$, $\lambda \in \mathbf{G}_m$ *and* $f(x, y) \in k[x, y]$ *with (degree of* $f(x, y)$*)* $\leq m$.

Proof. As in Section 2, Umemura [U3], the above birational automorphisms operate biregularly on J'_m. By Lemma (4.4), the rank of $\mathrm{Aut}^0_{\mathbf{P}^2} J'_m$ is 1 and the unipotent radical of $\mathrm{Aut}^0_{\mathbf{P}^2} J'_m$ is $H^0(\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(m))$ which is given by $\{z \mapsto z + f(x, y) \mid f(x, y) \in k[x, y], \deg f(x, y) \leq m\}$. Therefore by the exact sequence (4.9), $\mathrm{Aut}^0 J'_m$ is exhausted by the birational automorphisms in Corollary (4.10).

Remark (4.10.1). We can prove Proposition (4.8) by another method as follows. The algebraic group G of operations in Corollary (4.10) operates regularly on J'_m. The dimension of the group G coincides with $h^0(J'_m, \mathbf{T})$ by Lemma (4.3). Thus $G = \mathrm{Aut}^0 J'_m$ hence $\mathrm{Aut}^0 J'_m$ keeps the fibration.

Proposition (4.11). *If* $m, n \geq 1$, $\mathrm{Aut}^0 \mathrm{L}'_{m,n}$ *respects the fibration of the line bundle* $\mathrm{L}'_{m,n}$ *over* $\mathbf{P}^1 \times \mathbf{P}^1$, i.e. *there exists an exact sequence;*

(4.11.1) $$1 \to \mathrm{Aut}^0_{\mathbf{P}^1 \times \mathbf{P}^1} \mathrm{L}'_{m,n} \to \mathrm{Aut}^0 \mathrm{L}'_{m,n} \to \mathrm{Aut}^0 \mathbf{P}^1 \times \mathbf{P}^1 \to 1.$$

The semi-simple part of $\mathrm{Aut}^0 \mathrm{L}'_{m,n}$ *is* $\mathfrak{sl}_2 \times \mathfrak{sl}_2$.

Proof. Let U_{m+1}, U_{n+1} be irreducible $\mathfrak{sl}_2$-modules of dimension $m+1$, $n+1$ respectively. Then as in Proposition (4.8) (see also Section 2, [U3]), $(U_{m+1} \otimes U_{n+1})$ $(\mathrm{SL}_2 \times \mathrm{SL}_2)$ operates transitively on $\mathrm{L}'_{m,n}$. Thus the semi-simple part of $\mathrm{Aut}^0 \mathrm{L}'_{m,n}$ contains $\mathfrak{sl}_2 \times \mathfrak{sl}_2$. Since $U_{m+1} \otimes U_{n+1}$ is abelian, unipotent and of dimension $(m+1)(n+1) \geq 4$, the $\mathfrak{sl}_2 \times \mathfrak{sl}_2$-module $U_{m+1} U_{n+1}$ can not be a Lie subalgebra of the Lie algebras of Lemma (4.7). Therefore, $U_{m+1} \otimes U_{n+1}$ is in the unipotent radical U of $\mathrm{Aut}^0 \mathrm{L}'_{m,n}$. Let U_Z be the center of U. If $U_{m+1} \otimes U_{n+1}$ is not contained in the center U_Z, then $(U_{m+1} \otimes U_{n+1}) U_Z$ is an $\mathrm{SL}_2 \times \mathrm{SL}_2$-invariant vector subgroup of U since U_Z is a normal subgroup of $\mathrm{Aut}^0 \mathrm{L}'_{m,n}$. Since

$$U_{m+1}\otimes U_{n+1}\subsetneqq(U_{m+1}\otimes U_{n+1})U_Z,$$

$(U_{m+1}\otimes U_{n+1})U_Z$ is not an irreducible $SL_2\times SL_2$-module. Let

$$(U_{m+1}\otimes U_{n+1})\oplus W=(U_{m+1}\otimes U_{n+1}).U_Z.$$

Let $v\in U_{m+1}\otimes U_{n+1}$ be a highest weight vector and w a highest weight vector of one of the irreducible factors of W. Then

$$V=\left\{(av,\ bw)\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & y\\ 0 & 1\end{pmatrix}\,\middle|\, a,b,x,y\in k\right\}$$

is an algebraic subgroup of $(U_{m+1}\otimes U_{n+1})U_Z(SL_2\times SL_2)$ and V is abelian. As in Section 2, [U3], a subgroup

$$\left\{(av,\ 0)\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & y\\ 0 & 1\end{pmatrix}\,\middle|\, a,x,y\in k\right\}$$

has an open orbit on $L'_{m,n}$. Therefore V has an open orbit on $L'_{m,n}$ which is absurd since V is abelian and of dimension ≥ 4 (see Lemma (1.8), [U3]). Thus $U_{m+1}\otimes U_{n+1}$ should coincide with U_Z. Thus letting H be a closed subgroup of $\mathrm{Aut}^0 L'_{m,n}$ such that $(\mathrm{Aut}^0 L'_{m,n})/H\simeq L'_{m,n}$, we get a morphism

$$\begin{aligned}(\varphi,f)\colon (\mathrm{Aut}^0 L'_{m,n},\ L'_{m,n})&=(\mathrm{Aut}^0 L'_{m,n},\ (\mathrm{Aut}^0 L'_{m,n})/H)\\ &\to(\mathrm{Aut}^0 L'_{m,n},\ (\mathrm{Aut}^0 L'_{m,n})/(U_{m+1}\otimes U_{n+1})H).\end{aligned}$$

$$(\mathrm{Aut}^0 L'_{m,n})/(U_{m+1}\otimes U_{n+1})H=(U_{m+1}\otimes U_{n+1})\backslash L'_{m,n}\simeq \mathbf{P}^1\times\mathbf{P}^1$$

(cf. Section 2, [U3]). We have an exact sequence

$$\mathbf{1}\to\mathrm{Aut}_{\mathbf{P}^1\times\mathbf{P}^1}L'_{m,n}\to\mathrm{Aut}^0 L'_{m,n}\to\mathrm{Aut}^0\mathbf{P}^1\times\mathbf{P}^1\to\mathbf{1}.$$

$\mathrm{Aut}^0_{\mathbf{P}^1\times\mathbf{P}^1}L'_{m,n}$ consists of $\mathbf{A}^1$-bundle automorphisms over $\mathbf{P}^1\times\mathbf{P}^1$. Hence the unipotent radical of $\mathrm{Aut}_{\mathbf{P}^1\times\mathbf{P}^1}L'_{m,n}$ is abelian and coincides with

$$H^0(\mathbf{P}^1\times\mathbf{P}^1,\ p_1^*\mathcal{O}(m)\otimes p_1^*\mathcal{O}(n))\simeq U_{m+1}\otimes U_{n+1}.$$

The reductive part of $\mathrm{Aut}_{\mathbf{P}^1\times\mathbf{P}^1}L'_{m,n}$ is $\mathbf{G}_m$ (see Umemura [U3]). This completes the proof of the Proposition.

Corollary (4.12). $\mathrm{Aut}^0 L'_{m,n}$ *consists of the following birational automorphisms*:

$$x\mapsto\frac{ax+b}{cx+d},\quad y\mapsto\frac{d'x+b'}{c'x+d'},\quad z\mapsto\frac{\lambda z+f(x,y)}{(cx+d)^m(c'x+d')^n},$$

where $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \in \mathrm{SL}_2$, $\lambda \in \mathbf{G}_m$, $f(x, y) \in k[x, y]$ *such that* $\deg_x f(x, y) \leq m$, $\deg_y f(x, y) \leq n$ ($\deg_a g$ *denotes the degree of the polynomial g in a*).

Proof. As in [U3], we see the above automorphisms are regular automorphisms of $\mathrm{L}'_{m,n}$. By Lemma (1.21), [U3] the rank of $\mathrm{Aut}^0{}_{\mathbf{P}^1\times\mathbf{P}^1}\mathrm{L}'_{m,n}$ is 1, solvable and it is easy to see the unipotent radical of $\mathrm{Aut}_{\mathbf{P}^1\times\mathbf{P}^1}\mathrm{L}'_{m,n}$ is $H^0(\mathbf{P}^1\times\mathbf{P}^1, p_1^*\mathcal{O}_{\mathbf{P}^1}(m)\otimes p_2^*\mathcal{O}_{\mathbf{P}^1}(n))$ which is given by

$$\{z \mapsto z+f(x, y) \mid f(x, y) \in k[x, y] \mid \deg_x f(x, y) \leq m,\ \deg_y f(x, y) \leq n\}.$$

Therefore exact sequence (4.11.1) shows the automorphisms given in the Corollary exhaust $\mathrm{Aut}^0\mathrm{L}'_{m,n}$.

We defined $\mathrm{F}'_{m,n}$ as the total space of the vector bundle $\mathcal{O}_{\mathbf{P}^1}(m)\oplus\mathcal{O}_{\mathbf{P}^1}(n)$ over $\mathbf{P}^1$. Thus $\mathrm{F}'_{m,n}$ is considered as a line bundle over $\mathcal{O}_{\mathbf{P}^1}(n)$ which is nothing but F'_m. Hence we have fibrations $\mathrm{F}'_{m,n}\to\mathrm{F}'_n\to\mathbf{P}^1$.

Proposition (4.13). *If* $m>n\geq 1$, $\mathrm{Aut}^0\mathrm{F}'_{m,n}$ *respects the fibration* $\mathrm{F}'_{m,n}\to\mathrm{F}'_n\to\mathbf{P}^1$, i.e. *there exist exact sequences*;

$$\mathbf{1}\to\mathrm{Aut}^0_{\mathrm{F}'_n}\mathrm{F}'_{m,n}\to\mathrm{Aut}^0\mathrm{F}'_{m,n}\to\mathrm{Aut}^0\mathrm{F}'_n\to\mathbf{1},$$
$$\mathbf{1}\to\mathrm{Aut}^0_{\mathbf{P}^1}\mathrm{F}'_n\to\mathrm{Aut}^0\mathrm{F}'_n\to\mathrm{Aut}^0\mathbf{P}^1\to\mathbf{1}.$$

The semi-simple part of $\mathrm{Aut}^0\mathrm{F}'_{m,n}$ *is* $\mathfrak{sl}_2$.

Proof. Let U_{m+1}, U_{n+1} be the irreducible SL_2-modules of dimension $m+1$, $n+1$. Then $(U_{m+1}\oplus U_{n+1})\mathrm{SL}_2$ operates transitively on $\mathrm{F}'_{m,n}$ (cf. Section 2, [U3]). $\mathrm{F}'_{m,n}$ is the total space of the vector bundle $\mathcal{O}_{\mathbf{P}^1}(m)\oplus\mathcal{O}_{\mathbf{P}^1}(n)$ hence obtained by gluing two $\mathbf{A}^3$'s: Let $\mathcal{U}_1=\mathbf{A}^3$, $\mathcal{U}_2=\mathbf{A}^3$ and (x_1, y_1, z_1), (x_2, y_2, z_2) be coordinate systems on $\mathcal{U}_1$ and $\mathcal{U}_2$. We identify $x_2=x_1^{-1}$, $y_2=x_1^{-m}y_1$, $z_2=x_1^{-n}z_1$. As in Umemura [U3], the operation of $(U_{m+1}\oplus U_{n+1})\mathrm{SL}_2$ on $\mathrm{F}'_{m,n}$ is given by:

$$(4.14)\qquad x_1\mapsto\frac{ax_1+b}{cx_1+d},\quad y_1\mapsto\frac{y_1+f(x_1)}{(cx_1+d)^m},\quad z_1=\frac{z_1+g(x_1)}{(cx_1+d)^n},$$

where $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2$, $f(x_1)$, $g(x_1) \in k[x_1]$ with $\deg f(x_1)\leq m$ and $\deg g(x_1) \leq n$. In particular the operation of $U_{m+1}\oplus U_{n+1}$ is given by $x_1\mapsto x_1$, $y_1\mapsto y_1+f(x_1)$, $z_1\mapsto z_1+g(x_1)$ where $f(x_1)$, $g(x_1) \in k[x_1]$ and satisfy the condition above on the degree and the projection $\mathrm{F}'_{m,n}\to\mathbf{P}^1$ is the quotient by the operation of $U_{m+1}\oplus U_{n+1}$.

The semi-simple part of $\mathrm{Aut}^0(\mathrm{F}'_{m,n})$ is one of the Lie algebras of

Lemma (4.7). Since $U_{m+1} \oplus U_{n+1}$ is abelian and of dimension $(m+1)+(n+1) \geq 5$, $U_{m+1} \oplus U_{n+1}$ can not be contained in the semi-simple part of $\mathrm{Aut}^0(\mathrm{F}'_{m,n})$ and $\mathrm{Aut}^0(\mathrm{F}'_{m,n})$ has the non-trivial unipotent radical U. Let U_Z be the center of U. Now we need lemmas.

Lemma (4.15). *U_Z has not an open orbit on* $\mathrm{F}'_{m,n}$.

Proof. If U_Z had an open orbit, by Corollary (1.13), Umemura [U3], $\mathrm{F}'_{m,n}$ would be isomorphic to $\mathbf{A}^3$ which is absurd since $\mathrm{F}'_{m,n}$ contains $\mathbf{P}^1$ as the 0-section of the vector bundle $\mathrm{F}'_{m,n}$ over $\mathbf{P}^1$.

Lemma (4.16). *The semi-simple part of* $\mathrm{Aut}^0(\mathrm{F}'_{m,n})$ *is not* $\mathfrak{sl}_3$.

Proof. By Lemma (4.15) the dimension of the U_Z-orbits is 2 or 1. Notice that all the U_Z-orbits have the same dimension since U_Z is normal. Assume SL_3 acts on $\mathrm{F}'_{m,n}$. Let us put $G=\mathrm{Aut}^0(\mathrm{F}'_{m,n})$ and denote by H the stabilizer group at a point of $\mathrm{F}'_{m,n}$ so that $G/H \simeq \mathrm{F}'_{m,n}$. We have a morphism $(\varphi, f)\colon (G, G/H) \to (G, G/U_Z H)$ of algebraic operations. If the dimension of the U_Z-orbits is 2, the dimension of $G/U_Z H$ is 1 and we have an exact sequence:

$$\begin{array}{c} 1 \to N \to \mathrm{Aut}^0 G/H \to \mathrm{Aut}^0 G/U_Z H \\ \hphantom{1 \to N \to \mathrm{Aut}^0 G/H \to{}} \cap \\ \hphantom{1 \to N \to \mathrm{Aut}^0 G/H \to{}} \mathrm{Cr}_1 = \mathrm{PGL}_2 \end{array}$$

where N denotes the kernel which consists of the automorphisms of the fibre bundle $f\colon G/H \to G/U_Z H$. Since the fibre of f is $U_Z H/H$ which is a homogeneous space under the vector group U_Z thus is isomorphic to $\mathbf{A}^2$, SL_3 does not operate on the fibre. Therefore SL_3 is not contained in N. But SL_3 is not contained in $\mathrm{Aut}^0 G/U_Z H \subset \mathrm{Cr}_1 = \mathrm{PGL}_2$ either. If the dimension of U_Z-orbits is 1, the dimension of $G/U_Z H$ is 2 and we have an exact sequence:

$$\begin{array}{c} 1 \to N \to \mathrm{Aut}^0 G/H \to \mathrm{Aut}^0 G/U_Z H, \\ \hphantom{1 \to N \to \mathrm{Aut}^0 G/H \to{}} \cap \\ \hphantom{1 \to N \to \mathrm{Aut}^0 G/H \to{}} \mathrm{Cr}_2 \end{array}$$

where N denotes the kernel which consists of the automorphisms of the fibre bundle $f\colon G/H \to G/U_Z H$. Since the fibre of f is $\mathbf{A}^1$, SL_3 is not contained in N and hence SL_3 operates non-trivially on $G/U_Z H$. Since $G/U_Z H$ is rational by a Theorem of Zariski, it follows from Umemura [U3] $(\mathrm{SL}_3, G/U_Z H)$ is isomorphic to $(\mathrm{SL}_3, \mathbf{P}^2)$ and G/H is an $\mathbf{A}^1$-bundle over $\mathbf{P}^2$. Since $H^1(\mathbf{P}^2, L)=0$ for any line bundle L, an $\mathbf{A}^1$-bundle over $\mathbf{P}^2$ is a line bundle and hence $G/H = \mathrm{F}'_{m,n}$ contains $\mathbf{P}^2$. But $\mathrm{F}'_{m,n}$ is a vector bundle over $\mathbf{P}^1$. Let $\pi\colon \mathrm{F}'_{m,n}\to\mathbf{P}^1$ be the projection. Since there is no non-trivial morphism of $\mathbf{P}^2$ to $\mathbf{P}^1$, $\mathbf{P}^2$ should be contained in a fibre which is isomorphic to $\mathbf{A}^2$. This is a contradiction and the Lemma is proved.

In view of Lemma (4.7) and Lemma (4.16) we may assume the semi-simple part of $\mathrm{Aut}^0\mathrm{F}'_{m,n}$ is $\mathfrak{sl}_2$ or $\mathfrak{sl}_2\times\mathfrak{sl}_2$. Neither $U_{m+1}.\mathfrak{sl}_2$ nor $U_{n+1}.\mathfrak{sl}_2$ can be embedded in the semi-simple part of $\mathrm{Aut}^0\mathrm{F}'_{m,n}$ which is isomorphic to $\mathfrak{sl}_2$ or $\mathfrak{sl}_2\times\mathfrak{sl}_2$. Hence $U_{m+1}\oplus U_{n+1}$ is in the unipotent radical U of $\mathrm{Aut}^0(\mathrm{F}'_{m,n})$. When we speak of SL_2, it means SL_2 at the beginning of the Proof of Proposition (4.13). Namely its operation is given by

$$x_1\mapsto(ax_1+b)/(cx_1+d),\ y_1\mapsto y_1/(cx_1+d)^n,\quad z_1\mapsto z_1/(cx_1+d)^m.$$

We show one of U_{m+1} and U_{n+1} is in the center U_Z. For otherwise, let $W\subset U_Z$ be an irreducible factor of SL_2-module U_Z and $u\in U_{m+1}$, $v\in U_{n+1}$ and $w\in W$ be highest weight vectors. A vector group

$$\left\{(au,\,bv,\,cw)\begin{pmatrix}1 & d\\ 0 & 1\end{pmatrix}\middle|\, a,b,c,d\in k\right\}\subset(U_{m+1}\oplus U_{n+1})W\mathrm{SL}_2$$

of dimension 4 operates transitively on $\mathrm{F}'_{m,n}$. This is impossible by Lemma (1.8), Umemura [U3]. Now we prove U_{n+1} is not in the center U_Z and hence U_{m+1} is in the center. In fact the following vector group $\mathscr{V}$ is in $\mathrm{Aut}^0\mathrm{F}'_{m,n}$. $x_1\mapsto x_1$, $y_1\mapsto y_1$, $z_1\mapsto z_1+a(x_1)y_1$ where $a(x_1)\in k[x_1]$ and $\deg a(x_1)\le m-n$. It follows from the definition, $\mathscr{V}$ is SL_2-invariant and irreducible. The dimension of $\mathscr{V}$ is $m-n+1\ge 2$. Therefore $\mathscr{V}\mathrm{SL}_2$ can not be embedded in neither $\mathrm{SL}_2\times\mathrm{SL}_2$ nor SL_2 and $\mathscr{V}$ is in U. It follows from the definition, U_{n+1} does not commute with $\mathscr{V}$. Thus U_{n+1} does not lie in the center U_Z. If the center U_Z has 2-dimensional orbit, it contains another SL_2-irreducible factor $\mathscr{U}$ than U_{m+1} such that $\mathscr{U}\oplus U_{m+1}$ has 2-dimensional orbit. Let $u\in U_{n+1}$, $v\in U_{m+1}$ and $w\in\mathscr{U}$ be highest weight vectors then

$$\mathfrak{A}=\left\{(au,\,bv,\,cw)\begin{pmatrix}1 & d\\ 0 & 1\end{pmatrix}\middle|\, a,b,c,d\in k\right\}$$

is a vector group of dimension 4. $\mathfrak{A}$ has an open orbit which contradicts Lemma (1.8) Umemura [U3]. Therefore all the U_Z-orbits are 1-dimensional and coincide with U_{m+1}-orbits. Let us put $G=\mathrm{Aut}^0\mathrm{F}'_{m,n}$ and let H be the stabilizer at a point of $\mathrm{F}'_{m,n}$ so that $G/H\simeq\mathrm{F}'_{m,n}$. We have a morphism of algebraic operations $(G, G/H)\to(G, G/U_ZH)$ and $G/U_ZH\simeq U_{m+1}\backslash\mathrm{F}'_{m,n}\simeq\mathrm{F}'_n$. We have an exact sequence:

$$1\to N\to G\to\mathrm{Aut}^0\mathrm{F}'_n\to 1.$$

The last exact sequence is proved in [U3]. N is the subgroup of G consisting of the automorphisms of the $\mathbf{A}^1$-bundle $G/H \to G/U_Z H$ and hence N is solvable. The semi-simple part of $\mathrm{Aut}^0 \mathrm{F}'_m$ is $\mathfrak{sl}_2$ (Umemura [U2]).

Corollary (4.17). *For* $m \geq n \geq 1$, $\mathrm{Aut}^0 \mathrm{F}'_{m,n}$ *consists of the following birational automorphisms:*

$$x \mapsto \frac{ax+b}{cx+d}, \quad y \mapsto \frac{\lambda y + f(x)}{(cx+d)^n}, \quad z \mapsto \frac{\mu z + \sum_{j=0}^{l} y^{l-i} \varphi_{r+n_i}(x)}{(cx+d)^m},$$

where $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2$, $\lambda, \mu \in \mathbf{G}_m$, $f(x) \in k[x]$, $\varphi_{r+n_i}(x) \in k[x]$, $\deg f(x) \leq n$, $\deg \varphi_{r+n_i}(x) \leq r+n_i$ $(0 \leq i \leq l)$, $m = ln + r$ $(l, r \in \mathbf{Z},\ l \geq 0,\ 0 \leq r \leq n-1)$.

Proof. Notice that the unipotent radical of $\mathrm{Aut}^0_{\mathrm{F}'_n} \mathrm{F}'_{m,n}$ is $H^0(\mathrm{F}'_n, \mathcal{O}'_{\mathrm{F}_n}(m))$ which is isomorphic to, by the degeneracy of the spectral sequence,

$$H^0\left(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(m) \otimes \left(\bigoplus_{i=0}^{\infty} \mathcal{O}(-in)\right)\right)$$

hence to

$$\bigoplus_{i=0}^{l} H^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(m-in)).$$

Now argue as in Corollary (4.10).

Proposition (4.18). *The semi-simple part of* $\mathrm{Aut}^0 \mathrm{F}'_{m,m}$ *is* $\mathfrak{sl}_2 \times \mathfrak{sl}_2$ *for* $m \geq 1$.

Proof. As in the $\mathrm{F}'_{m,n}$ case, $(U_{m+1} \oplus U_{m+1})\mathrm{SL}_2$ operates on $\mathrm{F}'_{m,m}$ and the operation is given as in Corollary (4.17). Since $\dim(U_{m+1} \oplus U_{m+1}) = 2(m+1) \geq 4$, as at the beginning of the Proof of Proposition (4.13), $U_{m+1} \oplus U_{m+1}$ is not contained in the semi-simple part of $\mathrm{Aut}^0 \mathrm{F}'_{m,m}$. The same argument as in Lemma (4.16) shows the semi-simple part of $\mathrm{Aut}^0 \mathrm{F}'_{m,m}$ is $\mathfrak{sl}_2$ or $\mathfrak{sl}_2 \times \mathfrak{sl}_2$. Using the same local coordinate system on $\mathrm{F}'_{m,m}$ as in the Proof of Proposition (4.13), one more SL_2 operates on $\mathrm{F}'_{m,m}$: $x \mapsto x$, $y \mapsto \alpha y + \beta z$, $z \mapsto \gamma y + \delta z$, $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{SL}_2$. Thus $\mathrm{SL}_2 \times \mathrm{SL}_2$ operates on $\mathrm{F}'_{m,m}$ and the Lemma is proved.

Proposition (4.19). *The automorphism group* $\mathrm{Aut}^0 \mathrm{F}'_{m,m}$ *keeps the fibration* $\mathrm{F}'_{m,m} \to \mathbf{P}^1$ *for* $m \geq 1$.

Proof. As in Lemma (4.15), the unipotent radical U of $\mathrm{Aut}^0 \mathrm{F}'_{m,m}$ has not an open orbit. Since $U_{m+1} \oplus U_{m+1}$, which is in U by Lemma (4.18), has 2-dimensional orbits on $\mathrm{F}'_{m,m}$, the $U_{m+1} \oplus U_{m+1}$-orbits coincide with U-orbits. Now the proposition follows.

Corollary (4.20). *For* $m \geq 1$, $\mathrm{Aut}^0 \mathrm{F}'_{m,m}$ *consists of the following birational automorphisms*:

$$x \mapsto \frac{ax+b}{cx+d}, \quad y \mapsto \frac{a'y+b'z+f(x)}{(cx+d)^m}, \quad z \mapsto \frac{c'y+d'z+g(x)}{(cx+d)^m},$$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2$, $\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \in \mathrm{GL}_2$, $f(x), g(x) \in k[x]$ *with* $\deg f(x), \deg g(x) \leq m$.

Proof. As in preceding cases, the above birational automorphisms operate biregularly on $\mathrm{F}'_{m,m}$. It is sufficient to show $\mathrm{Aut}^0_{\mathbf{P}^1} \mathrm{F}'_{m,m}$ coincides with $\mathscr{G} = \left\{ (y, z) \mapsto (a'y+b'z+f(x), c'y+d'z+g(x)) \,\middle|\, \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \in \mathrm{GL}_2, f(x), g(x) \in k[x], \deg f(x), \deg g(x) \leq m \right\}$. In fact, the last group $\mathscr{G}$ induces on each fibre the group of the affine transformations of 2 variables. We need

Lemma (4.21). *Let G be a (connected) algebraic group operating on* $\mathbf{A}^2$. *If G contains all the affine transformations, then G coincides with the group of the affine transformations.*

Proof of Lemma. Since the affine transformation group is primitive $(G, \mathbf{A}^2)$ is primitive (cf. [U1]). It follows from Umemura [U1] $(G, \mathbf{A}^2)$ is a suboperation of $(\mathrm{PGL}_3, \mathbf{P}^2)$. Now the Lemma follows.

It follows from Lemma (4.21) and (1.21), Umemura [U3], the reductive part of $\mathrm{Aut}^0_{\mathbf{P}^1} \mathrm{F}'_{m,m}$ is GL_2 and hence coincides with that of G. As for the unipotent radical, argue as in Corollary (4.10).

Here are some properties of $\mathrm{Aut}^0(\mathrm{E}'^{l}_m; \mathrm{F}'_m)$ which will be used later.

Lemma (4.22). *Let X be a variety and L a line bundle with* $h^0(X, L) < \infty$. *Let $Y \to X$ be* $\mathbf{A}^1$-*bundle defined by a non-trivial extension* $0 \to \mathcal{O}_X \to E \to L^{-1} \to 0$. *Then* $\mathrm{Aut}^0_X Y$ *is isomorphic to the vector group* $H^0(X, L)$.

Proof. $\mathrm{Aut}^0_X Y$ operates on each fibre as affine transformation group of 1 variable. Thus $\mathrm{Aut}^0_X Y$ is of rank at most 1 by Lemma (1.12) Umemura [U3] and solvable. But a torus operates on an affine bundle if and only if the defining extension splits. Hence $\mathrm{Aut}^0_X Y$ is unipotent. As its operation on each fibre is a translation, the unipotent group $\mathrm{Aut}^0_X Y$ is a vector group. Let $\bigcup_{i \in I} U_i = X$ be a covering, $\{a_{ij} \in \Gamma(U_i \cap U_j, \mathcal{O}^*_X)\}_{i,j \in I}$

a cocycle for L and $\left\{\begin{pmatrix} a_{ij} & b_{ij} \\ 0 & 1 \end{pmatrix} \middle| b_{ij} \in \Gamma(U_i \cap U_j, \mathcal{O}_X)\right\}_{i,j\in I}$ a cocycle for X. To give an element of the unipotent radical is equivalent to giving $\left\{\begin{pmatrix} 1 & c_i \\ 0 & 1 \end{pmatrix} \middle| c_i \in \Gamma(U_i, \mathcal{O}_X)\right\}_{i\in I}$ such that

$$\begin{pmatrix} a_{ij} & b_{ij} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} c_i & 0 \\ 0 & 1 \end{pmatrix}=\begin{pmatrix} c_j & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a_{ij} & b_{ij} \\ 0 & 1 \end{pmatrix},$$

i.e. a section $\{c_i\}_{i\in I}$ of L.

Corollary (4.23). $\mathrm{Aut}^0_{\mathrm{F}'_m}\mathrm{E}'^l_m$ *is the vector group* $H^0(\mathrm{F}'_m, \mathcal{O}(lm-2))$. *Let* U *be the unipotent radical of* $\mathrm{Aut}^0(\mathrm{E}'^l_m; \mathrm{F}'_m)$. *Then we get an exact sequence*:

$$\begin{array}{ccc}
\mathbf{1}\to\mathrm{Aut}^0_{\mathrm{F}'_m}\mathrm{E}'^l_m\to U\to & (\textit{unipotent radical of}\ \mathrm{Aut}^0\,\mathrm{F}'_m) & \to\mathbf{1}. \\
\| & \wr| & \\
H^0(\mathrm{F}'_m, \mathcal{O}(lm-2)) & V_{m+1} & \\
\wr| & & \\
\bigoplus_{j\geq 0} H^0(\mathbf{P}^1, \mathcal{O}(lm-2-jm)) & &
\end{array}$$

The semi-simple part of $\mathrm{Aut}^0(\mathrm{E}'^l_m; \mathrm{F}'_m)$ *is* SL_2 *and has an open orbit on* E'^l_m. *There is no 2-dimensional* SL_2*-orbit on* E'^l_m *covering the* SL_2*-open orbit on* F'_m.

Proof. It follows from definition we have

$$\mathbf{1}\to\mathrm{Aut}_{\mathrm{F}'_m}\mathrm{E}'^l_m\to\mathrm{Aut}^0(\mathrm{E}'^l_m; \mathrm{F}'_m)\to\mathrm{Aut}^0\,\mathrm{F}'_m\to\mathbf{1}.$$

$\mathrm{Aut}_{\mathrm{F}'_m}\mathrm{E}'^l_m$ is solvable and a semi-simple part of $\mathrm{Aut}^0\,\mathrm{F}'_m$ is SL_2. Except for the last assertion, Corollary now follows from Lemma (4.22). As for the last assertion, assume SL_2 had not an open orbit on E'^l_m. As SL_2 has an open orbit on F'_m isomorphic to SL_2/U_m, where

$$U_m=\left\{\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \in \mathrm{SL}_2 \,\middle|\, a^m=1\right\},$$

SL_2 would have 2-dimensional orbit Y isomorphic to SL_2/U_n covering SL_2/U_m on E'^l_m with $n\,|\,m$,

$$U_n=\left\{\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \in \mathrm{SL}_2 \,\middle|\, a^n=1\right\} \quad \text{(cf. [U3])}.$$

Then the pull back of E'^l_m by $\varphi\colon \mathrm{SL}_2/U_n\to\mathrm{SL}_2/U_m\subset\mathrm{F}'_m$ would have a section and would be a line bundle. But this is impossible as we have inclusions,

$$\begin{tikzcd}
H^1(\mathrm{SL}_2/U_n, \mathcal{O}(lm-2)) \arrow[d, "\wr"'] & H^1(\mathrm{SL}_2/U_m, \mathcal{O}(lm-2)) \arrow[l, "\varphi^*"'] \arrow[d, "\wr"] \\
\bigoplus_{k=-\infty}^{k=\infty} H^1(\mathbf{P}^1, \mathcal{O}(lm-2-nk)) & \bigoplus_{k=-\infty}^{\infty} H^1(\mathbf{P}^1, \mathcal{O}(lm-2-km)) \arrow[l, hook'] \\
& \bigoplus_{k=0}^{\infty} H^1(\mathbf{P}^1, \mathcal{O}(lm-2-km)) \arrow[u, phantom, "\cup"] \arrow[d, "\wr"] \\
& H^1(\mathrm{F}'_m, \mathcal{O}(lm-2)).
\end{tikzcd}$$

Here for an integer $i \geq 0$,

$$\mathrm{SL}_2/U_i \left(U_i = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \in \mathrm{SL}_2\, a^i = 1 \right\} \right)$$

is F'_i-(0-section) and $\pi\colon \mathrm{SL}_2/U_i \to \mathrm{SL}_2/B = \mathbf{P}^1$, $B = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \in \mathrm{SL}_2 \right\}$ is an affine morphism and hence the spectral sequence for π degenerates. We know $\pi_* \mathcal{O} \simeq \bigoplus_{i=-\infty}^{\infty} \mathcal{O}(ki)$.

In Section 3 we defined E'^l_m, $\mathrm{F}_{m,n}$ $\cdots$ for a wider range of indices than in Section 2. We show for small indices the operations are contained in one of the conjugacy classes of Theorem (2.2).

Lemma (4.24). *For* $m \geq 2$, $(\mathrm{Aut}^0(\mathrm{E}'^1_m, \mathrm{F}'_m), \mathrm{F}'_m)$ *is contained in*

$$(\mathrm{Aut}^0 \mathrm{F}'_{m-1,m-1}, \mathrm{F}'_{m-1,m-1}).$$

Proof. In view of Corollary (4.2) and Lemma (4.3), it is sufficient to show E'^1_m is isomorphic to $\mathrm{F}'_{m-1,m-1}$. Let V_1 and V_{m-1} denote irreducible SL_2-modules of dimension 2 and m. We identify V_i with the vector space of homogeneous polynomials of degree i in u and v. The tensor product $V_1 \otimes V_{m-1}$ is an $\mathrm{SL}_2 \times \mathrm{SL}_2$-module. Let $G =$ a semi-direct product $(V_1 \otimes V_{m-1}) . \mathrm{SL}_2 \times \mathrm{SL}_2$, $G_\delta =$ the subgroup $(V_1 \otimes V_{m-1}) . \Delta(\mathrm{SL}_2)$, $G_2 =$ the subgroup $(V_1 \otimes V_{m-1})(\mathbf{1} \times \mathrm{SL}_2)$ of G, where Δ is the diagonal morphism $\Delta\colon \mathrm{SL}_2 \to \mathrm{SL}_2 \times \mathrm{SL}_2$. Let V' be the vector subspace of $V_1 \otimes V_{m-1}$ spanned by $u \otimes u^{n-i} v^i$, $v \otimes u^{n-i} v^i$, $1 \leq i \leq n$. Let B be the lower triangular Borel subgroup. We finally put $H = V' . (\mathrm{SL}_2 \times B) \subset G$, $H \cap G_2 = H_2$ and $H \cap G_\delta = H_\delta$. We have inclusions $G_2/H_2 \subset G/H$, $G_\delta/H_\delta \subset G/H$. If we consider $V = V_1 \otimes V_{m-1}$-orbits, we have fibrations $G/H \to G/V.H \simeq \mathrm{SL}_2/B$, $G_2/H_2 \to G_2/V.H_2 \simeq \mathrm{SL}_2/B$ and $G_\delta/H_\delta \to G_\delta/VH_\delta \simeq \mathrm{SL}_2/B$. They are $\mathbf{A}^2$-bundles over $\mathrm{SL}_2/B \simeq \mathbf{P}^1$ and hence the inclusions are isomorphism in particular G_2/H_2 is isomorphic to G_δ/H_δ. It follows from the argument of [U3], G_2/H_2 is isomorphic to $\mathrm{F}'_{m-1,m-1}$. It remains to prove

Sublemma (4.24.1). *G_δ/H_δ is isomorphic to* E'^{1}_{m}.

Proof of Sublemma. Considering the decomposition $V_1 \otimes V_{m-1} \simeq V_m \oplus V_{m-2}$, we get $G_\delta = (V_m \oplus V_{m-2}) \ldotp \mathrm{SL}_2$. The explicit decomposition shows, if we put $\mathscr{G} = V_m \ldotp \mathrm{SL}_2 \subset G_\delta$ and $\mathscr{H} = \mathscr{G} \cap H_\delta$, then

$$\mathscr{H} = \langle u^{m-2}v^2, u^{m-3}v^3, \cdots, v^m \rangle B.$$

We have an inclusion $\mathscr{G}/\mathscr{H} \subset G_\delta/H_\delta$. The same argument as above shows that $\mathscr{G}/\mathscr{H}$, G_δ/H_δ are $\mathbf{A}^2$-bundle over $\mathbf{P}^1$. Hence the inclusion is an isomorphism. Thus we have to show $\mathscr{G}/\mathscr{H}$ is isomorphic to E'^{1}_{m}. To this end we describe $\mathscr{G}/\mathscr{H}$ explicitly. $\left\{\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}(yu^n + zu^{n-1}v) \mid x, y, z \in k\right\}$ is a closed subgroup of $\mathscr{G}$ and its orbit $\left\{\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}(yu^n + zu^{n-1}v)\mathscr{H} \mid x, y, z \in k\right\}$ is 3-dimensional and isomorphic to $\mathbf{A}^3$ by mapping $(x, y, z) \in \mathbf{A}^3$ to $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}(yu^n + zu^{n-1}v)\mathscr{H}$. The orbit

$$\left\{\begin{pmatrix} 1 & 0 \\ x' & 1 \end{pmatrix}(z'uv^{n-1} + y'v^n)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\mathscr{H} \mid x', y', z' \in k\right\}$$

is also isomorphic to $\mathbf{A}^3$ by

$$(x', y', z') \in \mathbf{A}^3 \to \begin{pmatrix} 1 & 0 \\ x' & 1 \end{pmatrix}(y'uv^{n-1} + z'v^n)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\mathscr{H}.$$

By considering the fibration $\mathscr{G}/\mathscr{H} \to \mathrm{SL}_2 B$, we conclude $\mathscr{G}/\mathscr{H}$ is covered by the 2 affine orbits

$$\left\{\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}(yu^n + zu^{n-1}v)\mathscr{H} \mid x, y, z \in k\right\}$$

and

$$\left\{\begin{pmatrix} 1 & 0 \\ x' & 1 \end{pmatrix}(z'uv^{n-1} + y'v^n)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\mathscr{H} \mid x, y, z \in k\right\}.$$

Let now examine how the 2 open sets are glued to get the whole space $\mathscr{G}/\mathscr{H}$. For this purpose it is sufficient to solve the following equation:

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}(yu^n + zu^{n-1}v)\mathscr{H} = \begin{pmatrix} 1 & 0 \\ x' & 1 \end{pmatrix}(z'uv^{n-1} + y'v^n)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\mathscr{H} \tag{4.24.2}$$

under the hypothesis $x, x' \neq 0$. (4.24.2) is equivalent to

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}(yu^n + zu^{n-1}v) = \begin{pmatrix} 1 & 0 \\ x' & 1 \end{pmatrix}(z'uv^{n-1} + y'v^n)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} a & 0 \\ c & a^{-1} \end{pmatrix} \cdot \tag{4.24.3}$$

$\cdot$(a homogeneous polynomial of degree n in u, v and of degree $\leq n-2$ in u).

The right side of (4.24.3) is equal to

$$(4.24.4)\qquad \begin{pmatrix}1 & 0\\ x' & 1\end{pmatrix}\begin{pmatrix}c & a^{-1}\\ -a & 0\end{pmatrix}\left\{\begin{pmatrix}0 & -a^{-1}\\ a & c\end{pmatrix}(z'uv^{n-1}+y'v^n)\begin{pmatrix}0 & -a^{-1}\\ a & c\end{pmatrix}^{-1}\right.$$
$$\left.\cdot(\text{a polynomial of degree} \le n-2 \text{ in } u)\right\}.$$

Hence by (4.24.3), $\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}=\begin{pmatrix}1 & 0\\ x' & 1\end{pmatrix}\begin{pmatrix}c & a^{-1}\\ -a & 0\end{pmatrix}$. Therefore, $x=1/x'$, $c=1$, $a=x'$. (4.24.4) is thus equal to

$$(4.24.5)\qquad \begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}(z'x'v(-x'^{-1}u+v)^{n-1}+y'(-x'^{-1}u+v)^n).$$
$$\cdot(\text{a polynomial of degree} \le n-2 \text{ in } u).$$

By (4.24.3), we conclude

$$yu^n+zu^{n-1}v=z'x'v(-x'^{-1}u+v)^{n-1}+y'(-x'^{-1}u+v)^n$$
$$+\text{terms of degree} \le n-2 \text{ in } u.$$

Therefore, $y=y'(-x'^{-1})^n$, $z=-z'x'^{(2-n)}+n(-x')^{(1-n)}.v$. If we replace x, x' by $-x$ and $-x'$, we identify $\mathbf{A}^3$'s with coordinates (x, y, z), (x', y', z') by

$$(4.24.6)\qquad \begin{aligned} &x=1/x', \qquad y=y'(1/x')^n,\\ &z=-z'(1/x')^{(n-2)}+ny'(1/x')^{(n-1)}.\end{aligned}$$

(4.24.6) shows the fibration p is an $\mathbf{A}^1$-bundle defined by the exact sequence (which is evidently SL_2-equivariant because SL_2 operates on $W.SL_2/\mathscr{H}$ keeping the fibration p);

$$0\to\mathcal{O}_{F'_n}\to E\to\mathcal{O}_{F'_n}(2-n)\to 0,$$

where $\mathcal{O}_{F'_n}(2-n)=p^*\mathcal{O}_{\mathbf{P}^1}(2-n)$, $p\colon F'_n\to\mathbf{P}^1$ is the projection. We show the $\mathbf{A}^1$-bundle over F'_n defined by (4.24.6) is not isomorphic to the line bundle $\mathcal{O}_{F'_n}(2-n)$ over F'_n. In fact if it were isomorphic to $\mathcal{O}_{F'_n}(n-2)$, then there would exist polynomials $a(x, y)$, $a'(x', y')$, $b(x, y)$, $b'(x', y')$ such that $a(x, y)$ and $a'(x', y')$ never vanish over $\mathbf{A}^2$ and such that

$$(4.24.7)\qquad \begin{pmatrix}-\left(\frac{1}{x'}\right)^{(n-2)} & 0\\ 0 & 1\end{pmatrix}=\begin{pmatrix}a(x, y) & b(x, y)\\ 0 & 1\end{pmatrix}\begin{pmatrix}-\left(\frac{1}{x'}\right)^{(n-2)} & ny'\left(\frac{1}{x'}\right)^{(n-1)}\\ 0 & 1\end{pmatrix}$$
$$\cdot\begin{pmatrix}a'(x', y') & b'(x', y')\\ 0 & 1\end{pmatrix}$$

where $x=1/x'$, $y=y'(1/x')^n$.

The condition on $a(x, y)$, $a'(x', y')$ implies they are constants a and a'. Thus we get from (4.24.7) $a=a'^{-1}$ and

$$(4.24.8) \qquad -ab'(x', y')\left(\frac{1}{x'}\right)^{(n-2)} + any'\left(\frac{1}{x'}\right)^{(n-1)} + b(x, y) = 0.$$

Since we have relations $x=1/x'$, $y=y'(1/x')^n$, (4.24.8) for polynomials $b'(x', y')$, $b(x, y)$ is impossible. Now the Sublemma follows from (3.8).

Lemma (4.25). $(\mathrm{Aut}^0_{\mathrm{F}_1'}\mathrm{E}_1'^2, \mathrm{E}_1')$ *is contained in* (PSO_4, *quadric* $\subset \mathbf{P}^4$).

Proof. As in Corollary (4.23), we have an SL_2-exact sequence for the unipotent radical U:

$$\begin{array}{c} 0 \to \bigoplus_{j=0}^{\infty} H^0(\mathrm{F}_1'^2, \mathcal{O}(-j)) \to U \to U_Z \to 0. \\ \quad \Big| \wr \\ U_1 \end{array}$$

If U were abelian, as we shall see in Proposition (5.30.7) the transformation space would be isomorphic to $\mathrm{F}_{1,2}'$. Hence U is not abelian. Using the notation of Bourbaki [Bk] let us consider a Lie subalgebra

$$\left\{ \begin{bmatrix} A & 0 & 0 \\ y & 0 & 0 \\ \begin{matrix} c & 0 \\ 0 & -c \end{matrix} & 2s^t y & -s^t A s \end{bmatrix} \in \mathfrak{so}_5 \,\middle|\, A \in M_2,\ y \in \mathbf{C}^2,\ c \in \mathbf{C} \right\} \text{ of } \mathfrak{so}_5.$$

Let G be the corresponding subgroup which is algebraic. Letting $\mathfrak{p} = \{(b_{ij}) \in \mathfrak{so}_5 \mid b_{21}=b_{31}=b_{41}=b_{51}=0$ (this implies $b_{51}=b_{52}=b_{53}=b_{54}=0)\}$ and P be the corresponding parabolic subgroup. The quadric in $\mathbf{P}^4$ of the Lemma is isomorphic to SO_5/P. G has an open orbit on SO_5/P and the unipotent radical of G is the non-abelian 3-dimensional unipotent group. Let $Z=$ the center of the unipotent radical $\tilde{U}$ of G. Then we have a morphism $(G, G/P\cap G) \to (G, G/(P\cap G)Z)$. The latter defines $(\mathrm{Aut}^0\mathrm{F}_1', \mathrm{F}_1')$ and the SL_2 exact sequence $0 \to Z \to \tilde{U} \to U/Z \simeq U_2 \to 0$. This $\mathbf{A}^1$-bundle $G/P\cap G \to \mathrm{F}_1'$ is not a line bundle. For otherwise the unipotent radical U of $\mathrm{Aut}^0(G/P\cap G, \mathrm{F}_1')$ is abelian. Consequently $(G, G/P\cap G)$ is isomorphic to $(\mathrm{Aut}^0\mathrm{E}_1'^2, \mathrm{E}_1'^2)$.

Let $\pi: C_1 \to C_2$ be an étale 2-covering of a non-singular (open) rational curve C_2. We always assume C_1 irreducible. We defined the algebraic operation (PGL_2, X_π) when genus $C_1 \geq 1$. But we can define the operation (PGL_2, X_π) even if genus $C_1 = 0$.

Proposition (4.26). *Let $\pi\colon C_1 \to C_2$ and $\pi'\colon C_1' \to C_2'$ be two étale 2-coverings of non-singular rational curves C_2 and C_2'. Then* (PGL_2, X_π) *and* (PGL_2, $X_{\pi'}$) *are isomorphic as law chunks of algebraic operation if π and π' are birationally equivalent, i.e. there exist birational maps $g\colon C_1 \cdots \to C_1'$ and $h\colon C_2 \cdots \to C_2'$ with $h \circ \pi = \pi' \circ g$.*

Proof. The "if" part is trivial. We prove the "only if" part. Let $(\varphi, f)\colon (\mathrm{PGL}_2, X_\pi) \to (\mathrm{PGL}_2, X_{\pi'})$ be an isomorphism of law chunks of algebraic operation. Since f is birational, there exist open subsets $U \subset C_2$ and $U' \subset C_2'$ such that f gives a PGL_2-equivariant biregular isomorphism between $p^{-1}(U)$ and $p'^{-1}(U')$ where $p\colon X_\pi \to C_2$, $p'\colon X_{\pi'} \to C_2'$ are projections (Rosenlicht [R]). f induces an isomorphism $f'\colon U \rightrightarrows U'$ since U, U' are PGL_2-quotients. Thus we may assume $C_2 = C_2'$ and $(\varphi, f)\colon (\mathrm{PGL}_2, X_\pi) \to (\mathrm{PGL}_2, X_{\pi'})$ is an isomorphism of algebraic operations. We show that p has local sections. In fact, let $B \subset \mathrm{PGL}_2$ be a Borel subgroup. Then B has 2-dimensional orbits on X_π. Let W be the union of 2-dimensional B-orbits on X_π. W is an open set of X_π. Since p is flat, $p(W)$ is open. $W \to p(W)$ is the quotient with B-operation and hence $W \to p(W)$ has local section because B is solvable [R]. In particular p has a local section. Let us put $s' = f \circ s$. s' is a local section of p'. Replacing $C_2 = C_2'$ by a smaller open set, we may assume s and s' are sections. Let $j\colon C_2 \to \mathrm{PGL}_2/\mathrm{D}_\infty = \{\text{subgroup conjugate to the diagonal torus } T\}$, $x \in C_2 \to$ stabilizer at $s(x)$. $\mathrm{PGL}_2/\mathrm{D}_\infty$ has an étale 2-covering $\mathrm{PGL}_2/T \to \mathrm{PGL}_2/\mathrm{D}_\infty$. As we assume that ($\mathrm{PGL}_2$, X_π) is not trivially fibred over C_2, the fibre product $C_2 \times_{\mathrm{PGL}_2/\mathrm{D}_\infty} \mathrm{PGL}_2/T$ is irreducible and an étale 2-covering of C_2. Since $X_\pi \times_{C_2} C_1 \simeq (\mathrm{PGL}_2/T) \times C_1$ by definition, the section s defines a section $\tilde{s}\colon C_1 \to X_\pi \times_{C_2} C_1 \simeq (\mathrm{PGL}_2/T) \times C_1$. Thus we get a morphism $\tilde{j}\colon C_1 \to \mathrm{PGL}_2/T$ by putting $\tilde{j}(x) = p_1 \circ \tilde{s}(x) \in \mathrm{PGL}_2/T$ for $x \in C_1$, where p_1 is the projection $(\mathrm{PGL}_2/T) \times C_1 \to \mathrm{PGL}_2/T$. Obviously the diagram

$$\begin{array}{ccc} C_1 & \xrightarrow{\tilde{j}} & \mathrm{PGL}_2/T \\ {\scriptstyle \pi}\downarrow & & \downarrow \\ C_2 & \xrightarrow{j} & \mathrm{PGL}_2/\mathrm{D}_\infty \end{array}$$

is commutative hence we get a C_2-morphism $C_1 \to C_2 \times_{\mathrm{PGL}_2/\mathrm{D}_\infty} \mathrm{PGL}_2/T$. Since they are étale 2-coverings of C_2, C_1 is C_2-isomorphic to

$$C_2 \times_{\mathrm{PGL}_2/\mathrm{D}_\infty} \mathrm{PGL}_2/T.$$

Using s' for s, we conclude C_1' is C_2'-isomorphic to

$$C_2' \times_{\mathrm{PGL}_2/\mathrm{D}_\infty} \mathrm{PGL}_2/T = C_2 \times_{\mathrm{PGL}_2/\mathrm{D}_\infty} \mathrm{PGL}_2/T.$$

Hence C_1 is isomorphic to C_1' over $C_2 = C_2'$ as desired.

It follows from Proposition (4.26)

Corollary (4.27). *Let $\pi: C_1 \to C_2$ be an étale 2-covering of a curve C_2 (with C_1 irreducible and C_2 non-singular rational). Let g be the genus of C_1. Assume $g \geq 1$. Then the operation* (PGL_2, X_π) *is effectively and completely parametrized by the moduli space of hyperelliptic* (*or elliptic if* $g=1$) *curves of genus g.*

Lemma (4.28). *Let π: $C_1 \to C_2$ be an étale 2-covering with C_1 rational and C_2 non-singular. Then the conjugacy class of* (PGL_2, X_π) *is contained in that of* (PGL_4, $\mathbf{P}^3$).

Proof. Let V_3 be an irreducible SL_2-module of dimension 3. We identify V_3 with the vector space of homogeneous polynomials of degree 2 in x, y. (SL_2, V_3) is almost effective and defines by (PGL_2, V_3). This operation is linear hence contained in (PGL_4, $\mathbf{P}^3$). We show (PGL_2, V_3) is isomorphic to (PGL_2, X_π) as law chunks of algebraic operation. The discriminant $D(f) = b^2 - 4ac$ for $f(x, y) = ax^2 + bxy + cy^2$ is SL_2-invariant. (SL_2, V_3-$\{D=0\}$) consists of 2-dimensional orbits isomorphic to SL_2/T and the quotient is given by D: (SL_2, V_3-$\{D=0\}$)$\to$($\mathbf{1}$, $\mathbf{C}^*$), $f(x, y) \mapsto D(f)$. We show (PGL_2, V_3-$\{D=0\}$) is isomorphic to (PGL_2, X_π) as law chunk of algebraic operations. If we take

$$T = \left\{\begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}\right\} \subset \mathrm{SL}_2, \qquad T = \{g \in \mathrm{SL}_2 \mid g(xy) = xy\}.$$

Therefore $\mathrm{SL}_2/T = \{f(x, y) \in V_3 \mid D(f) = 1\}$. Letting PGL_2 operate on $\mathbf{C}^* \times \mathrm{PGL}_2/T$ through the second factor, consider a PGL_2-equivariant map ψ: $\mathbf{C}^* \times \mathrm{PGL}_2/T = \mathbf{C}^* \times \{f \in V_3 \mid D(f) = 1\} \to V_3$-$\{D=0\}$, $\psi((c, f)) = cf$ for $(c, f) \in \mathbf{C}^* \times \{f \in V_3 \mid D(f) = 1\}$. We thus get a commutative diagram:

$$\begin{array}{ccc}
\mathbf{C}^* \times \mathrm{PGL}_2/T & \xrightarrow{\psi} & V_3\text{-}\{D=0\} \\
{\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle D} \\
\mathbf{C}^* & \longrightarrow & \mathbf{C}^*. \\
c & \longmapsto & c^2
\end{array}$$

We notice that for $(c, f), (c', f') \in \mathbf{C}^* \times \{f \in V_3 \mid D(f) = 1\}$, $\psi((c, f)) = \psi((c', f'))$ if and only if $c = c', f = f'$ or $c = -c', f = -f'$. Let π': $\mathbf{C}^* \to \mathbf{C}^*$, $\pi'(c) = c^2$. (PGL_2, $X_{\pi'}$) is defined by the descent datum on $\mathbf{C}^* \times$

PGL_2/T in (3.9). We identify PGL_2/T with $\{f \in V_3 \mid D(f)=1\}$ by $gT \to g\cdot xy$. By this identification the descent data in (3.9) is $(c, f) \to (-c, -f)$. Therefore $(PGL_2, V_3\text{-}\{D=0\})$ is isomorphic to $(PGL_2, X_{\pi'})$ as algebraic operations. The Lemma follows from Proposition (4.26).

Lemma (4.29). $(\mathrm{Aut}^0 F'_{1,1}, F'_{1,1})$ *is contained in* $(PGL_4, \mathbf{P}^3)$,

Proof. It follows from Proposition (4.19) and Corollary (4.20), $\mathrm{Aut}^0 F'_{1,1}$ is isogeneous to the semi-direct product $(U_2 \otimes U_2)(\mathbf{G}_m \times SL_2 \times SL_2)$, where U_2 is the irreducible SL_2-module of dimension 2 and $U_2 \otimes U_2$ is regarded as a $\mathbf{G}_m$-module of weight 1. Let us consider a subgroup $\mathscr{G} = \{(a_{ij}) \in GL_4 \mid a_{13}=a_{14}=a_{23}=a_{24}=0\}$ of GL_4. Let us put $P=\{(a_{ij}) \in GL_4 \mid a_{21} =a_{31}=a_{41}=0\}$. Then let us prove the open $\mathscr{G}$-orbit $\mathscr{G}P \subset GL_4/P \simeq \mathbf{P}^3$ is isomorphic to $F'_{1,1}$. In fact, $\mathscr{G}' = \left\{\begin{pmatrix} A & 0 \\ C & 1_2 \end{pmatrix} \in GL_4 \mid A \in SL_2,\ C \in M_{2\times 2}\right\}$ has the same open orbit as $\mathscr{G}P$ on $\mathbf{P}^3$. $\mathscr{G}'P = \mathscr{G}'/\mathscr{G}' \cap P$ is isomorphic to $F'_{1,1}$ as in Section 2 [U3]. The image of $\mathscr{G}$ in PGL_4 operates on $\mathscr{G}P$ as $(\mathrm{Aut}^0 F'_{1,1}, F'_{1,1})$ and Lemma is proved.

Lemma (4.30). *The conjugacy classes of the operations* $(\mathbf{G}_m \times SL_2 \times SL_2, \mathbf{G}_m \times SL_2 \times SL_2/H_{n,-1})$ *and* $(\mathbf{G}_m \times SL_2 \times SL_2, \mathbf{G}_m \times SL_2 \times SL_2/H_{1,-n})$ $(n \geqq 1)$ *are contained in that of* $(\mathrm{Aut}^0 F'_{n,n}, F'_{n,n})$. *In particular they are contained in one of the conjugacy classes of Theorem* (2.1).

Proof. Let us put $G = \mathbf{G}_m \times SL_2 \times SL_2$. The rational map $G \times \mathbf{A}^3 \to \mathbf{A}^3$ sending $\left(t, \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}; x, y, z\right) \in \mathbf{G}_m \times SL_2 \times SL_2 \times \mathbf{A}^3$ to

$$\left(\frac{ax+b}{cx+d}, \frac{t(a'y+b'z)}{(cx+d)^n}, \frac{t(c'y+d'z)}{(cx+d)^n}\right)$$

defines a law chunk of algebraic operation (G, X_n). The stabilizer at $(0, 0, 1)$ is $K_n = \left\{\left(t, \begin{pmatrix} a & 0 \\ c & a^{-1} \end{pmatrix}, \begin{pmatrix} a' & 0 \\ c' & a'^{-1} \end{pmatrix}\right) \in G \mid t = a^{-n}a'\right\}$. By Corollary (4.20), we thus obtained a morphism of law chunks of algebraic operations $(\varphi, f)\colon (G, G/K_n) \to (\mathrm{Aut}^0 F'_{-n,-n}, F'_{-n,-n})$ with f birational, which is necessarily a morphism of algebraic operations since $(G, G/K_n)$ is a homogeneous space. The homogeneous spaces $(G, G/H_{n,-1})$ and $(G, G/H_{1,-n})$ are both isomorphic to $(G, G/H_n)$. The last assertion follows from the definition and Lemma (4.29).

§ 5. Proof of the first assertion of main Theorem (2.2)

Theorem (2.2) has been proved either if G is primitive [U1] or if G is not of de Jonquières type but imprimitive [U2]. Therefore it is sufficient to prove the first assertion of the main theorem under the assumption that G is of de Jonquières type. Hence throughout this section, we assume that G is of de Jonquières type. We verify the theorem by dividing into two cases i.e., *the generically transitive case* and *the generically intransitive case*. Moreover each case is divided into several subcases.

Classification of generically transitive algebraic groups in Cr_3.
This case is divided into two subcases; (i) *the group is reductive.* (ii) *the group is not reductive.*

Case (i). *G is reductive.*
Let G be a reductive group in Cr_3. Then by 1.6, Corollaire 1, Demazure [D], the rank of G is at most 3. Let $(G, G/H)$ be a realization of G. Then there exists an isogeny $\varphi: \tilde{G}\to G$ such that $\tilde{G}$ is isomorphic to the direct product $\mathbf{G}_m^r\times\tilde{G}_{ss}$, where $\tilde{G}_{ss}$ is semi-simple and simply connected and such that the restriction of φ onto $\mathbf{G}_m^r\times 1$ is an isomorphism. Hence if we put $\tilde{H}=\varphi^{-1}(H)$, then $(\tilde{G}, \tilde{G}/\tilde{H})$ satisfies the following conditions.

(5.1) (1) $\tilde{G}\simeq\mathbf{G}_m^r\times\tilde{G}_{ss}$ *where* $\tilde{G}_{ss}$ *is semi-simple and simply connected,*
(2) $(\tilde{G}, \tilde{G}/\tilde{H})$ *is almost effective, of de Jonquières type and* $\dim\tilde{G}/\tilde{H}=3$.
(3) $\mathbf{G}_m^r\times 1$ *operates effectively on* $\tilde{G}/\tilde{H}$.

Let us determine all the algebraic operations $(\tilde{G}, \tilde{G}/\tilde{H})$ satisfying the conditions (5.1).

(5.2) *Here is the list of all the reductive groups* $\tilde{G}$ *satisfying the condition* (5.1) (1) *and* rank $\tilde{G}\le 3$:
(i) rank 3

SL_4, $\widetilde{\mathrm{SO}_7}$, Sp_6, $\widetilde{\mathrm{SO}_6}$,
$\mathrm{SL}_2\times\mathrm{SL}_3$, $\mathrm{SL}_2\times\widetilde{\mathrm{SO}_5}$, $\mathrm{SL}_2\times\mathrm{G}_2$,
$\mathbf{G}_m\times\mathrm{SL}_3$, $\mathbf{G}_m\times\widetilde{\mathrm{SO}_5}$, $\mathbf{G}_m\times\mathrm{G}_2$
$\mathrm{SL}_2\times\mathrm{SL}_2\times\mathrm{SL}_2$, $\mathbf{G}_m\times\mathrm{SL}_2\times\mathrm{SL}_2$, $\mathbf{G}_m\times\mathbf{G}_m\times\mathrm{SL}_2$,
$\mathbf{G}_m\times\mathbf{G}_m\times\mathbf{G}_m$

(ii) rank 2

SL_3, $\widetilde{\mathrm{SO}_5}$, G_2, $\mathrm{SL}_2\times\mathrm{SL}_2$, $\mathbf{G}_m\times\mathrm{SL}_2$, $\mathbf{G}_m\times\mathbf{G}_m$

(iii) rank 1

SL_2, $\mathbf{G}_m$,

where $\tilde{G}$ *denotes the universal covering group of* G.

Since $\mathbf{G}_m\times\mathbf{G}_m$ and $\mathbf{G}_m$ can not be transitive, we can erase them from the list.

Lemma (5.3). *Let G be a simple algebraic groups with* rank $G=n$. *If G is contained in the Cremona group* Cr_n *of n-variables, then G is not of de Jonquières type.*

Proof. As rank $G=n$, an operation of a maximal torus of G on a variety of dimension $\leq n-1$ is not almost effective (Demazure [D])) and hence an operation of G is not almost effective either. Since G is simple, it follows that any operation of G on a variety of dim $\leq n-1$ is trivial. Let (G, X) be a realization of G. Since rank $G=n$, by Demazure [D] the operation of maximal torus of G on X is generically transitive and hence the operation (G, X) is generically transitive. Hence we may assume (G, X) is a homogeneous space $(G, G/H)$. Since, as we have seen, G acts only trivially on any variety of dimension $\leq n-1$, $(G, G/H)$ is not of de Jonquières type (Proposition 2.2, [U2]).

By Lemma (5.3), we can erase SL_4, $\widetilde{\mathrm{SO}_7}$, Sp_6, $\widetilde{\mathrm{SO}_6}$ in the list (5.2).

Lemma (5.4). *If $\tilde{G}$ is isomorphic to* $\mathrm{SL}_2\times\widetilde{\mathrm{SO}_5}$, $\mathbf{G}_m\times\widetilde{\mathrm{SO}_5}$ *or* $\widetilde{\mathrm{SO}_5}$, *there is no closed subgroup $\tilde{H}$ satisfying the conditions* (5.1).

Proof. Let us first assume $\tilde{G}=\widetilde{\mathrm{SO}_5}$. If there were a closed subgroup $\tilde{H}$ of $\tilde{G}$ satisfying the conditions (5.1), then there would be a closed subgroup $\tilde{K}$ of $\tilde{G}$ such that dim $\tilde{G}/\tilde{K}=2$ or 1. By a theorem of Enriques (cf. Umemura [U3]), $\widetilde{\mathrm{SO}_5}$ never operates non-trivially on any two dimensional variety. Hence we may assume dim $\tilde{G}/\tilde{K}=1$. But this is impossible, because there is no non-trivial morphism of algebraic groups $\widetilde{\mathrm{SO}_5}\to\mathrm{Aut}(\tilde{G}/\tilde{K})\subset\mathrm{Aut}\,\mathbf{P}^1$. Since the cases $\tilde{G}=\mathrm{SL}_2\times\widetilde{\mathrm{SO}_5}$, $\tilde{G}=\mathbf{G}_m\times\widetilde{\mathrm{SO}_5}$ are treated similarly, the proof is given only when $\tilde{G}=\mathrm{SL}_2\times\widetilde{\mathrm{SO}_5}$. If there were a closed subgroups $\tilde{H}$, $\tilde{K}$ of $\tilde{G}$ satisfying the conditions (5.1). Since $\widetilde{\mathrm{SO}_5}$ operates trivially on any variety of dimension ≤ 2, we get $(\varphi, f)\colon(\mathrm{SL}_2\times\widetilde{\mathrm{SO}_5}, \tilde{G}/\tilde{H})\to(\mathrm{SL}_2\times\widetilde{\mathrm{SO}_5}, \tilde{G}/\tilde{K})\to(\mathrm{SL}_2, \tilde{G}/\tilde{K})$ and $\widetilde{\mathrm{SO}_5}$ operates on the fibre of f. But since the dimension of the fibres of $f\leq 2$, $\widetilde{\mathrm{SO}_5}$ operates trivially also on the fibres and hence $\widetilde{\mathrm{SO}_5}$ operates trivially on $\tilde{G}/\tilde{H}$ which contradicts the almost effectivity of the operation $(\tilde{G}, \tilde{G}/\tilde{H})$.

Lemma (5.5). *If $\tilde{G}$ is isomorphic to* $\mathrm{SL}_2\times\mathrm{G}_2$, $\mathbf{G}_m\times\mathrm{G}_2$, *or* G_2, *there is no closed subgroup $\tilde{H}$ satisfying the conditions* (5.1).

Proof is the same as for Lemma (5.4).

Now, we determine all the closed subgroups $\tilde{H}$ satisfying the conditions (5.1) of the remaining groups of the list (5.2).

$$\tilde{G}=\mathrm{SL}_2\times\mathrm{SL}_3. \tag{5.6}$$

Our first aim is to show $\tilde{H}$ is parabolic. Since $(\tilde{G}, \tilde{G}/\tilde{H})$ is of de Jonquières type, there exists a morphism of homogeneous spaces (Id, f): $(\tilde{G}, \tilde{G}/\tilde{H})\to(\tilde{G}, \tilde{G}/\tilde{K})$. If $\dim \tilde{G}/\tilde{K}=1$, necessarily we have a morphism (p_1, f): $(\mathrm{SL}_2\times\mathrm{SL}_3, \mathrm{SL}_2\times\mathrm{SL}_3/\tilde{H})\to(\mathrm{SL}_2, \mathrm{SL}_2/B)$ where B is a Borel subgroup of SL_2. Hence we get an extension:

$$\mathbf{1}\to\mathbf{1}\times\mathrm{SL}_3\cap\tilde{H}\to\tilde{H}\xrightarrow{p_1}B. \tag{5.7}$$

$\mathrm{SL}_3\times\mathbf{1}$ operates on the fibre of p_1 which is 2-dimensional. By a Theorem of Enriques [U3], a 2-dimensional operation of SL_3 is necessarily $(\mathrm{SL}_3, \mathbf{P}^2)$ hence the fibre of f is isomorphic to $\mathbf{P}^2$ and $\mathbf{1}\times\mathrm{SL}_3/\mathbf{1}\times\mathrm{SL}_3\cap\tilde{H}=\mathbf{P}^2$. Therefore $\mathbf{1}\times\mathrm{SL}_3\cap\tilde{H}$ is of dimension 6 and contains a solvable group of dimension 5. Hence the dimension of the image $p_1(\tilde{H})$ is equal to $\dim\tilde{H}-\dim\mathbf{1}\times\mathrm{SL}_2\cap\tilde{H}=2$. Thus we get an exact sequence

$$\mathbf{1}\to\mathbf{1}\times\mathrm{SL}_3\cap\tilde{H}\to\tilde{H}\xrightarrow{p_1}B\to\mathbf{1}. \tag{5.8}$$

$\tilde{H}$ contains a solvable group of dimension $5+2=7$ which is an extension of B by a maximal solvable group in $\mathbf{1}\times\mathrm{SL}_3\cap\tilde{H}$. A solvable subgroup of $\mathrm{SL}_2\times\mathrm{SL}_3$ of dimension 7 is a Borel subgroup of $\mathrm{SL}_2\times\mathrm{SL}_3$. Hence $\tilde{H}$ is parabolic.

If $\dim\tilde{G}/\tilde{K}=2$, we have either $(\mathrm{SL}_2\times\mathrm{SL}_3, \mathrm{SL}_2\times\mathrm{SL}_3/\tilde{H})\to(\mathrm{SL}_3, \mathbf{P}^2)$ or $(\mathrm{SL}_2\times\mathrm{SL}_3, \mathrm{SL}_2\times\mathrm{SL}_3/\tilde{H})\to(\mathrm{SL}_2, \mathrm{SL}_2/H')$ where H' is a closed subgroup of dimension 1. In the last case, $\mathbf{1}\times\mathrm{SL}_3$ operates non-trivially on the fibre which is a curve but this is impossible and hence the last case never occurs. Let us now examine the morphism $(\mathrm{SL}_2\times\mathrm{SL}_3, \mathrm{SL}_2\times\mathrm{SL}_3/\tilde{H})\to(\mathrm{SL}_3, \mathbf{P}^2)$ and show $\tilde{H}$ is solvable. As in the $(\mathrm{SL}_2, \mathbf{P}^1)$ case, we get an exact sequence

$$\mathbf{1}\to\mathrm{SL}_2\times\mathbf{1}\cap\tilde{H}\to\tilde{H}\xrightarrow{p_2}P$$

where P is a parabolic subgroup of SL_3 such that $\mathrm{SL}_3/P\simeq\mathbf{P}^2$. $\mathrm{SL}_2\times\mathbf{1}$ operates non-trivially on the fibre of $\mathrm{SL}_2\times\mathrm{SL}_3/\tilde{H}\to\mathbf{P}^2$. Hence $\mathrm{SL}_2\times\mathbf{1}/\mathrm{SL}_2\times\mathbf{1}\cap\tilde{H}\subset$ the fibre of $\mathrm{SL}_2\times\mathrm{SL}_3/\tilde{H}$ and consequently $\dim\mathrm{SL}_2\times\mathbf{1}\cap\tilde{H}=2$. The dimension of the image $p_2(\tilde{H})=6$ and hence p_2 is surjective and we get an exact sequence:

$$\mathbf{1}\to\mathrm{SL}_2\times\mathbf{1}\cap\tilde{H}\to\tilde{H}\to P\to\mathbf{1}.$$

For the same reason as in the case $\dim\tilde{G}/\tilde{K}=1$, $\tilde{H}$ is parabolic. Suppose that $\tilde{H}\subset\mathrm{SL}_2\times\mathrm{SL}_3$ satisfies the conditions (5.1). The $\tilde{H}$ is parabolic and of dimension 8. It follows $\tilde{H}$ is $B \times P \subset SL_2 \times SL_3$ where B is a Borel subgroup of SL_2 and P is a parabolic subgroup of SL_3 such that $SL_3/P \simeq \mathbf{P}^2$. We have thus proved.

Proposition (5.9). *Let* $(SL_2 \times SL_3,\ SL_2 \times SL_3/\tilde{H})$ *be an almost effective realization of a de Jonquières type group* G *of* Cr_3. *Then* G *is realized by* $(PGL_2 \times PGL_3,\ \mathbf{P}^1 \times \mathbf{P}^2)$.

(5.10) $$\tilde{G} = \mathbf{G}_m \times SL_3.$$

Let $\tilde{H}$ be a closed subgroup of $\tilde{G}$ satisfying the conditions (5.1). Since $(\tilde{G}, \tilde{G}/\tilde{H})$ is almost effective, contains no normal subgroup of positive dimension. In particular $\mathbf{G}_m \times \mathbf{1}$ is not contained in $\tilde{H}$. Let us put $\tilde{K} = \mathbf{G}_m \times \mathbf{1} \cdot \tilde{H}$. We get a morphism $(\tilde{G}, \tilde{G}/\tilde{H}) \to (\tilde{G}, \tilde{G}/\tilde{K})$. Since $\mathbf{G}_m \times \mathbf{1}$ operates trivially on $\tilde{G}/\tilde{K}$, we get a morphism

$$(p_2, f) \colon (\mathbf{G}_m \times SL_3,\ \mathbf{G}_m \times SL_3/\tilde{H}) \to (SL_3,\ \mathbf{G}_m \times SL_3/\tilde{K}).$$

By a Theorem of Enriques [U3], $\mathbf{G}_m \times SL_3/\tilde{K}$ is isomorphic to $SL_3/P \simeq \mathbf{P}^2$. Hence, we have a morphism

$$(p_2, f) \colon (\mathbf{G}_m \times SL_3,\ \mathbf{G}_m \times SL_3/\tilde{H}) \to (SL_3,\ SL_3/P).$$

The existence of the morphism (p_2, f) implies $p_2(\tilde{H}) \subset P$. Since $\mathbf{G}_m \times \mathbf{1}$ operates on $\mathbf{G}_m \times SL_3/\tilde{H}$ effectively by our assumption, $\mathbf{G}_m \times \mathbf{1} \cap \tilde{H} = \mathbf{1}$. Hence by counting the dimension, $p_2(\tilde{H}) = P$. Namely p_2 maps $\tilde{H}$ isomorphically onto P. We have thus proved

Proposition (5.11). *Let* $(\mathbf{G}_m \times SL_3,\ \mathbf{G}_m \times SL_3/\tilde{H})$ *be an almost effective realization of an algebraic group* G *of de Jonquières type contained in* Cr_3. *Then, there exists an integer* l *such that* $\tilde{H}$ *is, up to inner automorphism,*

$$\left\{ t^l \times \begin{pmatrix} t & x & y \\ 0 & & \\ 0 & \multicolumn{2}{c}{A} \end{pmatrix} \in \mathbf{G}_m \times SL_3 \,\middle|\, t \in k^*,\ x, y \in k,\ A \in GL_2(k) \right\}.$$

G is contained in $(\mathrm{Aut}^0 J'_{|l|},\ J'_{|l|})$ *for any* $l \in \mathbf{Z}$. *If* $|l| = 1$, *G is contained in* $(PGL_4,\ \mathbf{P}^3)$.

Proof. Except for the last two assertions, Proposition (5.11) is proved above. $\mathbf{G}_m \times SL_3/\tilde{H}$ is a principal $\mathbf{G}_m$-bundle of degree l over $\mathbf{P}^2$ and $\mathbf{G}_m \times SL_3$ is in the automorphism group of $J'_{|l|}$. The last assertion follows from the inclusion $(\mathrm{Aut}^0 J'_1,\ J'_1) \subset (PGL_4,\ \mathbf{P}^3)$ because J'_1 is (a blowing-up of $\mathbf{P}^3$ at a point P) $-$ (the inverse image of P) $= \mathbf{P}^3 - \{P\}$.

(5.12) $$\tilde{G} = SL_2 \times SL_2 \times SL_2$$

Let $\tilde{H}$ and $\tilde{K}$ be closed subgroups of $\tilde{G}$ satisfying the conditions (5.1). We shall show $\tilde{H}$ is solvable. If $\dim \tilde{G}/\tilde{K}=1$, then we have a morphism $(\varphi, f)\colon (\tilde{G}, \tilde{G}/\tilde{H}) \to (\mathrm{SL}_2, \mathrm{SL}_2/B)$. We may assume φ is the projection p_1 onto the first factor. $\mathbf{1}\times\mathrm{SL}_2\times\mathrm{SL}_2$ operates non-trivially on the fibres of f which are isomorphic to $B\times\mathrm{SL}_2\times\mathrm{SL}_2/\tilde{H}$. By a Theorem of Enriques [U3], the operation of $\mathbf{1}\times\mathrm{SL}_2\times\mathrm{SL}_2$ onto the fibres of f is $(\mathrm{SL}_2\times\mathrm{SL}_2, \mathrm{SL}_2/B\times\mathrm{SL}_2/B)$ where B is a Borel subgroup of SL_2. Therefore, we may assume $\mathbf{1}\times\mathrm{SL}_2\times\mathrm{SL}_2\cap\tilde{H}=\mathbf{1}\times B\times B$. It now follows from the exact sequence

$$\begin{array}{c} \mathbf{1}\to\mathbf{1}\times\mathrm{SL}_2\times\mathrm{SL}_2\cap\tilde{H}\to\tilde{H}\xrightarrow{p_1}B, \\ \| \\ \mathbf{1}\times B\times B \end{array}$$

$\tilde{H}$ is solvable. Let us now assume $\dim\tilde{G}/\tilde{K}=2$. By a Theorem of Enriques [U3], we have either (1) a morphism $(\varphi, f)\colon (\tilde{G}, \tilde{G}/\tilde{H})\to(\mathrm{SL}_2\times\mathrm{SL}_2, \mathrm{SL}_2/B\times\mathrm{SL}_2/B)$, where B is a Borel subgroup of SL_2 or (2) a morphism $(\varphi, f)\colon (\tilde{G}, \tilde{G}/\tilde{H})\to(\mathrm{SL}_2, \mathrm{SL}_2/H')$ where H' is a 1-dimensional subgroup of SL_2. We notice the second case never happens. For, if it happened, $\mathrm{Ker}\,\varphi\simeq\mathrm{SL}_2\times\mathrm{SL}_2$ cannot operate almost effectively on a fibre of f which is isomorphic to $\mathbf{P}^1$ hence $(\tilde{G}, \tilde{G}/\tilde{H})$ would not be almost effective. In the first case, we may assume φ is the projection p_{12} onto the first two factors. Then the similar argument as above shows the existence of exact sequence:

$$\begin{array}{c} \mathbf{1}\to\mathbf{1}\times\mathbf{1}\times\mathrm{SL}_2\cap\tilde{H}\to\tilde{H}\xrightarrow{p_2}B\times B\times\mathbf{1}\to\mathbf{1}, \\ \| \\ \mathbf{1}\times\mathbf{1}\times B \end{array}$$

and $\tilde{H}$ is solvable. Since $\dim\tilde{H}=6$, $\tilde{H}$ is, up to an inner automorphism, $B\times B\times B$ where B is a Borel subgroup.

Proposition (5.12). *Let* $(\mathrm{SL}_2\times\mathrm{SL}_2\times\mathrm{SL}_2, \mathrm{SL}_2\times\mathrm{SL}_2\times\mathrm{SL}_2/\tilde{H})$ *be an almost effective realization of an algebraic group G of de Jonquières type contained in* Cr_3. *Then G is effectively realized by* $(\mathrm{PGL}_2\times\mathrm{PGL}_2\times\mathrm{PGL}_2, \mathbf{P}^1\times\mathbf{P}^1\times\mathbf{P}^1)$

(5.13) $$\tilde{G}=\mathbf{G}_m\times\mathrm{SL}_2\times\mathrm{SL}_2.$$

Let $\tilde{H}$ be a closed subgroup of $\tilde{G}$ satisfying the conditions (5.1). Since $(\tilde{G}, \tilde{G}/\tilde{H})$ is almost effective, $\tilde{H}$ does not contain any normal subgroup of positive dimension. In particular, $\mathbf{G}_m\times\mathbf{1}\times\mathbf{1}$ is not contained in $\tilde{H}$. If we put $\tilde{K}=\mathbf{G}_m\times\mathbf{1}\times\mathbf{1}\cdot\tilde{H}$, then $\dim\tilde{G}/\tilde{K}=2$. We have a morphism

$(\tilde{G}, \tilde{G}/\tilde{H}) \rightarrow (\tilde{G}, \tilde{G}/\tilde{K})$ and $\mathbf{G}_m \times \mathbf{1} \times \mathbf{1}$ operates trivially on $\tilde{G}/\tilde{K}$ and consequently by a Theorem of Enriques, we get a morphism $(p_{23}, f)\colon (\tilde{G}, \tilde{G}/\tilde{H}) \rightarrow (\mathrm{SL}_2 \times \mathrm{SL}_2, \mathrm{SL}_2/B \times \mathrm{SL}_2/B)$, where B is a Borel subgroup of SL_2 and p_{23} is the projection onto the second and third factor of $\tilde{G}$. Since $\mathbf{G}_m \times \mathbf{1} \times \mathbf{1}$ operates effectively on $\tilde{G}/\tilde{H}$ and is contained in the center of $\tilde{G}$, $\mathbf{G}_m \times \mathbf{1} \times \mathbf{1} \cap \tilde{H} = 1$.

Thus we get an exact sequence;

$$\begin{array}{c} \mathbf{1} \rightarrow \mathbf{G}_m \times \mathbf{1} \times \mathbf{1} \cap \tilde{H} \rightarrow \tilde{H} \xrightarrow{p_{23}} B \times B. \\ \Vert \\ \mathbf{1} \end{array}$$

By counting the dimension, we know that p_{23} induces an isomorphism of $\tilde{H}$ onto $B \times B$.

Proposition (5.14). *Let* $(\mathbf{G}_m \times \mathrm{SL}_2 \times \mathrm{SL}_2, \mathbf{G}_m \times \mathrm{SL}_2 \times \mathrm{SL}_2/\tilde{H})$ *be an almost effective realization of an algebraic group G of de Jonquières type contained in* Cr_3 *as in* (5.1). *Then, there exist integers l_1, l_2 such that $\tilde{H}$ is, up to an inner automorphism,*

$$\left\{ t_1^{l_1} t_2^{l_2} \times \begin{pmatrix} t_1 & x \\ 0 & t_1^{-1} \end{pmatrix} \times \begin{pmatrix} t_2 & y \\ 0 & t_2^{-1} \end{pmatrix} \in \mathbf{G}_m \times \mathrm{SL}_2 \times \mathrm{SL}_2 \mid t_1, t_2 \in k^*, x, y \in k \right\}.$$

Interchanging the order of SL_2*-factors and replacing the parameter of* $\mathbf{G}_m$ *by its inverse if necessary, we may assume that $l_1 < 0 < l_2$ or $0 \le l_1 \le l_2$. Then G is contained in;*

(5.14.1) *the operation* (J6) *of Theorem* (2.2) *for* $m = l_2$, $n = l_1$ *if* $l_2 \ge 2$, $-2 \ge l_1$,

(5.14.2) *the operation* (J8) *of Theorem* (2.2) *for* $m = l_2$, $n = l_1$ *if* $l_2 \ge l_1 > 0$,

(5.14.3) *the operation* (J3) *for* $m = l_2$ *if* $l_2 \ge 2$, $l_1 = 0$,

(5.14.4) *the operation* (J1) *if* $l_1 = 0$, $l_2 = 1$,

(5.14.5) *the operation* (J2) *if* $l_1 = l_2 = 0$,

(5.14.6) *the operation* (J10) *if* $l_2 > 0 > l_1$ *and if only one of* $|l_1|$ *and* $|l_2| = 1$,

(5.14.7) *the operation* (P1) *if* $l_2 = 1$, $l_1 = -1$.

Proof. We proved above $\tilde{H}$ is written in the form of Proposition. As we saw above, G/H is a principal $\mathbf{G}_m$-bundle of bidegree (l_1, l_2) over $\mathrm{SL}_2 \times \mathrm{SL}_2/B \times B = \mathbf{P}^1 \times \mathbf{P}^1$ and G is a group of $\mathbf{G}_m$-bundle isomorphisms which can be extended to a group of line bundle of bidegree (l_1, l_2). Thus (5.14.1), (5.14.2), (5.14.6) and (5.14.7) follow from Lemma (4.29) and

(4.30). (5.14.3), (5.14.4) and (5.14.5) follow from [U3] because in these cases the operations are products.

(5.15) $$\tilde{G}=\mathbf{G}_m\times\mathbf{G}_m\times\mathrm{SL}_2$$

Let $\tilde{H}$ be a closed subgroup of $\tilde{G}$ satisfying the conditions (5.1). Since a normal subgroup $\mathbf{G}_m\times\mathbf{G}_m\times\mathbf{1}$, contained in the center of $\tilde{G}$, operates effectively on $\tilde{G}/\tilde{H}$, $\mathbf{G}_m\times\mathbf{G}_m\times\mathbf{1}\times\tilde{H}=\mathbf{1}$. Thus, we get an exact sequence

$$\begin{array}{c} 1\to\mathbf{G}_m\times\mathbf{G}_m\times\mathbf{1}\cap\tilde{H}\to\tilde{H}\xrightarrow{p_3}\mathrm{SL}_2. \\ \| \\ \mathbf{1} \end{array}$$

Hence p_3 induces an isomorphism of $\tilde{H}$ onto a closed subgroup of dimension 2 hence onto a Borel subgroup of SL_2. Thus we have shown

Proposition (5.16). *Let* $(\mathbf{G}_m\times\mathbf{G}_m\times\mathrm{SL}_2,\ \mathbf{G}_m\times\mathbf{G}_m\times\mathrm{SL}_2/\tilde{H})$ *be an almost effective realization of an algebraic group G of de Jonquières type contained in* Cr_3 *as in* (5.1). *Then, there exist integers l_1, l_2 such that $\tilde{H}$ is, up to an inner automorphism,*

$$\left\{t^{l_1}\times t^{l_2}\times\begin{pmatrix} t & x \\ 0 & t^{-1}\end{pmatrix}\in\mathbf{G}_m\times\mathbf{G}_m\times\mathrm{SL}_2 \,\middle|\, t\in k^*,\ x\in k\right\}.$$

Interchanging the order of $\mathbf{G}_m$ *factors and replacing the parameters of* $\mathbf{G}_m$ *by their inverses, we may assume* $0\le l_1\le l_2$. **G** *is contained in*

(5.16.1) *the operation* (J9) *of Theorem* (2.2) *for* $l_2=m$, $l_1=n$ *if* $2\le l_1<l_2$,

(5.16.2) *the operation* (J10) *of Theorem* (2.2) *for* $l_2=m=l_1=n$ *if* $2\le l_1=l_2$,

(5.16.3) *the operation* (J7) *of Theorem* (2.2) *for* $m=l_2$, *if* $l_2\ge 2$, $l_1=1$,

(5.16.4) *the operation* (P1) *of Theorem* (2.2) *for* $l_1=l_2=1$,

(5.16.5) *the operation* (J3) *of Theorem* (2.2) *for* $m=l_2$ *if* $l_2\ge 2$, $l_1\ge 0$,

(5.16.6) *the operation* (J1) *of Theorem* (2.2) *if* $l_2=1$, $l_1=0$,

(5.16.7) *the operation* (J2) *of Theorem* (2.2) *if* $l_1=l_2=0$.

Proof. The first assertion was proved above. $\tilde{G}/\tilde{H}$ is a principal $\mathbf{G}_m^2$-bundle over $\mathrm{SL}_2/B=\mathbf{P}^1$ of degree (l_1, l_2). G is a group of automorphisms of the principal $\mathbf{G}_m^2$-bundle hence can be extended to a group of automorphisms of F'_{l_2,l_1}. For some small values of l_1, l_2, F'_{l_1,l_2} is isomorphic to other varieties (cf. [U3] and Lemma (4.29)).

(5.17) $$\tilde{G}=\mathbf{G}_m\times\mathbf{G}_m\times\mathbf{G}_m.$$

There is nothing to prove in this case. $\mathbf{G}_m^3$ is always contained in $(\mathrm{PGL}_4, \mathbf{P}_3)$.

(5.18) $$\tilde{G}=\mathrm{SL}_3.$$

Let $\tilde{H}$ be closed subgroup of $\tilde{G}$ satisfying the conditions (5.1). Since $(\tilde{G}, \tilde{G}/\tilde{H})$ is of de Jonquières type, there exists a closed subgroup $\tilde{K}$ such that $\tilde{H}\subset\tilde{K}$, $\dim \tilde{G}/\tilde{K}=1, 2$ (cf. Corollary 2.8, [U2]). Since SL_3 does not operate on a 1-dimensional variety, $\dim \tilde{G}/\tilde{K}=2$. By a Theorem of Enriques, $\tilde{K}$ is a parabolic subgroup of SL_3 such that $\mathrm{SL}_3/\tilde{K}=\mathbf{P}^2$. We may assume $\tilde{K}=\left\{\begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix}\in \mathrm{SL}_3\right\}$. $\tilde{H}$ is contained in $\tilde{K}$ and of dimension 5.

Lemma (5.19). *$\tilde{H}$ is one of the following*:

(1) $B=\left\{\begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix}\in \mathrm{SL}_3\right\}.$

(2) *$n\geq 1$ is an integer.* $\left\{\begin{pmatrix} a_{11} & x & y \\ 0 & \multicolumn{2}{c}{\multirow{2}{*}{A}} \\ 0 & & \end{pmatrix}\in \mathrm{SL}_3 \mid x, y\in k,\ a_{11}^n=1\right\}.$

This group is denoted by W_n.

Proposition (5.20). *Let $(\mathrm{SL}_3, \mathrm{SL}_3/\tilde{H})$ be an almost effective realization of an algebraic subgroup G of de Jonquières type contained in Cr_3. Then, $\tilde{H}$ is, up to an inner automorphism, B or W_n in Lemma (5.19). G is contained in (J7) of Theorem (2.2) for $m=n$ when $\tilde{H}=W_n$ and $n\geq 2$. G is contained in (P1) when $\tilde{H}=W_1$.*

Proof. $\mathrm{SL}_3/W_n\to\mathrm{SL}_2/\tilde{K}=\mathbf{P}^2$ is a principal $\mathbf{G}_m$-bundle of degree n. The proof is similar to the preceding cases and omitted.

(5.21) $$\tilde{G}=\mathrm{SL}_2\times\mathrm{SL}_2.$$

Let $\tilde{H}$ be a closed subgroup of $\tilde{G}$ satisfying the conditions (5.1). We shall show that $\tilde{H}^0$ is solvable. Let h be a Lie algebra of $\tilde{H}$. If h is not solvable, h must contain $\mathfrak{sl}_2$. Since the dimension of h is 3, h coincides with $\mathfrak{sl}_2$ and we get an inclusion $\mathfrak{sl}_2\to\mathfrak{sl}_2\times\mathfrak{sl}_2$. Up to automorphism of $\mathfrak{sl}_2$, there are only 3 inclusions.

(1) $\varphi(x)=(x, x)$ $\quad x\in\mathfrak{sl}_2$.
(2) $\varphi_1(x)=(x, 0)$ $\quad x\in\mathfrak{sl}_2$.
(3) $\varphi_2(x)=(0, x)$ $\quad x\in\mathfrak{sl}_2$.

In the first case $(\tilde{G}, \tilde{G}/\tilde{H})$ is primitive and as studied in Umemura [U3].

In particular, $\tilde{H}$ does not satisfy the conditions (5.1). In cases (2) and (3), h is an ideal in $\mathfrak{sl}_2 \times \mathfrak{sl}_2$. Hence the operation $(\mathrm{SL}_2 \times \mathrm{SL}_2, \mathrm{SL}_2 \times \mathrm{SL}_2/\tilde{H})$ is not almost effective. Thus h should be solvable. We may assume that $\tilde{H}^0$ is contained in $B \times B$ where

$$B = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \mid a \in k^*, b \in k \right\}.$$

Since $\dim \tilde{H}^0 = 3$, $\tilde{H}^0$ can not be reductive. If the dimension of the unipotent part U of $\tilde{H}^0$ is equal to 1, then U is a 1-dimensional subgroup of $\left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} \mid x, y \in k \right\}$ and the dimension of the maximal torus of $\tilde{H}^0$ is 2. But since there is no 1-dimensional subgroup of

$$\left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} \mid x, y \in k \right\}$$

invariant under the inner automorphisms of a maximal torus, the dimension of U can not be 1. Therefore the dimension of U is equal to 2. Then there exist mutually coprime integers l, m such that

$$\tilde{H}^0 = \left\{ \begin{pmatrix} t^l & x \\ 0 & t^{-l} \end{pmatrix} \times \begin{pmatrix} t^m & y \\ 0 & t^{-m} \end{pmatrix} \mid t \in k^*, x, y \in k \right\}.$$

Let us consider an exact sequence

$$\mathbf{1} \to \mathbf{G}_m \xrightarrow{i} \mathbf{G}_m \times \mathbf{G}_m \xrightarrow{\pi} \mathbf{G}_m \to \mathbf{1},$$

where $i(t) = (t^l, t^m)$, $\pi((x, y)) = x^m y^{-l}$. Let L be a closed subgroup of $\mathbf{G}_m \times \mathbf{G}_m$ with $L^0 = \mathbf{G}_m$. Then there exists an integer s such that $L = \pi^{-1}(\{\zeta \in \mathbf{G}_m \mid \zeta^s = 1\})$. Thus

$$\tilde{H} = \left\{ \begin{pmatrix} t_1 & x \\ 0 & t_2^{-1} \end{pmatrix} \times \begin{pmatrix} t_2 & y \\ 0 & t_2^{-1} \end{pmatrix} \mid (t_1^m t_2^{-l})^s = 1 \right\}.$$

Therefore $(\tilde{G}, \tilde{G}/\tilde{H})$ is a suboperation of

$$\Big(\mathbf{G}_m \times \mathrm{SL}_2 \times \mathrm{SL}_2,\ \mathbf{G}_m \times \mathrm{SL}_2 \times \mathrm{SL}_2 \Big/ \left\{ t_1^{ms} t_2^{-ls} \times \begin{pmatrix} t_1 & x \\ 0 & t_2^{-1} \end{pmatrix} \times \begin{pmatrix} t_2 & y \\ 0 & t_2^{-1} \end{pmatrix} \mid t_1, t_2 \in k^*, x, y \in k \right\} \Big).$$

Proposition (5.22). *Let* $(\mathrm{SL}_2 \times \mathrm{SL}_2, \mathrm{SL}_2 \times \mathrm{SL}_2/\tilde{H})$ *be an almost effective realization of an algebraic group G of de Jonquières type contained in* Cr_3. *Then, for appropriate l_1, l_2, G is contained in the conjugacy class realized by the operation in Proposition* (5.14). *In particular, G is contained in one of the operations of Theorem* (2.2).

(5.23) $$\tilde{G}=\mathbf{G}_m\times \mathrm{SL}_2.$$

Let $\tilde{H}$ be a closed subgroup of $\tilde{G}$ satisfying the conditions (5.1). Since the operation of $\mathbf{G}_m\times\mathbf{1}$ on $\tilde{G}/\tilde{H}$ is effective by (5.1), if we consider the projection p_2: $\mathbf{G}_m\times\mathrm{SL}_2\to\mathrm{SL}_2$, the image $p_2(\tilde{H})$ is a closed subgroup of SL_2 isomorphic to $\tilde{H}$. Hence $p_2(\tilde{H})$ is a 1-dimensional closed subgroup of SL_2 and we may assume that $p_2(\tilde{H})$ is one of the following:

(1) $\left\{\begin{pmatrix} t & 0\\ 0 & t^{-1}\end{pmatrix}\in \mathrm{SL}_2 \mid t\in k^*\right\}.$

(2) $\left\{\begin{pmatrix} \zeta & x\\ 0 & \zeta^{-1}\end{pmatrix}\in \mathrm{SL}_2 \mid \zeta^n=1,\ x\in k\right\}.$ This group is denoted by U_n.

Proposition (5.24). *Let* $(\mathbf{G}_m\times\mathrm{SL}_2,\ \mathbf{G}_m\times\mathrm{SL}_2/\tilde{H})$ *be an almost effective realization of an algebraic group contained in* Cr_3. *Then,* $\tilde{H}$ *is, up to an inner automorphism, one of the following*:

(1) *Let m be an integer.*

$$\left\{t^m\times\begin{pmatrix} t & 0\\ 0 & t^{-1}\end{pmatrix}\in \mathbf{G}_m\times\mathrm{SL}_2 \mid t\in k^*\right\}.$$

(2) *Let d be a divisor of a positive integer n.*

$$\left\{t^d\times\begin{pmatrix} t & x\\ 0 & t^{-1}\end{pmatrix}\in \mathbf{G}_m\times\mathrm{SL}_2 \mid t^n=1,\ x\in k\right\}.$$

In the first case G is contained in $(\mathbf{G}_m\times\mathrm{SL}_2\times\mathrm{SL}_2,\ \mathbf{G}_m\times\mathrm{SL}_2\times\mathrm{SL}_2/\tilde{H}')$ in Proposition (5.13), where

$$\tilde{H}'=\left(t_1^m\times\begin{pmatrix} t_1 & x\\ 0 & t_1^{-1}\end{pmatrix}\times\begin{pmatrix} t_2 & y\\ 0 & t_2^{-1}\end{pmatrix}\in \mathbf{G}_m\times\mathrm{SL}_2\times\mathrm{SL}_2 \mid t_1,\ t_2\in k^*,\ x,\ y\in k\right\}.$$

The inclusion is given by

$$(t, A)\longmapsto(t, A, {}^tA^{-1})\quad\text{where } t\in\mathbf{G}_m,\ A\in\mathrm{SL}_2.$$

In the second case, we may assume $d>0$. $\tilde{G}/\tilde{H}\to\tilde{G}/\mathbf{G}_m$.

$$\tilde{H}=\mathrm{SL}_2\Big/\left\{\begin{pmatrix} t & x\\ 0 & t^{-1}\end{pmatrix}\mid t^n=1\right\}$$

is a principal $\mathbf{G}_m$-bundle and as in preceding cases we conclude

Proposition (5.24.1). *Let* $(\mathbf{G}_m\times\mathrm{SL}_2,\ \mathbf{G}_m\times\mathrm{SL}_2/\tilde{H})$ *be a realization of an algebraic group G of de Jonquières type in* Cr_3, *then G is contained in one of the operations of Theorem* (2.2).

(5.25) $$\tilde{G}=\mathrm{SL}_2.$$

In this case $\tilde{H}$ is a finite subgroup of SL_2. *It is well known that a finite subgroup of* SL_2 *is conjugate to one of the following* (*Blichfeld* [B1]).

(A) *Cyclic group.*

$$\left\{\begin{pmatrix}\varepsilon^m & 0\\ 0 & \varepsilon^{-m}\end{pmatrix} \mid \varepsilon=e^{2\pi i/N},\ m=0,1,\cdots,N-1\right\},\ N=1,2,3,\cdots.$$

(B) *(binary) Dihedral group* $\tilde{\mathrm{D}}_{2N}$. *A subgroup generated by*

$$\begin{pmatrix}\varepsilon^m & 0\\ 0 & \varepsilon^{-m}\end{pmatrix},\ \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix},\ \varepsilon=e^{2\pi i/N},\ m=0,1,\cdots,N-1,$$
$$N=2,4,6,8,\cdots.$$

(C) *Tetrahedral group.*

(D) *Octahedral group.*

(E) *Icosahedral group.*

We saw that in the tetrahedral case $(\mathrm{SL}_2, \mathrm{SL}_2/\Gamma)$ is contained in (P2) and that the last cases give (E1) and (E2). Thus we have to treat the *cyclic* case and the *dihedral* case.

(5.25.A) *$\tilde{H}$ is a cyclic group C_N of order N.*

Then the conjugacy class realized by $(\mathrm{SL}_2, \mathrm{SL}_2/C_N)$ *is contained in* Proposition (5.24.1) (1). In fact, it is sufficient to consider a morphism $\varphi\colon \mathrm{SL}_2\to \mathbf{G}_m\times \mathrm{SL}_2$ defined by $\varphi(x)=(1,x)$ for $x\in \mathrm{SL}_2$.

(5.25.B) $\tilde{H}$ is a (binary) dihedral group $\tilde{\mathrm{D}}_{2N}$ of order $2N$.

Proposition (5.26). (1) *If $N=2$, then the conjugacy class realized by* $(\mathrm{SL}_2, \mathrm{SL}_2/\tilde{D}_{2N})$ *is nothing but the conjugacy class effectively realized by* $(\mathrm{PGL}_2, \mathrm{PGL}_2/\Gamma)$ *where* $\Gamma=\left\{1, \begin{pmatrix}\sqrt{-1} & 0\\ 0 & \sqrt{-1}\end{pmatrix}\right\}$. *Thus this case is reduced to* (5.25.A).

(2) *If $N=4$, then the conjugacy class realized by* $(\mathrm{SL}_2, \mathrm{SL}_2/\tilde{\mathrm{D}}_{2N})$ *is contained in the operation* (J4) *of Theorem* (2.2).

(3) *If $N=6$, then the conjugacy class realized by* $(\mathrm{SL}_2, \mathrm{SL}_2/\tilde{\mathrm{D}}_{2Z})$ *is contained in* $(\mathrm{PGL}_4, \mathbf{P}^3)$.

Proof. The first assertion is trivial. Let V_3 be an irreducible SL_2-module of degree 3 and hence V_3 is isomorphic to the vector space of all the homogeneous polynomials $f(x,y)$ in x, y of degree 2. Let $F\colon 0\subset(xy)\subset(x^2+y^2, xy)\subset V_3$ be a flag. SL_2 operates on the flag variety X of V_3. The stabilizer at F is $\mathrm{D}_{2\times 4}$ and the flag variety X is isomorphic to $\mathrm{SL}(V_3)$ /a Borel subgroup of $\mathrm{SL}(V_3)$. Therefore we have a morphism of operation

(φ, f): $(\mathrm{SL}_2, \mathrm{SL}_2/\tilde{\mathrm{D}}_{2\times 4}) \to (\mathrm{SL}_3, \mathrm{SL}/B)$ with f birational. Thus the second assertion is proved. To prove the third, consider the vector space E of homogeneous polynomials of 2 variables x, y of degree 3 as above. The vector space E is an irreducible representation of degree 4 of SL_2. The orbit of $x^3+y^3 \in \mathbf{P}(E)$ is isomorphic to $\mathrm{SL}_2/\tilde{\mathrm{D}}_{2\times 6}$ which is an open subset of $\mathbf{P}(E) \simeq \mathbf{P}^3$. Thus the second assertion is proved.

We have shown in Umemura [U3] that, if $\tilde{H}$ is the tetrahedral, octahedral or icosahedral subgroup of SL_2, then $(\mathrm{SL}_2, \mathrm{SL}_2/\tilde{H})$ is not of de Jonquières type.

Proposition (5.27). *Let* $(\mathrm{SL}_2, \mathrm{SL}_2/\Gamma)$ *be an almost effective realization of an algebraic group G of de Jonquières type in* Cr_3. *Then G is contained in one of the operations of Theorem* (2.2).

It now follows from what we have done from (5.1) to (5.27)

Conclusion (5.28). *Let $(G, G/H)$ be a realization of a reductive algebraic G of the Jonquières type in* Cr_3. *Then G is contained in one of the operations of Theorem* (2.2).

Case (ii). *G is not reductive.*

Let $(G, G/H)$ be a realization of G.

Case (5.29). *The center U_Z of the unipotent radical of G has a 2-dimensional orbit on G/H.*

We have a morphism of homogeneous spaces $(G, G/H) \to (G, G/U_Z H)$ and $\dim G/U_Z H = 1$. Hence we get a morphism of homogeneous spaces (φ, f): $(G, G/H) \to (\mathrm{PGL}_2, \mathbf{P}^1)$.

Subcase (5.30.1). *φ is surjective.*

Let N be the kernel of φ and N^0 its connected component of 1. We have an exact sequence

$$\begin{array}{ccccccccc} \mathbf{1} & \to & N & \to & G & \xrightarrow{\varphi} & \mathrm{PGL}_2 & \to & \mathbf{1} \\ & & \uparrow\cup & & \| & & \uparrow & & \\ \mathbf{1} & \to & N^0 & \to & G & \longrightarrow & G/N^0 & \to & \mathbf{1}. \end{array}$$

The kernel of the morphism $G/N^0 \to \mathrm{PGL}_2$ is N/N^0. Hence $|N/N^0| \leq 2$. If $|N/N^0| = 2$, φ factors through the degree 2 covering $\mathrm{SL}_2 \to \mathrm{PGL}_2$. Anyhow, by taking an isogeny $\tilde{G} \to G$ of degree at most 2, there exists an almost effective realization $(\tilde{G}, \tilde{G}/\tilde{H})$ such that the center $\tilde{U}_Z$ of the unipotent radical of $\tilde{G}$ has a 2-dimensional orbit on $\tilde{G}/\tilde{H}$, there exists a morphism of homogeneous spaces $(\tilde{\varphi}, \tilde{f})$: $(\tilde{G}, \tilde{G}/\tilde{H}) \to (\mathrm{SL}_2, \mathbf{P}^1)$ and such that the kernel $\tilde{\varphi} = \tilde{N}$ is connected. By Corollary (1.13), Umemura [U3], the operation of $\tilde{N}$ on each fibre is through affine transformation of 2 variables. Now, by Lemma (1.21), Umemura [U3], the reductive part of $\tilde{N}$ is a subgroup of the linear part GL_2 of the affine transformation group of 2 variables and $U_Z = U$.

Let us determine the SL_2-module U_Z. Since we have $\tilde{\varphi}$: $\tilde{G} \to \mathrm{SL}_2$, by Levi's Theorem $\tilde{G}$ contains a semi-direct product $U_Z.\mathrm{SL}_2$ and the orbit of $U_Z.\mathrm{SL}_2$ coincides with the whole space $\tilde{G}/\tilde{H}$. For, if we consider the fibration $\tilde{f}$: $\tilde{G}/\tilde{H} \to \tilde{G}/\tilde{U}_Z\tilde{H} = \mathbf{P}^1$, $\tilde{U}_Z$-orbit of $\tilde{H}$ is the fibre $\tilde{f}^{-1}(\tilde{U}_Z\tilde{H})$ and from the definition of the subgroup SL_2 of $\tilde{G}$, the morphism $(\tilde{U}_Z.\mathrm{SL}_2)\cdot\tilde{H} \to \mathbf{P}^1$ induced by f is surjective.

Lemma (5.30.2). SL_2-*module* U_Z *has at most* 2 *irreducible components.*

Proof. Let us put $\mathscr{H} = \tilde{H} \cap U_Z.\mathrm{SL}_2$. Then we have an exact sequence

$$1 \to U_2 \cap \mathscr{H} \to \mathscr{H} \xrightarrow{\varphi|\mathscr{H}} \mathrm{SL}_2.$$

Since U_Z has a 2-dimensional orbit, the image $\varphi(\mathscr{H})$ is a 2-dimensional subgroup of SL_2 hence a Borel subgroup B of SL_2. We may assume $B = \left\{\begin{pmatrix} * & 0 \\ * & * \end{pmatrix} \in \mathrm{SL}_2\right\}$. Thus the above exact sequence becomes

$$1 \to U_Z \cap \mathscr{H} \to \mathscr{H} \to B \to 1. \tag{5.30.3}$$

Let $U_Z = U_1 \oplus U_2 \oplus \cdots \oplus U_r$ be an irreducible decomposition. Let $v_i \in U_i$, $1 \leq i \leq r$ be a highest weight vector with respect to the upper triangular Borel subgroup and the diagonal Cartan subgroup. v_i does not belong to $U_Z \cap \mathscr{H}$. For, if $v_i \in U_Z \cap \mathscr{H}$, then $U_i \subset U_Z \cap \mathscr{H}$ since $U_Z \cap \mathscr{H}$ is B invariant by the exact sequence (5.30.3). On the other hand, since $(U_Z.\mathrm{SL}_2, U_Z.\mathrm{SL}_2/\mathscr{H})$ is almost effective, $U_Z \cap \mathscr{H}$ contains no normal subgroup of positive dimension of $U_Z.\mathrm{SL}_2$. In particular, U_i can not be a submodule of $U_Z \cap \mathscr{H}$. Assume $r \geq 3$ and let $L = \{\lambda_1 v_1 + \lambda_2 v_2 \mid \lambda_1, \lambda_2 \in k\}$. We shall show L-orbit of $\tilde{H} \in \tilde{G}/\tilde{H}$ is 2-dimensional. In fact, assume L-orbit of $\tilde{H} \in \tilde{G}/\tilde{H}$ is not 2-dimensional, then $\dim L/U_Z \cap \mathscr{H} \leq 1$. Namely v_i and v_2 are not linearly independent mod $U_Z \cap \mathscr{H}$. Since $v_i \notin U_Z \cap \mathscr{H}$ and since L and $U_Z \cap \mathscr{H}$ are invariant by the Cartan subgroup, if v_1 and v_2 have different weights then v_1 and v_2 are linearly independent mod $U_Z \cap \mathscr{H}$. If v_1 and v_2 have the same weight and there exist $\lambda_1, \lambda_2 \in k$ such that $\lambda_1 v_1 + \lambda_2 v_2 \neq 0$ and $\lambda_1 v_1 + \lambda_2 v_2 \in U_Z \cap \mathscr{H}$, then $\lambda_1 v_1 + \lambda_2 v_2$ is a highest vector of an irreducible SL_2-submodule of U_Z and $U_Z \cap \mathscr{H}$ contains a normal of subgroup of positive dimension of $U_Z.\mathrm{SL}_2$ which contradicts the almost effectivity of $(U_Z.\mathrm{SL}_2, U_Z.\mathrm{SL}_2/\mathscr{H})$. Let $W=\left\{\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix} \mid x \in k\right\}$. Then r is the dimension of W-invariant subspace V of U_Z and $L\subset V$. $V.W$ is an abelian closed subgroup. Since L-orbit of $\tilde{H}$ is 2-dimensional and contained in a fibre of f and W-orbit of $\tilde{H}$ is 1-dimensional and horizontal, $L.W$-orbit of $\tilde{H} \in \tilde{G}/\tilde{H}$ is 3-dimensional. Since $L.W\subset V.W$, $V.W$ has a 3-dimensional orbit. As $V.W$ is abelian, by Lemma (1.8), Umemura [U3], $3=\dim V.W=r+1$ hence $r=2$ which contradicts our assumption $r\geq 3$.

Notations being as in the proof of Lemma (5.30.2), let n_i be the highest weight of U_i and $U_i=\sum_{\alpha=0}^{n_i} W^{(i)}_{n_i-2\alpha}$ be the decomposition into the eigen-spaces of the Cartan subgroup where the weight of $W^{(i)}_{n_i-2\alpha}$ is $n_i-2\alpha$.

Lemma (5.30.4). *$\mathscr{H}$ is, up to an automorphism, one of the following*:

(i) *If* $r=1$, $\left(\sum_{\alpha=2}^{n_1} W^{(1)}_{n_1-2\alpha}\right).B(\subset U_1.\mathrm{SL}_2)$, $(n_1\geq 1)$

(ii) *If* $r=2$,

(*a*) $\left\{\left(\sum_{\alpha=1}^{n_1} W^{(1)}_{n_1-2\alpha}\right)+\left(\sum_{\alpha=1}^{n_2} W^{(2)}_{n_2-2\alpha}\right)\right\}.B$, $(n_1\geq n_2\geq 0)$

or

(b) $n_1-2=n_2\geq 0$, *$W.B$ where W is a vector subspace of $U_1\oplus U_2$ generated by* $\left(\sum_{\alpha=2}^{n_1} W^{(1)}_{n_1-2\alpha}+\sum_{\alpha=1}^{n_2} W^{(2)}_{n_2-2\alpha}\right)$ *and a vector u_1+u_2 with $u_1 \in W^{(1)}_{n_1-2}$, $u_2 \in W^{(2)}_{n_2}$ with u_1, $u_a\neq 0$.*

Proof. The proof is a refinement of the proof of Lemma (2.7), Umemura [U3]. By (5.30.3), $\mathscr{H}$ is connected. Therefore it is sufficient to determine its Lie algebra. We shall treat only the case $r=1$ because the case $r=2$ is proved similarly. The Lie algebra of $\mathscr{H}$ is a subalgebra of $\mathfrak{u}_z+\mathfrak{b}$ where $\mathfrak{u}_z$, $\mathfrak{b}$ are the Lie algebras of U_Z and B. Let

$$u=\sum_{\alpha=0}^{n_1} W_{n_1-2\alpha}$$

be a direct sum decomposition of C-eigen-spaces where the weight of $W_{n_1-2\alpha}$ is $n_1-2\alpha$. Then it is easy to see, $(\sum_{\alpha=2}^{n_1} W_{n_1-2\alpha})+\mathfrak{b}$ is the only $\mathfrak{b}$-invariant Lie subalgebra $\mathfrak{h}$ of $\mathfrak{u}+\mathfrak{b}$ of codimension 3 in $\mathfrak{u}+\mathfrak{sl}_2$ such that $\dim \mathfrak{u}/\mathfrak{u}\cap\mathfrak{h}=2$ (cf. Umemura [U3]).

Remark (5.30.5). In Lemma (5.30.4), the group in (b) is uniquely determined up to automorphism of $U_Z.\mathrm{SL}_2$. In fact let $0\neq\alpha$, $\beta \in k$ and replace u_1+u_2 by $\alpha u_1+\beta u_2$, then we have an automorphism ψ of SL_2-module $U_Z=U_1\oplus U_2$ defined by $\psi(x_1, x_2)=\alpha x_1+\beta x_2$.

Proposition (5.30.6). *Notations being as in Lemma* (5.30.4), *the conjugacy class realized by* $(\tilde{G}, \tilde{G}/\tilde{H})$ *is contained in*

(a) *the operation* (J1) $(\mathrm{PGL}_3\times\mathrm{PGL}_2, \mathbf{P}^2\times\mathbf{P}^1)$ *of Theorem* (2.2) *for case* (i) *and* $n_1=1$,

(b) *the operation* $(\mathrm{Aut}^0 \mathrm{F}'_{n_1-1, n_1-1}, \mathrm{F}'_{n_1-1, n_1-1})$ *hence in* (P1) *or* (J10) *of Theorem* (2.2) *either for case* (i), $n_1\geq 2$ *or for case* (ii) (b).

Proof. If we are in case (i), $n_1=1$, since U_Z is normal the operation of $\tilde{G}$ respects the fibration $\pi: U_{\bullet}\mathrm{SL}_2/B\to\mathrm{SL}_2/B=\mathbf{P}^1$ which is the trivial rank 2 vector bundle over $\mathbf{P}^1$. By Corollary (1.13), Umemura [U3], the Ker φ operates on the fibre of π as affine transformations. Therefore the operation of $\tilde{G}$ is extended to $\mathbf{P}^2\times\mathbf{P}^1$ and G is contained in the operation (J1) (see also Lemma (1.21), [U3]). The operation in case (1) is contained in the operation in case (2) (b) and they both have the same transformation space. Thus in view of Corollary (4.2) and Lemmas (4.3), (4.29), it is sufficient to show the transformation space $U_Z{}_{\bullet}\mathrm{SL}_2/\mathscr{H}$ is isomorphic to $\mathrm{F}'_{n_1-1, n_1-1}$. This is proved as Lemma (4.24).

Proposition (5.30.7). *We keep the notation of Lemma* (5.30.4) *and we assume that we are in case* (ii) (a) *with* $n_1\geq n_2\geq 0$.

(1) *If* $n_1>n_2\geq 2$, *the conjugacy class realized by* $(\tilde{G}, \tilde{G}/\tilde{H})$ *is contained in* (J9) $(\mathrm{Aut}^0 \mathrm{F}'_{n_1, n_2}, \mathrm{F}'_{n_1, n_2})$.

(2) *If* $n_1>n_2=1$, *the conjugacy class realized by* $(\tilde{G}, \tilde{G}/\tilde{H})$ *is contained in* (J7) $(\mathrm{Aut}^0 \mathrm{J}'_{n_1}, \mathrm{J}'_{n_1})$.

(3) *If* $n_1=n_2\geq 2$, *the conjugacy class realized by* $(\tilde{G}, \tilde{G}/\tilde{H})$ *is contained in* (J10) $(\mathrm{Aut}^0 \mathrm{F}'_{n_1, n_1}, \mathrm{F}'_{n_1, n_1})$.

(4) *If* $n_1=n_2=1$, *the conjugacy class realized by* $(\tilde{G}, \tilde{G}/\tilde{H})$ *is contained in* (P1) $(\mathrm{PGL}_4, \mathbf{P}^3)$.

(5) *If* $n_1=n_2=0$, *the conjugacy class realized by* $(\tilde{G}, \tilde{G}/\tilde{H})$ *is contained in* (J1) $(\mathrm{PGL}_3\times\mathrm{PGL}_2, \mathbf{P}^2\times\mathbf{P}^1)$.

Proof. The transformation space $\tilde{G}/\tilde{H}$ coincides with $U_Z{}_{\bullet}\mathrm{SL}_2/\mathscr{H}$ and as in the proof of Proposition (5.30.6) (see also Umemura [U3]). We can prove the homogeneous space $U_Z{}_{\bullet}\mathrm{SL}_2/\mathscr{H}$ is isomorphic to F'_{n_1, n_2}. The assertion (1) follows from Corollary (4.2) and Lemma (4.3). As for the second assertion, let $\mathrm{J}_m=\mathbf{P}(\mathcal{O}_{\mathbf{P}^2}\oplus\mathcal{O}_{\mathbf{P}^2}(-m))$, $\mathcal{O}_{\mathrm{J}_m}(1)$ the tautological line bundle on J_m and J'_m J_m minus the negative section of $\mathrm{J}_m/\mathbf{P}_2$. Since $\mathrm{F}'_1\simeq\mathbf{P}^2-(\text{a point})\subset\mathbf{P}^2$ and $\mathrm{Pic}\,\mathbf{P}^2\to\mathrm{Pic}\,\mathrm{F}'_1$ is an isomorphism, we have an open immersion

$$\begin{array}{ccc} \mathrm{F}'_{n_1,1} & \hookrightarrow & \mathrm{J}_{n_1} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ \mathrm{F}'_1 & \hookrightarrow & \mathbf{P}^2. \end{array}$$

We want to show that $(\mathrm{Aut}^0 \mathrm{F}'_{n_1,1}, \mathrm{F}'_{n_1,1})$ can be extended to $(\mathrm{Aut}^0 \mathrm{F}'_{n_1,1}, \mathrm{J}_{n_1})$. If there exists a very ample line bundle $\mathscr{L}$ on J_{n_1} such that the restriction induces an isomorphism $H^0(\mathrm{J}_{n_1}, \mathscr{L}) \simeq H^0(\mathrm{F}'_{n_1,1}, \mathscr{L} \mid \mathrm{F}'_{n_1,1})$, since $\mathrm{Aut}^0 \mathrm{F}'_{n_1,1}$ is a linear group (Theorem 3.2, Umemura [U2]), it operates on the linear system $\mathbf{P}(H^0(\mathrm{F}'_{n_1,1}, \mathscr{L} \mid \mathrm{F}'_{n_1,1}) \simeq \mathbf{P}(H^0(\mathrm{J}_{n_1}, \mathscr{L}))$ hence the operation of $\mathrm{Aut}^0 \mathrm{F}'_{n_1,1}$ on $\mathrm{F}'_{n_1',1}$ can be extended to an operation on J_{n_1} and the assertion (2) is proved. Now we look for such a very ample line bundle. In the following, it becomes clear that for any line bundle $\mathscr{M}$ on J_{n_1}, we have

$$H^0(\mathrm{J}'_n, \mathscr{M}) \rightrightarrows H^0(\mathrm{F}'_{n_1,1}, \mathscr{M} \mid \mathrm{F}_{n_1,1}).$$

We look for a very ample line bundle. Since $(\mathscr{O}_{\mathbf{P}^2} \oplus \mathscr{O}_{\mathbf{P}^2}(-n_1)) \otimes \mathscr{O}_{\mathbf{P}^2}(n_1+1)$ is ample, it follows from Hartshorne [H] $\mathscr{O}_{\mathrm{J}_{n_1}}(1) \otimes p^* \mathscr{O}_{\mathbf{P}^2}(n_1+1)$ is ample. Hence there exists a positive integer k such that

$$\mathscr{L} = \mathscr{O}_{\mathrm{J}_{n_1}}(k) \otimes p^* \mathscr{O}_{\mathbf{P}^2}((n_1+1)k)$$

is very ample. Now it is sufficient to show $H^0(\mathrm{J}_{n_1}, \mathscr{L}) \rightrightarrows H^0(\mathrm{F}'_{n_1,1}, \mathscr{L})$. Since the morphism of restriction is injective, we must show the both vector space have the same dimension. Since the inclusion $\mathrm{F}'_{n_1,1} \subset \mathrm{J}_{n_1}$ factors through $\mathrm{F}'_{n_1,1} \subset \mathrm{J}'_{n_1} \subset \mathrm{J}_{n_1}$ and the codimension of $\mathrm{J}'_{n_1} - \mathrm{F}'_{n_1,1}$ in J'_{n_1} is 2, $H^0(\mathrm{J}'_{n_1}, \mathscr{L}) \rightrightarrows H^0(\mathrm{F}'_{n_1,1}, \mathscr{L})$. Thus we have to show $\dim H^0(\mathrm{J}'_{n_1}, \mathscr{L}) = \dim H^0(\mathrm{J}_{n_1}, \mathscr{L})$. In fact, since the morphism $\mathrm{J}'_{n_1} \to \mathbf{P}^2$ is affine, we have

$$\begin{aligned} H^0(\mathrm{J}_{n_1}, \mathscr{L}) &= H^0(\mathrm{J}'_n, p^* \mathscr{O}_{\mathbf{P}^2}((n_1+1)k)) \\ &\simeq H^0(\mathbf{P}^2, \mathscr{O}_{\mathbf{P}^2}((n_1+1)k) \otimes p_* \mathscr{O}_{\mathrm{J}'_{n_1}}) \\ &\simeq H^0\Big(\mathbf{P}^2, \mathscr{O}_{\mathbf{P}^2}((n_1+1)k) \otimes \sum_{l=0}^{\infty} \mathscr{O}_{\mathbf{P}^2}(-ln_1)\Big) \\ &\simeq \sum_{l=0}^{\infty} H^0(\mathbf{P}^2, \mathscr{O}_{\mathbf{P}^2}((n_1+1)k - ln_1)) \\ &\simeq \sum_{l=0}^{k} H^0(\mathbf{P}^2, \mathscr{O}_{\mathbf{P}^2}((n_1+1)k - l_{n_1})). \end{aligned}$$

On the other hand,

$$\begin{aligned} H^0(\mathrm{J}_n, \mathscr{L}) &\simeq H^0(\mathbf{P}^2, S^k(\mathscr{O}_{\mathbf{P}^2} \oplus \mathscr{O}_{\mathbf{P}^2}(-n_1)) \otimes \mathscr{O}_{\mathbf{P}^2}((n_1+1)k)) \\ &\simeq \sum_{l=0}^{k} H^0(\mathbf{P}^2, \mathscr{O}_{\mathbf{P}^2}((n_1+1)k - n_1 l)). \end{aligned}$$

Hence the second assertion is proved. The assertion (3) follows from Corollary (4.2) and Lemma (4.3). The assertion (4) is proved as (3) combined with Lemma (4.29). It remains to prove the last assertion. As we noticed above, the reductive part of $\tilde{N}$ is contained in the linear part of the affine transformation group of 2 variables and $U_Z = U$ (cf. Lemma (1.21), Umemura [U3]).

Subcase (5.31.1). *φ of* (5.29) *is not surjective.*

The image of φ is isomorphic to the group of affine transformations of 1 variable GTA_1, $\mathbf{G}_a$ or $\mathbf{G}_m$. Let us first assume $\mathrm{Im}(\varphi) = \mathrm{GTA}_1$. Then, we have an exact sequence

$$\begin{array}{ccccccccc} \mathbf{1} & \to & N & \to & G & \xrightarrow{\varphi} & \mathrm{GTA}_1 & \to & \mathbf{1}. \\ & & & & \cup & & \cup & & \\ & & & & K & \simeq & \mathbf{G}_a & & \end{array}$$

By Sublemma (2.30), Umemura [U3], φ splits over the unipotent part of GTA_1: there exists a closed subgroup K of G mapped isomorphically onto the unipotent part of GTA_1. Since $G/U_Z H \simeq \mathbf{A}^1$, the orbit of the closed commutative group $U_Z . K$ coincides with G/H. In particular, the dimension of $U_Z = 2$ by Lemma (1.8), Umemura [U3] and $U_Z . K \simeq G/H$. By Umemura [U3], U_Z is the unipotent radical of N^0. But since $\mathbf{G}_a$ has no non-trivial étale covering, N is connected. Let $K . U_Z \simeq \mathbf{G}_a \times \mathbf{G}_a \times \mathbf{G}_a$ and use the coordinate on $\mathbf{G}_a \times \mathbf{G}_a \times \mathbf{G}_a$.

The operation of $U_Z . K$ on $G/H \simeq \mathbf{G}_a \times \mathbf{G}_a \times \mathbf{G}_a$ is given by $(x, y, z) \mapsto (x+a, y+b, z+c)$. The operation of the unipotent radical of N is, by Section 2, Umemura [U3], $(x, y, z) \mapsto (x, y+f(x), z+g(x))$ where f, g are polynomials in x and their degrees are bounded. By Lemma (1.21), Umemura [U3], the reductive part of N is GL_2 or $\mathbf{G}_m \times \mathbf{G}_m$, $\mathbf{G}_m$ and they operate on $U_Z \simeq k^2$ naturally, therefore on $\mathbf{A}^1 \times \mathbf{A}^2 = \mathbf{A}^3$ linearly: in GL_2 case for example, $(x, y, z) \mapsto (x, ay+cz, by+dz)$ for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2$. It remains only a torus mapped injectively by φ. For this purpose, we write the operation of maximal torus on the radical $K . U_Z$ which is linear. Thus the operation $(G, G/H)$ is a suboperation of (J1) or (J10) of Theorem (2.2) (cf. Section 2, Umemura [U3]). The cases $\mathrm{Im}\,\varphi = \mathbf{G}_a$ and $\mathrm{Im}\,\varphi = \mathbf{G}_m$ are treated similarly.

It follows from (5.30.1) and (5.31.1).

Conclusion (5.32). *Let $(G, G/H)$ be a realization of an algebraic group G of de Jonquières type. If G is not reductive and if the center of the unipotent radical has a* 2*-dimensional orbit, then G is contained in one of the operations of Theorem* (2.2).

Case (5.33). *The center U_Z of the unipotent radical has only* 1*-dimensional orbits on G/H.*

We have a morphism of homogeneous spaces (Id, f): $(G, G/H)\to(G, G/U_ZH)$. The algebraic operation $(G, G/U_ZH)$ determines a morphism $\varphi: G\to\mathrm{Aut}\, G/U_ZH\subset\mathrm{Autbirat}\ G/U_ZH$. Since G is linear, G is rational and the algebraic surface G/U_ZH is unirational. Therefore, G/U_ZH is rational and Autbirat G/U_ZH is non-canonically isomorphic to the Cremona group Cr_2 of 2 variables. Hence we can apply the results of Umemura [U3] to $(\varphi(G), G/U_ZH)$.

Proposition (5.34). *If* $G/U_ZH\simeq\mathbf{P}^2$, *the conjugacy class of* $(G, G/H)$ *is contained in* (J1), (P1) *or in* (J7) *of Theorem* (2.2) *according as* $\dim U_Z=1$, $\dim U_Z=3$, *or* $\dim U_Z=n>3$.

Proof. We can prove the proposition as in Section 2, Umemura [U3]. But we give a different proof. $G/H\to G/U_ZH=\mathbf{P}^2$ is an $\mathbf{A}^1$-bundle over $\mathbf{P}^2$. Since $H^1(\mathbf{P}^2, \mathscr{L})=0$ for any line bundle $\mathscr{L}$ over $\mathbf{P}^2$, the affine $\mathbf{A}^1$-bundle $G/H\to G/U_ZH$ comes from a $\mathbf{G}_m$-bundle, i.e. $G/H\to G/U_ZH$ is a line bundle. Thus there exists an integer n such that $\mathrm{J}'_n\underset{\mathbf{P}^2}{\simeq}G/H$ where $\mathrm{J}'_n=\mathrm{Spec}\,(\bigoplus_{l=0}^{\infty}\mathcal{O}_{\mathbf{P}^2}(-l_n))$. Since the positive dimensional group U_Z operates transitively on each fibre of $G/H\to G/U_ZH$. n should be non negative. If $n=0$, $(G, G/H)$ is a suboperation of (J1). If $n\geq 1$, by Corollary (4.2) and Lemma (4.3) $(G, G/H)$ is a suboperation of $(\mathrm{Aut}^0\,\mathrm{J}'_n, \mathrm{J}'_n)$ and $(\mathrm{Aut}^0\,\mathrm{J}'_1, \mathrm{J}'_1)$ is contained in $(\mathrm{PGL}_4, \mathbf{P}^3)$ in $(\mathrm{PGL}_4, \mathbf{P}^3)$ as in the proof of Proposition (5.30.16) (2).

Proposition (5.35). *If* $G/U_ZH\simeq\mathbf{P}^1\times\mathbf{P}^1$, *the conjugacy class of* $(G, G/H)$ *is contained in* (J1), (J2), (J3), *or* (J8) *of Theorem* (2.2).

Proof. $G/H\to G/U_ZH=\mathbf{P}^1\times\mathbf{P}^1$ is an $\mathbf{A}^1$-bundle over $\mathbf{P}^1\times\mathbf{P}^1$ hence defined by an exact sequence:

$$0\to\mathcal{O}_{\mathbf{P}^1\times\mathbf{P}^1}\to E\to\mathcal{O}_{\mathbf{P}^1\times\mathbf{P}^1}(l, m)\to 0 \tag{5.35.1}$$

where $\mathcal{O}_{\mathbf{P}^1\times\mathbf{P}^1}(l, m)$ denotes $p_1^*\mathcal{O}_{\mathbf{P}^1}(l)\otimes p_2^*\mathcal{O}_{\mathbf{P}^1}(m)$. The extension (5.35.1) is $\mathrm{SL}_2\times\mathrm{SL}_2$ homogeneous. Let us determine such extensions. The extensions are parametrized by $H^1(\mathbf{P}^1\times\mathbf{P}^1, \mathcal{O}(-l, -m))$. Since we have $\mathrm{SL}_2\times\mathrm{SL}_2$-isomorphism

$$\begin{aligned}H^1(\mathbf{P}^1\times\mathbf{P}^1, \mathcal{O}(-l, -m))\simeq{}& H^0(\mathbf{P}^1, \mathcal{O}(-l))\otimes H^1(\mathbf{P}^1, \mathcal{O}(-m))\\ &+H^1(\mathbf{P}^1, \mathcal{O}(-l))\otimes H^0(\mathbf{P}^1, \mathcal{O}(-m)).\end{aligned}$$

The SL_2-modules $H^1(\mathbf{P}^1, \mathcal{O}(-m))$, $H^1(\mathbf{P}^1, \mathcal{O}(-l))$ are dual to $H^0(\mathbf{P}^1, \mathcal{O}(m-2))$, $H^0(\mathbf{P}^1, \mathcal{O}(l-2))$. Since SL_2-module $H^0(\mathbf{P}^1, \mathcal{O}(n))$ is irreducible if non-zero, $H^1(\mathbf{P}^1\times\mathbf{P}^1, \mathcal{O}(l, m))$ contains a non-zero $\mathrm{SL}_2\times\mathrm{SL}_2$-invariant element if and only if either $m=2$ and $l=0$ or $m=0$ and $l=2$.

Therefore the extension (5.35.1) is one of the following:

(1) trivial, i.e. $\mathbf{A}^1$-bundle $G/H \to G/U_Z H = \mathbf{P}^2$ is a line bundle.

(2) the non-trivial extension:

$$0 \to \mathcal{O} \to E \to \mathcal{O}(2, 0) \to 0,$$

(3) the non-trivial extension:

$$0 \to \mathcal{O} \to E \to \mathcal{O}(0, 2) \to 0.$$

Using the notation of Section 3, in the first case, $G/H \underset{\mathbf{P}^1 \times \mathbf{P}^1}{\simeq} \mathrm{L}'_{l,m}$. We may assume $l \geq m$. Since the algebraic group U_Z operates transitively on each fibre of $G/H \to \mathbf{P}^1 \times \mathbf{P}^1$, $l \geq m \geq 0$. If $l, m \geq 1$, $(G, G/H)$ is a suboperation of (J8) by Corollary (4.2) and Lemma (4.3). If $l \geq 2$, $m=0$, $(G, G/H)$ is contained in (J3) by Corollary (4.2) since $G/H \simeq \mathrm{F}'_l \times \mathbf{P}^1$. If $l=1$, $m=0$ $(G, G/H)$ is contained in (J1) since $G/H \simeq \mathrm{F}'_1 \times \mathbf{P}^1$ and $(\mathrm{Aut}^0(\mathrm{F}'_1 \times \mathbf{P}^1), \mathrm{F}'_1 \times \mathbf{P}^1) \subset (\mathrm{PGL}_3 \times \mathrm{PGL}_2, \mathbf{P}^2 \times \mathbf{P}^1)$. If $l=m=0$, $(G, G/H)$ is contained in (J2) by Lemm (4.4). Since the second and the third cases are symmetric, we treat only the second case. The non-trivial extension is explicitly written

$$0 \to \mathcal{O} \to \mathcal{O}(1, 0) \oplus \mathcal{O}(1, 0) \to \mathcal{O}(2, 0) \to 0.$$

This shows G/H is $\mathbf{P}(\mathcal{O}(1, 0) \oplus \mathcal{O}(1, 0))$ — a section and the operation of G on G/H can be extended to an operation over $\mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1$.

Proposition (5.36). *If $G/U_Z H \simeq \mathrm{F}'_m$ with $m \geq 0$, then the conjugacy class realized by $(G, G/H)$ is contained in one of the conjugacy classes of Theorem* (2.2).

Proof. If $G/U_Z H \simeq \mathrm{F}'_m$ with $m \geq 0$ and if the $\mathbf{A}^1$-bundle $G/H \to G/HU_Z \simeq \mathrm{F}'_m$ is not a line bundle, then the transformation space is isomorphic to E'^p_m for certain integer p. For Pic $\mathrm{F}'_m \simeq \mathbf{Z}$ and a line bundle over F'_m is isomorphic to $\pi^* \mathcal{O}_{\mathbf{P}^1}(a) = \mathcal{O}_{\mathrm{F}_m}(a)$ where $\pi \colon \mathrm{F}'_m \to \mathbf{P}^1$ is the projection. The $\mathbf{A}^1$-bundle $G/H \to G/HU_Z$ is defined by an exact sequence

$$0 \to \mathcal{O}_{\mathrm{F}'_m} \to E \to \mathcal{O}_{\mathrm{F}'_m}(b) \to 0. \tag{5.36.1}$$

The extensions are parametrized by $H^1(\mathrm{F}'_m, \mathcal{O}_{\mathrm{F}'_m}(-b))$ and the $\mathbf{A}^1$-bundle $G/H \to G/U_Z H$ is homogeneous. It follows from Umemura [U3] there exists a finite cover $\tilde{G}$ of $\varphi(G)$ such that $\tilde{G}$ has a $\mathcal{O}_{\mathrm{F}'_m}(1)$-linearized action and hence $\tilde{G}$ acts on $H^i(\mathrm{F}'_m, \mathcal{O}_{\mathrm{F}'_m}(j))$ for $j \in \mathbf{Z}$. The exact sequence (5.36.1) or the corresponding $\mathbf{A}^1$-bundle defined by $\lambda \in H^1(\mathrm{F}'_m, \mathcal{O}_{\mathrm{F}'_m}(b))$ is homogeneous if and only if λ is a $\tilde{G}$-eigen vector. Since the spectral sequence for π degenerates as π is affine, we have a $\tilde{G}$-isomorphism

$$H^1(\mathbf{P}^1, \pi_*\mathcal{O}_{\mathrm{F}'_m}(-b)) \simeq H^1(\mathrm{F}'_m, \mathcal{O}_{\mathrm{F}'_m}(-b)),$$

where $\tilde{G}$ acts on $H^1(\mathbf{P}^1, \pi_*\mathcal{O}_{\mathrm{F}'_m}(-b))$ through $(\tilde{G}, G/U_Z H) \to (\tilde{G}, \mathbf{P}^1) \to (\mathrm{SL}_2, \mathbf{P}^1)$.

$$\begin{aligned} H^1(\mathbf{P}^1, \pi_*\mathcal{O}_{\mathrm{F}'_m}(-b)) &= H^1(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(-b) \otimes \pi_*\mathcal{O}_{\mathrm{F}'_m}) \\ &= H^1(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(-b) \oplus (\bigoplus_{c \geq 0} \mathcal{O}_{\mathbf{P}^1}(-cm))) \\ &= \bigoplus_{c \geq 0} H^1(\mathbf{P}^1, \mathcal{O}(-b-cm)). \end{aligned}$$

$H^1(\mathbf{P}^1, \mathcal{O}(-b-cm))$ is dual to $H^0(\mathbf{P}^1, \mathcal{O}(b+cm-2))$ which is an irreducible SL_2-module of highest weight $b+cm-2$. Therefore a homogeneous $\mathbf{A}^1$-bundle which is not isomorphic to a line bundle, exists if and only if there exists an integer $p \geq 0$ such that $b+pm-2=0$. In that case by definition our variety G/H is isomorphic to E'^p_m and it follows from Lemma (4.22) $pm \geq 2$ since non-trivial U_Z operates. The group G is contained in one of the operations of Theorem (2.2) by Corollary (4.2), Lemma (4.3), Theorem (4.1) and Lemmas (4.24), (4.25).

If $m \geq 0$ and $G/H \to G/U_Z H$ is a line bundle $\mathrm{F}_{m,n}$ over F'_m. If m and n are not exceptional appear in Theorem (2.2), then Proposition follows from Corollary (4.2) and Lemma (4.3). When their values are exceptional argue as in Proposition (5.35) or in Lemma (4.4) and Lemma (1.21) in [U3].

Subcase (5.37). *Let us study the case where* $\varphi(G)$ *in* (5.33) *is reductive and not solvable.*

$\varphi(G)$ is an algebraic subgroup of the Cremona group of 2 variables. Then by Umemura [U3] $\varphi(G)$ *is almost effectively realized by one of the following*:

(1) $(\mathrm{PGL}_3, \mathbf{P}^2)$,

(2) $(\mathrm{PGL}_2 \times \mathrm{PGL}_2, \mathbf{P}^1 \times \mathbf{P}^1)$,

(3) $(\mathbf{G}_m \times \mathrm{PGL}_2, \mathbf{P}^1 \times \mathbf{P}^1)$,

(4) $(\mathbf{G}_m \times \mathrm{SL}_2, \mathbf{G}_m \times \mathrm{SL}_2/K_n)$, where

$$K_m = \left\{ \left(t^m, \begin{pmatrix} t & b \\ 0 & t^{-1} \end{pmatrix} \right) \in \mathbf{G}_m \times \mathrm{SL}_2 \,|\, t \in k^*,\ b \in k \right\}$$

and m *is an integer* ≥ 1,

(5) $(\mathrm{SL}_2, \mathrm{SL}_2/U_m)$, *where* $U_m = \left\{ \begin{pmatrix} \zeta & b \\ 0 & \zeta^{-1} \end{pmatrix} \,|\, \zeta^m = 1,\ b \in k \right\}$ *and* m *is an integer* ≥ 1,

(6) $(\mathrm{SL}_2, \mathrm{SL}_2/\mathbf{G}_m)$,

(7) $(\mathrm{SL}_2, \mathrm{SL}_2/\mathrm{D}_\infty)$, *where* D_∞ *is an algebraic subgroup generated by* $\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$, $t \in k^*$ *and* $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

Proposition (5.37.1). *If* $\varphi(G)$ *is realized by one of the operations* (1), (2), $\cdots$, (7), *then* G *is contained in one of the operations of Theorem* (2.2).

Proof. The case (1), (2) are treated in Propositions (5.34) and (5.35). Now we give a proof for operation (5). Since the projection $G/H \to G/U_Z H$ is an $\mathbf{A}^1$-bundle, $(\mathrm{Ker}\,\varphi)^0$ is solvable and the dimension of a maximal torus is at most 1 (cf. Lemma (1.21), Umemura [U3]). The unipotent part of $(\mathrm{Ker}\,\varphi)^0$ is abelian (see Umemura [U3]). For the same reason as in [U3], we may assume $(\mathrm{Ker}\,\varphi)^0$ is unipotent. Assume now that $\varphi(G)$ is realized by $(\mathrm{SL}_2, \mathrm{SL}_2/U_m)$. The exact sequence $0 \to \mathrm{Ker}\,\varphi \to G \to \varphi(G) \to \mathbf{1}$ gives a new exact sequence

$$0 \to U \to U_{\bullet}\mathrm{SL}_2 \xrightarrow{p} \mathrm{SL}_2 \to \mathbf{1}$$

and the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & U \arrow[r] \arrow[d] & U_{\bullet}\mathrm{SL}_2 \arrow[r, "p"] \arrow[d, "\psi"] & \mathrm{SL}_2 \arrow[r] \arrow[d] & \mathbf{1} \\ 0 \arrow[r] & \mathrm{Ker}\,\varphi \arrow[r] & G \arrow[r] & \varphi(G) \arrow[r] & \mathbf{1} \end{tikzcd} \tag{5.37.2}$$

where p is the projection and ψ is an isogeny. We set $\tilde{G} = U_{\bullet}\mathrm{SL}_2$. Let us determine $\tilde{H} = \psi^{-1}(H)$. $(\tilde{G}, \tilde{G}/\tilde{H})$ is an almost effective realization of G. Let $U = V_1 \oplus V_2 \oplus \cdots \oplus V_s$ be a decomposition of SL_2-module U into the direct sum of irreducible modules V_i with highest weight n_i and $n_1 \leq n_2 \leq \cdots \leq n_s$. We may assume $p(\tilde{H}) = U_m$. In fact $p(\tilde{H}) \subset U_m$ is obvious and if $p(\tilde{H}) \subsetneq U_m$, $\varphi(G)$ would be realized by $(\mathrm{SL}_2, \mathrm{SL}_2/U_{m'})$ with $1 \leq m' < m$. Let us notice $n_1 < n_2 < \cdots < n_s$ and $n_i - n_j \equiv 0 \bmod m$ for $1 \leq i, j \leq m$. In fact since the dimension of any U_Z-orbit is 1, the dimension of $U/U \cap \tilde{H}$ is 1. Moreover $U \cap \tilde{H}$ is $p(\tilde{H}) = U_m$-invariant. Since the operation $(U_{\bullet}\mathrm{SL}_2, U_{\bullet}\mathrm{SL}_2/\tilde{H})$ is almost effective, $\tilde{H}$ contains no normal subgroup of positive dimension of $U_{\bullet}\mathrm{SL}_2$. Therefore for any $1 \leq i \leq s$, the dimension of any V_i-orbit is 1 and $V_i \cap \tilde{H}$ is the unique $\left\{\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}\right\}$-invariant subspace of codimension 1 of V_i. More precisely let B be a Borel subgroup of SL_2 consisting of upper triangle matrices and C a Cartan subgroup of diagonal matrices. Let n_i be the highest weight of V_i. Hence

$$V_i = \mathbf{C}f^{(i)}_{ni} \oplus \mathbf{C}f^{(i)}_{n_1-2} \oplus \cdots \oplus \mathbf{C}f^{(i)}_{-ni}$$

where $f_l^{(i)}$ is a C-eigenvector of weight l. $V_i \cap \tilde{H} = \mathbf{C}f_{ni}^{(i)} \oplus \cdots \oplus \mathbf{C}f_{-ni+2}^{(i)}$. Thus $U \cap \tilde{H}$ contains a subspace

$$V' = \bigoplus_{i=1}^{s} \mathbf{C}f_{ni}^{(i)} \oplus \cdots \oplus \mathbf{C}f_{-ni+2}^{(i)}.$$

If there were an $1 \leq i \leq s-1$ such that $n_i = n_{i+1}$. Since $(V_{n_i} \oplus V_{n_{i+1}}) \cap \tilde{H}$ is of codimension 1 in $V_{n_i} \oplus V_{n_{i+1}}$, a non-zero linear combination $af_{-n_i}^{(i)} + bf_{-n_{i+1}}^{(i+1)}$ should be in $(V_{n_i} \oplus V_{n_{i+1}}) \cap \tilde{H}$. Then an U_m-invariant subspace generated by $U_m(af_{-n_i}^{(i)} + bf_{-n_{i+1}}^{(i+1)})$ should be in $(V_{n_i} \oplus V_{n_{i+1}}) \cap \tilde{H}$. But $H_m(af_{ni}^{(i)} + bf_{ni+1}^{(i+1)})$ is an irreducible SL_2-submodule of lowest weight $-n_i = -n_{i+1}$. Thus $U \cap \tilde{H}$ would contain a normal subgroup of positive dimension. This contradicts the almost effectiveness of the operation $(U_{\bullet}\mathrm{SL}_2, U_{\bullet}\mathrm{SL}_2/\tilde{H})$. If there were integers $1 \leq j, k \leq s$ such that $n_j \not\equiv n_k \bmod m$, since $V_{n_j} \oplus V_{n_k} \cap \tilde{H}$ is of codimension 1 in $V_{n_j} \oplus V_{n_k}$, a non-zero linear combination $af_{-n_i}^{(j)} + bf_{-n_k}^{(k)}$ should be in $(V_{n_j} \oplus V_{n_k}) \cap \tilde{H}$. Operating $\begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix} \in H_m$, we conclude $a\zeta^{-n_j}f_{-n_j}^{(j)} + b\zeta^{-n_k}f_{-n_k}^{(k)}$ is in $(V_{n_j} \oplus V_{n_k}) \cap \tilde{H}$. Therefore $f_{-n_j}^{(i)}$ and $f_{-n_k}^{(k)}$ would be in $(V_{n_j} \oplus V_{n_k}) \cap \tilde{H}$ hence the U_m-invariant subspace generated by $U_m f_{-n_j}^{(j)}$ or $U_m f_{-n_k}^{(k)}$ would be contained in $(V_{n_j} \oplus V_{n_k}) \cap \tilde{H}$. Since these U_m-invariant subspaces are SL_2-invariant, this contradicts the almost effectiveness of the operation. Since we have an isomorphism of the group $U_{\bullet}\mathrm{SL}_2$ by multiplying a scalar $a_i \neq 0$ on each U_i $(1 \leq i \leq s)$, we may assume $U \cap \tilde{H}$ is spanned by

$$\left(\bigoplus_{i=1}^{s} \mathbf{C}f_{ni}^{(i)} \oplus \mathbf{C}f_{ni-2}^{(i)} \oplus \cdots \oplus \mathbf{C}f_{-ni+2}^{(i)}\right), \quad f_{-ni}^{(i)} - f_{-ni+1}^{(i+1)} \quad (1 \leq i \leq s-1).$$

This space uniquely determined when we fix m and $(n_1, n_2, \cdots, n_s)$, is denoted by U'.

U_1 being $\left\{\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \mid b \in k\right\}$, we get

(5.37.3) $$\tilde{H}^0 = U'_{\bullet}U.$$

or

(5.37.4) $\tilde{H}^0$ coincides with the inverse image by ψ of a codimension 1 subgroup of $W \subset U/U'_{\bullet}U_1$ where $\psi_i U_{\bullet}U_1 \to (U/U')_{\bullet}U_1$ is the natural projection.

If we are in (5.37.3), $\tilde{H} = U'_{\bullet}U_m$. In fact, as $\psi(\tilde{H}) = U'_m$, there exists an element $\left(v, \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix}\right) \in \tilde{H}$ such that ζ is a primitive m-th root of unity. By adding an element of $U \cap \tilde{H}$, we may assume $v = \sum_{i=1}^{s} a_i f_{-ni}^{(i)}$. There exists an integer $l \geq 1$ such that $\left(v, \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix}\right)^{ml} \in \tilde{H}^0$. Thus,

$$\left(v, \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix}\right)^{ml} = \left(\sum_{i=1}^{s} a_i f^{(i)}_{-n_i}, \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix}\right)^{ml} = \left(ml \sum_{i=1}^{s} \zeta^{-n_i ml} a_i f^{(i)}_{-n_i}\right),$$

$\begin{pmatrix} \zeta^{ml} & 0 \\ 0 & \zeta^{-ml} \end{pmatrix} = \left(ml \sum_{i=1}^{s} a_i f^{(i)}_{-n_i}, \mathrm{I}_2\right) \in \tilde{H}^0$. Therefore $mlv \in U \cap \tilde{H} = U'$, $v \in U \cap \tilde{H} = U'$ and consequently $\left(0, \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix}\right) \in \tilde{H}$.

If we are in (5.37.4), considering the weight we conclude $n_1 \equiv n_2 \equiv \cdots \equiv n_s \equiv -2 \bmod m$. Furthermore we may assume

$$W = \left\{\left(a_1 \bar{f}_{-n_1} + a_2 \bar{f}_{-n_2} + \cdots + a_s \bar{f}_{-n_s}, \begin{pmatrix} \zeta & b \\ 0 & \zeta \end{pmatrix}\right) \right.$$
$$\left. \in U/U'_{\cdot} U_m \mid a_1 = a_2 = \cdots = a_s = b \in k, \zeta^m = 1\right\},$$

since we have automorphisms of SL_2-module U of multiplying non-zero constants c_i on each factor U_i and since $\tilde{H}$ contains no normal subgroup of positive dimension. By the same argument, in case (5.37.3), we have $\tilde{H}$ coincides with the inverse image by ψ of a codimension 1 subgroup $W \subset U/U'_{\cdot} U_m$ containing $\left\{\begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix} \mid \zeta^m = 1\right\}$ where ψ: $U_{\cdot} U_m \to (U/U')_{\cdot} U_m$ is the natural projection.

Let us study operation (5) satisfying (5.37.3). As we have seen above under (5.37.3) $\tilde{H}$ is determined when we fix the representation U and m. As in Corollary (4.17), on $\mathrm{F}_{m,n}$ $(n \geq m \geq 1)$ operates the following group $\mathscr{G}$

$$x' = \frac{ax+b}{cx+d}, \qquad y' = \frac{y}{(cx+d)^m},$$
$$z' = \frac{z + y^k \varphi_l(x) + y^{k-1} \varphi_{m+l}(x) + \cdots + \varphi_n(x)}{(cx+d)^n}$$

where $\varphi_i(x)$ is a polynomial of degree $\leq i$, $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2$ and k is an integer with $n = km + l$, $0 \leq l < n$. The group $\mathscr{G}$ contains SL_2 as

$$x' = \frac{ax+b}{cx+d}, \quad y' = \frac{y}{(cx+d)^m}, \quad z' = \frac{z}{(cx+d)^n}.$$

The unipotent radical $\mathscr{U}$ of $\mathscr{G}$ consists of the following transformations and has 1-dimensional orbits;

$$x' = x, \quad y' = y, \quad z' = z + y^k \varphi_l(x) + y^{k-1} \varphi_{m+l}(x) + \cdots + \varphi_n(x).$$

The SL_2-module $\mathscr{U}$ is decomposed into the direct sum $\mathscr{U} = \bigoplus_{i=0}^{k} \mathscr{U}_{im+l}$, $\mathscr{U}_{im+l}$ being the irreducible SL_2-module of dimension $im+l+1$ consisting of the operations $x'=x$, $y'=y$, $z'=z+y^{(k-i)}\varphi_{im+l}(x)$. The stabilizer group at $(x, y, z)=(0, 1, 1)$ is $\left\{(u, g)\in \mathscr{U}\,\mathrm{SL}_2 \,|\, g=\begin{pmatrix}\zeta & 0\\ c & \zeta\end{pmatrix}, \zeta^m=1,\ u\colon x=x', y=y', z'=z+y^k\varphi_l(x)+y^{k-1}\varphi_{m+l}(x)+\cdots+\varphi_n(x) \text{ with } 1=1+\varphi_l(0)+\varphi_{m+l}(0)+\cdots+\varphi_n(0)\right\}$. Namely $\mathscr{G}$ satisfies (5.37.3). This shows, if we take $l\equiv n_1\equiv n_2\equiv\cdots\equiv n_s \bmod m$ and choose $n>n_s$, the conjugacy class of the operation $(\tilde{G}, \tilde{G}/\tilde{H})$ is contained in the operation of $\mathscr{G}$ hence in an operation of Theorem (2.2) if $m\geq 2$. For $m=1$, argue as in Proposition (5.35).

Now let us study operation (5) in (5.37) with (5.37.4). We shall show in this case the conjugacy class of the operation is contained in $(\mathrm{Aut}^0\,(\mathrm{E}'^l_m, \mathrm{F}'_m), \mathrm{F}'_m)$ for a big integer l. It follows from Corollary (4.23), the semi-simple part of $\mathrm{Aut}^0(\mathrm{E}'^l_m, \mathrm{F}'_m)$ is SL_2 and the unipotent radical is SL_2-module $\bigoplus_{j=0}^{\infty} H^0(\mathbf{P}^1, \mathscr{O}(lm-2-jm))$. Take l big enough so that SL_2-module U is contained in $\bigoplus_{j=0}^{\infty} H^0(\mathbf{P}^1, \mathscr{O}(lm-2-jm))$. This is possible as we have $n_1\equiv n_2\equiv\cdots\equiv n_s\equiv -2 \bmod m$. Consider the operation of $\bigoplus_{j=0}^{\infty} H^0(\mathbf{P}^1, \mathscr{O}(lm-2-jm)).\mathrm{SL}_2$ on E'^l_m. Since SL_2 has no 2-dimensional orbit on E'^l_m covering the SL_2-open orbit on F'_m,

$$\bigoplus_{j=0}^{\infty} H^0(\mathbf{P}^1, \mathscr{O}(lm-2-jm)).\mathrm{SL}_2$$

satisfies (3.37.4). Hence the conjugacy class of G is contained in the conjugacy class of $(\mathrm{Aut}^0\mathrm{E}'^l_m, \mathrm{E}'^l_m)$.

Operation (4) is treated similarly hence we omit the proof for operation (4). We can prove the assertion for operation (3) similarly. Here is a sketch. As above, U is abelian, we may assume $\mathrm{Ker}\,\varphi$ is unipotent and we have a diagram,

$$\begin{array}{ccccccc}
0\to & U & \to & U.(\mathbf{G}_m\times \mathrm{SL}_2) & \xrightarrow{p} & \mathbf{G}_m\times\mathrm{SL}_2 & \to \mathbf{1}\\
 & \downarrow & & \downarrow{\scriptstyle\psi} & & \downarrow & \\
0\to & \mathrm{Ker}\,\varphi & \to & G & \longrightarrow & \varphi(G) & \to \mathbf{1}.
\end{array}$$

Let $U=V_1\oplus V_2\oplus\cdots\oplus V_s$ be a decomposition of SL_2-module U into the direct sum of irreducible modules V_i with highest weight n_i and $n_1\leq n_2\leq\cdots\leq n_s$. By Schur's Lemma $\mathbf{G}_m$ operates on each V_i by weight d_i. The argument for operation (5) applied to this case gives $n_1=n_2=\cdots=n_s$ and d_i are different each other. And so on to conclude G is contained in $(\mathrm{Aut}^0\,\mathrm{L}'_{m,n}, \mathrm{L}'_{m,n})$ where $n=n_1=n_2=\cdots=n_s$ and m is a sufficiently large integer.

Let us treat operation (6). Let $m\geq n\geq 0$ be integers and V_m, V_n the irreducible SL_2-modules with highest weight m, n. $V_m\otimes V_n$ is an $\mathrm{SL}\times\mathrm{SL}_2$-

module. The semi-direct product $(V_m \otimes V_n)_{\textstyle .}(\mathrm{SL}_2 \times \mathrm{SL}_2)$ operates on $\mathrm{L}'_{m,n}$ and $\mathrm{L}'_{m,n}$ is a homogeneous space of this algebraic group (cf. Corollary (4.12)). To see this, we notice that $p_i: \mathbf{P}^1 \times \mathbf{P}^1 \to \mathbf{P}^1$ being the projections $(i=1, 2)$, $\mathrm{SL}_2 \times \mathrm{SL}_2$-module $H^0(\mathbf{P}^1 \times \mathbf{P}^1, p_1^* \mathcal{O}_{\mathbf{P}^1}(m) \otimes p_2^* \mathcal{O}_{\mathbf{P}^2}(n))$ is isomorphic to $H^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(m)) \otimes H^0(\mathbf{P}^1. \mathcal{O}_{\mathbf{P}^1}(n))$ by the Künneth formula. Hence $V_m \otimes V_n$ operates on the fibre space $\mathrm{L}'_{m,n} \to \mathbf{P}^1 \times \mathbf{P}^1$. $\mathrm{SL}_2 \times \mathrm{SL}_2$ also operates on $\mathrm{L}'_{m,n}$ as we have seen in Corollary (4.12). In terms of local coordinate the operation is written;

$$x' = \frac{a_1 x + b_1}{c_1 x + d_1}, \quad y' = \frac{a_2 y + b_2}{c_2 y + d_2}, \quad z' = \frac{z + F(x, y)}{(c_1 x + d_1)^m (c_2 y + d_2)^n}.$$

$\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \in \mathrm{SL}_2 \times \mathrm{SL}_2$, $F(x, y) \in k[x, y]$ such that the degree of $F(x, y)$ in $x \leq m$, the degree of $F(x, y)$ in $y \leq n$. The stabilizer group at $(0, 0, 0)$ is $K = \{\Phi_{F\textstyle .} B \times B \mid F(0, 0) = 0\}$ where B is a Borel subgroup consisting of the lower triangular matrices and Φ_F is the transformation sending $x \mapsto x$, $y \mapsto y$, $z \mapsto z + F(x, y)$. The weight of constant term transformation Φ_c, $c \in k$ is (m, n) with respect to a Cartan subgroup $\left\{ \begin{pmatrix} t_1 & \\ & t_1^{-1} \end{pmatrix} \times \begin{pmatrix} t_2 & \\ & t_2^{-1} \end{pmatrix} \right\} \subset B \times B$. Since $G/H \to G/U_Z H$ is an $\mathbf{A}^1$-bundle, $(\mathrm{Ker}\,\varphi)^0$ is solvable and the dimension of a maximal torus is at most 1 (see Lemma (1.21), Umemura [U3]). The unipotent radical of $(\mathrm{Ker}\,\varphi)^0$ is abelian (see Umemura [U3]). For the same reason as above, we may assume $(\mathrm{Ker}\,\varphi)^0$ is unipotent. The same argument as above gives us an isogeny $\psi: U_{\textstyle .}\mathrm{SL}_2 \to G$. Let $\tilde{H} = \psi^{-1}(H)$ and modulo finite group we embed $(\tilde{G}, \tilde{G}/\tilde{H})$ into the above automorphism group of $\mathrm{L}'_{m,n}$ for suitable m, n. It follows from our assumption, letting $p: U_{\textstyle .}\mathrm{SL}_2 \to \mathrm{SL}_2$ be the projection $p(\tilde{H}) = \mathbf{G}_m$. We may assume $\mathbf{G}_m$ is the diagonal subgroup of SL_2. Since we have $\mathbf{1} \to U \cap \tilde{H} \to \tilde{H} \to \mathbf{G}_m \to \mathbf{1}$, a maximal torus of $\tilde{H}$ is 1-dimensional and we may assume $\tilde{H} = (U \cap \tilde{H})_{\textstyle .}\mathbf{G}_m$. Let $U = V_1 \oplus V_2 \oplus \cdots \oplus V_s$ be a decomposition of SL_2-module U into the direct sum of irreducible modules V_i of highest weight $n_1 \leq n_2 \leq \cdots \leq n_s$. By the same argument as in the proof for operation (5) $n_1 < n_2 < \cdots < n_s$. Since $\tilde{H} \cap U$ is $\mathbf{G}_m$-invariant subgroup of codimension 1 of U, all the weight except for at most 1 appear in $\tilde{H} \cap U$. Let l be the weight such that l appears in $U/\tilde{H} \cap U$. Since $(\tilde{G}, \tilde{G}/\tilde{H})$ is almost effective, $\tilde{H}$ contains no normal subgroup of positive dimension hence the weight l appears in V_i, $1 \leq i \leq s$. Therefore $n_i \equiv n_j \bmod 2$. Let $i: \mathrm{SL}_2 \hookrightarrow \mathrm{SL}_2 \times \mathrm{SL}_2$, $i(g) = (g, {}^t g^{-1})$, $g \in G$. $V_m \otimes V_n$ is considered as an SL_2-module by i. By counting the dimension of eigenspaces, we conclude, as SL_2-module, $V_m \otimes V_n$ is isomorphic to the direct sum $V_{(m+n)} \oplus V_{(m+n-2)} \oplus \cdots \oplus V_{(m-n)}$, where $V_{(j)}$ denote the irreducible SL_2-module with highest weight j.

The weight of constant term transformation, which is (m, n) as $\mathrm{SL}_2\times \mathrm{SL}_2$-module, is $m-n$. Now we take m, n so that $m+n\geq n_s\geq n_1\geq l=m-n$. Then $U=V_1\oplus\cdots\oplus V_s$ is an SL_2-submodule of $V_m\otimes V_n$ hence $U_{\bullet}\mathrm{SL}_2$ is a subgroup of $(V_m\otimes V_n)$ $(\mathrm{SL}_2\otimes\mathrm{SL}_2)$. The intersection of the stabilizer group K and $U_{\bullet}\mathrm{SL}_2$ may not be $\tilde{H}$ but as in the proof for operation (5) by replacing $\tilde{H}$ by an automorphim of the SL_2-module U, we may assume $\tilde{H}=K\cap U_{\bullet}\mathrm{SL}_2$ as desired.

Let us give an outline of the proof of for operation (7) in (3.57). As above we treat the case where there exists an isogeny $\psi\colon \tilde{G}=U_{\bullet}\mathrm{SL}_2\to G$. Let $\tilde{H}=\psi^{-1}(H)$. U is abelian as before and $U\cap\tilde{H}$ is D_∞-invariant. Therefore $U=V_{n_1}\oplus\cdots\oplus V_{n_s}$ with $n_1<\cdots<n_s$ as above. But the weight appearing in $U/U\cap\tilde{H}$ is necessarily 0, n_i's are even and $n_i\equiv n_j$ mod 4 since $\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\in \mathrm{D}_\infty$. By the same argument as for operation (6) we may assume $\tilde{H}=(U\cap\tilde{H})\mathrm{D}^\infty$. Let us embed SL_2 in SL_3 by the degree 3 irreducible representation. Let W be the natural irreducible representation of SL_3 of degree 3. As in 2 variable case, J'_m is a homogeneous space of $S^m(W)_{\bullet}\mathrm{SL}_3$ where $S^m(W)$ is the m-th symmetric power of W. In terms of local coordinate, the operation of $S^m(W)_{\bullet}\mathrm{SL}_2$ is written;

$$x'=\frac{ax+by+c}{gx+hy+i},\quad y'=\frac{dx+ey+f}{gx+hy+i},\quad z'=\frac{z+F(x,y)}{(gx+hy+i)^m}$$

where $F(x, y)$ is a polynomial of degree $\leq m$. We can thus determine the stabilizer group K of $S^m(W)_{\bullet}\mathrm{SL}_2$ at $(0, 0, 0)$. By the above inclusion $\mathrm{SL}_2\hookrightarrow\mathrm{SL}_3$, $S^m(W)$ is an SL_2-module. By counting the dimension of the $\mathbf{G}_m$-eigen-spaces, we get Sublemma (5.37.5). As SL_2-module, $S^m(W)$ is isomorphic to $V_{2m}\oplus V_{(2m-4)}\oplus\cdots$.

By taking as m a sufficiently large even or odd number according as $n_1\equiv 0$ or $\equiv 2$ mod 4, U is an SL_2-submodule of $S^m(W)$ hence $U_{\bullet}\mathrm{SL}_2$ is a subgroup of $S^m(W)_{\bullet}\mathrm{SL}_2\times\mathrm{SL}_2$. Then by replacing $\tilde{H}$ by an automorphism of SL_2-module U, we may assume $\tilde{H}=K\cap U_{\bullet}\mathrm{SL}_2$ q.e.d.

Next case that we want to treat is

Subcase (5.38). *The algebraic group $\varphi(G)\subset \mathrm{Aut}\, G/U_Z H$ is solvable.*

Proposition (5.38.1). *If $\varphi(G)$ is solvable, then G is contained in one of the conjugacy class of Theorem* (2.2).

Proof. The proposition is proved by a case by case verification for $\varphi(G)$. Since the proofs are similar, we give the proofs for 2 cases: $\varphi(G)$ reductive and $\varphi(G)$ unipotent.

Lemma (5.38.2). *If $\varphi(G)$ is solvable and reductive, then $\varphi(G)=\mathbf{G}_m\times\mathbf{G}_m$ and the conjugacy class of $(G, G/H)$ is contained in one of the conjugacy classes of the operations of Theorem* (2.2).

Proof. $\varphi(G)$ is a torus in $\operatorname{Aut} G/U_Z H$ which operates generically transitively hence $\varphi(G)=\mathbf{G}_m\times\mathbf{G}_m$. The morphism $(G, G/H)\to(G, G/U_Z H)$ induces an exact sequence:

$$\mathbf{1}\to N\to G\xrightarrow{\varphi}\mathbf{G}_m\times\mathbf{G}_m\to\mathbf{1},$$

where $N=\operatorname{Ker}\varphi$. Let T be a maximal torus of G. Then T is mapped surjectively onto $\mathbf{G}_m\times\mathbf{G}_m$ by φ and therefore there exists a 2-dimensional torus $T'\subset T$ which is mapped surjectively onto $\mathbf{G}_m\times\mathbf{G}_m$. Since N^0 operates on each fibre of f which is isomorphic to $\mathbf{A}^1$, N^0 is solvable hence G is solvable. Let us first assume rank $G=2$. The unipotent radical U of G is contained in N and G is the semi-direct product $U.T'$ (cf. Borel [B]). Since $U\subset N$ and the fibre of f is $\mathbf{A}^1$, U is abelian (cf. Umemura [U3]). Let $W\subset U$ be a 1-dimensional T'-invariant subspace. Thus the group $W.T'$ operates transitively on G/H because the orbit WH is a fibre of f and $T'.H$ is 2-dimensional in the horizontal direction. Since $\dim W.T'=3$, $W.T'\cap H=H'$ is a finite group. By Borel [B], we may assume $H'\subset T'$. Then $G/H\simeq W.T'/H'\simeq W\times(T'/H')$. Taking an appropriate isomorphism $T'\simeq\mathbf{G}_m\times\mathbf{G}_m$, we may assume

$$H'=\{(\xi_1,\xi_2)\in\mathbf{G}_m\times\mathbf{G}_m\mid\xi_1^m=1,\ \xi_2^m=1\}.$$

Let (z, x', y') be the natural coordinate system on $W.T'\simeq W.\mathbf{G}_m\times\mathbf{G}_m$, then $(z, x, y)=(z, x'^m, y'^n)$ is a coordinate system on $W.T'/H'$. The operation of T' is $(z, x, y)\mapsto(t_1^a t_2^b z, t_1^m x, t_2^n y)$ for $(t_1, t_2)\in\mathbf{G}_m\times\mathbf{G}_m=T'$. Let us describe the operation of U. Then, U is a unipotent algebraic subgroup of $\mathbf{A}^1$-bundle automorphism of G/H over $G/U_Z H$. We get a commutative diagram:

$$\begin{array}{ccc}
G/H & \xrightarrow{\quad f\quad} & G/U_Z H\\
\wr\| & & \wr\|\\
W\times T'/H' & & T'/H'\\
\wr\| & & \wr\|\\
\mathbf{A}^1\times\mathbf{A}'^1\times\mathbf{A}'^1 & \xrightarrow{\text{projection}} & \mathbf{A}'^1\times\mathbf{A}'^1,
\end{array}$$

where $\mathbf{A}'^1=\mathbf{A}^1$-a point. U is abelian hence isomorphic to a product of copies of $\mathbf{G}_a$. Now it is easy to see changing the fibre coordinate if necessary, an operation of a product of copies of $\mathbf{G}_a$ on $\mathbf{A}^1$-bundle $\mathbf{A}^1\times \mathbf{A}'^1\times\mathbf{A}'^1\to\mathbf{A}'^1\times\mathbf{A}'^1$ is contained in the following group for a sufficiently big integer N:

$$(z, x, y)\longmapsto(z+f(x, y), x, y)$$

where $f(x, y)\in k[x, y]$ and $\deg f(x, y)\leq N$ (see Section 2, Umemura [U3]. The remaining case is rank $G=3$. In the proof for rank $G=2$, when we take eigen-space W, we choose W so that W is a T-eigen-space. Then the above proof works similarly.

Lemma (5.38.3). *If $\varphi(G)$ is unipotent, then the conjugacy class of G is contained in one of the conjugacy classes of the operations of Theorem* (2.2).

Proof. Let us put $G=U$, $U_2=\varphi(G)$ and $N=U_1$. As in the proof of Proposition (5.37.1), we may assume U_1 to be unipotent. We have an exact sequence:

(5.38.4) $$\mathbf{1}\to U_1\to U\xrightarrow{\varphi} U_2\to\mathbf{1}.$$

Let W_1 be a 1-dimensional subgroup of U_2 contained in the center of U_2. By Umemura [U3], there exists a 1-dimensional subgroup W_2 of U_2 such that the semi-direct product $W_1.W_2$ operates transitively on G/U_ZH which is isomorphic to $\mathbf{A}^2$. By Umemura [U3], the extension (5.38.4) splits over W_1 and W_2. Let s_i be a section of $\varphi^{-1}(W_i)\xrightarrow{\varphi} W_i$. Let $U'=U_1.s_1(W_1).s_2(W_2)$. Then U' is a closed subgroup of U and operates transitively on G/H. Let $H'=U'\cap H$. Then the exact sequence (5.38.4) gives

(5.38.5) $$\begin{array}{ccccc} \mathbf{1}\to U_1\longrightarrow & U' & \longrightarrow & \mathbf{G}_a\times\mathbf{G}_a\to 0 \\ & \Vert & & \wr\vert & \\ & U_1.s_1(W_1).s_2(W_2) & \xrightarrow{\varphi} & W_1.W_2\to 0. & \end{array}$$

Since $W_1.W_2$ operates effectively on $G/U_ZH\simeq\mathbf{A}^2$, $H^1\subset U_1$. The map $\mathbf{A}^1\times\mathbf{A}^1\times\mathbf{A}^1\simeq\mathbf{A}^1\times\mathbf{A}^1\times(U_1/H')\to(s_1(W_1).s_2(W_2).U_1)/H'=U'/H'=U/H$ given by $(x, y, u_2H')\mapsto s_1(x)s_2(y)u_2H$, is an isomorphism of $\mathbf{A}^1$-bundle over $\mathbf{A}^2$ (see the diagram

(5.38.6) $$\begin{array}{ccc} \mathbf{A}^2\times\mathbf{A}^1 & \longrightarrow & U/H \\ \downarrow & & \downarrow \\ \mathbf{A}^2 & \simeq & U/U_ZH). \end{array}$$

Thus it defines a coordinate system on U/H. By the Proof of Lemma (4.4) there exists an integer N such that the operation of U_1 on U/H is contained in $(x, y, z)\mapsto(x, y, z+f(x, y))$ where $f(x, y) \in k[x, y]$, $\deg f \leq N$. To show Lemma (5.38.3), it is sufficient to prove that there exists an algebraic group L contained in Cr_3 such that for any $\mathbf{G}_a$ contained in U, it is contained in L. It is sufficient to check this for those $\mathbf{G}_a$'s generating U. Let $M \simeq \mathbf{G}_a$ be a subgroup of U. If M is not contained in U_1, then $\varphi\colon M \to \varphi(M)$ is an isomorphism because we are in characteristic 0. Hence by the isomorphism (5.38.6), the operation of M on U/H is given by $(x, y, z) \mapsto (g(\lambda, x, y), h(\lambda, x, y, z))$ with $g(\lambda, x, y) \in \mathbf{A}^2$, $h(\lambda, x, y, z) \in \mathbf{A}^1$. Therefore by the Proof of Proposition (2.26), Umemura [U3], there exists an integer N' independent of the algebraic group $M \subset U$ such that the operation of $\varphi(M)$ on $U/U_Z H \simeq \mathbf{A}^2$, $(x, y)\mapsto g(\lambda, x, y)$ is contained in the following group:

$$(x, y)\longmapsto(x, y+i(y)) \qquad i(y) \in k[y],\ \deg i(y) \leq N'.$$

As $h(\lambda, x, y, z)$ is an affine bundle morphism, $h(\lambda, x, y, z)=a(x, y, \lambda)z+b(z, y, \lambda)$ where $(x, y) \in \mathbf{A}^2$, $\lambda \in M \simeq \mathbf{G}_a$ and $a(x, y, \lambda) \in k[x, y, \lambda]$, $b(x, y, \lambda) \in k[x, y, \lambda]$. Since $h(\lambda, x, y, z)$ is an automorphism for any $\lambda \in M \simeq \mathbf{G}_a$, $a(x, y, \lambda)$ is invertible. Hence $a(x, y, \lambda)$ is a constant independent of x, y, λ. It follows now from $h(\lambda+\lambda', x, y, z) = h(\lambda, x, y, z) \circ h(\lambda', x, y, z)$ that $a(x, y, \lambda)=1$. Hence for a sufficiently large integer N'', M is contained in the algebraic group generated by:

$$(x, y, z)\longmapsto(x, y+f(x), z)$$
$$(x, y, z)\longmapsto(x, y, z+b(x, y))$$

where $f(x) \in k[x]$, $b(x, y) \in k[x, y]$ such that

$$\deg f(x), \qquad \deg b(x, y) \leq N''.$$

Since we have to check for finitely many M's, combining what we have done for U_2 with the above result, for a sufficiently large integer N, U is contained in an algebraic group generated by:

$$(x, y, z)\longmapsto(x, y+f(x), z)$$
$$(x, y, z)\longmapsto(x, y, z+b(x, y))$$

where $f(x) \in k[x]$, $b(x, y) \in k[x, y]$

$$\deg f(x), \qquad \deg b(x, y) \leq N.$$

Lemma now follows from Corollary (4.17).

(5.39) *$\varphi(G)$ is neither solvable nor reductive.*

Since the unipotent radical U'_Z of $\varphi(G)$ is normal, we have on G/U_ZH either the orbits of U'_Z are 1-dimensional or U'_Z operates transitively on G/U_ZH.

Sublemma (5.39.1). *If the orbits of U'_Z on G/U_ZH are 1-dimensional, then G/U_ZH is isomorphic to F'_m $(m=0, 1, 2\cdots)$. The conjugacy class of G is contained in one of the conjugacy classes of the operations of Theorem* (2.2).

Proof. The first assertion is proved in Umemura [U3]. Let us briefly review it. Let $G'=\varphi(G)$ and $G'/H'=G/U_ZH$. Then we get a morphism $f'\colon G'/H'\to G'/U'_ZH'$ and the fibre of f' is $\mathbf{A}^1$. f' defines an morphism of algebraic operations $(\varphi', f')\colon (G', G'/H')\to(\mathrm{PSL}_2, \mathbf{P}^1)$. Since f' is an $\mathbf{A}^1$-bundle, $(\mathrm{Ker}\,\varphi')^0$ is solvable. Therefore, as we assume G' is not solvable, φ' is surjective (cf. [U3]). Now it is easy to see the affine bundle $f'\colon G'/H'\to\mathbf{P}^1$ is isomorphic to $\mathrm{F}'_m\to\mathbf{P}^1$ for a certain $m\geq 0$. The second assertion is proved by Proposition (5.36).

Sublemma (5.39.2). *If U'_Z operates transitively on G/U_ZH, $(\varphi(G), G/U_ZH)$ is the special affine transformation group or the affine transformation group of* 2 *variables and the group G is contained in one of the conjugacy classes of the operations of Theorem* (2.2).

Proof. In fact by Corollary (1.13), Umemura [U3]. $(\varphi(G), G/U_ZH)$ is contained in the affine transformation group of 2 variables, $U_Z\simeq\mathbf{G}_a^2$ and putting $G'=\varphi(G)$ and $G'/H'=G/U_ZH$, we get H' is the reductive part of G' and the representation $H'\to\mathrm{GL}(U'_Z)$ is faithful. Hence, $H'=\mathrm{SL}_2$ or GL_2 because we assume G' is not solvable and the first assertion is proved. Let us limit ourselves to the case where G' is the special affine transformation group of 2 variables. The affine transformation group case is treated similarly. Then, we are given an exact sequence:

$$\mathbf{1}\to N\to G\xrightarrow{\varphi} U'_Z.\mathrm{SL}_2\to\mathbf{1} \tag{5.39.3}$$

where N is the kernel of φ. Since $U_Z.\mathrm{SL}_2$ is simply connected, N is connected. N is solvable, the unipotent radical of N is abelian and the rank of N is at most equal to 1 (Lemma (1.21), Umemura [U3]). We assume for simplicity N is unipotent. The case rank $N=1$ is treated similarly. Since $\dim N/N\cap H=1$, $\varphi(H)$ is of dimension 3 and contained in the stabilizer of a point of the operation $(U'_Z.\mathrm{SL}_2, U'_Z.\mathrm{SL}_2/\mathrm{SL}_2)$ and hence we may asssume $\varphi(H)\subset\mathrm{SL}_2$ and hence $\varphi(H)=\mathrm{SL}_2$. Therefore we have an exact sequence:

$$1 \to N \cap H \to H \xrightarrow{\varphi} \mathrm{SL}_2 \to 1.$$

In particular H is connected, $N \cap H$ is an SL_2-invariant subspace of codimension 1 of N and we may assume $H=(N\cap H).\mathrm{SL}_2$. Let $N=\bigoplus_{i=0}^{r} V_i$ be a decomposition into a direct sum of irreducible modules of SL_2-module N. Then as we have seen above, there exists an $0 \leq i \leq r$ such that $\dim V_i=1$. When we define a coordinate system on $N/N\cap H$, we use a coordinate system z on V_i. Let $U'_Z=\mathbf{G}_a \oplus \mathbf{G}_a$ and take sections s_1 on $\mathbf{G}_a \oplus 0$ and s_2 on $0 \oplus \mathbf{G}_a$ of φ.

$$\begin{array}{ccccccc} \mathbf{A}^2 \times \mathbf{A}^1 & \xrightarrow{\sim} & (N/N\cap H)\times \mathbf{G}_a \times \mathbf{G}_a & \xrightarrow{\sim} & V_i \times (\mathbf{G}_a \times \mathbf{G}_a) & \xrightarrow{\sim} & G/H \\ ((x, y), z) & \longmapsto & (z, x, y) & \longmapsto & (z, x, y) & \longmapsto & z s_1(x) s_2(y) H \end{array}$$

By using this coordinate system, as in the Proof of Lemma (5.38.3), the unipotent radical U of G is contained in the following algebraic group $\mathscr{G}_n$:

$$(x, y, z) \longmapsto (x+a, y+b, z+f(x, y))$$

where $a, b \in \mathbf{G}_a, f(x, y) \in k[x, y]$ and $\deg f(x, y) \leq n$. Let us describe the operation of semi-simple part of G which is isomorphic to SL_2. We may assume, by the definition of the coordinate system on G/H, that SL_2 operation on G/H is given by: $(x, y, z) \mapsto (ax+by, cx+dy, h(g, z, x, y))$ for $g=\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2$ where h is a regular function on $\mathrm{SL}_2 \times \mathbf{A}^3$. Since SL_2 operation is an automorphism group of trivial $\mathbf{A}^1$-bundle $G/H \to G/U_Z H$, $h(g, z, x, y)$ is an affine transformation: there exist regular functions $a(g, x, y), b(g, x, y)$ on $\mathrm{SL}_2 \times \mathbf{A}^2$ such that $a(g, x, y)$ is invertible (i.e. $a(g, x, y)^{-1}$ is also regular) and such that $h(g, z, x, y)=a(g, x, y)z+b(g, x, y)$. The regularity of $a(g, x, y)^{-1}$ shows $a(g, x, y)$ does not depend on x, y hence we write $a(g)=a(g, x, y)$. Since $(x, y, z) \mapsto (ax+by, cx+dy, h(g, z, x, y))$ for $g \in \mathrm{SL}_2$ is an operation, $\left\{\begin{pmatrix} a(g) & b(g, x, y) \\ 0 & 1 \end{pmatrix}\right\}_{g \in \mathrm{SL}_2}$ is a cocycle. In particular $g \mapsto a(g)$ is a representation of SL_2 and hence $a(g)=1$. Now we have show SL_2 operation is written as follows:

$$(x, y, z) \longmapsto (ax+by, cx+dy, z+\sum_{\text{finite sum}} a_{i,j}(g) x^i y^j).$$

Hence taking N sufficiently big, G is contained in the algebraic group $\mathscr{G}_N$ generated by the following:

$$(x, y, z) \longmapsto (ax+by, cx+dy, z)$$

$$(x, y, z) \longmapsto (x+\xi, y+\eta, z+f(x, y))$$

where $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2$, $\xi, \eta \in \mathbf{G}_a$, $f(x, y) \in k[x, y]$. Sublemma now follows from Corollary (4.10).

Conclusion (5.40). *Let $(G, G/H)$ be a realization of an algebraic group G of de Jonquières type in* Cr_3. *If G is not reductive and if an (or equivalently any) orbit of the center of the unipotent radical of G is* 1-*dimensional, then G is contained in one of the conjugacy classes of the operation of Theorem* (2.2).

In fact this is a result of Subcases (5.37), (5.38) and (5.39).
It follows from Conclusions (5.28), (5.32) and (5.40).

Conclusion (5.41). *Let G be an algebraic group of de Jonquières type in* Cr_3. *If G is generically transitive, then G is contained in one of the conjugacy classes of the operations of Theorem* (2.2).

Classification of even generically intransitive algebraic groups in Cr_3.

We examine the generically intransitive case. Let (G, X) be a realization of an algebraic group $G \in \mathrm{Cr}_3$.

Case (5.42). *G has a 2-dimensional orbit on X.*

If G has a 2-dimensional orbit on X, then there exists a G-invariant non-empty open subset X' of X such that all the G-orbits on X' are 2-dimensional (see Borel [B]). As also (G, X') realizes G, we may assume that all the G-orbits on X are 2-dimensional. Let U_Z denote the center of the unipotent radical U of G.

Subcase (5.42.1). *U_Z has a 2-dimensional orbit.*

Replacing X by a suitable open set, we may assume by a Theorem of Rosenlicht [R] that the quotient $f: X \to G\backslash X$ exists. As we assume the existence of a 2-dimensional U_Z-orbit, there exists a non-empty open set W of $G\backslash X$ such that on each fibre $f^{-1}(w)$, U_Z has a 2-dimensional orbit for $w \in W$. Replacing X by $f^{-1}(W)$, we may assume that on any G-orbit on X, U_Z has a 2-dimensional orbit. We can now apply Corollary (1.13), Umemura [U3], to the operations of G on G-orbits on X. Here are some conclusions: (1) The G-orbits coincide with the U_Z-orbits, (2) $G\backslash X = U_Z\backslash X$, (3) The fibre $f^1(w)$ is isomorphic to $\mathbf{A}^2$ for $w \in G\backslash X$, (4) Since the unipotent radical U operates on each fibre $\mathbf{A}^2$ of f through translation, U is abelian and $U = U_Z$. Let us show that $\mathbf{A}^2$-bundle $f: X \to G\backslash X$ is locally trivial (of course for the Zariski topology). In fact, let w be a point of $G\backslash X$ which we fix. We can find a 2-dimensional subgroup $U'\subset U_Z$ such that U' operates transitively on $f^{-1}(w)$. Then, there exists a non-empty open set $W\subset G\backslash X$ such that U' operates on each fibre $f^{-1}(w)$ transitively for $w\in W$. Therefore on W the affine bundle f is a principal U'-bundle. Since U' is abelian, the principal U'-bundle $f\colon f^1(W)\to W$ is locally trivial. Now we may assume $f\colon X\to G\backslash X$ is trivial; $X=\mathbf{A}^2\times(G\backslash X)$ and $f\colon \mathbf{A}^2\times(G\backslash X)\to G\backslash X$ is a projection. Let $G=U_{\bullet}G_r$ where G_r is a reductive part of G. It follows from Lemma (1.13) and Lemma (1.21) [U3] that G_r operates effectively on each fibre and G_r is a subgroup of the affine transformation group of 2 variables and hence G_r is isomorphic to $\mathbf{G}_m$, $\mathbf{G}_m\times\mathbf{G}_m$, SL_2, GL_2. We shall show there exists a section s of f such that $s(w)$ is a fixed point of G_r. Let z be a point of $\mathbf{A}^2$. Then $\tilde{s}(w)=(z,w)\in\mathbf{A}^2\times(G\backslash X)=X$ is a section of f. The section $\tilde{s}$ does not satisfy our requirement. Hence we modify $\tilde{s}$ to obtain s. Let $N(G_r)$ be the normalizer of G_r in G and $\Phi\colon G\backslash X\to G/N(G_r)$ be a morphism defined by $\Phi(w)=$(stabilizer at $\tilde{s}(w)$) for $w\in G\backslash X$. It is easy to see that there exists a normal subgroup U'' contained in U such that G is a semi-direct product $G=U''_{\bullet}N(G_r)$ because G_r is reductive. Therefore $\pi\colon G\to G/N(G_r)$ has a section and we can find a morphism $\tilde{\Phi}\colon G/X\to G$ such that the diagram

$$\begin{tikzcd} G\backslash X \arrow[r, "\tilde{\Phi}"] \arrow[dr, "\Phi"'] & G \arrow[d, "\pi"] \\ & G/N(G_r) \end{tikzcd}$$

commutes. Let us now define a section s of f by $s(w)=\tilde{\Phi}(w)^{-1}{}_{\bullet}\tilde{s}(w)$. Then it is evident that s is a section of f and $s(w)$ is a fixed point of G_r. Let $w\in G\backslash X$. There exists a subgroup $U'\subset U$ such that U' is 2-dimensional, U' is G_r-invariant and such that $f^{-1}(w)$ is a U'-orbit. The operation of U' on $f^{-1}(w)$ is necessarily effective since U' has no subgroup of dimension 0 as we are in characteristic 0. As we have seen above, there exists a non-empty open set W of $G\backslash X$ such that for any $w'\in W$, $f^{-1}(w)$ is a U'-orbit. Replacing X by $f^{-1}(w)$, we may assume $f^{-1}(w)$ is a U'-orbit for $w\in G\backslash X$. Then we determine a coordinate system on X:

$$U'\times(G\backslash X)\xrightarrow{\sim}X$$
$$(v,x)\longmapsto v_{\bullet}s(x)$$

In terms of this coordinate system, the operation of G_r is expressed as follows; for $g\in G_r$, $g(vs(x))=gv(g^{-1}g)s(x)=(gvg^{-1})(gs(x))=(gvg^{-1})s(x)$ since $G_r\subset$ stabilizer at $s(x)$. Namely, the operation of $g\in G_r$ sends (v,x) to (gvg^{-1},x). If we choose an appropriate coordinate system on U'

(a base of the vector group U') the operation of G_r is given:

(i) If G_r is isomorphic to $\mathbf{G}_m$, $(y, z, x) \mapsto (ty, z, x)$, $t \in \mathbf{G}_m$;

(ii) if G_r is isomorphic to $\mathbf{G}_m \times \mathbf{G}_m$, $(y, z, x) \mapsto (t_1^l y, t_2^m z, x)$ for $(t_1, t_2) \in \mathbf{G}_m$, where l, m are mutually prime natural numbers;

(iii) if G_r is isomorphic to SL_2, $(y, z, x) \mapsto (ay+bz, cy+dz, x)$ for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2$;

(iv) if G_r is isomorphic to GL_2, $(y, z, x) \mapsto (ay+bz, cy+dz, x)$ for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2$.

As in the Proof of Proposition (5.38.1), there exist finite dimensional k-vector subspaces L, M of the vector space $H^0(G\backslash X, 0)$ of regular functions on $G\backslash X$ such that the operation of the unipotent radical U on X is contained in the algebraic group:

$$(y, z, x) \longmapsto (y+f(x), z+g(x), x), \qquad f \in L, \ g \in M$$

If $G\backslash X = \mathbf{P}^1$, the operation (G, X) is a suboperation of (J1) of Theorem (2.2). If $G\backslash X$ is not complete, then $G\backslash X$ is considered as an open subset of $\mathbf{A}^1$. Let us fix an immersion $G\backslash X \hookrightarrow \mathbf{A}^1$. There exists a regular function φ on $G\backslash X$ such that φ^{-1} is also regular and such that $\varphi L = L'$, $\varphi M = M'$ are regular on $\mathbf{A}^1$. Then, let us take a new coordinate system $(\varphi(x)y, \varphi(x)z, x)$ instead of (y, z, x) and we take as x a coordinate on $\mathbf{A}^1$. The new coordinate system is denoted by (y, z, x) again. Then the operation of G_r is expressed as in the old coordinate and the operation of U is contained in the algebraic group:

$$(y, z, x) \longmapsto (y+f(x), z+g(x), x), \qquad f \in L', \ g \in M'$$

Hence U is contained in the algebraic group:

$$(y, z, x) \longmapsto (y+f(x), z+g(x), x) \quad \text{where } f, g \text{ are polynomial in } x \text{ with their degrees bounded.}$$

Therefore (G, X) is contained in (J9) by Corollary (4.17).

Subcase (5.42.2). *The dimension of any U_Z-orbit is* 1.

Let (G, X) be a realization. As in (5.42.1), we may assume there exists an effective algebraic operation (G', Y) and a morphism of algebraic operations (φ, f): $(G, X) \to (G', Y)$ such that $f: X \to Y$ is an $\mathbf{A}^1$-bundle and the fibres $f^{-1}(y)$ are U_Z-orbits for $y \in Y$. Since (G, X) is generically intransitive, (G', Y) is also generically intransitive. As Y is dominated by

X, Y is rational. It follows from Subcase (C-intr-1), Umemura [U3], either (1) (G', Y) is isomorphic to $(\mathrm{PSL}_2, \mathbf{P}^1 \times W)$ where W is a rational curve and PSL_2 operates on the first factor or (2) G' is solvable. Let us treat first the case G' is solvable. Then G' is isomorphic to one of the following: $\mathbf{G}_m$, $\mathbf{G}_a^r . \mathbf{G}_m$, $\mathbf{G}_a^r$ where r is a positive integer (cf. Umemura [U3]). Since the other cases are handled similarly, we assume $G' \simeq \mathbf{G}_a^r . \mathbf{G}_m$. Replacing Y by a G'-invariant Zariski open set, we may assume that the quotient of Y by G' exists: There exists a morphism of algebraic operations (ψ, g): $(G', Y) \to (\mathbf{1}, Z)$ such that $g: Y \to Z$ is an $\mathbf{A}^1$-bundle. By the Lüroth theorem, Z is a rational curve. If we replace Z by a Zariski open set, we may assume $Y = \mathbf{A}^1 \times Z$, Z is non-singular and not complete. There exists a finite dimensional k-vector subspace $V \subset k[Z]$ such that the operation of G' on $\mathbf{A}^1 \times Z$ is contained in the algebraic subgroup generated by the following:

$$(y, x) \longmapsto (y + f(x), x) \qquad \text{for } (y, x) \in \mathbf{A}^1 \times Z,$$
$$(y, x) \longmapsto (\lambda y, x) \qquad \text{for } (y, x) \in \mathbf{A}^1 \times Z, \text{ where } t(x) \in k[Z],\ \lambda \in \mathbf{G}_m.$$

To describe the operation (G, X) we need

Lemma (5.42.2.1). *Any $\mathbf{A}^1$-bundle over $Y = \mathbf{A}^1 \times Z$ is trivial. In particular $X \simeq \mathbf{A}^1 \times Y$.*

Proof. Let us first show that Pic $Y = 0$. In fact since Z is open rational curve, there exists an open immersion $i: Z \subset \mathbf{A}^1$ hence an open immersion ι: $Y = \mathbf{A}^1 \times Z \subset \mathbf{A}^1 \times \mathbf{A}^1$. Since Y is non-singular, a line bundle over Y is defined by a Weil divisor. Therefore, the map ι^*: Pic $(\mathbf{A}^1 \times \mathbf{A}^1) \to$ Pic $(\mathbf{A}^1 \times Z)$ is surjective. Since Pic $(\mathbf{A}^1 \times \mathbf{A}^1) = 0$, Pic $(\mathbf{A}^1 \times Z) = 0$. An $\mathbf{A}^1$-bundle over Y is now defined by an exact sequence: $0 \to \mathcal{O}_Y \to E \to \mathcal{O}_Y \to 0$. Since Y is affine, the extension splits and the lemma is proved.

We have shown $X = \mathbf{A}^1 \times Y = \mathbf{A}^1 \times \mathbf{A}^1 \times Z$. As in the Proof of Proposition (5.38.1), there exist k-vector subspaces $V' \subset k(\mathbf{A}^1 \times Z)$ such that the algebraic group G is contained in the algebraic group:

$$(z, y, x) \longmapsto (z + f(y, x), y + g(x), x),$$
$$(z, y, x) \longmapsto (\lambda z, \mu y, x) \qquad \text{for } (x, y, z) \in \mathbf{A}^1 \times \mathbf{A}^1 \times Z = X,$$
$$\text{where } (\lambda, \mu) \in \mathbf{G}_m^2,\ f(y, x) \in V',\ g(x) \in V.$$

There exists a regular function $\psi(x)$ on Z such that $\psi(x)^{-1}$ is also regular on Z and such that $\psi(x) V' \subset k[\mathbf{A}^1 \times \mathbf{A}^1]$ and $\psi(x) V \subset k[\mathbf{A}^1]$. Applying the coordinate change,

$$(z, y, x) \longmapsto (\psi(x)z, \psi(x)y, x).$$

There exists an integer n such that the operation (G, X) is contained in the following algebraic subgroup of the automorphism group of $\mathbf{A}^1 \times \mathbf{A}^1 \times \mathbf{A}^1$:

$$(z, y, x) \longmapsto (z + f(y, x), y + g(x), x)$$
$$(z, y, x) \longmapsto (\lambda z, \mu y, z) \quad \text{for } (z, y, x) \in \mathbf{A}^3 \text{ where } f(y, x) \in k[y, x],$$
$$g(x) \in k[x], (\lambda, \mu) \in \mathbf{G}_m^2 \text{ such that } \deg f(y, x), \deg g(x) \leq n.$$

Therefore (G, X) is contained in (J9) by Corollary (4.17). If $(G', Y) = (\mathrm{PSL}_2, \mathbf{P}^1 \times W)$, X is an $\mathbf{A}^1$-bundle over $\mathbf{P}^1 \times W$. Since $\mathrm{Pic}\,(\mathbf{P}^1 \times W) \simeq \mathrm{Pic}\,\mathbf{P}^1$, a line bundle on $\mathbf{P}^1 \times W$ is the pull-back of a line bundle on $\mathbf{P}^1$. Let $\mathscr{L} \simeq p^*\mathcal{O}(n)$ be a line bundle on $\mathbf{P}^1 \times W$ where $p: \mathbf{P}^1 \times W \to \mathbf{P}^1$ is the projection. $H^1(\mathbf{P}^1 \times W, \mathscr{L}) \simeq H^0(W, \mathcal{O}) \otimes H^1(\mathbf{P}^1, \mathcal{O}(n))$. Therefore an affine bundle A over $\mathbf{P}^1 \times W$ is a line bundle if and only if its restriction $A \mid \mathbf{P}^1 \times w$ is a line bundle over $\mathbf{P}^1 \times w \simeq \mathbf{P}^1$ for any point $w \in W$. By Proposition (2.10), [U3], $X \mid \mathbf{P}^1 \times w$ is a line bundle on $\mathbf{P}^1$ for any point $w \in W$. Thus X is a line bundle L over $\mathbf{P}^1 \times W$. By the same proposition the operation of SL_2 on each $X \mid \mathbf{P}^1 \times w$ is linear.

Sublemma (5.42.2.2). *Let A be a line bundle over a variety Z. We assume* SL_2 *operates on Z.* (SL_2*-linearization of A is by definition an* SL_2*-operation on a line bundle A covering the* SL_2*-operation on Z.) Z has at most* 2 SL_2*-linearizations.*

Proof. For $g \in \mathrm{SL}_2$, we denote by T_g the automorphism $x \mapsto gx$ of X. To give an SL_2-linearization is equivalent to give an isomorphism $\psi_g: A \xrightarrow{\sim} T_g^* A$ of line bundles satisfying $\psi_{gg'} = T_g^* \psi_{g'} \circ \psi_g$ for any $g, g' \in \mathrm{SL}_2$. Let ψ_{1g}, ψ_{2g} be 2 linearizations. Then consider the map

$$\Phi: X \times \mathrm{SL}_2 \to \mathscr{H}om_{\mathcal{O}_X}(A, A) = \mathcal{O}_X^*, \quad (x, g) \mapsto (x, \psi_{1g}(x)\psi_{2g}(x)^{-1}).$$

On each fibre Φ is a homomorphism. $\Phi(x)$ should be ± 1 on each fibre hence Φ is ± 1. It follows from the Proof of the Sublemma.

Corollary (5.42.2.3). *The notations being as in the Sublemma, let* $\{\psi_g\}$, $g \in \mathrm{SL}_2$ *be a linearization then the other linearization is* $\{-\psi_g\}$, $g \in \mathrm{SL}_2$.

Let us now continue the study of (G, X). The semi-simple part of G is SL_2 and it operates on X linearly as we saw above. Thus it follows from Corollary (5.42.2.3) and Proposition (2.11), [U3], choosing a suitable local coordinate on X, SL_2 operation on X is given by

$$t \mapsto t, \quad y \mapsto \frac{ax+b}{cx+d}, \quad z \mapsto \frac{\pm z}{(cz+d)^m}.$$

By the argument in the Proof of Proposition (5.38.1) shows the operation of U on X is given by $t \mapsto t$, $y \mapsto y$, $z \mapsto z+f(t, y)$, where $f(t, y) \in$ a finite dimensional vector subspace M of $H^0(T, \mathcal{O}_T)[y]$. Then by a suitable $\psi(t) \in H^0(T, \mathcal{O}_T)$ we change the coordinate $(t, y, z) \mapsto (t, y, \psi(t)z)$ so that $M \subset k[t, y]$ (see Proof of Proposition (5.38.1)). Notice in this new coordinate, SL_2-operation on X has same expression. Thus G is contained in (J3) by Proposition (2.11), [U3].

Subcase (5.42.3). *G is reductive* (*and G has a 2-dimensional orbit*).

Let (G, X) be a realization and we may assume X consists of 2-dimensional orbits. It follows from Lemma (1.21), [U3] that G can be embedded in Cr_2. Hence it follows from [U3], G is finitely covered by one of the following algebraic group: (i) SL_3, (ii) $\mathrm{SL}_2 \times \mathrm{SL}_2$, (iii) $\mathbf{G}_m \times \mathrm{SL}_2$, (iv) $\mathbf{G}_m \times \mathbf{G}_m$, (v) SL_2. We know, a torus in Cr_3 is, up to conjugacy, contained in $(\mathbf{G}_m^3, \mathbf{G}_m^3)$ hence in $(\mathrm{PGL}_4, \mathbf{P}^3)$. Therefore, we may assume G is not a torus. Let us first assume G is covered by SL_3. Under this assumption, we can argue as in Subcase (C-intr-1), [U3]. We shall determine almost effective operation (SL_3, X) such that X is a union of the 2-dimensional orbit of SL_3. Replacing X by a smaller Zariski open set if necessary, we may assume there exists a morphism of algebraic operations $(\varphi, f)\colon (\mathrm{SL}_3, X) \to (\mathbf{1}, Y)$ such that f is dominant and Y is a rational curve. Since 2-dimensional SL_2-orbit is isomorphic to $\mathbf{P}^2$, f is a $\mathbf{P}^2$-bundle. Since the Brauer group of a curve Y is trivial, the $\mathbf{P}^2$-bundle f is locally trivial. Replacing Y by a Zariski open set, we may assume f is trivial, namely $X \simeq Y \times \mathbf{P}^2$. Let $s\colon Y \to X$ be a section and P is a parabolic subgroup of SL_3 consisting of the matrices (a_{ij}) such that $a_{1i}=0$ for $i=2, 3$. Therefore $\mathrm{SL}_3/P \simeq \mathbf{P}^2$. Let us recall the following.

Lemma (5.42.4). (i) *Let P' be a parabolic subgroup of* SL_3 *such that* $\mathrm{SL}_3/P' \simeq \mathbf{P}^2$. *There exists an automorphism φ of* SL_3 *such that* $\varphi(P) = P'$.

(ii) *Let $\mathcal{G}$ be the automorphism group of* SL_3. *We have an exact sequence*

$$\mathbf{1} \to \textit{Inner automorphism group} \to \mathcal{G} \to \mathbf{Z}/2\mathbf{Z} \to 0.$$

(iii) $\{\varphi \in \mathcal{G} \mid \varphi(P) = P\} = \{\mathrm{Int}(g) \mid g \in P$ *where* $\mathrm{Int}(g)(x) = gxg^{-1}$, $x \in \mathrm{SL}_3\}$.

(iv) $\mathcal{G}/\mathrm{Int}(P) = \{$*Parabolic subgroups* $P' \mid \mathrm{SL}_3/P' \simeq \mathbf{P}^2\}$.

(v) *The fibration* $\mathcal{G} \to \mathcal{G}/\mathrm{Int}(P)$ *is locally trivial.*

Proof. If $\mathrm{SL}_3/P' \simeq \mathbf{P}^2$ hence we have an isomorphism h: $\mathrm{SL}_3/P' \simeq \mathrm{SL}_3/P$, then there exists an automorphism f: $\mathrm{SL}_3 \to \mathrm{SL}_3$ making the diagram

$$\begin{tikzcd} \mathrm{SL}_3 \arrow[r, "f"] \arrow[d] & \mathrm{SL}_3 \arrow[d] \\ \mathrm{SL}_3/P' \arrow[r, "h"] & \mathrm{SL}_3/P \end{tikzcd}$$

commutative, since $\mathrm{Aut}\,\mathbf{P}^2 = \mathrm{PSL}_3$. This proves (i). The assertion (iii) is a linear algebra. The assertions (ii) is well-known (cf. Bourbaki and Borel [B]). Assertion (iv) is a consequence of (i). $\mathscr{G}$ (hence laso $\mathscr{G}/\mathrm{Int}\,P$) has 2 connected components isomorphic to SL_3 (resp. SL_3/P). We have to show the fibration $\mathrm{SL}_3 \to \mathrm{SL}_3/P$ is locally trivial. This is quite elementary. As $\mathrm{SL}_3 \to \mathrm{SL}_3/P \simeq \mathbf{P}^2$ is given by $(a_{ij}) \mapsto (a_{11}, a_{21}, a_{31}) \in \mathbf{P}^2$, for example on $\mathbf{A}^2 = \{(x_0, x_1, x_2) \in \mathbf{P}^2 \mid x_0 = 1\}$ a section is given by

$$(1, x_1, x_2) \longmapsto \begin{pmatrix} 1 & 0 & 0 \\ x_1 & 1 & 0 \\ x_2 & 0 & 1 \end{pmatrix}. \qquad \text{q.e.d.}$$

Let us now consider the map F: $Y \to \{$parabolic subgroups $P' \mid \mathrm{SL}_3/P' \simeq \mathbf{P}^2\} = \mathscr{G}/\mathrm{Int}\,(P)$ defined as follows: for $y \in Y$, $y \mapsto \psi^{-1}$ (stabilizer at $s(y)$, where s: $Y \mapsto X$ is the section (cf. Lemma (5.42.4))). It is a linear algebra $\mathscr{G}/\mathrm{Int}\,(P) = \{$Parabolic subgroups conjugate to $P\} \coprod$ (Parabolic subgroup conjugate to ${}^tP\}$. Since Y is connected, $F(Y)$ is contained in one of the connected component. From now on we assume $F(Y) \subset \{$Parabolic subgroups conjugate to $P\}$, since the other case is treated similarly. Then we have a factorization F: $Y \to \mathrm{SL}_3/P \subset \mathscr{G}/\mathrm{Int}\,(P)$. It follows from Lemma (5.42.4) (v), by replacing a smaller open set if necessary, we may assume there exists a morphism $\tilde{F}$: $Y \to \mathrm{SL}_3$ such that the diagram

$$\begin{tikzcd} & \mathrm{SL}_3 \arrow[d, "p"] \\ Y \arrow[ur, "\tilde{F}"] \arrow[r, "F"] & \mathrm{SL}_3/P \end{tikzcd}$$

is commutative. For $g \in \mathrm{SL}_3$, we denote by $L_g: X \to X$ the translation by g; for $x \in X$, $L_g(x) = gx$. Let $(\mathbf{1} \times \mathrm{SL}_3, Y \times \mathrm{SL}_3/P) = (\mathrm{SL}_3, Y \times \mathrm{SL}_3/P)$ be the product operation. Let f': $V \times \mathrm{SL}_3/P \to X$ be a morphism defined by $f'((y, gp)) = g\tilde{F}(y)^{-1}s(y)$. The morphism f' is well-defined and is an isomorphism. Then (Id, f'): $(\mathrm{SL}_3, Y \times \mathrm{SL}_3/P) \to (\mathrm{SL}_3, X)$ is an isomorphism of algebraic operations. Hence we have proved

Lemma (5.42.5). *In Subcase* (5.42.3), *if we assume G is finitely covered* *by* SL_3, *G is contained in the conjugacy class of* $(\mathrm{PGL}_2\times\mathrm{PGL}_3, \mathbf{P}^1\times\mathbf{P}^2)$.

In a similar way, we can prove

Lemma (5.42.6). *In Subcase* (5.42.3), *if we assume G is finitely covered by* $\mathrm{SL}_2\times\mathrm{SL}_2$, *G is contained in the conjugacy class of* $(\mathrm{PGL}_2\times\mathrm{PGL}_2\times\mathrm{PGL}_2, \mathbf{P}^1\times\mathbf{P}^1\times\mathbf{P}^1)$.

Let us now study the case G is finitely covered by $\mathbf{G}_m\times\mathrm{SL}_2$. $\mathbf{G}_m\times\mathrm{SL}_2$ operates on X. In view of Lemma (1.21), [U3] and Lemma (2.22), [U3], we may assume the stabilizer at a point of X is conjugate to the subgroup $K_n=\left\{\left(t^n, \begin{pmatrix} t & 0 \\ x & t^{-1}\end{pmatrix}\right)\in \mathbf{G}_m\times\mathrm{SL}_2 \,|\, t\in k^*,\ x\in k\right\}$ for an integer n. We see below the morphism $f\colon X\to Y$ has a rational section. In fact we may assume as before X consists of only 2-dimensional orbits of G. The group $\mathbf{G}_m\times\mathrm{SL}_2$ operates on S and we may assume a 2-dimensional subgroup $K=\left\{\left(t, \begin{pmatrix} 1 & y \\ 0 & 1\end{pmatrix}\right)\in \mathbf{G}_m\times\mathrm{SL}_2 \,|\, t\in k^*,\ y\in k\right\}$ has a 2-dimensional orbit since the stabilizer of a point is conjugate to K_n. Let X' be the union of the 2-dimensional orbits of K on X. Then X' is a non-empty Zariski open set of X. Since f is flat, $f(X')$ is an open set of Y which we denote by Y'. Y' is the quotient of X' with respect to the action of K. Since K is solvable, the restriction of f to X' (hence f itself) has a section locally on Y'. Therefore replacing Y by an open subset U and X by $f^{-1}(U)$, we may assume $f\colon X\to Y$ has a section s. Let

$$Z=\{(g, Y)\in(\mathbf{G}_m\times\mathrm{SL}_2)\times Y \,|\, gs(y)=s(y)\}.$$

Then Z is a group scheme on Y parametrizing the stabilizer at $s(y)$, $y\in Y$. Let us put $Z'=Z\cap(1\times\mathrm{SL}_2)\times Y$. Z_y is conjugate to $K_{n(y)}$ for an integer $n(y)$ depending on $y\in Y$. We want to show that the number $n(y)$ is constant on an open set of Y hence we may assume on Y by replacing by the open set of Y. Since the absolute value $|n(y)|$ is the number of the connected components of $Z'_{s(y)}$, $|n(y)|$ is upper semi-continuous and constant on a non-empty Zariski open set of Y. For an integer $n\neq 0$, K_n and K_{-n} are images one another of an outer automorphism i of $\mathbf{G}_m\times\mathrm{SL}_2$; $i(t, y)=(t^{-1}, g)$ for $(t, g)\in\mathbf{G}_m\times\mathrm{SL}_2$. Hence the quotient variety $\mathbf{G}_m\times\mathrm{SL}_2/K_n$ is isomorphic to $\mathbf{G}_m\times\mathrm{SL}_2/K_{-n}$. But it is easy to see they are not conjugate. Let $y_0\in Y$ and $Z_{s(y_0)}$ be conjugate to K_{n_0}, $n_0\neq 0$, Let us put $\mathscr{G}=(\mathbf{Z}/2\mathbf{Z})\times\mathrm{PSL}_2=\mathrm{Aut}\,\mathbf{G}_m\times\mathrm{Inner\ Aut}\,\mathrm{SL}_2\subset\mathrm{Aut}\,(\mathbf{G}_m\times\mathrm{SL}_2)$. An element of $\mathbf{G}_m\times\mathrm{SL}_2$ defines an inner automorphism and hence we have a morphism $\psi\colon\mathbf{G}_m\times\mathrm{SL}_2\to\mathscr{G}$. Let $\mathscr{H}_{n_0}=\{\varphi\in\mathscr{G}\,|\,\varphi(K_{n_0})\subset K_{n_0}\}$. Let $\mathscr{H}_{n_0}=\{\varphi\in\mathscr{G}\,|\,\varphi(K_{n_0})\subset K_{n_0}\}$. Then $\mathscr{H}_{n_0}=\psi(K_{n_0})$. We have an isomorphism {subgroups of $\mathbf{G}_m$

$\times \mathrm{SL}_2$ conjugate to K_{n_0} or $K_{-n_0}\} \simeq \mathscr{G}\mathscr{H}_{n_0}$. Since $\mathscr{H}_{n_0}$ is connected $\mathscr{G}/\mathscr{H}_{n_0}$ has two connected components {subgroups conjugate to K_n} and {subgroups conjugate to K_{-n}}. $\mathscr{G} \to \mathscr{G}/\mathscr{H}_{n_0}$ is locally trivial since $\mathscr{H}_{n_0}$ is solvable. Let $F: Y \to \mathscr{G}/\mathscr{H}_{n_0}$ be defined as follows: $F(y) = \varphi \bmod \mathscr{H}_{n_0}$ such that $\varphi(\mathscr{H}_{n_0}) = Z_{s(y)}$ for $y \in Y$. Since $\mathscr{G}/\mathscr{H}_{n_0}$ has two connected components, $n(y)$ is constant. Now argue as in Lemma (5.42.5), using $\mathbf{G}_m \times \mathrm{SL}_2$ instead of $\mathscr{G}$ when $n_0 = 0$, to conclude that G is almost effectively realized by $(\mathbf{1} \times (\mathbf{G}_m \times \mathrm{SL}_2), \mathbf{P}^1 \times \mathrm{Aut}^0 F'_m)$, $m = |n_0|$, hence contained in one of the conjugacy classes of Theorem (2.2).

It remains to study the subcase where G is finitely covered by SL_2. As we noticed in [U3], there are up to conjugacy 3 types of closed subgroups of dimension 1: (i) $U_n = \left\{\begin{pmatrix} a & b \\ 0 & a^{-1}\end{pmatrix} \in \mathrm{SL}_2 \,|\, a^n = 1,\ a \in k^*,\ b \in k\right\}$, $n = 1, 2, 3 \cdots$, (ii) $T = \left\{\begin{pmatrix} a & 0 \\ 0 & a^{-1}\end{pmatrix} \in \mathrm{SL}_2 \,|\, a \in k^*\right\}$, (iii) $\mathrm{D}_\infty =$ the subgroup of SL_2 generated by T and $\begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix}$. As before SL_2 operates on X and we may assume that X consists of 2-dimensional orbits and that there exists a morphism of algebraic operations (φ, f): $(\mathrm{SL}_2, X) \to (\mathbf{1}, Y)$ where f is a quotient map onto a rational curve Y. We can prove by the same argument as in the preceding subcase that f has a rational section and hence we may assume f has a section s. Then we define a group scheme Z over Y: $Z = \{(g, y) \in \mathrm{SL}_2 \times Y \,|\, gs(y) = s(y)\}$. Z is a subgroup scheme of dimension 1 over Y of $\mathrm{SL}_2 \times Y$. Then there exists a non-empty Zariski open subset U of Y such that we have either $(Z_y)^0$ is unipotent for all $y \in Y$ or $(Z_y)^0$ is a torus for all $y \in Y$. Therefore we may assume either $(Z_y)^0$ is unipotent for all $y \in Y$ or $(Z_y)^0$ is a torus for all $y \in Y$. In the first case, the same argument as in the preceding subcase allows us to conclude:

(i) $(Z_y)^0$ is conjugate to $U_{n(y)}$ and the number $n(y)$ is independent of $y \in Y$,

(ii) The normalizer of U_n in SL_2 is a Borel subgroup

$$B = \left\{\begin{pmatrix} a & b \\ 0 & a^{-1}\end{pmatrix} \middle|\, a \in k^*,\ b \in k\right\},$$

(iii) The projection p: $\mathrm{SL}_2 \to \mathrm{SL}_2/B$ is locally trivial,

(iv) Letting F: $Y \to \mathrm{SL}_2/B$ be the same morphism as in the preceding subcase, we can locally lift F to $\tilde{F}$ so that we have a commutative diagram

$$\begin{tikzcd} & \mathrm{SL}_2 \arrow[d, "p"] \\ Y \arrow[ur, dashed, "\tilde{F}"] \arrow[r, "F"'] & \mathrm{SL}_2/B, \end{tikzcd}$$

(v) Replacing F by $\tilde{F}$, we may assume $\tilde{F}$ is globally defined and hence the conjugacy class of (G, X) is the same as that of $(\mathbf{1}\times\mathrm{SL}_2, \mathbf{P}^1\times \mathrm{SL}_2/U_n)$. Therefore the conjugacy class of (G, X) is contained in the conjugacy class of $(\mathrm{SL}_2\times\mathrm{Aut}^0\mathrm{F}'_n, \mathbf{P}^1\times\mathrm{F}'_n)$ hence in one of the conjugacy classes of Theorem (2.2).

We assume that $(Z_y)^0$ is a torus for $y\in Y$. We want to argue as above. The only difference is that the normalizer of T or D_∞ is D_∞ hence the projection $\mathrm{SL}_2\to\mathrm{SL}_2/\mathrm{D}_\infty=\mathrm{SL}_2/$the normalizer of T or D_∞ does not have a local section for the Zariski topology. This subtle difference gives rise to an interesting family (J11) of maximal algebraic groups contained in Cr_3. Let us argue however in a similar way but carefully. We may assume that we have either (a) Z_y is conjugate to T for all $y\in Y$ or (b) Z_y is conjugate to D_∞ for all $y\in Y$. The normalizer of T or of D_∞ is D_∞. Let F: $Y\to\mathrm{SL}_2/\mathrm{D}_\infty$ be defined as follows: for $y\in Y$, $F(y)HF(y)^{-1}=Z_{s(y)}$ where H is T or D_∞ according as (a) or (b). The projection p: $\mathrm{SL}_2\to\mathrm{SL}_2/\mathrm{D}_\infty$ does not have a rational section but it has a section over an étale 2-covering $\mathrm{SL}_2/\mathbf{G}_m\to\mathrm{SL}_2/\mathrm{D}_\infty$. Thus there exists a étale 2-covering q: $\tilde{Y}\to Y$ such that $F\circ q$ has a local lifting $\tilde{F}$ for the Zariski topology:

$$\begin{tikzcd}
 & & \mathrm{SL}_2 \arrow[d, "p"] \arrow[r] & \mathrm{SL}_2/\mathbf{G}_m \arrow[dl] \\
\tilde{Y} \arrow[urr, "\tilde{F}"] \arrow[r, "q"] & Y \arrow[r, "F"] & \mathrm{SL}_2/\mathrm{D}_\infty. &
\end{tikzcd}$$

We may assume $\tilde{F}$ is regular on $\tilde{Y}$. Hence by the same argument as above, the operation of SL_2 over the fibre space f: $X\to Y$ is isomorphic to the product operation if we pull it back over $\tilde{Y}$. Namely the operation (SL_2, X) is given by the product operation plus a descent data through q. Let us classify the descent data. We notice, first of all, that when we proved $(\mathrm{SL}_2, X\times_Y\tilde{Y})$ is isomorphic to the product operation, so far as the isomorphism $(\tilde{\varphi}, \tilde{f})$: $(\mathrm{SL}_2, \tilde{Y}\times_Y X)\rightrightarrows(\mathrm{SL}_2, \tilde{Y}\times\mathrm{SL}_2/H)$ is concerned, $\tilde{\varphi}$ is the identity automorphism of SL_2. Therefore the descent data is given such an automorphism. Since q: $\tilde{Y}\to Y$ is an étale 2-covering, there is an involutive automorphism ι of $\tilde{Y}$ such that $\tilde{Y}/\langle\iota\rangle\simeq Y$. To give a descent data on the product operation $(\mathrm{SL}_2, \tilde{Y}\times\mathrm{SL}_2/H)$ is to give an involutive automorphism (Id, f') of algebraic operation $(\mathrm{SL}_2, \tilde{Y}\times\mathrm{SL}_2/H)$ such that f' covers the involution ι: $q\circ p_1=p_1\circ f'$. We need

Lemma (5.42.7). *The group of the automorphisms* (Id, f) *of the algebraic operation* $(\mathrm{SL}_2, \mathrm{SL}_2/T)$ *is isomorphic to* $\mathbf{Z}/2\mathbf{Z}$. *The group of the automorphisms* (Id, f) *of the algebraic operation* $(\mathrm{SL}_2, \mathrm{SL}_2/\mathrm{D}_\infty)$ *is isomorphic to* $\mathbf{1}$.

Proof. The first assertion was proved in Lemma (3.9.1) and the last assertion is proved by the same method.

Corollary (5.42.8). *If Z_y is conjugate to* D_∞ *for any $y \in Y$, the operation (G, X) is isomorphic to the product operation* $(\mathbf{1}\times \mathrm{PGL}_2,\ Y\times \mathrm{PGL}_2/\mathrm{D}_\infty)$ *where T is a Cartan subgroup of* PGL_2. *In particular the conjugacy class of (G, X) is contained in the conjugacy class of* $(\mathrm{PGL}_2\times \mathrm{PGL}_3,\ \mathbf{P}^1\times \mathbf{P}^2)$.

Proof. The lemma implies there is only trivial descent data on $\tilde{Y}\times \mathrm{SL}_2/\mathrm{D}_\infty$. For the inclusion $(\mathrm{PGL}_2, \mathrm{PGL}_2/\mathrm{D}_\infty)\hookrightarrow(\mathrm{PGL}_3, \mathbf{P}^2)$, we refer to Umemura [U3].

Corollary (5.42.9). *If Z_y is conjugate to T for any $y \in Y$, G is contained in one of the conjugacy classes of Theorem* (2.2).

Proof. If Z_y is conjugate to T for any $y \in Y$, then the only one non-trivial descent data on $(\mathbf{1}\times \mathrm{SL}_2,\ \tilde{Y}\times(\mathrm{SL}_2/T))$ is given by (Id, h): $(\mathbf{1}\times \mathrm{SL}_2,\ \tilde{Y}\times(\mathrm{SL}_2/T))\rightarrow(\mathbf{1}\times \mathrm{SL}_2,\ Y\times(\mathrm{SL}_2/T))$, $h(y, gT)=\left(\iota(y),\ g\begin{pmatrix} 0 & 1\\ -1 & 0\end{pmatrix}T\right)$ for $y \in \tilde{Y}$, $g \in \mathrm{SL}_2$ by Lemma (5.42.7). Corollary now follows from Proposition (4.26) and Lemma (4.28) since Y is rational by Lüroth's theorem.

Case (5.43). *G has at most* 1*-dimensional orbit.*

The last case to examine is the generically intransitive group of which the orbits are at most 1-dimensional. Let (G, X) be an algebraic operation realizing such a group in Cr_3. As in the preceding case, replacing X by a Zariski open subset, we may assume that (1) X consists of 1-dimensional orbits, (2) there exists a morphism of algebraic operation (φ, f): $(G, X)\rightarrow(1, Y)$ where Y is the quotient of X by G. Since X is rational, Y is rational by a theorem of Zariski [Z].

As in the case of generically intransitive groups in Cr_2 we can prove

Proposition (5.43.1). *Let G be an algebraic group contained in* Cr_3. *If a (or any) realization of G has at most* 1*-dimensional orbit, then G is contained in one of the conjugacy classes of Theorem* (2.2).

We sketch the proof. If G is reductive, as in Subcase (5.42.3) G is isomorphic to $\mathbf{G}_m$ or PGL_2. In the first case $\mathbf{G}_m$ is contained in $\mathbf{G}_m^3$ and in the second case if we replace Y be a Zariski open set the same argument as in Subcase (C-intr-1), [U3] shows G is realized by the product operation $(\mathbf{1}\times \mathrm{PGL}_2,\ Y\times \mathbf{P}^1)$. If G is unipotent, the fibration f: $X\rightarrow Y$ is locally trivial for the Zariski topology and we may assume f: $X\rightarrow Y$ is trivial. G is abelian since its operation on each orbit is abelian. G is nothing but a finite dimensional vector space V contained $H^0(Y, \mathcal{O}_Y)$ (cf. Lemma (4.4)). Replacing Y by a smaller Zariski open set, we may assume Y is a Zariski open set of $\mathbf{A}^2$. There exists a certain rational function f on $\mathbf{A}^2$ such that $f_{\cdot} V \subset H^0(\mathbf{A}^2, \mathcal{O}_{\mathbf{A}^2})$ (cf. Proposition (5.38.1)). By this twist, G is contained in (J7) of Theorem (2.2) for an appropriate n. If G is neither reductive nor unipotent, the operation of the unipotent radical is normalized as above and we have to control the operation of a torus. The dimension of a maximal torus is 1 by Lemma (1.21), [U3] and it is controlled as in Subcase (C-intr-2), [U3].

§ 6. Proof of the second assertion of the main theorem

Let us recall some basic facts on the inclusion of algebraic subgroups of Cr_3.

Proposition (6.1). *Let (G_1, X_1), (G_2, X_2) be realizations of conjugacy classes of algebraic subgroups of* Cr_3. *The conjugacy class of (G_1, X_1) is contained in the conjugacy class of (G_2, X_2) if and only if there exists a morphism (φ, f): $(G_1, X_1) \to (G_2, X_2)$ of law chunks of algebraic operation such that f is birational.*

See Proposition (1.10) Umemura [U3].

Let (G_1, X_1), (G_2, X_2) be algebraic operations and (φ, f): $(G_1, X_1) \to (G_2, X_2)$ be a morphism of law chunks of algebraic operation. Then there exists a non-empty G_1-invariant Zariski open set U of X_1 such that f is regular on U. Since a morphism of algebraic group germs φ: $G_1 \to G_2$ is necessarily regular by Weil [W], (φ, f): $(G_1, U) \to (G_2, X_2)$ induces a morphism of algebraic operations. We have thus proved

Proposition (6.2). *Let (G_1, X_1), (G_2, X_2) be realizations of conjugacy classes of algebraic subgroups of* Cr_3. *We assume (G_1, X_1) is a homogeneous space. Then the conjugacy class of (G_1, X_1) is contained in the conjugacy class of (G_2, X_2) if and only if there exists a morphism (φ, f): $(G_1, X_1) \to (G_2, X_2)$ of algebraic operations such that f is birational.*

Now we being the *proof.* In what follows we use the numbering $((a)$-$(b))$ *which says the proof that any operation of type* (a) *is not contained in any operation of type* (b) *of Theorem* (2.2). We have shown in our preceding papers [U1], [U2] that (P1), (P2), (E1) and (E2) determine maximal (conjugacy classes of) connected algebraic subgroups of Cr_3. Let us therefore begin with the proof that (J1) is not contained in other operations of the main theorem.

((J1)-(x)) In view of Proposition (6.2), it is evident that the conjugacy class of (J1) is not contained in any other conjugacy classes of the algebraic operations of the main theorem since (J1) is a homogeneous space and $\mathbf{P}^2\times\mathbf{P}^1$ is complete.

((J2)-(x)) For the same reason as ((J1)-(x)), the conjugacy class of (J2) is not contained in any conjugacy classes of the algebraic operations of the main theorem.

((J3)-(P1)) We know the conjugacy class of $(\mathrm{Aut}^0 \mathrm{F}'_m, \mathrm{F}'_m)$ is almost effectively realized by $(U_{m+1}\mathbf{G}_m\mathrm{SL}_2, \mathrm{F}'_m)$ where U_{m+1} is an irreducible SL_2-module of dimension $m+1$ and U_{m+1} is a $\mathbf{G}_m$-module of weight 1 (see Umemura [U3]). Therefore we have to show that there does not exist a morphism $(\varphi,f)\colon (\mathrm{SL}_2\times(U_{m+1}\mathbf{G}_m\mathrm{SL}_2), \mathbf{P}^1\times\mathrm{F}'_m)\to(\mathrm{PGL}_4, \mathbf{P}^3)$ of algebraic operations such that f is birational. Let us assume that such a morphism (φ,f) exists. This gives an almost effective operation of $\mathrm{SL}_2\times\mathrm{SL}_2$ on $\mathbf{P}^3$ hence a representation V of degree 4 of $\mathrm{SL}_2\times\mathrm{SL}_2$. $\mathrm{SL}_2\times\mathrm{SL}_2$-module V is a direct sum of irreducible modules and hence 4 is represented as a sum of positive numbers; $4=1+1+1+1$, $4=2+1+1$, $4=3+1$, $4=4$, $4=2+2$. The partition $4=1+1+1+1$ (or V is a trivial $\mathrm{SL}_2\times\mathrm{SL}_2$-module) is impossible because the operation $(\mathrm{SL}_2\times\mathrm{SL}_2, \mathbf{P}^3)$ is almost effective. For the same reason the partitions $4=2+1+1$ and $4=3+1$ are impossible. We show the partition $4=4$ (or V is irreducible) is impossible. If V is an irreducible $\mathrm{SL}_2\times\mathrm{SL}_2$-module then V is isomorphic to either $U_2\otimes U_2$, $k\otimes U_4$ or $U_4\otimes k$. The last two cases can never occur since the operation $(\mathrm{SL}_2\times\mathrm{SL}_2, \mathbf{P}^3)$ is almost effective. Thus we may assume V is isomorphic to $U_2\otimes U_2$. To distinguish the first factor from the second factor, we denote by U'_2 (resp. by U''_2) an irreducible $\mathrm{SL}_2\times\mathbf{1}$ (resp. $\mathbf{1}\times\mathrm{SL}_2$)-module of degree 2. In this notation V is isomorphic to $U'_2\otimes U''_2$. Then as $\mathfrak{sl}_2\times\mathfrak{sl}_2$-module, $\mathfrak{gl}_4=(U'_2\otimes U''_2)\otimes(U'_2\otimes U''_2)\simeq(U'_2\otimes U'_2)\otimes(U''_2\otimes U''_2)\simeq(k\oplus U'_3)\otimes(k\oplus U''_3)\simeq k\oplus k\otimes U''_3\oplus U'_3\otimes k\oplus U'_3\otimes U''_3$. Therefore $\mathfrak{sl}_3\simeq\mathfrak{pgl}_4$ is isomorphic to, as $\mathfrak{sl}_2\times\mathfrak{sl}_2$-module, $k\otimes U''_3\oplus U'_3\otimes k\oplus U'_3\otimes U''_3$. $\varphi_*(0\times(\text{Lie algebra of }0\times\mathbf{G}_m\times\mathbf{1}\text{ in }U_{m+1}\mathbf{G}_m\mathrm{SL}_2))$ is a 1-dimensional $\mathfrak{sl}_2\times\mathfrak{sl}_2$-submodule of $\mathfrak{pgl}_4$. But the decomposition above of $\mathfrak{sl}_2\times\mathfrak{sl}_2$-module $\mathfrak{pgl}_4$ does not contain a trivial $\mathfrak{sl}_2\times\mathfrak{sl}_2$-module of dimension 1. We verify now the partition $4=2+2$. Since the operation $(\mathrm{SL}_2\times\mathrm{SL}_2, \mathbf{P}^3)$ is almost effective, the only possible isomorphism is $V\simeq k\otimes U''_2\oplus U'_2\otimes k$. Then as $\mathfrak{sl}_2\times\mathfrak{sl}_2$-module $\mathfrak{gl}_4\simeq(k\otimes U''_2\oplus U'_2\otimes k)\otimes(k\otimes U''_2\oplus U'_2\otimes k)\simeq k\otimes(U''_2\otimes U''_2)\oplus U'_2\otimes U''_2\oplus U'_2\otimes U''_2\oplus(U'_2\otimes U'_2)\otimes k\simeq k\otimes(k\oplus U''_3)\oplus U'_2\otimes U''_2\oplus U'_2\otimes U''_2\oplus(k\oplus U'_3)\otimes k$. We identify the Lie algebra of U_{m+1} with U_{m+1} itself. Since $\varphi_*(0\times U_{m+1})$ is an irreducible $\mathfrak{sl}_2\times\mathfrak{sl}_2$-module of degree $m+1$ through the second factor, it follows from the decomposition of $\mathfrak{sl}_2\times\mathfrak{sl}_2$-module $\mathfrak{gl}_4$, $m=2$, $\varphi_*(0\times U_{m+1})\simeq k\otimes U''_3$. But $\varphi_*(0\times U_{m+1})$ is an abelian Lie subalgebra of $\mathfrak{sl}_4$ whereas $k\otimes U_3''$ is not abelian since it comes out from the decomposition $U_2''\otimes U_2''$. Therefore the partition $4=2+2$ is impossible too and the conjugacy class of (J3) is not contained in the conjugacy class of (P1).

((J3)-(P2)) Since as we have seen in ((J3)-(P1)) the rank of $\mathrm{PGL}_2\times \mathrm{Aut}^0\,\mathrm{F}_m'$ is 3 and that of SO_5 is 2, the conjugacy class of (J3) is not contained in the conjugacy class of (P2).

((J3)-(Ei)) Counting the dimension of the groups, we conclude the conjugacy class of (J3) is not contained in the conjugacy class of (Ei) for $i=1, 2$.

((J3)-(J1)) Since $\mathrm{Aut}^0\,\mathrm{F}_m'$ contains an abelian unipotent group of dimension $m+1$ (Umemura [U3]), $\mathrm{PGL}_2\times\mathrm{Aut}^0\,\mathrm{F}_m'$ contains an abelian unipotent group of dimension $m+2\geq 4$. But $\mathrm{PGL}_2\times\mathrm{PGL}_3$ can not contain an abelian unipotent group of dimension≥ 4.

((J3)-(J2)) We can argue as in ((J3)-(J1)).

((J3)-(J3)) It follows from Corollary (4.10) that (J3) for m is not a suboperation of (J3) for n with $n\neq m$.

((J3)-(J4)) The same argument as in ((J3)-(P1)) works.

((J3)-(J5)) Compare the dimension.

((J3)-(J6)) The same reason as ((J3)-(J5)).

((J3)-(J7)) Since the semi-simple part of $\mathrm{PGL}_2\times\mathrm{Aut}^0\,\mathrm{F}_m'$ is $\mathfrak{sl}_2\times\mathfrak{sl}_2$ and the semi-simple part of $\mathrm{Aut}^0\,\mathrm{J}_n'$ is $\mathfrak{sl}_3$ by [U3] and by Proposition (4.8), if there is an inclusion $(\mathrm{PGL}_2\times\mathrm{Aut}^0\,\mathrm{F}_m')\subset(\mathrm{Aut}^0\,\mathrm{J}_n', \mathrm{J}_n')$, $\mathfrak{sl}_2\times\mathfrak{sl}_2$ would be a Lie subalgebra of $\mathfrak{sl}_3$ which is a contradiction.

((J3)-(J8)) Since by [U3] the semi-simple part of $\mathrm{PGL}_2\times\mathrm{Aut}^0\,\mathrm{F}_m'$ is $\mathrm{SL}_2\times\mathrm{SL}_2$, if we have a morphism (φ, f): $(\mathrm{PGL}_2\times\mathrm{Aut}^0\,\mathrm{F}_m', \mathbf{P}^1\times\mathrm{F}_m')\to(\mathrm{Aut}^0\,\mathrm{L}_{p,q}', \mathrm{L}_{p,q}')$ of algebraic operations such that f is birational then φ maps surjectively the semi-simple part of $\mathrm{PGL}_2\times\mathrm{Aut}^0\,\mathrm{F}_m'$ to the semi-simple part of $\mathrm{Aut}^0\,\mathrm{L}_{p,q}'$. Therefore φ maps the unipotent radical W_1 of $\mathrm{PGL}_2\times\mathrm{Aut}^0\,\mathrm{F}_m'$ to the unipotent radical W_2 of $\mathrm{Aut}^0\,\mathrm{L}_{p,q}'$. All the W_1-orbits on $\mathbf{P}^1\times\mathrm{F}_m'$ is 1-dimensional and $\mathbf{P}^1\times\mathrm{F}_m'\to\mathbf{P}^1\times\mathrm{F}_m'/W_1\simeq\mathbf{P}^1\times\mathbf{P}^1$ is an $\mathbf{A}^1$-bundle. Similarly all the W_2-orbits on $\mathrm{L}_{p,q}'$ is 1-dimensional and $\mathrm{L}_{p,q}'\to\mathrm{L}_{p,q}'/W_2\simeq\mathbf{P}^1\times\mathbf{P}^1$. Thus f induces an isomorphism f: $\mathbf{P}^1\times\mathrm{F}_m'\to\mathrm{L}_{p,q}'$ of $\mathbf{A}^1$-bundles over $\mathbf{P}^1\times\mathbf{P}^1$. This is a contradiction as $m\geq 2$, $p\geq q>0$.

((J3)-(J9)) Since F'_m is a line bundle on $\mathbf{P}^1$, it contains $\mathbf{P}^1$ as the 0-section. Hence $\mathbf{P}^1 \times F'_m$ contains $\mathbf{P}^1 \times \mathbf{P}^1$. By Proposition (6.2), it is sufficient to show $\mathbf{P}^1 \times F'_m$ can not be embedded into $F'_{p,q}$. For this end, we show that $\mathbf{P}^1 \times \mathbf{P}^1$ is not contained in $F'_{p,q}$ if $p > q \geq 2$. In fact if $\mathbf{P}^1 \times \mathbf{P}^1$ were in $F'_{p,q}$, letting $\pi: F'_{p,q} \to \mathbf{P}^1$ be the projection, the intersection $\mathbf{P}^1 \times \mathbf{P}^1$ with the fibre of π would be a finite set since the fibre of π is affine and $\mathbf{P}^1 \times \mathbf{P}^1$ is projective. Thus $\mathbf{P}^1 \times \mathbf{P}^1$ would cover $\mathbf{P}^1$ finitely. This is a contradiction.

((J3)-(J10)) and ((J3)-(J′11)) The same argument as ((J3)-(J9)) gives the result.

((J3)-(J12)) Compare the dimension of the groups.

((J4)-(x)). Argue as ((J1)-(x)).

((J5)-(P1)) Assume that there exists a morphism (φ, f): $(\mathrm{PGL}_2, \mathrm{PGL}_2/\mathrm{D}_{2n}) \to (\mathrm{PGL}_4, \mathbf{P}^3)$ with f birational. Denoting by $\tilde{\mathrm{D}}_{2n}$ the inverse image $\pi^{-1}(\mathrm{D}_{2n})$ by the natural isogeny π: $\mathrm{SL}_2 \to \mathrm{SL}_2/\pm 1 = \mathrm{PGL}_2$, we have a $\tilde{\mathrm{D}}_{2n}$-invariant F of degree at most 3 (cf. Umemura [U2]) such that $\{g \in \mathrm{SL}_2 \mid gF = r_g F,\ r_g \in k\} = \tilde{\mathrm{D}}_{2n}$. Such F does not exist for $n \geq 4$.

((J5)-(P2)) Argue as in ((J5)-(P1)) but we look for a $\tilde{\mathrm{D}}_{2n}$-invariant as above of degree at most 4. This is possible only when $n=4$ and F is, up to constant factor, in the GL_2-orbit $\mathrm{GL}_2(x^4 - y^4)$ in the irreducible SL_2-module of homogeneous polynomial of degree 4 in x, y. Then SL_2-orbit $\mathrm{SL}_2\, F = \mathrm{SL}_2(x^4 - y^4)$ in $\mathbf{P}$ (homogeneous polynomials of degree 4) $= \mathbf{P}^4$ is not contained in a quadric in $\mathbf{P}^4$. In fact SL_2 operates on the generic homogeneous polynomial $a_0x^4 + a_1x^3y + a_2x^2y^2 + a_3xy^3 + a_4y^4$ and $i = a_2^2 - 3a_1a_3 + 12a_0a_4$ is the unique (up to constant) SL_2-invariant of degree 2 but i does not vanish on $x^4 - y^4$ (See Vol I, 70, Weber [Wb]).

((J5)-(E1)) and ((J5)-(E2)) Compare the stabilizers.

((J5)-(J1)) If there were a morphism (φ, f): $(\mathrm{PGL}_2, \mathrm{PGL}_2/\mathrm{D}_{2n}) \to (\mathrm{PGL}_3 \times \mathrm{PGL}_2, \mathbf{P}^2 \times \mathbf{P}^1)$ of algebraic operations with f birational, then there would be a non-trivial morphism (φ', f'): $(\mathrm{PGL}_2, \mathrm{PGL}_2/\mathrm{D}_{2n}) \to (\mathrm{PGL}_2, \mathbf{P}^1)$ of algebraic operations since there exists a morphism $(\mathrm{PGL}_3 \times \mathrm{PGL}_2, \mathbf{P}^3 \times \mathbf{P}^1) \to (\mathrm{PGL}_2, \mathbf{P}^1)$. The following Lemma gives a contradiction.

Lemma (6.3). *There is no non-trivial morphism* $(\mathrm{PGL}_2, \mathrm{PGL}_2/\mathrm{D}_{2n}) \to (\mathrm{PGL}_2, \mathrm{P}^1)$.

Assume the existence of such a morphism (φ', f'): $(\mathrm{PGL}_2, \mathrm{PGL}_2/\mathrm{D}_{2n}) \to (\mathrm{PGL}_2, \mathbf{P}^1)$. Then D_{2n} is contained in a Borel subgroup. Denoting by

$\tilde{D}_{2n}$ the inverse image under $\pi: SL_2 \to SL_2/\pm 1 = PGL_2$, the irreducible SL_2-module E of degree 2 so that $SL_2 = SL(E)$ is not an irreducible $\tilde{D}_{2n}$-module. Since $\tilde{D}_{2n}$ is finite, therefore $\tilde{D}_{2n}$ is diagonalized in SL_2 which is absurd.

((J5)-(J2)) and ((J5)-(J3)) Argue as in ((J5)-(J1)).

((J5)-(J4) Assume that there exists a morphism (φ, f): $(PGL_2, PGL_2/D_{2n}) \to (PGL_3, PGL_3/B)$ of algebraic operations with f birational. We may assume B consists of all the upper triangular matrices of PGL_3. Putting $P = \{(a_{ij}) \in PGL_3 \mid a_{12} = a_{13} = 0\}$, we get a morphism $(PGL_3, PGL_3/B) \to (PGL_3, PGL_3/P)$ of algebraic operations hence by composition (φ, f'): $(PGL_2, PGL_2/D_{2n}) \to (PGL_3, PGL_3/P)$. φ defines a representation E of degree 3 of SL_2. This representation should be irreducible. In fact letting $\pi: SL_2 \to SL_2/\pm 1 = PGL_2$ be the natural isogeny and $\tilde{D}_{2n} = \pi^{-1}(D_{2n})$, we have a morphism $(SL_2, SL_2/\tilde{D}_{2n}) \to (PGL_2, PGL_2/D_{2n})$ of algebraic operations hence by composition $(\tilde{\varphi}, \tilde{f})$: $(SL_2, SL_2/\tilde{D}_{2n}) \to (PGL_3, PGL_3/P)$. This morphism is nothing but $(SL_2, SL_2/\tilde{D}_{2n}) \to (SL_2, \mathbf{P}(E))$. Since $\tilde{f}$ is dominant, SL_2 has an open orbit on $\mathbf{P}(E)$. If E is not irreducible, then E is the direct sum of E_1 and E_2 where E_i's are irreducible SL_2-modules of degree i. Then the stabilizer at a point of the open SL_2-orbit in $\mathbf{P}(E)$ is a 1-dimensional unipotent subgroup of SL_2. Thus the existence of the morphism $(\tilde{\varphi}, \tilde{f})$ implies $\tilde{D}_{2n}$ is contained in a unipotent subgroup of SL_2 which is a contradiction and E must be irreducible. The transformation space PGL_3/B is the flag variety of E. We identify SL_2-module E with the SL_2-module of homogeneous polynomials of degree 2 in x, y: $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2$ sends x to $ax+cy$ and y to $bx+dy$. The stabilizer at a point $P = \{(x^2+y^2) \subset (x^2, y^2) \subset E\}$ of the flag variety is $\tilde{D}_{2\times 2}$ of SL_2. Hence $SL_2.P$ is an open SL_2-orbit on PGL_3/B = the flag variety of E. Therefore $SL_2.P$ coincides with $f(PGL_2/D_{2n})$ and $\tilde{D}_{2n}$ is conjugate to $\tilde{D}_{2\times 2}$ which is impossible.

((J5)-(J6)) Argue as ((J5)-(J1)).

((J5)-(J7)) Assume that there exists a morphism (φ, f): $(PGL_2, PGL_2/D_{2n}) \to (\mathrm{Aut}^0 J'_m, J'_m)$ of algebraic operation with f birational. As there exists a morphism $(\mathrm{Aut}^0 J'_m, J'_m) \to (PGL_3, \mathbf{P}^2)$ of algebraic operations by Proposition (4.8). We get a morphism (φ', f'): $(PGL_2/D_{2n}) \to (PGL_3, \mathbf{P}^2)$ of algebraic operations by composition. φ' defines a representation E of SL_2 of degree 3 and (φ', f') gives a morphism (φ'', f''): $(SL_3, SL_3/\tilde{D}_{2n}) \to (SL(E), \mathbf{P}(E))$. As in ((J5)-(J4)), E is irreducible. Let now W be the natural representation of degree 3 of SL_3: W is the vector space spanned by indeterminants u, v, w and $(\alpha_{ij}) \in SL_3$ sends $u \mapsto \alpha_{11}u + \alpha_{21}v + \alpha_{31}w$, $v \mapsto \alpha_{12}u$

$+\alpha_{22}v+\alpha_{32}w$, $w\mapsto\alpha_{13}u+\alpha_{23}v+\alpha_{33}w$. Let $S^m(W)$ be the m-th symmetric power of W, i.e., the vector space of the homogeneous polynomials in u, v, w of degree m which is an SL_3-module. We make $S^m(W)\,\mathbf{G}_m\,.\,\mathrm{SL}_3$-module by letting the vector space $S^m(W)\,\mathbf{G}_m$-module of weight 1. We need the semi-direct product $G_{(m)}=S^m(W)\,\mathbf{G}_m\,.\,\mathrm{SL}_3$ and let $H_m=$(homogeneous polynomials in u, v, w of degree m whose degree in $u\le m-1$)$\,.\,\mathbf{G}_m\,.\,P \subset G_{(m)}$ where $P=\{(a_{ij})\in \mathrm{SL}_3\,|\,a_{12}=a_{13}=0\}$. Then H_m is a subgroup of $G_{(m)}$ and as in two variable case [U3], $(\mathrm{Aut}^0\,\mathrm{J}'_m, \mathrm{J}'_m)$ is almost effectively realized $(G_{(m)}, G_{(m)}/H_m)$ (see also Corollary (4.10)). Now (φ, f): $(\mathrm{PGL}_2, \mathrm{PGL}_2/\mathrm{D}_{2n})\to(\mathrm{Aut}^0\,\mathrm{J}'_m, \mathrm{J}'_m)$ gives a morphism $(\tilde\varphi, \tilde f)$: $(\mathrm{SL}_2, \mathrm{SL}_2/\tilde{\mathrm{D}}_{2n})\to(G_{(m)}, G_{(m)}/H_m)$. By considering the translation by an element of $G_{(m)}$, we can replace $\tilde\varphi$ by a conjugacy in $G_{(m)}$. We may thus assume $\tilde\varphi(\mathrm{SL}_2)$ lies in $(0, 1, \mathrm{SL}_3)$ and $\tilde\varphi$: $\mathrm{SL}_2\to(0, 1, \mathrm{SL}_3)=\mathrm{SL}_3$ is given by

$$\tilde\varphi\left(\begin{pmatrix} a & b \\ c & d\end{pmatrix}\right)=\begin{pmatrix} ad+bc & 2ac & 2bd \\ ab & a^2 & b^2 \\ cd & c^2 & d^2\end{pmatrix}:$$

This is given by letting SL_2 operate on the homogeneous polynomials of degree 2 in x, y and taking a basis xy, x^2, y^2. $\tilde\varphi(\mathrm{SL}_2)$ has an open orbit on J'_m and we letting π: $\mathrm{J}'_m\to\mathbf{P}^2$ denote the projection, this orbit projected onto $\mathbf{P}^2$ is a $\tilde\varphi(\mathrm{SL}_2)=\mathrm{SL}_2$-open orbit on $\mathbf{P}^2=\mathbf{P}^2(kxy+kx^2+ky^2)$. Hence (the projection by π of the open orbit) $=\mathrm{SL}_2xy\subset\mathbf{P}^2(kxy+kx^2+ky^2)$. Therefore there exists a point P on $\pi^{-1}(xy)\in\mathrm{J}'_m$ such that SL_2P is open in J'_m. Equivalently since P is written as $(\lambda u^n, 1, 1)\,.\,H_m\in G_{(m)}/H_m$ for suitable $\lambda\in k$, the stabilizer at $P=(\lambda u^m, 1, 1)\tilde\varphi(\mathrm{SL}_2)(-\lambda u^m, 1, 1)\cap H_m$ is conjugate to $\tilde{\mathrm{D}}_{2n}$, in particular finite. But if m is even, $(\lambda u^m, 1, 1)\tilde\varphi\,.\,(\mathrm{SL}_2)(-\lambda u^m, 1, 1)\cap H_m=\mathrm{D}_{2\infty}=$one dimensional dihedral group and if $m=$ odd, $(\lambda u^m, 1, 1)\cdot\tilde\varphi(\mathrm{SL}_2)(-\lambda u^m, 1, 1)\cap H_m=\mathbf{G}_m$, $\mathrm{D}_{2\infty}$ according as $\lambda\neq 0$ or $\lambda=0$. This is a contradiction.

((J5)-(J8)), ((J5)-(J9)), ((J5)-(J10)), ((J5)-(J′11)) argue as ((J5)-(J1)) (see Propositions (4.13), (4.18) and Corollary (4.23).

((J5)-(J12)) (J5) is generically transitive whereas (J12) is intransitive even generically.

((J6)-(P1)) If there were a morphism (φ, f): $(G, G/H_{m,n})\to(\mathrm{PGL}_4, \mathbf{P}^4)$ with f birational, φ would give a representation E of $\mathbf{G}_m\times\mathrm{SL}_2\times\mathrm{SL}_2$ of degree 4. Since the kernel φ is finite, as $\mathrm{SL}_2\times\mathrm{SL}_2$-module E would be isomorphic to $V_2^{(1)}\oplus V_2^{(2)}$ or $V_2^{(1)}\otimes V_2^{(2)}$ where $V_2^{(1)}$ (resp. $V_2^{(2)}$) is the irreducible $\mathrm{SL}_2\times\mathbf{1}$ (resp. $\mathbf{1}\times\mathrm{SL}_2$)-module of degree 2. In the last case, the image of the center $\varphi(\mathbf{G}_m)$ would be scalar multiplications hence operate trivially on $\mathbf{P}^3$. This is absurd. Thus E would be isomorphic to $V_2^{(1)} \oplus V_2^{(2)}$. Since $V_2^{(1)}$, $V_2^{(2)}$ are irreducible non-isomorphic $\mathrm{SL}_2 \times \mathrm{SL}_2$-module, $\varphi(\mathbf{G}_m)$ would be diagonal. Let e_1, e_2 (resp. e_3, e_4) be a basis of $V_2^{(1)}$ (resp. $V_2^{(2)}$). With these basis, we may assume

$$\varphi(\mathrm{SL}_2 \times \mathrm{SL}_2) = \begin{bmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} & 0 \\ 0 & \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \end{bmatrix}, \quad \varphi(t) = \begin{pmatrix} t^r \mathrm{I}_2 & 0 \\ 0 & t^s \mathrm{I}_2 \end{pmatrix}$$

for $t \in \mathbf{G}_m \times \mathbf{1} \times \mathbf{1}$. A non-trivial subgroup $\{\zeta \in \mathbf{G}_m \mid \zeta^r = \zeta^s\}$ of $\mathbf{G}_m$ would operate trivially on $\mathbf{P}^3$. This is impossible because $\mathbf{G}_m \times \mathbf{1} \times \mathbf{1}$ operates effectively on $G/H_{m,n}$ and f is birational.

((J6)-(P2)) The rank of SO_5 is 2 but the rank of $\mathbf{G}_m \times \mathrm{SL}_2 \times \mathrm{SL}_2$ is 3.

((J6)-(E1)), ((J6)-(E2)) Compare the dimension of the groups.

((J6)-(J1)) Let (G', X') be an effective realization of (J6). Assume that there exists a morphism $(\varphi', f')\colon (G', X') \to (\mathrm{PGL}_2 \times \mathrm{PGL}_3, \mathbf{P}^1 \times \mathbf{P}^2)$ of algebraic operations with f' birational. Then there exists a morphism $(\varphi, f)\colon (G, G/H_{m,n}) \to (\mathrm{PGL}_2 \times \mathrm{PGL}_3, \mathbf{P}^1 \times \mathbf{P}^2)$ of algebraic operations with f birational. Then as other case is treated similarly, we may assume $\varphi((\lambda, A, B)) = (A, \varphi_2(\lambda, B)) \in \mathrm{PGL}_2 \times \mathrm{PGL}_3$ for $\lambda \in \mathbf{G}_m$, $A, B \in \mathrm{SL}_2$, where $\varphi_2\colon \mathbf{G}_m \times \mathrm{SL}_2 \to \mathrm{PGL}_3$ is a morphism and in the right hand side A is regarded as an element of PGL_2. The stabilizer of $\mathrm{PGL}_2 \times \mathrm{PGL}_3$ at a point of $\mathbf{P}^1 \times \mathbf{P}^2$ is a Borel subgroup. Therefore φ^{-1} (stabilizer) can not coincide with $H_{m,n}$, $m > 0 > n$.

((J6)-(J2)) Assume that there exists a morphism $(\varphi, f)\colon (G, G/H_{m,n}) \to (\mathrm{PGL}_2 \times \mathrm{PGL}_2 \times \mathrm{PGL}_2, \mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1)$ with f birational. Then we may assume $\varphi((\lambda, A, B)) = (\lambda, A, B) \in \mathrm{PGL}_2 \times \mathrm{PGL}_2 \times \mathrm{PGL}_2$ for $\lambda \in \mathbf{G}_m$, $A, B \in \mathrm{SL}_2$. Therefore φ^{-1}(the stabilizer of $\mathrm{PGL}_2 \times \mathrm{PGL}_2 \times \mathrm{PGL}_2$ at a point of $\mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1) = \varphi^{-1}$ (a Borel subgroup of $\mathrm{PGL}_2 \times \mathrm{PGL}_2 \times \mathrm{PGL}_2$) which never coincides with $H_{m,n}$, $m > 0 > n$.

((J6)-(J3)) Argue as ((J6)-(J1)) (see also Umemura [U3]).

((J6)-(J4)) Compare the rank of the groups.

((J6)-(J5)) Compare the dimension of the groups.

((J6)-(J7)) Assume that there exists a morphism $(\varphi, f)\colon (G, G/H_{m,n}) \to (\mathrm{Aut}^0 \mathrm{J}'_l, \mathrm{J}'_l)$ of algebraic operations with f birational. By Proposition

(4.8), we get a morphism (φ', f'): $(G, G/H_{m,n}) \to (\mathrm{PGL}_3, \mathbf{P}^2)$ of algebraic operations. Thus Ker φ' contains a subgroup K isogeneous to SL_2 which operates on the fibre $\mathrm{J}'_l \to \mathbf{P}^2$. But the fibre is $\mathbf{A}^1$ and the operation of SL_2 should be trivial. This is a contradiction.

((J6)-(J8)) The reductive part of $\mathrm{Aut}^0 \mathrm{L}'_{k,l}$ is isogeneous to $\mathbf{G}_m \times \mathrm{PGL}_2 \times \mathrm{PGL}_2$ by Proposition (4.11) and this group has an open orbit W on $\mathrm{L}'_{m,n}$. $(\mathbf{G}_m \times \mathrm{PGL}_2 \times \mathrm{PGL}_2, W)$ is almost effectively realized by $(\mathbf{G}_m \times \mathrm{SL}_2 \times \mathrm{SL}_2, \mathbf{G}_m \times \mathrm{SL}_2 \times \mathrm{SL}_2/H_{k,l})$ where

$$H_{k,l} = \left\{ \left(t_1^k t_2^l, \begin{pmatrix} t_1 & x \\ 0 & t_1^{-1} \end{pmatrix}, \begin{pmatrix} t_2 & y \\ 0 & t_2^{-1} \end{pmatrix} \right) \in \mathbf{G}_m \times \mathrm{SL}_2 \times \mathrm{SL}_2 \mid t_1, t_2 \in \mathbf{G}_m, x, y \in k \right\}.$$

The existence of an inclusion implies that of the isomorphism $(\mathbf{G}_m \times \mathrm{SL}_2 \times \mathrm{SL}_2, \mathbf{G}_m \times \mathrm{SL}_2 \times \mathrm{SL}_2/H_{m,n}) \cong (\mathbf{G}_m \times \mathrm{SL}_2 \times \mathrm{SL}_2, \mathbf{G}_m \times \mathrm{SL}_2 \times \mathrm{SL}_2/H_{k,l})$. Considering the orbits of the center, it implies also that they are isomorphic as principal $\mathbf{G}_m$-bundles over $\mathbf{P}^1 \times \mathbf{P}^1$ but this is impossible because $m \geqq 2$, $-2 \geqq n$ and $k > l \geqq 2$.

((J6)-(J9)) A semi-simple part of (J9) is $\mathfrak{sl}_2$ whereas that of (J6) is $\mathfrak{sl}_2 \times \mathfrak{sl}_2$.

((J6)-(J10)) It follows from Corollary (4.20) that a reductive part of $(\mathrm{Aut}^0 \mathrm{F}'_{k,k}, \mathrm{F}'_{k,k})$ is $\mathbf{G}_m \times \mathrm{SL}_2 \times \mathrm{SL}_2$. As in the proof of Lemma (4.30), $\mathbf{G}_m \times \mathrm{SL}_2 \times \mathrm{SL}_2$ has an open orbit on $\mathrm{F}'_{k,k}$ isomorphic to $(\mathbf{G}_m \times \mathrm{SL}_2 \times \mathrm{SL}_2, \mathbf{G}_m \times \mathrm{SL}_2 \times \mathrm{SL}_2/H_{k,-1})$. For the same reason as in ((J6)-(J8)), $(\mathbf{G}_m \times \mathrm{SL}_2 \times \mathrm{SL}_2, \mathbf{G}_m \times \mathrm{SL}_2 \times \mathrm{SL}_2/H_{m,n})$ $(m \geqq 2, -2 \geqq n)$ can not be isomorphic to $(\mathbf{G}_m \times \mathrm{SL}_2 \times \mathrm{SL}_2, \mathbf{G}_m \times \mathrm{SL}_2 \times \mathrm{SL}_2/H_{k,-1})$.

((J6)-(J′11)) Compare reductive parts (cf. Corollary (4.23)).

((J6)-(J12)) Compare the dimension of the groups.

((J7)-(P1)) Let E be the natural irreducible representation of degree 3 of SL_3. Let $S^m(E)$ be the m-th symmetric power of E. Letting $\mathbf{G}_m$ operate on $S^m(E)$ with weight 1, we see as in Section 2, [U3], the semi-direct product $S^m(E) . \mathbf{G}_m \times \mathrm{SL}_3$ operates on J'_m. The dimension of this algebraic group is $\binom{m+2}{m} + 9$ which is greater than or equal to 15 if $m \geq 2$. Thus if we have an inclusion of (J7) in (P1), comparing the dimension, we conclude $m = 2$ and no automorphism of $\mathbf{G}_m \times \mathrm{SL}_2 \times \mathrm{SL}_2$ mapping $H_{m,n}$ $(m > 0 > n)$ onto $H_{k,l}$ $(k \geq l > 0)$.

((J7)-(P2)) As we have seen above dim $\mathrm{Aut}^0 \mathrm{J}'_m \geq 15$, it is sufficient to compare the dimension.

((J7)-(E1)), ((J7)-(E2)), ((J7)-(J1)) and ((J7)-(J2)) Compare the dimension.

((J7)-(J3)) Compare semi-simple parts (cf. Proposition (4.8) and [U3]).

((J7)-(J4)), ((J7)-(J5)) and ((J7)-(J6)) Compare the dimension.

((J7)-(J8)), ((J7)-(J9)), ((J7)-(J10)), ((J7)-(J′11)) and ((J7)-(J12)) Compare semi-simple parts (cf. [U3], Propositions (4.13), (4.18) and Corollary (4.23).

((J8)-(P1)) Let U_{m+1}, U_{n+1} be the irreducible SL_2-modules of dimension $m+1$ and $n+1$. Letting $\mathbf{G}_m$ operate on $U_{m+1}\otimes U_{n+1}$ with weight 1, we see the semi-direct product $U_{m+1}\otimes U_{n+1}.(\mathbf{G}_m\times\mathrm{SL}_2\times\mathrm{SL}_2)$ operates almost effectively on $\mathrm{L}'_{m,n}$ (Corollary (4.12)). In particular $\mathrm{Aut}^0\mathrm{L}'_{m,n}$ cotains the vector group $U_{m+1}\otimes U_{n+1}$ of dimension $(m+1)(n+1)\geq 4$. Since SL_4 does not contain vector group of dimension ≥ 5, it is sufficient to show that there exists no inclusion when $m=n=1$. Assume that there exists a morphism (φ, f): $(\mathrm{Aut}^0\mathrm{L}'_{1,1}, \mathrm{L}'_{1,1})\to(\mathrm{PGL}_4, \mathbf{P}^4)$ with f birational. Then φ induces a morphism $U_2\otimes U_2.(\mathbf{G}_m\times\mathrm{SL}_2\times\mathrm{SL}_2)\to\mathrm{PGL}_4$. In particular, a representation E of degree 4 of $\mathrm{SL}_2\times\mathrm{SL}_2$. Since the operation of $\varphi(\mathrm{SL}_2\times\mathrm{SL}_2)$ on $\mathbf{P}^3$ is almost effective, the representation E is faithful modulo finite group. Thus E is isomorphic to either $U_2\otimes\mathbf{C}\oplus\mathbf{C}\otimes U_2$ or $U_2\otimes U_2$. If we consider φ induces a morphism $U_2\otimes U_2(\mathrm{SL}_2\times\mathrm{SL}_2)\to\mathrm{PGL}(E)$ of which the kernel is finite, E is isomorphic to $U_2\otimes\mathbf{C}\oplus\mathbf{C}\otimes U_2$ and the image $\varphi(U_2\otimes U_2.(\mathbf{G}_m\times\mathrm{SL}_2\times\mathrm{SL}_2))=\{(a_{ij})\in\mathrm{PGL}_4\,|\,a_{13}=a_{14}=a_{23}=a_{24}=0\}$. But the stabilizer of PGL_4 at a point Q of $\mathbf{P}^3$ is $P=\{(b_{ij})\in\mathrm{PGL}_4\,|\,a_{31}=a_{32}=a_{41}=a_{42}=0\}$. Counting the dimension of stabilizer, the orbit $\varphi((U_2\otimes U_2).(\mathbf{G}_m\times\mathrm{SL}_2\times\mathrm{SL}_2))Q$ is 3-dimensional subset of $\mathbf{P}^3$. But $(U_2\otimes U_2).\mathbf{G}_m\times\mathrm{SL}_2\times\mathrm{SL}_2\cap P$ contains $\left\{\begin{pmatrix}1 & 0\\ 0 & A\end{pmatrix}\,\middle|\,A\in\mathrm{SL}_2\right\}$. But the stabilizer of $(U_2\times U_2).\mathbf{G}_m\times\mathrm{SL}_2\times\mathrm{SL}_2$ at a point of the open orbit on $\mathrm{L}'_{m,n}$ is solvable (cf. Corollary (4.12)). This is a contradiction.

((J8)-(P2)) By Corollary (4.12) the rank of $\mathrm{Aut}^0\mathrm{L}'_{m,n}$ equals to 3. But the rank of PSO_5 is 2.

((J8)-(E1)) and ((J8)-(E2)) Compare the dimension of the groups.

((J8)-(J1)) and ((J8)-(J2)) We have seen in ((J8)-(P1)) $\mathrm{Aut}^0\mathrm{L}'_{m,n}$ contains a vector group of dimension 4. But $\mathrm{PGL}_3\times\mathrm{PGL}_2$ and $\mathrm{PGL}_2\times\mathrm{PGL}_2\times\mathrm{PGL}_2$ do not contain a vector group of dimension 4.

((J8)-(J3)) Since both groups have the same semi-simple parts, if we have a morphism (φ, f): $(\mathrm{Aut}^0 \mathrm{L}'_{m,n}, \mathrm{L}'_{m,n}) \to (\mathrm{PGL}_2 \times \mathrm{Aut}^0 \mathrm{F}'_l, \mathbf{P}^1 \times \mathrm{F}'_l)$ with birational f, φ map the unipotent radical to the unipotent radical. Since the fibrations $\mathrm{L}'_{m,n} \to \mathbf{P}^1$ and $\mathbf{P}^1 \times \mathrm{F}'_l \to \mathbf{P}^1 \times \mathbf{P}^1$ are quotient morphisms with respect to the unipotent radicals, f induces an isomorphism of line bundles $\mathrm{L}'_{m,n}$ and $\mathbf{P}^1 \times \mathrm{F}'_l$ over $\mathbf{P}^1 \times \mathbf{P}^1$, which is absurd.

((J8)-(J4)), ((J8)-(J5)) and ((J8)-(J6)) Compare the dimension of the groups.

((J8)-(J7)) Compare semi-simple parts.

((J8)-(J8)) Argue as ((J8)-(J3)).

((J8)-(J9)), ((J8)-(J10)) and ((J8)-(J'11)) Since $\mathrm{L}'_{m,n}$ is the total space of a line bundle over $\mathbf{P}^1 \times \mathbf{P}^1$ as we have seen, it contains $\mathbf{P}^1 \times \mathbf{P}^1$. But being $\mathbf{A}^2$-bundle over $\mathbf{P}^1$, $\mathrm{F}'_{m,n}$ and E'^{l}_{m} do not contain $\mathbf{P}^1 \times \mathbf{P}^1$.

((J8)-(J12)) Compare the dimension of the groups.

((J9)-(P1)) and ((J9)-(P2)) Let U_{m+1}, U_{n+1} be irreducible SL_2-module of dimension $m+1$ and $n+1$. Then the semi-direct product $(U_{m+1} \oplus U_{n+1}).\mathrm{SL}_2$ operates on $\mathrm{F}'_{m,n}$. Therefore $\mathrm{Aut}^0 \mathrm{F}'_{m,n}$ contains the vector group $U_{m+1} \oplus U_{n+1}$ of dimension ≥ 7. But in PGL_4 and SO_5, there is no vector group of dimension 7.

((J9)-(E1)) and ((J9)-(E2)) Compare the dimension of the groups.

((J8)-(J1)) and ((J9)-(J2)) Argue as ((J9)-(P1)).

((J9)-(J3)) By Umemura [U3], Section 2, $\mathrm{PGL}_2 \times \mathrm{Aut}^0 \mathrm{F}'_l$ is isogeneous to $\mathrm{SL}_2 \times (U_{l+1}.(\mathbf{G}_m \times \mathrm{SL}_2))$ where U_{l+1} is the irreducible SL_2-module of dimension $l+1$ and as $\mathbf{G}_m$-module U_{l+1} is of weight 1. $(U_{m+1} \oplus U_{n+1})\mathrm{SL}_2$ is simply connected and isogeneous to a closed subgroup of $\mathrm{Aut}^0 \mathrm{F}_{m,n}$. There is no morphism with finite kernel of $(U_{m+1} \oplus U_{n+1})\mathrm{SL}_2$ to $\mathrm{SL}_2 \times (U_{l+1}.(\mathbf{G}_m \times \mathrm{SL}_2))$ and hence $\mathrm{Aut}^0 \mathrm{F}'_{m,n}$ can not be embedded in $\mathrm{PGL}_2 \times \mathrm{Aut}^0 \mathrm{F}'_l$ for $m > n \geq 2$.

((J9)-(J4)), ((J9)-(J5)) and ((J9)-(J6)) Argue as in ((J9)-(P1)).

((J9)-(J7)) Since a semi-simple part of $\mathrm{Aut}^0 \mathrm{F}'_{m,n}$ is SL_2 and that of $\mathrm{Aut}\, \mathrm{J}'_l$ is SL_3, if there exists a morphism (φ, f): $(\mathrm{Aut}^0 \mathrm{F}'_{m,n}, \mathrm{F}'_{m,n}) \to (\mathrm{Aut}^0 \mathrm{J}'_l, \mathrm{J}'_l)$ with f birational, φ maps that subgroup U_{m+1} of $(U_{m+1} \oplus U_{m+1}).\mathrm{SL}_2$ (hence of $\mathrm{Aut}^0 \mathrm{F}'_{m,n}$) into the unipotnet radical of $\mathrm{Aut}^0 \mathrm{J}'_l$. By considering the quotients by U_{m+1} and the unipotent radical of $\mathrm{Aut}^0 \mathrm{J}'_l$, we get

$$\begin{tikzcd} \mathrm{F}'_{m,n} \arrow[r, hook, "f"] \arrow[d, "p"'] & \mathrm{J}'_l \arrow[d, "q"] \\ \mathrm{F}'_n \arrow[r, "f'"] & \mathbf{P}^2. \end{tikzcd}$$

Since the fibres of p and q are $\mathbf{A}^1$, they coincide and f' is an embedding. This is impossible as $n \geq 2$.

((J9)-(J8)) Argue as in ((J9)-(J7)). We conclude in this case F'_n is embedded in $\mathbf{P}^1 \times \mathbf{P}^1$ which is impossible as $n \geq 2$.

((J9)-(J9)) If there exists a morphism (φ, f): $(\mathrm{Aut}^0 \mathrm{F}'_{m,n}, \mathrm{F}'_{m,n}) \to (\mathrm{Aut}^0 \mathrm{F}'_{k,l}, \mathrm{F}'_{k,l})$ with f birational, φ maps a semi-simple part to a semi-simple part and unipotent radical to the unipotent radical because both groups have isomorphic semi-simple parts. Therefore considering the quotients by the unipotent radicals, we get

$$\begin{tikzcd} \mathrm{F}'_{m,n} \arrow[r, "f"] \arrow[d, "p"'] & \mathrm{F}'_k \arrow[d, "q"] \\ \mathbf{P}^1 \arrow[r, "f'"] & \mathbf{P}^1. \end{tikzcd}$$

As in ((J9)-(J8)) we conclude $\mathrm{F}'_{m,n}$ is isomorphic to $\mathrm{F}'_{k,l}$. Then by Proposition (4.13), $m=k$, $n=l$.

((J9)-(J10)) Argue as ((J9)-(J9)) (see Propositions (4.18) and (4.19)).

((J9)-(J'11)) Compare the ranks (see Proposition (4.13) and Corollary (4.23)).

((J9)-(J12)) Compare the dimension of the groups.

((J10)-(P1)) A semi-simple part of $\mathrm{Aut}^0 \mathrm{F}'_{m,m}$ is $\mathrm{SL}_2 \times \mathrm{SL}_2$ by Proposition (4.18) and an $\mathrm{SL}_2 \times \mathrm{SL}_2$-module $U_{m+1} \otimes U_2$ operates on $\mathrm{F}'_{m,m}$ hence the semi-direct product $(U_{m+1} \otimes U_2).\mathrm{SL}_2 \times \mathrm{SL}_2$ by Corollary (4.20). By letting $\mathbf{G}_m$ operates on $U_{m+1} \otimes U_2$ as scalar multiplication, $(U_{m+1} \otimes U_2) \cdot (\mathbf{G}_m \times \mathrm{SL}_2 \times \mathrm{SL}_2)$ operates on $\mathrm{F}'_{m,m}$. Now argue as in ((J8)-(P1)).

((J10)-(P2)) Compare the rank (cf. Corollary (4.20)).

((J10)-(E1)) and ((J10)-(E2)) Compare the dimension of the groups.

((J10)-(J1)) and ((J10)-(J2)) $\mathrm{Aut}^0 \mathrm{F}'_{m,m}$ contains $(U_{m+1} \otimes U_2)(\mathrm{SL}_2 \times \mathrm{SL}_2)$ by Corollary (4.20) hence a unipotent subgroup of dimension $2(m+2)$

≥ 8. But such a big unioptent group is not contained neither in $PGL_3 \times PGL_2$ nor in $PGL_2 \times PGL_2 \times PGL_2$.

((J10)-(J3)) The semi-simple part of $\mathrm{Aut}^0 F'_{m,m}$ is $SL_2 \times SL_2$ by Proposition (4.18) and that of $PGL_2 \times \mathrm{Aut}^0 F'_l$ is also $SL_2 \times SL_2$. If there exists a morphism (φ, f): $(\mathrm{Aut}^0 F'_{m,m}, F'_{m,m}) \to (PGL_2 \times \mathrm{Aut}^0 F'_l, \mathbf{P}^1 \times F'_l)$ with f birational, then the unipotent radical of $\mathrm{Aut}^0 F'_{m,m}$ is mapped to that of $PGL_2 \times \mathrm{Aut}^0 F'_l$. But the unipotent radical of $\mathrm{Aut}^0 F'_{m,m}$ has 2-dimensional orbit whereas that of $PGL_2 \times \mathrm{Aut}^0 F'_l$ has 1-dmensional orbit (cf. [U3]).

((J10)-(J4)) Compare semi-simple parts (cf. Proposition (4.18)).

((J10)-(J5)) and ((J10)-(J6)) Compare the dimension of the groups.

((J10)-(J7)) Compare semi-simple parts (cf. Propositions (4.8) and (4.18)).

((J10)-(J8)) Both groups have the same semi-simple parts $SL_2 \times SL_2$ hence the unipotent radical of $\mathrm{Aut}^0 F'_{m,m}$ is mapped to that of $\mathrm{Aut}^0 L'_{m,n}$. But the unipotent radical of $\mathrm{Aut}^0 F'_{m,m}$ has 2-dimensional orbits whereas that of $\mathrm{Aut}^0 L'_{m,n}$ has only 1-dimensional orbits which is a contradiction.

((J10)-(J9)) Compare semi-simple parts.

((J10)-(J10)) See Corollary (4.20).

((J10)-(J′11)) Compare semi-simple parts.

((J10)-(J12)) Compare the dimension.

((J′11))-(P1)) By exact sequence in Corollary (4.23) the unipotent radical of $\mathrm{Aut}^0(E'^l_m, F'_m)$ is of dimension $\geq lm-1+m+1=lm+m$. Thus the dimension of a maximal unipotent subgroup of $\mathrm{Aut}^0(E'^l_m, F'_m)$ is bigger than or equal to $lm+m+1$, since a semi-simple part of $\mathrm{Aut}^0(E'^l_m, F'_m)$ contains $\mathbf{G}_a$. Therefore if $lm+m+1>$ dimension of a maximal unipotent subgroup of $PGL_4=6$, there does not exist an inclusion. We have to check the case $lm+m \leq 5$. There are only two possible values of (l, m), $(4, 1)$ $(3, 1)$. If $l=4$, $m=1$, then by letting U be the unipotent radical of $\mathrm{Aut}^0(E'^4_1, F'_1)$, we have an exact sequence by Corollary (4.23),

$$\begin{array}{c} 0 \to H^0(F'_1, \mathcal{O}_{F'_1}(2)) \to U \to H^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(1)) \to 0 \\ \wr\Big| \\ H^0(\mathbf{P}^1, \mathcal{O}(2)) \oplus H^0(\mathbf{P}^1, \mathcal{O}(1)) \oplus H^0(\mathbf{P}^1, \mathcal{O}). \end{array}$$

Hence $\dim U=8$. Hence U can not be contained in PGL_4. If $l=3$,

$m=1$, the unipotent radical U of $\mathrm{Aut}^0(\mathrm{E}_1'^3, \mathrm{F}_1')$ has an SL_2-composition series;

$$\begin{array}{ccccccc} 0\to & H^0(\mathrm{F}_1', \mathcal{O}_{\mathrm{F}_1'}(1)) & \to U \to & H^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(1)) & \to 0. \\ & \wr| & & \wr| & \\ & H^0(\mathbf{P}^{-1}, \mathcal{O}(1))\oplus H^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}) & & U_2 & \\ & \wr| & & & \\ & U_2\oplus U_1 & & & \end{array}$$

It follows from the composition series that the only one possible way of embedding (mod finite kernel) $U_{\bullet}\mathrm{SL}_2$ into SL_4 is through the SL_2-module $V_2\oplus V_2\oplus V_1$. Then $U_{\bullet}\mathrm{SL}_2$ is identified with

$$\left\{\begin{pmatrix} \alpha & \beta & 0 & 0 \\ \gamma & \delta & 0 & 0 \\ a & b & 1 & 0 \\ c & d & e & 1 \end{pmatrix} \in \mathrm{SL}_4 \,\middle|\, \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{SL}_2,\ a, b, c, d, e \in k\right\}.$$

The $U_{\bullet}\mathrm{SL}_2$-open orbit on $\mathbf{P}^3$ is also a

$$(V_2\oplus V_2)\mathrm{SL}_2=\left\{\begin{pmatrix} \alpha & \beta & 0 & 0 \\ \gamma & \delta & 0 & 0 \\ a & b & 1 & 0 \\ c & a & 0 & 1 \end{pmatrix} \in \mathrm{SL}_4 \,\middle|\, \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{SL}_2,\ a, b, c, d \in k\right\}\text{-}$$

orbit of $P \in \mathrm{SL}_2/P \simeq \mathbf{P}^4$ where $P=\{(a_{ij}) \in \mathrm{SL}_4 \,|\, a_{21}=a_{31}=a_{41}=0\}$. Then $(V_2\oplus V_2)_{\bullet}\mathrm{SL}_{2\bullet}P \simeq (V_2\oplus V_2)_{\bullet}\mathrm{SL}_2/P\cap(V_2\oplus V_2)\mathrm{SL}_2$ which is isomorphic to $\mathrm{F}_{1,1}'$ (see Proposition (5.30.7)). Thus we get an F_1'-isomorphism $\mathrm{F}_{1,1}'\simeq U_{\bullet}\mathrm{SL}_2$-open orbit in $\mathbf{P}^3\simeq \mathrm{E}_1'^3$. This is a contradiction.

((J′11)-(P2)) As we have seen in ((J11)-(P1)) above the dimension of a maximal unipotent subgroup of $\mathrm{Aut}^0(\mathrm{E}_m'^l, \mathrm{F}_m')$ for $l, m=2$ or $m\geq 1, l\geq 3$ is at least 5. Therefore no inclusion.

((J′11)-(E1)) and ((J′11)-(E2)) Compare the dimension of the groups.

((J′11)-(J1)) and ((J′11)-(J2)) Compare the dimension of maximal unipotent subgroups.

((J′11)-(J3)) Argue as ((J9)-(J3)).

((J′11)-(J4)), ((J′11)-(J5)) and ((J′11)-(J6)) Compare the dimension of the groups.

((J’11)-(J7)) We have an exact sequence

$$\text{(a)}\quad \mathbf{1}\to N\to \mathrm{Aut}^0(\mathrm{E}'^l_m;\ \mathrm{F}'_m)\to \mathrm{Aut}^0\,\mathrm{F}'_m\to\mathbf{1}.$$

By Corollary (4.23), N coincides with $H^0(\mathrm{F}'_m, \mathcal{O}(lm-2))$. As SL_2-module $H^0(\mathrm{F}'_m, \mathcal{O}(lm-2))$ is isomorphic to, by the spectral sequence, $\bigoplus_{k=0}^{\infty} H^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}((l-k)m-2))$. If there exists a morphism (φ, f): $(\mathrm{Aut}^0\mathrm{E}'^l_m, \mathrm{E}'^l_m)\to(\mathrm{Aut}^0\mathrm{J}'_n, \mathrm{J}'_n)$ with f birational, then composing with the morphism (φ', f'): $(\mathrm{Aut}^0\mathrm{J}'_n, \mathrm{J}'_n)\to(\mathrm{PGL}_3, \mathbf{P}^2)$, we get a morphism (φ'', f''): $(\mathrm{Aut}^0\mathrm{E}'^l_m, \mathrm{E}'^l_m)\to(\mathrm{PGL}_3, \mathbf{P}^2)$. Since φ': $\mathrm{Aut}^0\mathrm{J}'_n\to\mathrm{PGL}_3$ gives the semi-simple part of $\mathrm{Aut}^0\mathrm{J}'_n$, $\varphi(\mathrm{SL}_2)\neq 1$.

Let us first assume $lm\geq 4$. Under this assumption, the image of the subgroup $H^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(lm-2))$ under φ'' is 0. In fact otherwise the semi-direct product $H^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(lm-2)).\mathrm{SL}_2$ is contained in SL_3 because SL_2-module $H^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(lm-2))$ is irreducible. But this is impossible if $lm\geq 4$. Therefore the image $\varphi'(H^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(lm-2)))$ is in the unipotent radical of $\mathrm{Aut}^0\mathrm{J}'_n$. Considering the quotients by $H^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(lm-2))$ and by the unipotent radical of $\mathrm{Aut}^0\mathrm{J}'_n$, since their orbits are $\mathbf{A}^1$, we get

$$\begin{array}{ccc} \mathrm{E}'^l_m & \longrightarrow & \mathrm{J}'_n \\ \downarrow & & \downarrow \\ \mathrm{F}'_m & \longrightarrow & \mathbf{P}^2. \end{array}$$

Thus E'^l_m is a line bundle over F'_m which is a contradiction. Now we treat the case $lm=3$, namely $m=1$ and $l=3$. The unipotent part of kernel N of (a) is $H^0(\mathrm{F}'_m, \mathcal{O}(lm-2))=H^0(\mathrm{F}'_m, \mathcal{O}(1))=H^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(1))\oplus H^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1})$ because the spectral sequence degenerates. Thus denoting by U the unipotent radical of $\mathrm{Aut}^0(\mathrm{E}'^3_1;\ \mathrm{F}'_1)$, we have an exact sequence

$$\begin{array}{ccccccc} 0\to & H^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(1))\oplus H^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}) & \to U\to & (\text{unipotent radical of } \mathrm{Aut}^0\,\mathrm{F}'_1) & \to 0. \\ & \Vert & & \Vert & \\ & N & & H^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(1)) & \end{array}$$

Let us put $W=H^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(1))$ of the kernel N.

If $\varphi''(W)=0$, we can argue as in the case $lm\geq 4$. We may thus assume $\varphi''(W)\neq 0$. Then $\varphi''((H^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1})\text{ of }N))=0$ by comparing the dimension of the unipotent radicals of a Borel subgroup of SL_3 and that of $\varphi''(\mathrm{Aut}^0(\mathrm{E}'^3_1, \mathrm{F}'_1))$. Then we can argue as for $lm\geq 4$.

((J′11)-(J8)) We have an exact sequence

$$\text{(b)}\quad 0\to R\to \mathrm{Aut}^0\,\mathrm{L}'_{r,s}\to \mathrm{PGL}_2\times\mathrm{PGL}_2\to\mathbf{1}$$

where R is the radical of $\mathrm{Aut}^0\mathrm{L}'_{r,s}$ by Proposition (4.11). Assume there exists a morphism $(\varphi, f)\colon (\mathrm{Aut}^0(\mathrm{E}'^l_m; \mathrm{F}'_m), \mathrm{E}'^l_m) \to (\mathrm{Aut}^0 \mathrm{L}'_{r,s}, \mathrm{L}'_{r,s})$ with f birational. As $lm > 2$, φ takes an SL_2-invariant subspace of the kernel N of (a) to R.

In fact otherwise, since

$$N \simeq \bigoplus_{\substack{j \le l \\ jm-2 \ge 0}} H^0(\mathbf{P}^1, \mathcal{O}(jm-2))$$

as SL_2-module, the semi-direct product $N_{\bullet}$(semi-simple part of $\mathrm{Aut}^0 \mathrm{E}'^l_m$) would be contained in $\mathrm{PGL}_2 \times \mathrm{PGL}_2$ (see Proposition (4.11)). If we consider $\varphi(N)$-orbits, their dimension is at most 1. Hence generically they coincides with R-orbits and we get isomorphism

$$\begin{tikzcd} \mathrm{E}'^l_m \arrow[r, hook] \arrow[d] & \mathrm{L}'_{r,s} \arrow[d] \\ \mathrm{F}'_m \arrow[r, hook] & \mathbf{P}^1 \times \mathbf{P}^1. \end{tikzcd}$$

But this is absurd as $m \geq 1$.

((J′11)-(J9)) As the both algebraic groups have the same semi-simple parts, if we have a morphism $(\varphi, f)\colon (\mathrm{Aut}^0 \mathrm{E}'^l_m, \mathrm{E}'^l_m) \to (\mathrm{Aut}^0 \mathrm{F}'_{s,t}, \mathrm{F}'_{s,t})$ with f birational, φ takes the unipotent radical to the unipotent radical. Since the unipotent radicals have only 2-dimensional orbits and the transformation spaces E'^l_m, $\mathrm{F}'_{s,t}$ are $\mathbf{A}^2$-bundle over $\mathbf{P}^1$, f is a $\mathbf{P}^1$-isomorphism. Let Z_E be the center of the unipotent radical of $\mathrm{Aut}^0(\mathrm{E}'^l_m, \mathrm{F}'_m)$ and Z_F be that of $\mathrm{Aut}^0 \mathrm{F}'_{s,t}$. If generically the Z_E-orbits under f coincides with Z_F-orbits, then as in ((J′11)-(J7)), considering the quotients, we get isomorphisms

$$\begin{tikzcd} \mathrm{E}'^l_m \arrow[r, "\sim"] \arrow[d] & \mathrm{F}'_{s,t} \arrow[d] \\ \mathrm{F}'_m \arrow[r, "\sim"] & \mathrm{F}'_t. \end{tikzcd}$$

This shows E'^l_m is a line bundle. Hence even generically the image $f(Z_E$-orbits) does not coincide with Z_F-orbits. Therefore the vector subgroup $Z_F.\varphi(Z_E)$ has a 2-dimensional orbit therefore $(Z_F.\varphi(Z_E))\,\mathrm{SL}_2$ operates on $\mathrm{F}'_{s,t}$ and has an open orbit. This open orbit is an $\mathbf{A}^2$-bundle over $\mathbf{P}^1$ (cf. Subcase (5.30.1)). Consequently it coincides with $\mathrm{F}'_{s,t}$. In other words $\mathrm{F}'_{s,t}$ is a homogeneous space under $(Z_F.\varphi(Z_E))\,\mathrm{SL}_2$. Let H be the stabilizer group at a point of $(Z_F.\varphi(Z_E))\,\mathrm{SL}_2$ so that $\mathrm{F}'_{s,t} \simeq (Z_F.\varphi(Z_E))_{\bullet}\mathrm{SL}_2/H$. By Subcase (5.30.1), Z_F and $\varphi(Z_E)$ are necessarily irreducible SL_2-modules. If we consider the quotient $(Z_F.\varphi(Z_E))/Z_F.H$ of $(Z_F.\varphi(Z_E))\,\mathrm{SL}_2/H$ with respect to the normal subgroup Z_F of $(Z_F.\varphi(Z_E))\,\mathrm{SL}_2/H$, the quotient

$(Z_F . \varphi(Z_E)) . \mathrm{SL}_2/Z_F H$ is isomorphic to F'_t. Therefore $Z_F \simeq U_{s+1}$, $\varphi(Z_E) \simeq U_{t+1}$ and H is identified with $(U'_{s+1} \oplus U'_{t+1}) . B$ where B is a Borel subgroup of SL_2 consisting of the upper triangular matrices and U'_{s+1}, U'_{t+1} are B-invariant subspace of codimension 1 of U_{s+1}, U_{t+1} (cf. (5.30.1)). Consider now the quotient with respect to Z_E or U_{t+1}, then we get $Z_E \backslash \mathrm{E}'^l_m \simeq \mathrm{F}'_m$ from the definition and $\varphi(Z_E) \backslash \mathrm{F}'_{s,t} \simeq (U_{s+1} \oplus U_{t+1}) \mathrm{SL}_2/U_{t+1} . H \simeq \mathrm{F}'_s$:

$$\begin{array}{ccc} \mathrm{E}'^l_m & \xrightarrow{\sim} & \mathrm{F}'_{s,t} \\ \downarrow & & \downarrow \\ \mathrm{F}'_m & \xrightarrow{\sim} & \mathrm{F}'_s . \end{array}$$

It follows from the form of H that, the morphism $\mathrm{F}'_{s,t} \to \mathrm{F}'_s$, which is $(U_{s+1} \oplus U_{t+1}) \mathrm{SL}_2/H \to (U_{s+1} \oplus U_{t+1}) \mathrm{SL}_2/U_{t+1} . H \simeq \mathrm{F}'_s$, is a line bundle over F'_s. This is absurd.

((J'11)-(J10)) We saw in Corollary (4.20) $\mathrm{Aut}^0 \mathrm{F}'_{n,n}$ is isogeneous to $(U_{n+1} \otimes U_2) \mathbf{G}^2_m . \mathrm{SL}_2 \times \mathrm{SL}_2$. The semi-simple part of $\mathrm{Aut}^0(\mathrm{E}'^l_m; \mathrm{F}'_m)$ is SL_2 and the unipotent radical U has SL_2-exact sequence

$$\text{(c)} \quad 0 \to N \to U \to U_{m+1} \to 0$$

induced from the morphism $\mathrm{E}'^l_m \to \mathrm{F}'_m$ and $N = \bigoplus_{j=0}^{\infty} H^0(X, \mathcal{O}(lm-2-jm))$. N contains an irreducible SL_2-module of dimension $lm-1 \geq 2$. Thus SL_2-composition series of U contains at least U_{m+1}, U_{lm-1} $(lm-1 \geq 2)$. Assume the existence of morphism $(\varphi, f) = (\mathrm{Aut}^0(\mathrm{E}'^l_m; \mathrm{F}'_m), \mathrm{E}'^l_m) \to (\mathrm{Aut}^0 \mathrm{F}'_{n,n}, \mathrm{F}'_{n,n})$ with f birational. Then φ maps semi-simple part SL_2 into $\mathrm{SL}_2 \times \mathrm{SL}_2$. We may assume φ is either $A \mapsto (A, A)$, $A \mapsto (A, 1)$ or $A \mapsto (1, A)$ since other cases (e.g. $A \mapsto (A, {}^tA^{-1})$) are treated similarly. In the last case, $\varphi(\mathrm{SL}_2)$ has only 2-dimensional orbits hence this case never occurs (Corollaries (4.20) and (4.23)). In the first case the SL_2-module $U_{n+1} \otimes U_2$ is isomorphic to $U_{n+1} \otimes U_{n-1}$ and in the second case the SL_2-module $U_{n+1} \otimes U_2$ is isomorphic to $U_{n+1} \oplus U_{n+1}$.

Let φ' be the canonical map $\mathrm{Aut}^0 \mathrm{F}'_{n,n} \to \mathrm{Aut}^0 \mathrm{F}'_{n,n}/\text{radical} \simeq \mathrm{PGL}_2 \times \mathrm{PGL}_2$. Then $\varphi' \circ \varphi$ defines an SL_2-exact sequence

$$\begin{array}{c} \mathrm{PGL}_2 \times \mathrm{PGL}_2 \\ \cup \\ 0 \to \mathrm{Ker}\, \varphi' \circ \varphi \to U \xrightarrow{\varphi' \circ \varphi} \varphi' \circ \varphi(U) \to 0, \end{array}$$

$\mathrm{Ker}\ \varphi' \circ \varphi \simeq \varphi(U) \cap$ (unipotent radical of $\mathrm{Aut}^0 \mathrm{F}'_{n,n}$). Since the diagonal $\mathfrak{sl}_2 \subset \mathfrak{sl}_2 \times \mathfrak{sl}_2$ is a maximal Lie subalgebra of $\mathfrak{sl}_2 \times \mathfrak{sl}_2$, we get

$$\varphi' \circ \varphi(U) = \begin{cases} \text{at most } \mathbf{G}_a & \text{if } A \mapsto (A, 1), \\ 0 & \text{if } A \mapsto (A, A). \end{cases}$$

Therefore the simple SL_2-modules in a composition series of

$$U = \text{ at most} \begin{cases} U_{n+1}, U_{n-1}, U_1 & \text{if } A \mapsto (A, 1), \\ U_{n+1}, U_{n+1} & \text{if } A \mapsto (A, A). \end{cases}$$

Let us first exclude $m=1$. In fact if $m=1$, $l \geq 3$, then by exact sequence (c) in an SL_2-composition series of U, U_{l-1}, U_{l-2}, $\cdots$, U_1, U_2 appear, which is impossible as we noticed above.

Let us now assume $l \geq 2$, $m \geq 2$. By the observation above and by (c) it is impossible unless (a) $(lm-1)=m+1$ or (b) $(lm-1)-(m+1)=\pm 1$ since $lm-1 \geq 2$, $m+1 \geq 3$. Notice $(lm-1)-(m+1)=-1$ never holds as $l=2$, $m=1$ is excluded.

Let us study case (a) $(l+1)m=2$, namely $l=m=2$ or $l=3$, $m=1$. The case $l=3$, $m=1$ was excluded above. If $l=m=2$, in an SL_2-composition series of U, U_3, U_2 and U_3 appear which contradicts our observation above. We pass to case (b) $(lm-1)-(m+1)=1$, namely $m=1$, $l=4$ or $m=3$, $l=2$. We can exclude $m=1$, $l=4$ as above. If $m=3$, $l=2$, by (c), U contains $U_5 \oplus U_2$ which can not be imbedded in $\mathrm{Aut}^0 F'_{n,n}$ as we have seen above.

(J'11)-(J'11)) See Corollary (4.23).

((J'11)-(J12)) Compare the dimension of the groups.

((J12)-(P1)) In (J12) SL_2 does not operate faithfully. Assume for an operation (PGL_2, X_π) of type (J12) the existence of morphism (φ, f): $(PGL_2, X_\pi) \to (PSL_4, \mathbf{P}^3)$ with f birational, φ defines an operation of SL_2 on $\mathbf{P}^3$ hence a representation E of degree 4 such that the center of SL_2 operates trivially on $\mathbf{P}^3$. E is isomorphic to one of the following; (i) V_4, (ii) $V_3 \oplus V_1$, (iii) $V_2 \oplus V_2$, (iv) $V_2 \oplus V_1 \oplus V_1$. We can exclude (iv) since in this case SL_2 operates effectively on $\mathbf{P}(E)$. In case (i) or (iii), SL_2 has an open orbit and hence these cases never occur. If $E \simeq V_3 \oplus V_1$, then $(SL_2, \mathbf{P}(V_3 \oplus V_1))$ is, as an algebraic law chunk of operation, equivalent to (SL_2, V_3). As the center of SL_2 operates trivially on V_3, (SL_2, V_3) gives an operation (PGL_2, V_3). It follows from Proposition (4.26) and Lemma (4.28) that (PGL_2, V_3) is not isomorphic to (PGL_2, X_π) in (J12) as law chunks of algebraic operation.

((J12)-(P2)) We argue as in ((J12)-(P1)). Assume for an operation of type (J12) the existence of morphism (φ, f): $(PGL_2, X_\pi) \to (PSO_5$, quadric $\subset \mathbf{P}^4)$ with f birational, φ defines a representation E of SL_2 of degree 5. E is isomorphic to one of the following: (i) V_5, (ii) $V_4 \oplus V_1$, (iii) $V_3 \oplus V_2$, (iv) $V_3 \oplus V_1 \oplus V_1$, (v) $V_2 \oplus V_2 \oplus V_1$. If $E \simeq \mathbf{P}(V_5)$, then $SL_2 \subset SO(E)$ since SL_2

has an invariant i of degree 2 of homogeneous polynomial of degree 4 (see Weber [Wb] vol. I, § 70). It follows from the explicit expression of the invariant i, SL_2 has an open orbit on the quadric$=\{i=0\}\subset \mathbf{P}(E)$ (see Umemura [U2]). If $E\simeq V_4\oplus V_1$, $V_3\oplus V_2$ or $V_2\oplus V_2\oplus V_1$, SL_2 operates effectively on $\mathbf{P}(E)$ hence on the quadric. If $E\simeq V_3\oplus V_1\oplus V_1$, we argue as in ((J12)-(JP)) for the case $E\simeq V_3\oplus V_1$.

((J12)-(E1)) and ((J12)-(E2)) (J12) is generically intransitive but (E1). (E2) are generically transitive and they all have the same transformation group.

((J12)-(J1)) Assume for an operation (PGL_2, X_π) of type (J12), there exists a morphism $(\varphi, f)\colon (\mathrm{PGL}_2, X_\pi)\to(\mathrm{PGL}_3\times\mathrm{PGL}_2, \mathbf{P}^2\times\mathbf{P}^1)$ of law chunks of algebraic operation with f birational. Replacing $\pi\colon C_2\to C_1$ by an smaller covering, we may assume (φ, f) is a morphism of algebraic operations. SL_2 operates on X_π through $\mathrm{SL}_2\to\mathrm{SL}_2/\pm1=\mathrm{PGL}_2$ hence we get a morphism $(\mathrm{SL}_2, X_\pi)\to(\mathrm{PGL}_3\times\mathrm{PGL}_2, \mathbf{P}^2\times\mathbf{P}^1)$ of algebraic operations. This gives an operation of SL_2 on $\mathbf{P}^2\times\mathbf{P}^1$. This operation comes from linear representations E_3, E_2 of SL_2 of degree 3 and 2, namely $(\mathrm{SL}_2, \mathbf{P}^2\times\mathbf{P}^1)=(\mathrm{SL}_2, \mathbf{P}(\mathrm{E}_3)\times\mathbf{P}(E_2))$. E_3 is not trivial. For otherwise, SL_2 has only 1-dimensional orbits on $\mathbf{P}^2\times\mathbf{P}^1$. E_2 is not trivial either. In fact, otherwise SL_2-orbits in $\mathbf{P}^2\times\mathbf{P}^1$ are contained in $\mathbf{P}^2\times x$ $(x\in\mathbf{P}^1)$. For any SL_2-module E_3, $(\mathrm{SL}_2, \mathrm{SL}_2/T)$ does not appear in $(\mathrm{SL}_2, \mathbf{P}(E_3))$; (i) if $E_3=V_3$, then the 2-dimensional SL_2-orbit in $(\mathrm{SL}_2, \mathbf{P}(E_3))$ is $(\mathrm{SL}_2, \mathrm{SL}_2/\mathrm{D}_\infty)$ (ii) if $E_3=V_2+V_1$, then the 2-dimensional SL_2-orbit in $(\mathrm{SL}_2, \mathbf{P}(E_3))$ is $(\mathrm{SL}_2, \mathrm{SL}_2/U_1)$ with $U_1=\{(a_{ij})\in\mathrm{SL}_2\mid a_{11}=a_{22}=1,\ a_{21}=0\}$. Since we have a morphism $(\mathrm{SL}_2, \mathbf{P}(E_3)\times\mathbf{P}(E_2))\to(\mathrm{SL}_2, \mathbf{P}(E_3))$. We have a dominant SL_2-equivariant morphism $X_\pi\to\mathbf{P}(E_3)$. Therefore a general SL_2-orbit on $\mathbf{P}(E_3)$ is dominated by $(\mathrm{SL}_2, \mathrm{SL}_2/T)$ and hence by above observation E_3 is irreducible. But if E_3 and E_2 are irreducible $(\mathrm{SL}_2\mathbf{P}(E_3)\times\mathbf{P}(E_2))$ has an open orbit. Contradiction.

((J12)-(J2)) Essentially the same as ((J′11)-(J1)). Argue as ((J′11)-(J1)). We get $(\mathrm{SL}_2, \mathbf{P}^1\times\mathbf{P}^1\times\mathbf{P}^1)$ which is defined by the irreducible SL_2-modules E_2, E_2', E_2'' of degree 2. If one of them, say E_2 were trivial, we would get SL_2-orbits on $\mathbf{P}^1\times\mathbf{P}^1\times\mathbf{P}^1$ are contained in $x\times\mathbf{P}^1\times\mathbf{P}^1$ $(x\in\mathbf{P}^1)$. Thus X_π would be trivially fibred over a rational curve. Therefore E_2, E_2', E_2'' are irreducible. Then $(\mathrm{SL}_2, \mathbf{P}^1\times\mathbf{P}^1\times\mathbf{P}^1)$ has an open SL_2-orbit.

((J12)-(J3)) Assume there exists a morphism $(\varphi, f)\colon(\mathrm{PGL}_2, X_\pi)\to(\mathrm{PGL}_2\times\mathrm{Aut}^0\,\mathrm{F}_m', \mathbf{P}^1\times\mathrm{F}_m')$ of law chunks of algebraic operation with f birational. As in ((J12)-(J1)) we may assume (φ, f) is a morphism of algebraic operations. By taking the projection we get a morphism

(φ', f'): $(\mathrm{PGL}_2, X_\pi) \to (\mathrm{Aut}^0 \mathrm{F}'_m, \mathrm{F}'_m)$ of algebraic operations with f' dominant. Hence a general PGL_2-orbit dominates a general $\varphi'(\mathrm{PGL}_2)$-orbit on F'_m. We show in [U2] a semi-simple part of $\mathrm{Aut}^0 \mathrm{F}'_m$ is SL_2. If $\varphi'(\mathrm{PGL}_2) \neq 1$, $\varphi'(\mathrm{PGL}_2)$ is a semi-simple part of $\mathrm{Aut}^0 \mathrm{F}'_m$ and has an open orbit on F'_m isomorphic to $(\mathrm{PGL}_2, \mathrm{PGL}_2/U_m)$. But there is no morphism (φ'', f''): $(\mathrm{PGL}_2$, PGL_2-orbit on $X_\pi) = (\mathrm{PGL}_2, \mathrm{PGL}_2/T) \to (\mathrm{PGL}_2, \mathrm{PGL}_2/U_m) = (\varphi'(\mathrm{PGL}_2)$, open $\varphi'(\mathrm{PGL}_2)$-orbit on $\mathrm{F}'_m)$. Hence $\varphi(\mathrm{PGL}_2) = 1$. But then PGL_2-orbit on X_π should be contained in $\mathbf{P}^1 \times x$ $(x \in \mathrm{F}'_m)$.

((J12)-(J4)) Assume there exists a morphism (φ, f): $(\mathrm{PGL}_2, X_\pi) \to (\mathrm{PGL}_3, \mathrm{PGL}_3/B)$ of law chunks of algebraic operation with f birational. We may assume (φ, f) is a morphism of algebraic operations as in ((J12)-(J1)). SL_2 operates on X_π and φ defines an SL_2-module E of degree 3. E is not irreducible. For otherwise, SL_2 has an open orbit on $\mathrm{PGL}(E)/B$. If $E = V_2 \oplus V_1$, we get a morphism $(\mathrm{PGL}_3, \mathrm{PGL}_3/B) \to (\mathrm{PGL}_3, \mathrm{PGL}_3/P) = (\mathrm{PGL}_3, \mathbf{P}(V_2 \oplus V_1))$. SL_2 has an open orbit isomorphic to $(\mathrm{SL}_2, \mathrm{SL}_2/U_1)$. Since $(\mathrm{SL}_2, \mathrm{SL}_2/T)$ does not dominate $(\mathrm{SL}_2, \mathrm{SL}_2/U_1)$. The assertion follows as in ((J12)-(J3)).

((J12)-(J5)) Obvious.

((J12)-(J6)) Assume there exists an inclusion. This is equivalent to the existence of a morphism (φ, f): $(\mathrm{SL}_2, X_\pi) \to (G, G/H_{m,n})$ of law chunks of algebraic operation with f birational. We may assume as above (φ, f) is a morphism of algebraic operations. Considering the projection (Id, p): $(G, G/H_{m,n}) \to (G, G/\mathbf{G}_m \times B \times B) = (G, \mathbf{P}^1 \times \mathbf{P}^1)$. $p: G/H_{m,n} \to \mathbf{P}^1 \times \mathbf{P}^1$ is a $\mathbf{G}_m$-bundle. The above morphism (φ, f) defines an SL_2-operation on $\mathbf{P}^1 \times \mathbf{P}^1$, thus 2 SL_2-modules E, E' of degree 2 so that SL_2-operation on $\mathbf{P}^1 \times \mathbf{P}^1$ is isomorphic to $(\mathrm{SL}_2, \mathbf{P}(E) \times \mathbf{P}(E'))$. We denote by π_1 the composite of p and the projection $P_1: \mathbf{P}^1 \times \mathbf{P}^1 \to \mathbf{P}^1$ onto the first factor. If one of the representations E, E', say E, were trivial, then $\varphi(\mathrm{SL}_2)$-orbit on $G/H_{m,n}$ would be contained in a fibre $\pi_1^{-1}(x)$ which is isomorphic to $(\mathbf{G}_m \times \mathrm{SL}_2, \mathbf{G}_m \times \mathrm{SL}_2/H_m) = (\mathbf{G}_m \times \mathrm{SL}_2, \mathrm{SL}_2/U_m)$, where

$$H_m = \left\{ (t, A) \in \mathbf{G}_m \times \mathrm{SL}_2 \,|\, A = \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}, \ t = a^m \right\}$$

(see the definition of $H_{m,n}$). Therefore $(\mathrm{SL}_2, \mathrm{SL}_2/T)$ would be in $(\mathrm{SL}_2, \mathrm{SL}_2/U_m)$. Thus E, E' are irreducible. Since a semi-simple part of G is $\mathrm{SL}_2 \times \mathrm{SL}_2$ and since the semi-simple part is given by the projection $(G, G/H_{m,n}) \to (G, \mathbf{P}^1 \times \mathbf{P}^1) \to (\mathrm{SL}_2 \times \mathrm{SL}_2, \mathbf{P}^1 \times \mathbf{P}^1)$, φ is determined, up to conjugacy in G once the SL_2-modules E, E' are fixed, this shows φ maps SL_2 to the semi-simple part diagonally. The diagonal SL_2 of $\mathrm{SL}_2 \times \mathrm{SL}_2$ as a subgroup of G has an open orbit on $H/H_{m,n}$.

((J12)-(J7)) Assume there exists a morphism (φ, f): $(\mathrm{PGL}_2, X_\pi)\to(\mathrm{Aut}^0 \mathrm{J}'_m, \mathrm{J}'_m)$ of law chunks of algebraic operations. As in ((J12)-(J1)), we may assume (φ, f) is a morphism of algebraic operations. By considering the projection $(\mathrm{Aut}^0 \mathrm{J}'_m, \mathrm{J}'_m)\to(\mathrm{PGL}_3, \mathbf{P}^2)$. (φ, f) defines a PGL_2-operation on $\mathbf{P}^2$ therefore a representation E of SL_2 of degree 3. E is not trivial since otherwise, SL_2-orbits on J'_m would be contained in fibres which are isomorphic to $\mathbf{A}^1$. $E=V_2\oplus V_1$ is impossible as in ((J12)-(J1)). Notice, since $(\mathrm{Aut}^0 \mathrm{J}'_m, \mathrm{J}'_m)\to(\mathrm{PGL}_3, \mathbf{P}^2)$ gives a semi-simple part of $\mathrm{Aut}^0 \mathrm{J}'_m$, the morphism φ is determined, up to conjugacy in $\mathrm{Aut}^0 \mathrm{J}'_m$ once SL_2-module E is fixed. If $E=V_3$, then $\varphi(\mathrm{SL}_2)$ has an open orbit on J'_m (see Corollary (4.10)).

((J12)-(J8)) Argue as in ((J12)-(J6)).

((J12)-(J9)) A semi-simple part of $\mathrm{Aut}^0 \mathrm{F}'_{m,n}$ is SL_2 and the general SL_2-orbit on $\mathrm{F}'_{m,n}$ is $(\mathrm{SL}_2, \mathrm{SL}_2/U_k)$ with $k=(m, n)$ by Corollary (4.20), where $U_k=\left\{\begin{pmatrix} a & b \\ 0 & d\end{pmatrix} \in \mathrm{SL}_2 \mid a^k=d^k=1\right\}$. But the general SL_2-orbit on X_π is $(\mathrm{SL}_2, \mathrm{SL}_2/\mathbf{G}_m)$ from the construction of X_π.

((J12)-(J10)) By Corollary (4.20), a semi-simple part of $\mathrm{Aut}^0 \mathrm{F}'_{m,n}$ is $\mathrm{SL}_2\times\mathrm{SL}_2$. The operation (PGL_2, X_π) defines an operation (SL_2, X_π) such that the center of SL_2 operates trivially. If there exists a morphism (φ, f): $(\mathrm{PGL}_2, X_\pi)\to(\mathrm{Aut}^0 \mathrm{F}'_{m,m}, \mathrm{F}'_{m,m})$ of law chunks of algebraic operations, φ defines a morphism $\tilde{\varphi}$: $\mathrm{SL}_2\to$(a semi-simple part). By taking conjugacy, we may assume φ defines a morphism $\tilde{\varphi}$: $\mathrm{SL}_2\to(\mathrm{SL}_2\times\mathrm{SL}_2)$ in Corollary (4.20). Since the operation of SL_2 by $\tilde{\varphi}$ on $\mathrm{F}'_{m,m}$ collapses the operation of the center as on X_π, $\tilde{\varphi}(A)=(A, 1)$ for $A \in \mathrm{SL}_2$ and m even. But then the general SL_2-orbit on $\mathrm{F}_{m,m}$ is $(\mathrm{SL}_2, \mathrm{SL}_2/U_m)$ by Corollary (4.20), where $U_m=\left\{\begin{pmatrix} a & b \\ 0 & d\end{pmatrix} \in \mathrm{SL}_2 \mid a^m=d^m=1\right\}$. But the general SL_2-orbit on X_π is $(\mathrm{SL}_2, \mathrm{SL}_2/\mathbf{G}_m)$ from the construction of X_π.

((J12)-(J′11)) A semi-simple part of $\mathrm{Aut}^0(\mathrm{E}'^{l}_m; \mathrm{F}'^{l}_m)$ is SL_2 and has an open orbit on E'^{l}_m by Corollary (4.23).

((J12)-(J12)) This is proved in Corollary (4.27).

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