Advanced Studies in Pure Mathematics 5, 1985
Foliations
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Dynamical Systems on Foliations and Existence Problem of Transverse Foliations

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§ 0. Introduction and statements of results

Let M be an n-dimensional C^∞ manifold and let $\mathscr{F}$ be a codimension one C^{r+1} foliation of M $(r \geqq 1)$. It is well known that there exists always a one dimensional foliation of M transverse to $\mathscr{F}$. Thus the following question on the integrability of 2-plane fields comes to the front:

Received April 16, 1984.

'Does there exist a 2-dimensional foliation of M transverse to $\mathscr{F}$ if $\mathscr{F}$ admits transverse 2-plane fields?'

Suppose that there exists a 2-dimensional C^{r+1} foliation $\mathscr{F}'$ of M which is transverse to $\mathscr{F}$, that is, at each point $x \in M$, the leaf L_x of $\mathscr{F}$ through x and the leaf L'_x of $\mathscr{F}'$ through x intersect transversely at x. Then the set of intersections of leaves of $\mathscr{F}$ and $\mathscr{F}'$ forms a one dimensional C^{r+1} foliation of M, denoted by $\mathscr{F} \cap \mathscr{F}'$, each leaf of which lies on a leaf of $\mathscr{F}$. In case M is orientable and both of $\mathscr{F}$ and $\mathscr{F}'$ are transversely orientable, the foliation $\mathscr{F} \cap \mathscr{F}'$ consists of orbits of a non-singular C^r vector field X on M such that each vector of X is tangent to a leaf of $\mathscr{F}$. In this context, dynamical systems on the foliation $\mathscr{F}$ come in the study of 2-dimensional foliations transverse to $\mathscr{F}$. The least dimension of M which we are interested in is 3, and foliations $\mathscr{F}$ and $\mathscr{F}'$ as above are both of codimension one in this case and can be treated on the same level.

In [6] we classified codimension one foliations transverse to the Reeb foliation $\mathscr{F}_R$ of the solid torus by studying non-singular vector fields on $\mathscr{F}_R$, and proved Theorem A below, making use of the classification mentioned above.

Let k be a non-trivial fibred knot in the 3-sphere S^3 and let $N(k)$ denote a tubular neighborhood of k. Let $\mathscr{F}$ be a codimension one foliation of S^3 which is the union of the Reeb foliation of $N(k)$ and the foliation of $S^3 - \operatorname{Int} N(k)$ obtained by turbulizing the interior of each fibre of $S^3 - \operatorname{Int} N(k) \to S^1$ in a collar of the boundary of $N(k)$. For the definition of the turbulization, see Section 4. Remark that $\mathscr{F}$ admits a transverse 2-plane field, since 2-plane bundles over S^3 are always trivial. The following is the first result on codimension one foliations of 3-dimensional manifolds admitting no transverse codimension one foliation (Tamura-Sato [6, Theorem 6]):

Theorem A. *Let $\mathscr{F}$ be the codimension one C^∞ foliation of the* 3-*sphere S^3 as above. Then there does not exist any codimension one C^r foliation of S^3 $(r \geqq 2)$ transverse to $\mathscr{F}$.*

Our results were developed by Nishimori [2]. He studied foliations transverse to various codimension one foliations generalizing the Reeb foliation and classified them. Theorem B below is a typical result on foliations admitting no transverse foliation in [2]. Let $D^2 - \operatorname{Int} D_1^2 - \operatorname{Int} D_2^2$ be a two punctured 2-disk and let

$$S^3 = (S^1 \times D^2) \cup (((D^2 - \operatorname{Int} D_1^2 - \operatorname{Int} D_2^2) \cup D_1^2 \cup D_2^2) \times S^1)$$

be the natural decomposition of the 3-sphere. And let $\mathscr{F}_0$ denote the codimension one C^∞ foliation of S^3 consisting of Reeb foliations of $S^1 \times D^2$, $D_1^2 \times S^1$ and $D_2^2 \times S^1$, and the codimension one foliation of $(D^2 - \mathrm{Int}\, D_1^2 - \mathrm{Int}\, D_2^2) \times S^1$ formed by turbulization to the same direction as these of $D_1^2 \times S^1$ and $D_2^2 \times S^1$. For the precise definition, see Section 4. Then the following theorem holds (Nishimori [2; Theorem 5]);

Theorem B. *Let $\mathscr{F}_0$ be the codimension one C^∞ foliation of the 3-sphere S^3 as above. Then there does not exist any codimension one C^r foliation of S^3 $(r \geqq 2)$ transverse to $\mathscr{F}_0$.*

Furthermore Nishimori proved the following interesting and beautiful theorem in his second paper on this subject [3, Theorem 6]:

Theorem C. *Let E be an orientable 3-dimensional C^∞ manifold which is the total space of a C^∞ bundle over S^1 with one punctured torus $T^2 - \mathrm{Int}\, D^2$ as fibre, and let $\mathscr{F}_\pi$ denote the codimension one C^∞ foliation of E formed by turbulizing the interior of each fibre in a collar of ∂E. Let $\phi: T^2 - \mathrm{Int}\, D^2 \to T^2 - \mathrm{Int}\, D^2$ be the monodromy map of this bundle and let*

$$\phi_*: H_1(T^2 - \mathrm{Int}\, D^2; Z) \to H_1(T^2 - \mathrm{Int}\, D^2; Z)$$

be the homomorphism induced by ϕ which is expressed by a conjugacy class of $SL(2; Z)$.

Then there exists a transversely orientable codimension one C^r foliation of E $(r \geqq 2)$ transverse to $\mathscr{F}_\pi$ if and only if

$$\mathrm{Trace}\, \phi_* \geqq 2.$$

We remark that codimension one foliations $\mathscr{F}_0$ in Theorem B and $\mathscr{F}_\pi$ in Theorem C admit both transverse 2-plane fields.

It was a common pattern of proofs of Theorems A, B and C that they needed firstly to classify codimension one foliations transverse to some specified codimension one foliations. However these classifications are hard tasks to describe and disturb the clear understanding of the meaning of these theorems.

The purpose of this paper is to carry on the study of non-singular vector fields on codimension one foliations of 3-manifolds and to give direct proofs for Theorems A, B and C so that they can reveal the obstruction to admit transverse foliations, in the frame of dynamical systems such as the compactification of vector fields (Section 6), asymptotic homology classes (Section 7) and the bifurcation of leaves (Section 9), without using any classifications.

The methods used in this paper enable us to prove the following theorem. Since the proof is contained in that of Theorem C given in

Sections 7 and 8, we state here the result without proof.

Theorem C′. *Let E be an orientable 3-dimensional C^∞ manifold which is the total space of a C^∞ bundle over S^1 with the torus T^2 as fibre, and let $\mathscr{F}$ denote the codimension one C^∞ foliation of E whose leaves are fibres of this bundle. Let $\phi: T^2 \to T^2$ be the monodromy map of this bundle and let $\phi_*: H_1(T^2; \boldsymbol{Z}) \to H_1(T^2; \boldsymbol{Z})$ be the induced homomorphism.*

Then there exists a transversely orientable codimension one C^r foliation of E $(r \geqq 2)$ transverse to $\mathscr{F}$ if and only if

$$\operatorname{Trace} \phi_* \geqq 2.$$

As a direct consequence of the main lemma (Theorem 10.1) to prove Theorem B, the following theorem will be obtained:

Theorem D. *Every 3-dimensional C^∞ manifold has a codimension one C^∞ foliation which does not admit any codimension one C^r foliation $(r \geqq 2)$ transverse to it.*

The phenomena of geometric dynamics appeared in this study may be considered as a new object of the study of dynamical systems.

The author thanks Koichi Yano, Toshiyuki Nishimori and Atsushi Sato for helpful comments.

§ 1. Some elementary properties of non-singular vector fields on the torus

In this section we recall some properties of non-singular vector fields on the torus by Reinhart [5] and give proofs for the convenience of the reader.

Let Q' be a subset of the 2-dimensional euclidean space $\boldsymbol{R}^2$ and let Y' be a non-singular continuous vector field on Q'. Then, by identifying $\boldsymbol{R}^2$ and the tangent space $T_y(\boldsymbol{R}^2)$ of $\boldsymbol{R}^2$ at $y \in \boldsymbol{R}^2$ naturally, a continuous map

$$\bar{f}(Y'): Q' \to S^1$$

is defined by

$$(\bar{f}(Y'))(y) = Y'(y)/|Y'(y)|.$$

Let Q be a subset of the torus T^2 and let Y be a non-singular continuous vector field on Q. Then a continuous map

$$f(Y): Q \to S^1$$

is defined as follows. Let $\tilde{\pi}: \boldsymbol{R}^2 \to T^2$ be a covering map such that $\tilde{\pi}(p, q) = \tilde{\pi}(p+n, q+m)$ for $(p, q) \in \boldsymbol{R}^2$ and arbitrary integers n, m. For a point $z \in Q$, let $\hat{z} \in \boldsymbol{R}^2$ be a lift of z, that is $\tilde{\pi}(\hat{z}) = z$, and let $v \in T_{\hat{z}}(\boldsymbol{R}^2)$ be the tangent vector at $\hat{z}$ such that $d\tilde{\pi}(v) = Y(z)$. We define

$$(f(Y))(z) = v/|v|.$$

Obviously $f(Y)$ is well defined and continuous. The homotopy class of $f(Y)$ is uniquely determined independent of the choice of covering maps $\boldsymbol{R}^2 \to T^2$.

Now let C be an oriented simple closed C^r curve $(r \geqq 1)$ on the torus and let Y_C be the unit tangent vector field on C. Then the following lemma holds ([5, Theorem 1]):

Lemma 1.1. *If C is not homologous to zero, then the degree of $f(Y_C): C \to S^1$ is zero.*

Proof. We take an imbedding $g_0: S^1 \to T^2$ such that $g_0(S^1) = C$. Let $g_1: S^1 \to T^2$ be an imbedding homotopic to g_0 such that $g_1(S^1)$ is the image of a line in $\boldsymbol{R}^2$ by $\tilde{\pi}$.

Consider the covering map

$$\tilde{\pi}': S^1 \times \boldsymbol{R} \to T^2$$

corresponding to the subgroup of $\pi_1(T^2)$ generated by $\{g_0\}$. Then there exist lifts $\tilde{g}_0$, $\tilde{g}_1$: $S^1 \to S^1 \times \boldsymbol{R}$ of g_0 and g_1:

$$\tilde{\pi}' \circ \tilde{g}_0 = g_0, \qquad \tilde{\pi}' \circ \tilde{g}_1 = g_1.$$

We can take $\tilde{g}_0$ and $\tilde{g}_1$ so that

$$\tilde{g}_0(S^1) \cap \tilde{g}_1(S^1) = \phi.$$

Then $\tilde{g}_0(S^1) \cup \tilde{g}_1(S^1)$ bounds an annulus in $S^1 \times \boldsymbol{R}$. This implies that $\tilde{g}_0$ and $\tilde{g}_1$ are isotopic in $S^1 \times \boldsymbol{R}$ and, thus, g_0 and g_1 are regularly homotopic in T^2.

Let $g_t: S^1 \to T^2$ $(0 \leqq t \leqq 1)$ be a regular homotopy between g_0 and g_1, and let $f_t: S^1 \to S^1$ $(0 \leqq t \leqq 1)$ be a continuous map defined by

$$f_t(\theta) = \frac{dg_t}{d\theta}(\theta) \Big/ \left| \frac{dg_t}{d\theta}(\theta) \right| \quad (\theta \in S^1).$$

Since f_1 is a constant map, the degree of f_0 is zero. This shows that the degree of $f(Y_C)$ is zero. Thus this lemma is proved.

In the following homology groups $H_*(\ )$ denote always the integral homology groups unless the coefficients are specified.

The following proposition due to Reinhart [5, Corollary 3] is used in Section 3:

Proposition 1.2. *Let X be a non-singular C^r vector field on the torus $(r\geqq 1)$. If X has no closed orbit, then the homomorphism*

$$(f(X))_*\colon H_1(T^2)\to H_1(S^1)$$

induced by the continuous map $f(X)\colon T^2\to S^1$ is a zero map.

Proof. As is well known there exists an oriented simple closed curve C_0 transverse to X. Let x_0 be a point of C_0 and let $\varphi(t, x_0)$ denote the orbit of X through x_0. If $\{\varphi(t, x_0); t>0\}\cap C_0=\phi$, then the ω-limit set of $\varphi(t, x_0)$ is a closed orbit by the Poincaré-Bendixson theorem, since the compactification of T^2-C_0 by adding two points is homeomorphic to the 2-sphere S^2. This contradicts the assumption. Thus we have

$$\{\varphi(t, x_0); t>0\}\cap C_0\neq\phi.$$

Therefore, as is easily verified, there exists a non-singular C^r vector field X' on the torus obtained from X by modifying vectors near C_0 such that X' satisfies the following:

(a) $f(X')\colon T^2\to S^1$ is homotopic to $f(X)$.
(b) The first intersection of the positive orbit $\{\varphi(t, x); t>0\}$ with C_0 is x_0.

Let C_1 denote the oriented simple closed curve formed by the orbit of X' through x_0 and let $[C_1]$ denote the homology class represented by C_1. Then, by Lemma 1.1, we have

$$(f(X'))_*([C_1])=0.$$

Furthermore, let X_0 denote the unit tangent vector field on C_0. Then, as is easily verified, two maps $f(X_0), f(X')\,|\,C_0\colon C_0\to S^1$ are homotopic. This implies by Lemma 1.1 that

$$(f(X'))_*([C_0])=0.$$

Since $[C_0]$ and $[C_1]$ generate $H_1(T^2)$, the homomorphism $(f(X'))_*$, thus $(f(X))_*$, is a zero map. This proves the proposition.

§ 2. Reeb components

Let M be an n-dimensional C^∞ manifold with or without boundary. A codimension q C^r foliation of M is denoted by a set $\mathscr{F}$ of leaves. In case $\partial M \neq \phi$, we understand that, for each connected component N_i of ∂M, the restriction $\mathscr{F}|N_i = \{$connected components of $L \cap N_i$; $L \in \mathscr{F}\}$ of $\mathscr{F}$ to N_i is a codimension q or $q-1$ C^r foliation.

In the following sections, we fix an orientation on the circle S^1.

The Reeb foliation of the solid torus $S^1 \times D^2$ is the codimension one C^∞ foliation constructed by turbulizing $\{\theta\} \times \text{Int } D^2$ ($\theta \in S^1$) in a collar of the boundary $S^1 \times D^2$ (Fig. 1). In the following, a point $e^{2\pi\theta i}$ of S^1 is simply denoted by θ.

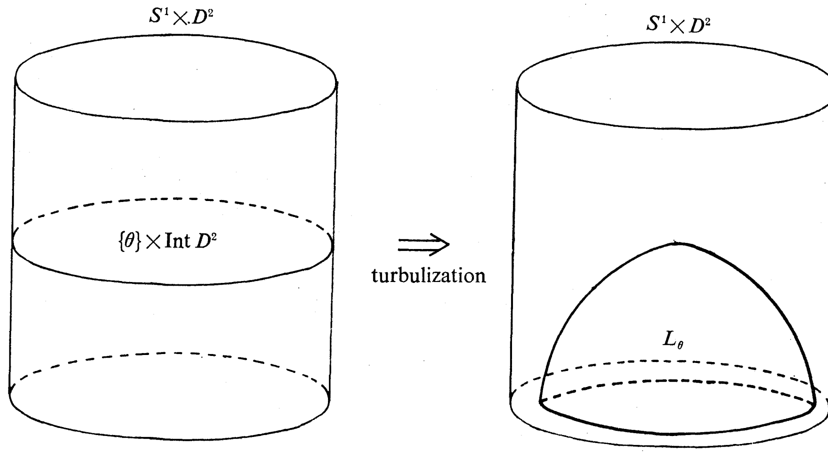


Fig. 1.

In case the turbulization is taken in the minus (resp. plus) direction of S^1, the Reeb foliation is called the *plus Reeb foliation* (resp. *minus Reeb foliation*) of $S^1 \times D^2$ and is denoted by $\mathscr{F}_R^{(+)}$ (resp. $\mathscr{F}_R^{(-)}$). (Fig. 2). The plus Reeb foliation $\mathscr{F}_R^{(+)}$ (resp. minus Reeb foliation $\mathscr{F}_R^{(-)}$) has a contracting holonomy with respect to the compact leaf $T^2 = S^1 \times \partial D^2$ in the minus (resp. plus) direction of $S^1 \times \{*\}$. The leaf of $\mathscr{F}_R^{(\pm)}$ obtained from $\{\theta\} \times \text{Int } D^2$ is denoted by L_θ. Thus $\mathscr{F}_R^{(\pm)} = \{L_\theta;\ \theta \in S^1\} \cup \{T^2\}$.

Since the Reeb foliations $\mathscr{F}_R^{(\pm)}$ are given objects in this paper, we may assume that leaves of $\mathscr{F}_R^{(\pm)}$ have a normalized form, that is, they are symmetric with respect to $S^1 \times \{0\}$ and $\{\theta\} \times D^2$ is tangent exactly to one leaf L_θ of $\mathscr{F}_R^{(+)}$ (resp. $\mathscr{F}_R^{(-)}$) at one point $(\theta, 0)$ for each $\theta \in S^1$.

The Reeb foliation of the annulus $S^1 \times D^1$ is a codimension one C^r foliation ($r \geqq 1$) constructed by turbulizing $\{\theta\} \times \text{Int } D^1$ ($\theta \in S^1$) in collars of $S^1 \times \{-1\}$ and $S^1 \times \{1\}$. In case the turbulization is taken in the minus

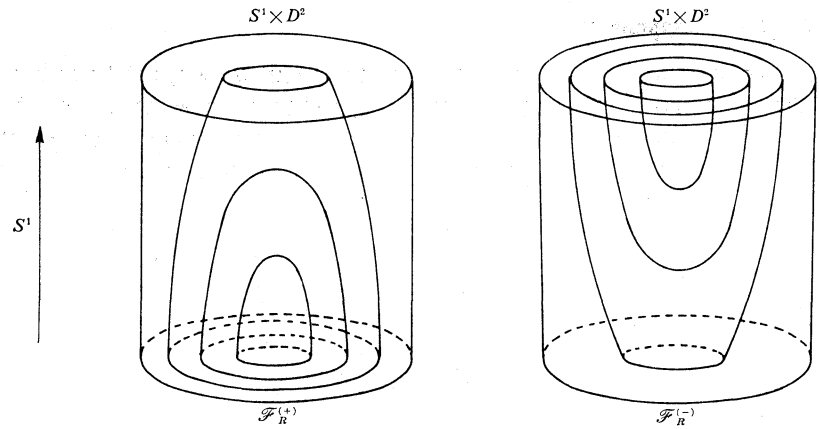


Fig. 2.

(resp. plus) direction of S^1, in other words, compact leaves $S^1\times\{-1\}$ and $S^1\times\{1\}$ have both contracting holonomy in the minus (resp. plus) direction of S^1, the Reeb foliation of $S^1\times D^1$ is called a *plus Reeb foliation* (resp. *minus Reeb foliation*) and is denoted by $\bar{\mathscr{F}}_R^{(+)}$ (resp. $\bar{\mathscr{F}}_R^{(-)}$) (Fig. 3).

A codimension one C^r foliation ($r\geqq 1$) of the annulus $S^1\times D^1$ constructed by turbulizing $\{\theta\}\times \mathrm{Int}\, D^2$ ($\theta\in S^1$) in collars of $S^1\times\{-1\}$ and $S^1\times\{1\}$ so that the directions are different for $S^1\times\{-1\}$ and $S^1\times\{1\}$ is called a *slope foliation* and is denoted by $\bar{\mathscr{F}}_S$ (Fig. 3). The leaf of $\bar{\mathscr{F}}_R^{(\pm)}$ or $\bar{\mathscr{F}}_S$ obtained from $\{\theta\}\times \mathrm{Int}\, D^1$ is denoted by $\bar{L}_\theta$. Then $\bar{\mathscr{F}}_R^{(\pm)}$ and $\bar{\mathscr{F}}_S$ are $\{\bar{L}_\theta;\ \theta\in S^1\}\cup\{S^1\times\{-1\},\ S^1\times\{1\}\}$.

In this paper, plus and minus Reeb foliations $\bar{\mathscr{F}}_R^{(\pm)}$ and slope foliations $\bar{\mathscr{F}}_S$ of the annulus appear as foliations formed by orbits of non-singular vector fields which are restrictions of vector fields tangent to

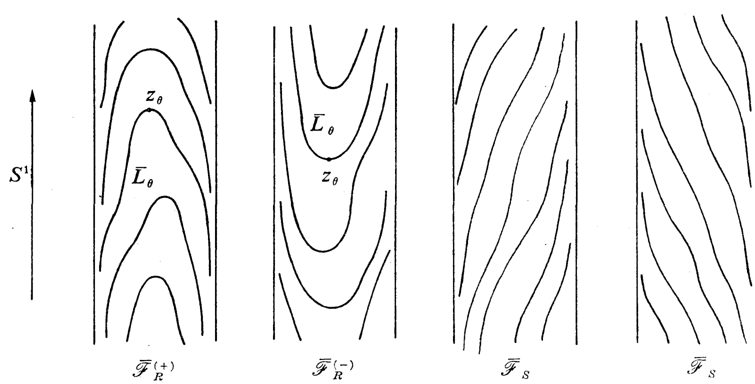


Fig. 3.

codimension one foliations of 3-dimensional manifolds. The only condition we assume for them is the following:

(2.1) Each noncompact leaf $\bar{L}_\theta$ of plus and minus Reeb foliations $\mathscr{F}_R^{(\pm)}$ of $S^1 \times D^1$ is tangent to $\{\theta\} \times D^1$ at exactly one point, say z_θ, and is transverse to $\{\theta'\} \times D^1$ ($\theta' \in S^1$) if $\theta \neq \theta'$ (Fig. 3). Each noncompact leaf $\bar{L}_\theta$ of slope component $\mathscr{F}_S$ is always transverse to $\{\theta'\} \times D^1$ ($\theta' \in S^1$).

A foliated C^r I-bundle of $S^1 \times D^1$, that is, a codimension one C^r foliation of the annulus $S^1 \times D^1$ ($r \geqq 1$) whose leaves are transverse to $\{\theta\} \times D^1$ for any $\theta \in S^1$, consists of compact leaves and a countable number of codimension one foliations isomorphic to slope components (Fig. 4).

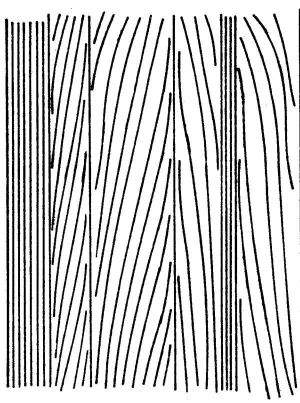


Fig. 4.

The following lemma will be used in Section 3.

Lemma 2.2. *Let $\mathscr{F}$ be one of $\mathscr{F}_R^{(+)}$, $\mathscr{F}_R^{(-)}$ and $\mathscr{F}_S$, and let a proper C^r imbedding*

$$\bar{g}\colon D^1 \to S^1 \times D^1$$

with $\bar{g}(-1) \in S^1 \times \{-1\}$, $\bar{g}(1) \in S^1 \times \{1\}$ be given. Then there exists a proper C^r imbedding

$$\bar{g}_0\colon D^1 \to S^1 \times D^1$$

satisfying the following conditions:

(i) $\bar{g}_0(-1) = \bar{g}(-1)$, $\bar{g}_0(1) = \bar{g}(1)$.

(ii) *$\bar{g}_0$ is isotopic to $\bar{g}$ fixing the end points $\bar{g}_0(-1)$, $\bar{g}_0(1)$.*

(iii) (a) *In case $\mathscr{F} = \mathscr{F}_R^{(+)}$ or $\mathscr{F}_R^{(-)}$, the curve $\bar{g}_0(D^1)$ is transverse to leaves of $\mathscr{F}$ except one leaf, say $\bar{L}_{\theta_0}$, and is tangent to $\bar{L}_{\theta_0}$ at exactly one point z_{θ_0}. $\bar{L}_{\theta_0}$ is tangent to $\bar{g}_0(D^1)$ from the minus side (resp. plus side) in case $\mathscr{F} = \mathscr{F}_R^{(+)}$ (resp. $\mathscr{F}_R^{(-)}$)* (Fig. 5).

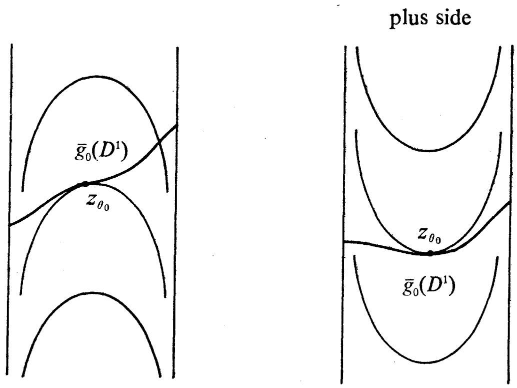


Fig. 5.

(b) *In case* $\mathscr{F}=\mathscr{F}_S$, *the curve* $\bar{g}_0(D^1)$ *is transverse to leaves of* $\mathscr{F}$.

Proof. First suppose that $\mathscr{F}=\mathscr{F}_R^{(+)}$ or $\mathscr{F}_R^{(-)}$. It is obvious that a proper imbedding

$$\bar{g}' \colon D^1 \to S^1 \times D^1$$

such that $\bar{g}'(D^1)=\{\theta\}\times D^1$, satisfies the condition corresponding to the condition (iii) for $\bar{g}_0$ (Fig. 6).

By modifying this imbedding $\bar{g}'$ in collars of $S^1\times\{-1\}$ and $S^1\times\{1\}$ as written by broken curves in Fig. 6 if necessary, we obtain an imbedding $\bar{g}_0$ with desired property.

The proof for the case $\mathscr{F}=\mathscr{F}_S$ is similar. Thus this lemma is proved.

Lemma 2.3. *Let* $\mathscr{F}$ *be one of* $\mathscr{F}_R^{(+)}$, $\mathscr{F}_R^{(-)}$ *and* $\mathscr{F}_S$, *and let* $\hat{g}\colon S^1\times D^1 \to S^1\times D^1$ *be a* C^r *diffeomorphism. Then there exists a* C^r *diffeomorphism*

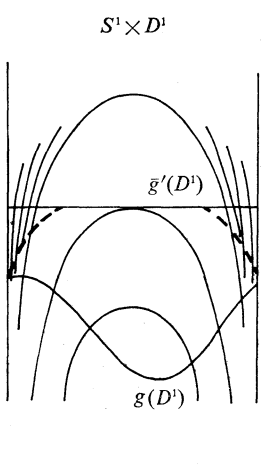


Fig. 6.

$$\hat{g}_0 \colon S^1 \times D^1 \to S^1 \times D^1$$

satisfying the following conditions:

(i) *For each point $\theta \in S^1$, it holds that*

$$\hat{g}_0(\theta, -1) = \hat{g}(\theta, -1), \qquad \hat{g}_0(\theta, 1) = \hat{g}(\theta, 1),$$

and that $\hat{g}\,|\,\{\theta\} \times D^1$ and $\hat{g}_0\,|\,\{\theta\} \times D^1$ are isotopic fixing their end points.

(ii) (a) *In case $\bar{\mathscr{F}} = \bar{\mathscr{F}}_R^{(+)}$ or $\bar{\mathscr{F}}_R^{(-)}$, each curve $\hat{g}_0(\{\theta\} \times D^1)$ $(\theta \in S^1)$ is transverse to leaves of $\bar{\mathscr{F}}$ except one leaf, say $\bar{L}_{\theta'}$, and is tangent to $\bar{L}_{\theta'}$ at exactly one point $z_{\theta'}$.*

(b) *In case $\bar{\mathscr{F}} = \bar{\mathscr{F}}_S$, each curve $\hat{g}_0(\{\theta\} \times D^1)$ $(\theta \in S^1)$ is transverse to leaves of $\bar{\mathscr{F}}$.*

Proof. We can prove this lemma by the argument used in the proof of Lemma 2.1. The details are left to the reader.

Let $T^2 = S^1 \times \partial D^2$ be the torus which is the boundary of the solid torus $S^1 \times D^2$. Recall that S^1 and $S^1 = \partial D^2$ are oriented. We denote by α and β the homology classes of $H_1(T^2)$ represented by the longitude $S^1 \times \{*\}$ and the meridian $\{**\} \times \partial D^2$ respectively, where $** \in S^1$, $* \in \partial D^2$.

Let X be a non-singular C^r vector field $(r \geq 1)$ on T^2. Denote by $\bar{\mathscr{F}}$ the codimension one C^r foliation of T^2 consisting of orbits of X. We assume that $\bar{\mathscr{F}}$ has a compact leaf, say L_{comp}, and that the homology class $[L_{comp}]$ represented by L_{comp} with the orientation induced from $X\,|\,L_{comp}$ is $a\alpha + b\beta$ with $a \neq 0$.

Let L_λ $(\lambda \in \Lambda)$ denote the compact leaves of $\bar{\mathscr{F}}$ and let U_σ $(\sigma \in \Sigma)$ be the connected components of $T^2 - \bigcup_{\lambda \in \Lambda} L_\lambda$. Then the boundary of a connected component U_σ consists of compact leaves, say L_σ and L'_σ, where it may happen that $L_\sigma = L'_\sigma$. We give L_σ and L'_σ the orientations induced from X.

For a point $z \in U_\sigma$, the α-limit set and the ω-limit set of z are contained in $L_\sigma \cup L'_\sigma$ by the Poincaré-Bendixson theorem. Furthermore, as is easily verified, the α-limit set of z is one of L_σ and L'_σ, and the ω-limit set of z is the other. The α-limit and the ω-limit sets of z do not depend on the choice of the point z. We assume here that L_σ is the α-limit set and L'_σ is the ω-limit set.

In the following we denote by $\bar{P}_1 \colon S^1 \times \partial D^2 \to S^1$ the projection onto the first factor.

If the degree of $\bar{P}_1\,|\,L_\sigma \colon L_\sigma \to S^1$ (resp. $\bar{P}_1\,|\,L'_\sigma \colon L'_\sigma \to S^1$) is $|a|$ and the degree of $\bar{P}_1\,|\,L'_\sigma \colon L'_\sigma \to S^1$ (resp. $\bar{P}_1\,|\,L_\sigma \colon L \to S^1$) is $-|a|$ with respect to orientations of L'_σ, L'_σ and S^1, then the restriction $\bar{\mathscr{F}}\,|\,\bar{U}_\sigma$ of $\bar{\mathscr{F}}$ to $\bar{U}_\sigma$ is said to be a *plus Reeb component* (resp. *minus Reeb component*) of $\bar{\mathscr{F}}$, and furthermore, if the degrees of $\bar{P}_1\,|\,L_\sigma \colon L_\sigma \to S^1$ and $\bar{P}_1\,|\,L'_\sigma \colon L'_\sigma \to S^1$ are

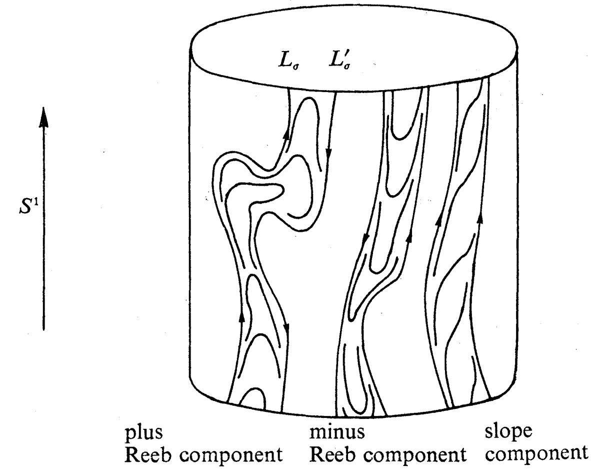


Fig. 7.

the same, then the restriction $\mathscr{F} \mid \bar{U}_\sigma$ of $\mathscr{F}$ to $\bar{U}_\sigma$ is said to be a *slope component* of $\mathscr{F}$ (Fig. 7).

As is easily verified, there exists a C^r diffeomorphism

$$g_\sigma \colon S^1 \times D^1 \to \bar{U}_\sigma$$

with the following properties (i), (ii):

(i) The degree of the map $\bar{P}_1 \circ g_\sigma \colon S^1 \times \{0\} \to S^1$ is positive.

(ii) g_σ is a leaf preserving map from a plus Reeb foliation, a minus Reeb foliation or a slope foliation of $S^1 \times D^1$ to $\mathscr{F} \mid U_\sigma$ according as $\mathscr{F} \mid U_\sigma$ is a plus Reeb component, a minus Reeb component or a slope component.

A union of slope components and compact leaves of $\mathscr{F}$ is said to be an *I-bundle component* if the union of the underlying space is connected.

It is obvious that the number of $\sigma \in \Sigma$ such that $\mathscr{F} \mid U_\sigma$ is a plus or a minus Reeb component is finite.

We denote the plus Reeb components and the minus Reeb components in $\mathscr{F}$ by $\mathscr{F} \mid K_i^{(+)}$ $(i=1, 2, \cdots, p)$ and $\mathscr{F} \mid K_i^{(-)}$ $(i=1, 2, \cdots, q)$ respectively, where $K_i^{(+)}$ and $K_i^{(-)}$ are closed subsets of T^2. And thus, the restriction of $\mathscr{F}$ to the closure of $T^2 - \bigcup_{i=1}^p K_i^{(+)} - \bigcup_{i=1}^q K^{(-)}$ consists of a finite number of I-bundle components of $\mathscr{F}$.

§ 3. Non-singular vector fields on the Reeb foliation of the solid torus

Let X be a non-singular C^r vector field $(r \geq 1)$ on the plus Reeb foliation $\mathscr{F}_R^{(+)}$ of the solid torus $S^1 \times D^2$, that is, X is a non-singular C^r vector field on $S^1 \times D^2$ such that the vector $X(z)$ of X at $z \in S^1 \times D^2$ is tangent to the leaf L_z of $\mathscr{F}_R^{(+)}$ containing z.

Let $\tau(S^1\times D^2)$ be the tangent bundle of $S^1\times D^2$ and let $\tau(\mathscr{F}_R^{(+)})$ be the tangent bundle of $\mathscr{F}_R^{(+)}$, that is to say, $\tau(\mathscr{F}_R^{(+)})$ is an orientable 2-plane bundle over $S^1\times D^2$ consisting of vectors of $\tau(S^1\times D^2)$ tangent to leaves of $\mathscr{F}_R^{(+)}$. Since the classifying space for orientable 2-plane bundles is $P^\infty(\boldsymbol{C})$, $\tau(\mathscr{F}_R^{(+)})$ is a trivial bundle. Thus the existence of a non-singular vector field X as above and 2-plane fields transverse to $\mathscr{F}_R^{(+)}$ is obvious.

In the following we denote by $P\colon S^1\times D^2-(S^1\times\{0\})\to T^2$ the projection defined by

$$P(x, y)=(x, y/|y|).$$

Thus the map $P|(L_\theta-\{(\theta, 0)\})\colon L_\theta-\{(\theta, 0)\}\to T^2$ is locally diffeomorphic.

Lemma 3.1. *Let $f(X|T^2)\colon T^2\to S^1$ be the continuous map defined in Section* 1. *Then, for a simple closed curve $\{\theta_0\}\times\partial D^2$ in T^2, we have*

$$(f(X|T^2))_*([\{\theta_0\}\times\partial D^2])\neq 0.$$

Proof. Consider the non-singular C^r vector field $X|(\{\theta_0\}\times\partial D^2)$ on $\{\theta_0\}\times\partial D^2$. Now suppose that

$$(f(X|T^2))_*([\{\theta_0\}\times\partial D^2])=0.$$

Then the map $f(X|\{\theta_0\}\times\partial D^2)\colon\{\theta_0\}\times\partial D^2\to S^1$ is null homotopic. On the other hand, for the unit tangent vector field Y_0 on the simple closed curve $\{\theta_0\}\times\partial D^2$ with a specified orientation in T^2, the continuous map

$$f(Y_0)\colon\{\theta_0\}\times\partial D^2\to S^1$$

is null homotopic by Lemma 1.1. Thus two continuous maps $f(X|\{\theta_0\}\times\partial D^2)$ and $f(Y_0)$ are homotopic.

Denote $D^2(r)=\{(x, y)\in\boldsymbol{R}^2;\ x^2+y^2\leqq r^2\}$. Then we may assume that $\{\theta_0\}\times\partial D^2(1-\varepsilon)$ is a simple closed curve of a noncompact leaf L of $\mathscr{F}_R^{(+)}$ for sufficiently small $\varepsilon>0$. Since $f(X|\{\theta_0\}\times\partial D^2)$ and $f(Y_0)$ are homotopic, as is easily verified, we can define a continuous family Y'_t $(0\leqq t\leqq 1)$ of non-singular continuous vector fields on $\{\theta_0\}\times\partial D^2(1-\varepsilon)$ such that $Y'_0=X|(\{\theta_0\}\times\partial D^2(1-\varepsilon))$ and that Y'_1 is a unit tangent vector field of $\{\theta_0\}\times\partial D^2(1-\varepsilon)$, by making use of the homeomorphism $P|(\{\theta_0\}\times\partial D^2(1-\varepsilon))\colon\{\theta_0\}\times\partial D^2(1-\varepsilon)\to\{\theta_0\}\times\partial D^2$, where P is the projection defined above.

By considering $L=\boldsymbol{R}^2$, two continuous maps

$$\bar{f}(Y'_0), \bar{f}(Y'_1)\colon\{\theta_0\}\times\partial D^2(1-\varepsilon)\to S^1$$

are defined as in Section 1. These two continuous maps are homotopic.

On the other hand, it is obvious that the degree of $\bar{f}(Y_0')$ is zero and the degree of $\bar{f}(Y_1')$ is 1. This is a contradiction. Thus this lemma is proved.

The following proposition is due to Davis-Wilson [1, Corollary 4.2]:

Proposition 3.2. *Let X be as above. Then the non-singular C^r vector field $X|T^2$ on the compact leaf T^2 of $\mathscr{F}_R^{(+)}$ has at least one closed orbit.*

Proof. It follows from Lemma 3.1 that $(f(X|T^2))_*$ is not a zero map. By Proposition 1.2, this implies that the C^r vector field $X|T^2$ on T^2 has a closed orbit. Thus this proposition is proved.

In fact $X|T^2$ has at least two closed orbits (see Lemma 3.4).

Let $\mathscr{F}_X$ denote the one dimensional C^r foliation of the solid torus $S^1\times D^2$ whose leaves are orbits of X. Then the restriction of $\mathscr{F}_X$ to each leaf L of the plus Reeb foliation $\mathscr{F}_R^{(+)}$ is a codimension one C^r foliation of L. By Proposition 3.2, the codimension one C^r foliation $\mathscr{F}_X|T^2$ of T^2 has at least one compact leaf, say L_{comp}.

Lemma 3.3. *Let $[L_{comp}]$ be the homology class represented by L_{comp}. Then it holds that*

$$[L_{comp}]=a\alpha+b\beta, \quad a\neq 0,$$

where α, β are generators of $H_1(T^2)$ defined in Section 2.

Proof. Suppose that $a=0$, thus $[L_{comp}]=\pm\beta$. Then, by Lemma 1.1, $(f(X|T^2))_*(\beta)=0$. This contradicts Lemma 3.1. Thus this lemma is proved.

Since $\mathscr{F}_X|T^2$ has a compact leaf L_{comp} with $[L_{comp}]=a\alpha+b\beta$, $a\neq 0$, as was observed in Section 2, there exist closed subsets $K_1^{(+)}, K_2^{(+)}, \cdots, K_p^{(+)}, K_1^{(-)}, K_2^{(-)}, \cdots, K_q^{(-)}$ of T^2 such that $\mathscr{F}_X|K_i^{(+)}$ $(i=1, 2, \cdots, p)$ are plus Reeb components and $\mathscr{F}_X|K_i^{(-)}$ $(i=1, 2, \cdots, q)$ are minus Reeb components, and that the restriction $\mathscr{F}_X|T^2$ to the closure of $T^2-\bigcup_{i=1}^p K_i^{(+)}-\bigcup_{i=1}^q K_i^{(-)}$ is a disjoint union of finite number of I-bundle components.

Lemma 3.4. *The number $p+q$ is an even integer.*

Proof. Two compact leaves bounding $K_i^{(+)}$ or $K_i^{(-)}$ have different directions and two compact leaves bounding a slope component have the same direction with respect to the directions induced from $X|T^2$. This implies that $p+q$ is even.

Lemma 3.5. *Let $[L_{comp}]=a\alpha+b\beta, a\neq 0$, as in Lemma 3.3. Then there exists a C^r imbedding*

$$g: S^1 \to T^2$$

satisfying the following conditions (Fig. 8):

(i) *The homology class* $[g(S^1)]$ *represented by* $g(S^1)$ *is* $\pm\beta$.

(ii) $g(S^1)$ *is transverse to leaves of* $\mathscr{F}_X | T^2$ *except* $|a|(p+q)$ *points, say* $z_{1,j}, z_{2,j}, \cdots, z_{p,j}, z'_{1,j}, z'_{2,j}, \cdots, z'_{q,j}$ $(j=1, 2, \cdots, |a|)$, *such that*

$$z_{i,j} \in \operatorname{Int} K_i^{(+)} \quad (i=1, 2, \cdots, p; j=1, 2, \cdots, |a|),$$
$$z'_{i,j} \in \operatorname{Int} K_i^{(-)} \quad (i=1, 2, \cdots, q; j=1, 2, \cdots, |a|),$$

and a leaf of $\mathscr{F}_X | K_i^{(+)}$ (resp. $\mathscr{F}_X | K_i^{(-)}$) *is tangent to* $g(S^1)$ *at* $z_{i,j}$ (*resp.* $z'_{i,j}$) *from the minus side* (*resp. plus side*) *of* $g(S^1)$.

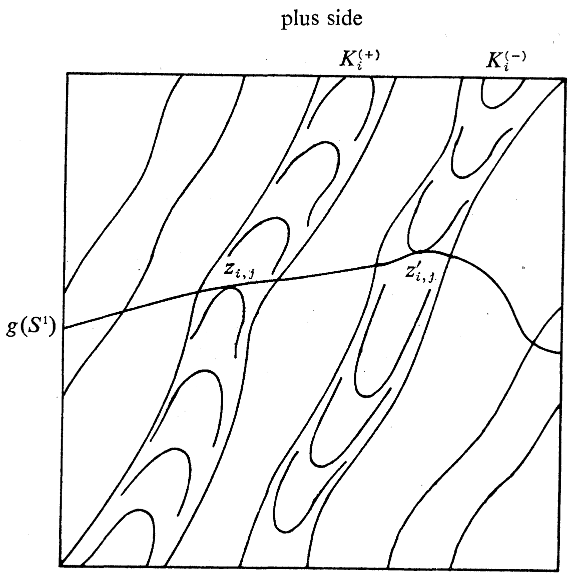


Fig. 8.

Proof. Let L_{comp} be a compact leaf of $\mathscr{F}_X | T^2$ and let $g_1: S^1 \to T^2$ be an imbedding such that $g_1(S^1) = \{\theta\} \times \partial D^2$. Then the algebraic intersection number of L_{comp} and $g_1(S^1)$ is $\pm|a|$. The manifold obtained from T^2 by cutting along L_{comp} is an annulus. Thus, by an argument using Schoenflies theorem, it follows that there exists an imbedding $g_2: S^1 \to T^2$ isotopic to g_1 such that $g_2(S^1)$ intersects with L_{comp} at $|a|$ points. By a similar argument we have an imbedding $g_3: S^1 \to T^2$ isotopic to g_2 such that $g_3(S^1)$ intersects with each compact leaf of $\mathscr{F}_X | T^2$ at $|a|$ points.

Let $\mathscr{F}_X | K$ denote one of plus Reeb components and minus Reeb components of $\mathscr{F}_X | T^2$, where K is a closed subset of T^2. Then $g_3(S^1) \cap K$

consists of disjoint $|a|$ simple curves, say $\bar{g}^{(i)}: (D^1, \partial D^1) \to (K, \partial K)$ $i=1, 2, \cdots, |a|$. Now we apply Lemma 2.2 to each $\bar{g}^{(i)}$. Then as a union of these simple curves, we obtain an imbedding $g: S^1 \to T^2$ we are looking for. Thus this lemma is proved.

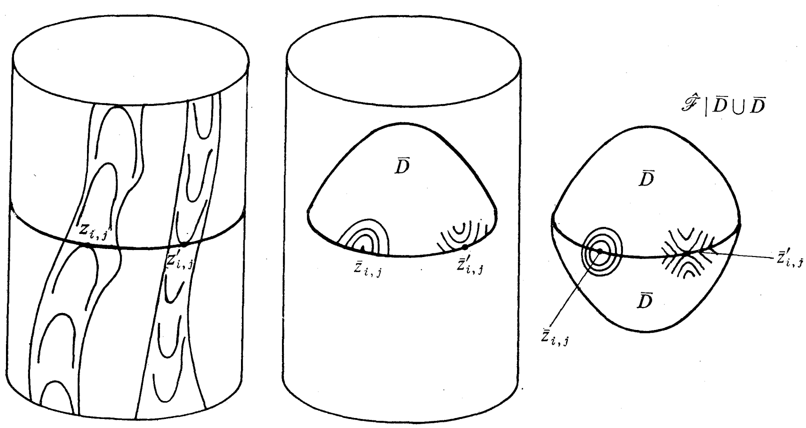
For the imbedding g as in Lemma 3.5, the leaves of $\mathscr{F}_X | T^2$ at the plus side of the curve $g(S^1)$ form a family of concentric half circles with center $z_{i,j}$ near $z_{i,j}$ and an upper part of a family of conforcal parabolas near $z'_{i,j}$ (Fig. 9, (a)).

According to the condition (i) of Lemma 3.5, for a noncompact leaf L_θ of $\mathscr{F}_R^{(+)}$, there exists a simple closed C^r curve C of L_θ situated very close to $g(S^1)$ in $S^1 \times D^2$. Let $\bar{D}$ denote the compact subset of L_θ bounded by C which is C^r diffeomorphic to the 2-disk.

Consider the C^r vector field $X | \bar{D}$ on $\bar{D}$. Since C is very close to $g(S^1)$, the vectors of $X | \bar{D}$ are tangent to the boundary $\partial \bar{D}$ of $\bar{D}$ at exactly $|a|(p+q)$ points $\bar{z}_{i,j}$ $(i=1, 2, \cdots, p; j=1, 2, \cdots, |a|)$, $\bar{z}'_{i,j}(i=1, 2, \cdots, q; j=1, 2, \cdots, |a|)$ such that $\bar{z}_{i,j}$ (resp. $\bar{z}'_{i,j}$) is very close to $z_{i,j}$ (resp. $z'_{i,j}$) and, furthermore, the orbits of $X | \bar{D}$ form a family of concentric half circles with center $\bar{z}_{i,j}$ near $\bar{z}_{i,j}$ and an upper part of conforcal parabolas near $\bar{z}'_{i,j}$ (Fig. 9, (b)).

We define a codimension one C^0 foliation $\hat{\mathscr{F}}$ of the double $\bar{D} \cup \bar{D}$ of $\bar{D}$ with $|a|(p+q)$ singular points $\bar{z}_{i,j}$ $(i=1, 2, \cdots, p; j=1, 2, \cdots, |a|)$, $\bar{z}'_{i,j}$ $(i=1, 2, \cdots, q; j=1, 2, \cdots, |a|)$ by the double of the orbits of the vector field $X | \bar{D}$. The index of the singularity of $\hat{\mathscr{F}}$ at $\bar{z}_{i,j}$ (resp. at $\bar{z}'_{i,j}$) is 1 (resp. -1) (Fig. 9, (c)). Thus, since $\bar{D} \cup \bar{D}$ is homeomorphic to the 2-sphere, we have

$$|a|(p-q)=2.$$



(a) (b) (c)

Fig. 9.

The integer $p-q$ is even by Lemma 3.4. Thus the following proposition holds. (Compare with Davis-Wilson [1]).

Proposition 3.6. *Let $\mathscr{F}_R^{(+)}$, X, $\mathscr{F}_X$, L_{comp}, $a\alpha+b\beta$, p and q be as above. Then we have*

$$|a|=1, \quad p-q=2.$$

A codimension one C^r foliation $\mathscr{F}$ of the torus $(r\geqq 1)$ having compact leaves L_λ $(\lambda \in \Lambda)$ with the homology class $[L_\lambda]=a\alpha+b\beta$, $a\neq 0$, is said to be *normalized* if the following conditions are satisfied (Fig. 10):

(i) Every L_λ is the image of a line in $\boldsymbol{R}^2$ by the projection $\tilde{\pi}$: $\boldsymbol{R}^2\to T^2$.

(ii) For every noncompact leaf L of a plus or a minus Reeb component of $\mathscr{F}$, the leaf L is transverse to $\{\theta\}\times S^1$ $(\theta \in S^1)$ except one, say $\{\theta_L\}\times S^1$, and is tangent to $\{\theta_L\}\times S^1$ at exactly one point.

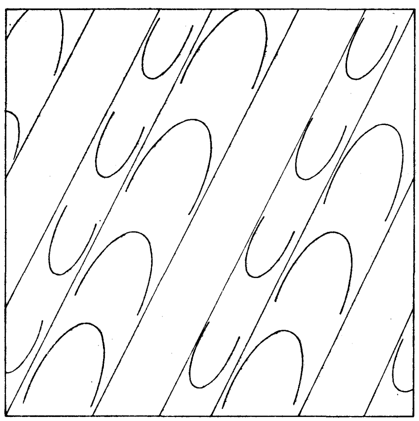


Fig. 10.

Now we have the following proposition:

Proposition 3.7. *Let $\bar{X}$ be a non-singular C^r vector field $(r\geqq 1)$ on T^2 with a closed orbit C such that the homology class $[C]=a\alpha+b\beta$, $a\neq 0$, and let $\bar{\mathscr{F}}$ denote the codimension one C^r foliation of T^2 formed by orbits of $\bar{X}$. Then there exist a normalized codimension one C^r foliation $\bar{\mathscr{F}}_0$ of T^2 and a C^r diffeomorphism*

$$g\colon T^2\to T^2$$

satisfying the following conditions:

(i) *g is an isomorphism between $\bar{\mathscr{F}}$ and $\bar{\mathscr{F}}_0$.*

(ii) *g is isotopic to the identity.*

This proposition can be proved by using Lemmas 2.2, 2.3 and 3.5. We omit the details.

§ 4. Non-singular vector fields on turbulized foliations $\mathscr{F}_\pi^\varepsilon$ of punctured surface bundles over the circle

Let $\Sigma_g(m)$ denote a compact 2-dimensional C^∞ manifold obtained from the closed orientable surface Σ_g of genus g by deleting m disjointly imbedded 2-disks $D_1^2, D_2^2, \cdots, D_m^2$:

$$\Sigma_g(m)=\Sigma_g-\bigcup_{i=1}^m \operatorname{Int} D_i^2.$$

Let E be a compact connected orientable 3-dimensional C^∞manifold with boundary and let $\pi\colon E\to S^1$ be a C^∞ fibering over the circle S^1 with fibre $\Sigma_g(m)$, where $m\geqq 1$. Thus E is constructed as follows. Suppose that

$$\phi\colon \Sigma_g(m)\to\Sigma_g(m)$$

is an orientation preserving C^∞ diffeomorphism. Then E is a quotient space obtained from the product space $I\times\Sigma_g(m)$ by identifying $(0, y)$ and $(1, \phi(y))$ for $y\in\Sigma_g(m)$ and the projection π is the map $\pi(t, y)=t \pmod 1$ for $t\in I$, $y\in\Sigma_g(m)$.

The boundary ∂E of E consists of disjoint union of tori, say $T_1^2, T_2^2, \cdots, T_s^2$. In case $\phi(\partial D_i^2)=\partial D_i^2$ $(i=1, 2, \cdots, m)$, we have $s=m$.

Recall that an orientation is specified on the circle S^1. In the following a point $e^{2\pi i\theta}$ of S^1 is simply denoted by $\theta\in\boldsymbol{R}$ and the orientation of S^1 is compatible with the natural orientation of $\boldsymbol{R}$.

We choose a set of generators α_k, β_k of $H_1(T_k^2)$ $(k=1, 2, \cdots, s)$ so that

$$(\pi\,|\,T_k^2)_*\alpha_k=c_k[S^1],\quad c_k>0,$$
$$(\pi\,|\,T_k^2)_*\beta_k=0,$$

where $[S^1]$ denotes the homology class represented by S^1. Then we have $\sum_{k=1}^s c_k=m$.

A turbulization of the base space S^1 in the minus or the plus direction induces a turbulization of the boundary of T_k^2 $(k=1, 2, \cdots, s)$.

Let ε be a function defined on the set $\{1, 2, \cdots, s\}$ whose values are 1 or -1. We let $\mathscr{F}_\pi^\varepsilon$ denote a codimension one C^∞ foliation of E obtained by turbulizing the interior of each fibre of $\pi\colon E\to S^1$ in the sign $(-\varepsilon(k))$ direction in a collar of ∂T_k^2 for $k=1, 2, \cdots, s$, similarly as to construct $\mathscr{F}_R^{(\pm)}$ in Section 2 (Fig. 11).

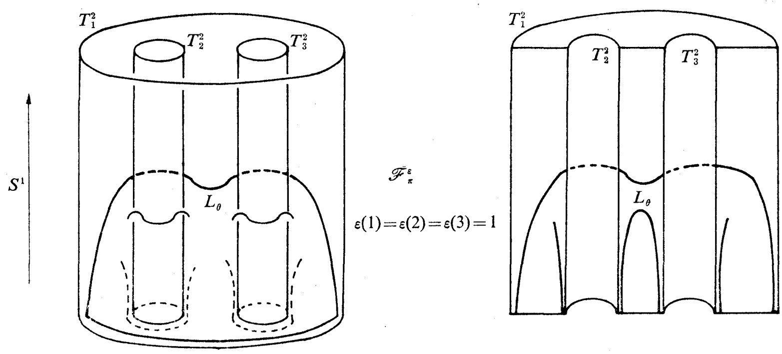


Fig. 11.

Let L_θ $(0\leqq\theta\leqq 1)$ denote a noncompact leaf of $\mathscr{F}_\pi^\varepsilon$ obtained from the interior of $\pi^{-1}(\theta)$ by the turbulization. Then we have

$$\mathscr{F}_\pi^\varepsilon=\{L_\theta;\, \theta\in S^1\}\cup\{T_k^2;\, k=1,2,\cdots,s\}.$$

Let X be a non-singular C^r vector field $(r\geqq 1)$ on the codimension one foliation $\mathscr{F}_\pi^\varepsilon$ of E, that is to say, the vector $X(z)$ of X at $z\in E$ is tangent to the leaf of $\mathscr{F}_\pi^\varepsilon$ containing z. Let $\mathscr{F}_X$ denote the one dimensional C^r foliation of E whose leaves are orbits of X.

Suppose that the restriction $\mathscr{F}_X|T_k^2$ of $\mathscr{F}_X$ to T_k^2 has a compact leaf, say $\bar{L}_k$. We denote by $[\bar{L}_k]$ the homology class of $H_1(T_k^2)$ represented by $\bar{L}_k$ with the orientation induced from X. Then, if $(\pi|T_k^2)_*([\bar{L}_k])\neq 0$, plus Reeb components, minus Reeb components and slope components of $\mathscr{F}_X|T_k^2$ can be defined by using the projection $\pi|T_k^2\colon T_k^2\to S^1$ as in the case of the plus Reeb foliation $\mathscr{F}_R^{(+)}$ of the solid torus (Section 3).

We define integers a_k, b_k, p_k and q_k $(k=1,2,\cdots,s)$ as follows:

(4.1) (1) In case $\mathscr{F}_X|T_k^2$ has a compact leaf, say $\bar{L}_k$, let

$$[\bar{L}_k]=a_k\alpha_k+b_k\beta_k.$$

Furthermore, in case $a_k\neq 0$, let p_k and q_k denote the number of plus Reeb components $\mathscr{F}_X|K_{k,i}^{(+)}$ $(i=1,2,\cdots,p_k)$ and the number of minus Reeb components $\mathscr{F}_X|K_{k,i}^{(-)}$ $(i=1,2,\cdots,q_k)$ contained in $\mathscr{F}_X|T_k^2$ respectively, and, in case $a_k=0$ thus $[\bar{L}_k]=\pm\beta_k$, let $p_k=q_k=0$.

(2) In case $\mathscr{F}_X|T_k^2$ has no compact leaf, let

$$a_k=b_k=p_k=q_k=0.$$

Then the following proposition generalizing Proposition 3.6 holds (Tamura-Sato [6; Proposition 2], Nishimori [2; Proposition 4.3]):

Proposition 4.2. *Let E, $\mathscr{F}^{\varepsilon}_{\pi}$, X, $\bar{\mathscr{F}}_X$, a_k, b_k, c_k, p_k and q_k be as above. Then we have*

$$\sum_{k=1}^{s} \varepsilon(k) c_k |a_k| (p_k - q_k) = 2(2-2g-m).$$

Proof. First suppose that $\bar{\mathscr{F}}_X | T^2_k$ is as (4.1) (1). We take a C^r imbedding

$$g_k\colon S^1 \to T^2_k$$

as follows:

(i) If $a_k \neq 0$, g_k is an imbedding g as in Lemma 3.5 for $\alpha_k = \alpha$, $\beta_k = \beta$, $a_k = a$, $b_k = b$, $p_k = p$ and $q_k = q$.

(ii) If $a_k = 0$, g_k is an imbedding such that $g_k(S^1) = \bar{L}_k$.

Then, in the case of (i), the leaves of $\bar{\mathscr{F}}_X | T^2_k$ are transverse to $g_k(S^1)$ except $|a_k|(p_k + q_k)$ points $z_{k,1,j}, z_{k,2,j}, \cdots, z_{k,p_k,j}, z'_{k,1,j}, z'_{k,2,j}, \cdots, z'_{k,q_k,j}$ $(j = 1, 2, \cdots, |a_k|)$, and a noncompact leaf of $\bar{\mathscr{F}}_X | T^2_k$ contained in the plus Reeb component $\bar{\mathscr{F}}_X | K^{(+)}_{k,i}$ (resp. minus Reeb component $\bar{\mathscr{F}}_X | K^{(-)}_{k,i}$) is tangent to $g(S^1)$ at $z_{k,i,j}$ (resp. at $z'_{k,i,j}$) for $i = 1, 2, \cdots, |a_k|$ from the minus side (resp. plus side) of $g(S^1)$ (Fig. 12).

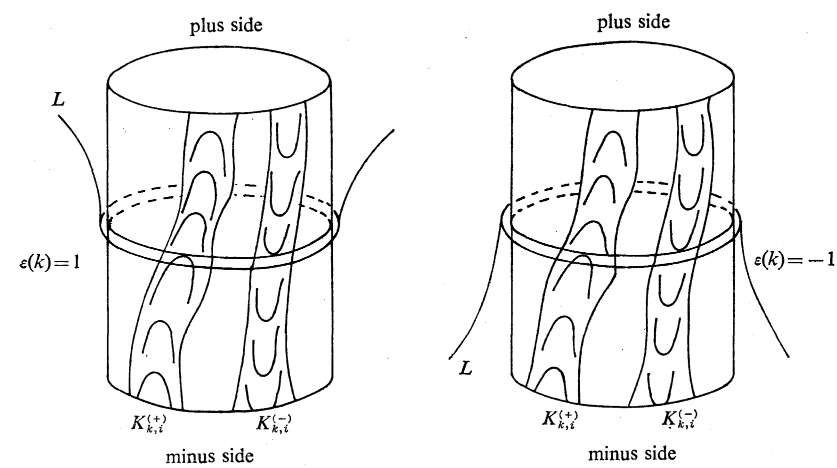


Fig. 12.

In the case of (ii), as is easily verified, there exists a non-singular C^r vector field $Y^{(k)}$ on T^2_k such that the map $f(Y^{(k)})\colon T^2 \to S^1$ is homotopic to the map $f(X | T^2_k)\colon T^2 \to S^1$ and the vectors of $Y^{(k)} | g_k(S^1)$ are transverse to $g_k(S^1)$, where $f(Y^{(k)})$ and $f(X | T^2_k)$ are maps defined in Section 1.

Next suppose that $\bar{\mathscr{F}}_X | T^2_k$ has no compact leaves ((4.1) (2)). Then we let

$$g_k: S^1 \to T_k^2$$

be a C^r imbedding such that $g_k(S^1)$ is a connected component of the intersection of T_k^2 with a fibre of π. By Proposition 1.2, the homomorphism

$$(f(X|T_k^2))_*: H_1(T_k^2) \to H_1(S^1)$$

is a zero map. Thus, as is easily verified, there exists a non-singular continuous vector field $Y^{(k)}$ on T_k^2 such that the map $f(Y^{(k)}): T^2 \to S^1$ is homotopic to $f(X|T_k^2)$ and that the vectors of $Y^{(k)}|g_k(S^1)$ are transverse to $g_k(S^1)$.

Now let L be a noncompact leaf of $\mathscr{F}_\pi^\varepsilon$. Then, as is easily verified, we can take simple closed C^r curves $C_l^{(k)}$ $(k=1, 2, \cdots, s;\ l=1, 2, \cdots, c_k)$ in L such that $C_l^{(k)}$ $(l=1, 2, \cdots, c_k)$ are very close to $g_k(S^1)$ and that the union of $C_l^{(k)}$ $(k=1, 2, \cdots, s;\ l=1, 2, \cdots, c_k)$ bounds a compact subset of L, say Σ. Let us consider the vector field $X|\Sigma$ on Σ. In case $\mathscr{F}_X|T_k^2$ is as (4.1) (1) and $a_k \neq 0$, the vectors of $X|\Sigma$ are transverse to $C_l^{(k)}$ $(l=1, 2, \cdots, c_k)$ except $|a_k|(p_k+q_k)$ points $\bar{z}_{k,i,j,l}$ $(i=1, 2, \cdots, p_k;\ j=1, 2, \cdots, |a_k|)$ and $\bar{z}'_{k,i,j,l}$ $(i=1, 2, \cdots, q_k;\ j=1, 2, \cdots, |a_k|)$ such that $\bar{z}_{k,i,j,l}$ and $\bar{z}'_{k,i,j,l}$ are very close to $z_{k,i,j,l}$ and $z'_{k,i,j,l}$ respectively, and that the orbits of $X|\Sigma$ form a family of concentric half circles with center $\bar{z}_{k,i,j,l}$ (resp. an upper part of conforcal parabolas) near $\bar{z}_{k,i,j,l}$ and an upper part of conforcal parabolas (resp. a family of concentric half circles with center $\bar{z}'_{k,i,j,l}$) near $\bar{z}'_{k,i,j,l}$ if $\varepsilon(k)=1$ (resp. $\varepsilon(k)=-1$). (Fig. 12).

We let Π be a subset of $\{1, 2, \cdots, s\}$ such that $k \in \Pi$ if and only if $\mathscr{F}_X|T_k^2$ is as (4.1) (1), $a_k=0$, or as (4.1) (2). Then by modifying the vector field $X|\Sigma$ in collars $c(C_l^{(k)})$ of $C_l^{(k)}$ for $k \in \Pi$ making use of the vector field $Y^{(k)}$, we obtain a non-singular C^r vector field X_0 on Σ with the following properties:

(i) $X_0|(\Sigma - \bigcup_{k\in\Pi} c(C_l^{(k)})) = X|(\Sigma - \bigcup_{k\in\Pi} c(C_l^{(k)}))$.

(ii) Vectors of X_0 are transverse to $C_l^{(k)}$ for $k \in \Pi$.

We define a codimension one C^0 foliation $\hat{\mathscr{F}}$ of the double $\Sigma \cup \Sigma$ of Σ with $\sum_{k=1}^{s} c_k|a_k|(p_k+q_k)$ singular points $\bar{z}_{k,i,j,l}$ $(k \notin \Pi,\ i=1, 2, \cdots, p_k;\ j=1, 2, \cdots, |a_k|;\ l=1, 2, \cdots, c_k)$, $\bar{z}'_{k,i,j,l}$ $(k \notin \Pi,\ i=1, 2, \cdots, q_k;\ j=1, 2, \cdots, |a_k|;\ l=1, 2, \cdots, c_k)$ by the double of the orbits of the vector field X_0. Since the indices of the singularities at $\bar{z}_{k,i,j,l}$ and these at $\bar{z}'_{k,i,j,l}$ are 1 and -1 (resp. -1 and 1) if $\varepsilon(k)=1$ (resp. $\varepsilon(k)=-1$), this proposition is proved by considering the Euler number of $\Sigma \cup \Sigma$.

Now let $\pi: E_1 \to S^1$ be a C^∞ fibering over S^1 with $\Sigma_g(1) = \Sigma_g - \mathrm{Int}\, D^2$ as fibre and an orientable 3-dimensional manifold E_1 as total space. Then $\partial E_1 = T^2$. Let α and β be generators of $H_1(\partial E_1)$ such that

$$\pi_*(\alpha) = [S^1], \quad \pi_*(\beta) = 0.$$

Let $\mathscr{F}_{\pi}$ denote a codimension one C^r foliation $\mathscr{F}_{\pi}^{\varepsilon}$ of E_1 for which $\varepsilon(1)=1$. Then the following proposition is a direct consequence of Proposition 4.2 ([6, Proposition 2]):

Proposition 4.3. *Let E_1, $\mathscr{F}_{\pi}$, α and β be as above, and let X be a non-singular C^r vector field ($r\geqq 1$) on $\mathscr{F}_{\pi}$. Then the following hold:*

(i) *The C^r vector field $X|\partial E_1$ has at least one closed orbit.*

(ii) *The homology class represented by a closed orbit is $a\alpha+b\beta$, $a\neq 0$.*

(iii) *Let p and q be the numbers of plus Reeb components and minus Reeb components in the codimension one foliation formed by the orbits of $X|\partial E_1$. Then it holds that*

$$|a|(p-q)=2(1-2g).$$

§ 5. Transverse foliations

Let M be an n-dimensional C^∞ manifold and let $\mathscr{F}$ be a codimension q C^r foliation of M ($r\geqq 1$). A codimension q' C^r foliation $\mathscr{F}'$ of M with $q+q'\leqq n$ is said to be *transverse* to $\mathscr{F}$, denoted by $\mathscr{F}\overline{\pitchfork}\mathscr{F}'$, if, at each point $x\in M$, the leaf L_x of $\mathscr{F}$ through x and the leaf L'_x of $\mathscr{F}'$ through x intersect transversely at x, that is, $T_x(L_x)+T_x(L'_x)=T_x(M)$.

Let

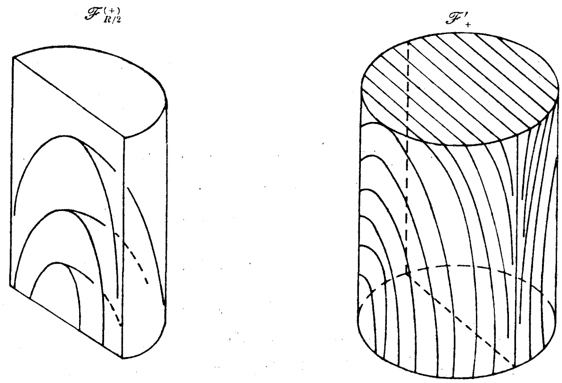
$$\mathscr{F}\Cap\mathscr{F}'=\{\text{connected components of } L\cap L';\ L\in\mathscr{F},\ L'\in\mathscr{F}'\}.$$

Then $\mathscr{F}\Cap\mathscr{F}'$ is obviously a codimension $q+q'$ C^r foliation of M.

Let D^2_+ denote the half 2-disk $\{(x, y)\in D^2;\ y\geqq 0\}$ and let $\mathscr{F}^{(+)}_{R/2}$ denote the restriction of the plus Reeb foliation $\mathscr{F}^{(+)}_{R}$ of $S^1\times D^2$ to $S^1\times D^2_+$. Let $\mathscr{F}'_+$ denote the codimension one C^∞ foliation of $S^1\times D^2$ obtained from two copies of $\mathscr{F}^{(+)}_{R/2}$ by identifying two copies of compact subsets $(S^1\times D^2_+)\cap\partial D^2$ (Fig. 13, (b)). Then $\mathscr{F}'_+$ is transverse to $\mathscr{F}^{(+)}_{R}$. $\mathscr{F}^{(+)}_{R/2}$ is called the *plus half Reeb foliation* of $S^1\times D^2_+$ (Fig. 13, (a)).

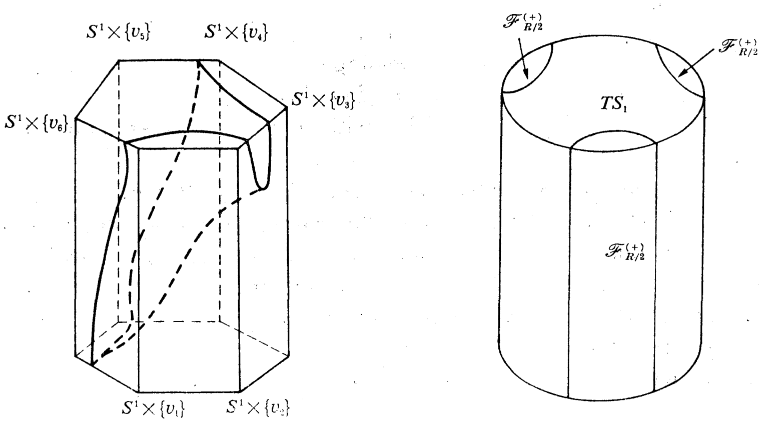
$\mathscr{F}^{(+)}_{R/2}$ and TS_1 below are codimension one C^r 'foliations' of 3-dimensional C^∞ manifolds with corner. Let H denote a hexagon with vertices $v_1, v_2, \cdots, v_6$. Let TS_1 denote a codimension one C^∞ 'foliation' of $S^1\times H$ consisting of 3 compact leaves $S^1\times\overline{v_1v_2}$, $S^1\times\overline{v_3v_4}$, $S^1\times\overline{v_5v_6}$ and noncompact leaves such that they are of the same form and one of them is as Fig. 14, (a). TS_1 is called the *TS component of type* 1. Let $\mathscr{F}'$ be a codimension one C^∞ foliation of $S^1\times D^2$ consisting of TS_1 and three copies of $\mathscr{F}^{(+)}_{R/2}$ by identifying $S^1\times\overline{v_iv_{i+1}}$ with the compact leaf of $\mathscr{F}^{(+)}_{R/2}$ for $i=1, 3, 5$. Then $\mathscr{F}'$ is transverse to $\mathscr{F}^{(+)}_{R}$ (Fig. 14, (b)). For details, see Tamura-Sato [6].

In a previous paper [6], the classification of codimension one C^∞



(a) (b)

Fig. 13.



(a) (b)

Fig. 14.

foliations transverse to the Reeb foliation of the solid torus is completed. And the classifications of codimension one C^∞ foliations transverse to codimension one C^∞ foliations of 3-manifolds of more general types are obtained by Nishimori [2].

Let E, $\mathscr{F}^\varepsilon_\pi$ and T^2_k be as in Section 4. We remark that $\mathscr{F}^\varepsilon_\pi$ is transversely orientable. Suppose that $\mathscr{F}'$ is a transversely orientable codimension one C^r foliation of E transverse to $\mathscr{F}^\varepsilon_\pi$ $(r\geq 2)$. Then $\bar{\mathscr{F}}=\mathscr{F}^\varepsilon_\pi \cap \mathscr{F}'$ is the one dimensional C^r foliation of E consisting of C^r simple curves of leaves of $\mathscr{F}^\varepsilon_\pi$. For a leaf L of $\mathscr{F}^\varepsilon_\pi$, the restriction $\bar{\mathscr{F}}\,|\,L$ of $\bar{\mathscr{F}}$ to L is a codimension one C^r foliation of L.

Since $\mathscr{F}^\varepsilon_\pi$ and $\mathscr{F}'$ are transversely orientable, we can give consistent orientations on elements of $\mathscr{F}$. Let X denote the vector field on E consisting of unit tangent vectors of curves belonging to $\mathscr{F}$. Then X is a non-singular C^{r-1} vector field on E tangent to (leaves of) $\mathscr{F}_\pi^\varepsilon$. $\mathscr{F}$ consists of orbits of X.

A C^{r-1} vector field X on $\mathscr{F}_\pi^\varepsilon$ obtained from transverse foliation $\mathscr{F}'$ as above is said to be *transversely integrable*.

By Propositions 3.6, 4.2 and 4.3, we have the following proposition.

Proposition 5.1. *Let* $\mathscr{F}=\mathscr{F}_\pi^\varepsilon \cap \mathscr{F}'$ *be as above.*

(I) *Let* α_k, β_k, a_k, b_k, c_k, p_k *and* q_k *be as in Section* 4 *for* $\mathscr{F}\,|\,\partial E$. *Then the equation of Proposition* 4.2 *holds.*

(II) *Let* E_1 *be as in Section* 4 *and let* α, β, a, p *and* q *be as in Section* 4 *for* $\mathscr{F}\,|\,\partial E_1$. *Then the following hold*:

(i) $\mathscr{F}\,|\,\partial E_1$ *has at least one compact leaf.*

(ii) $|a|(p-q)=2(1-2g)$.

In particular, $|a|=1$, $p-q=2$ *in case* $E_1=S^1\times D^2$.

Let $V=\{V(z);\, z\in E\}$ be a non-singular C^{r-1} vector field ($r\geqq 2$) on E satisfying the following conditions (Fig. 15):

(5.2) (i) $|V(z)|=1$ $(z\in E)$ (with respect to a Riemannian metric on E).

(ii) Each $V(z)$ is tangent to the leaf of $\mathscr{F}'$ containing z.

(iii) Each $V(z)$ is transverse to the leaf of $\mathscr{F}_\pi^\varepsilon$ containing z and, in case z is contained in a noncompact leaf L_θ ($\theta\in S^1$), $V(z)$ is towards the minus direction of S^1. (The latter condition implies that $V(z)$ is inward (resp. outward) if $z\in T_k^2$ and $\varepsilon(k)=1$ (resp. $\varepsilon(k)=-1$).)

(iv) $d\pi(V(z))=0$ if $z\in T_k^2$ $(k=1, 2, \cdots, s)$.

The existence of such a C^{r-1} vector field V is obvious.

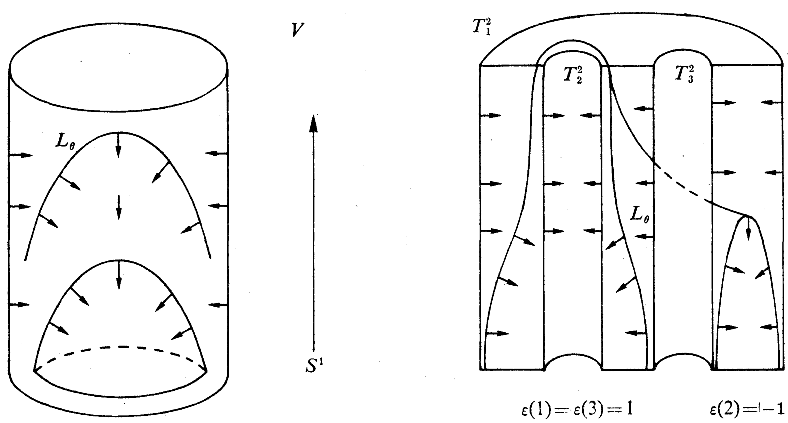


Fig. 15.

Let $\varphi(t, z)$ denote the integral curve of V through z :$\varphi(0, z)=z$. For two noncompact leaves L_θ, $L_{\theta'}$ of $\mathscr{F}_\pi^\varepsilon$ $(\theta, \theta' \in S^1)$, we define a C^r map

$$\Phi_{\theta',\theta}: L_\theta \to L_{\theta'} \quad (0 \leqq \theta' < \theta \leqq 1)$$

by that $\Phi_{\theta',\theta}(z)$ is the first intersection of the positive orbit $\{\varphi(t, z); t>0\}$ through z with $L_{\theta'}$ for $z \in L_\theta$. Then it is clear that $\Phi_{\theta',\theta}$ is a C^r diffeomorphism and maps each leaf of $\bar{\mathscr{F}} \mid L_\theta$ onto a leaf of $\bar{\mathscr{F}} \mid L_{\theta'}$. Therefore $\bar{\mathscr{F}} \mid L_\theta$ and $\bar{\mathscr{F}} \mid L_{\theta'}$ are always isomorphic.

Now let α_k, β_k, a_k and b_k $(k=1, 2, \cdots, s)$ be as in Section 4 for $\bar{\mathscr{F}} \mid \partial E$. Suppose that $a_k \neq 0$ for $k=1, 2, \cdots, s$. Then, by Proposition 3.7, there exist a normalized codimension one C^r foliation $\bar{\mathscr{F}}_0^{(k)}$ of T_k^2 and a C^r diffeomorphism $g^{(k)}: T^2 \to T^2$ isotopic to the identity such that $g^{(k)}$ is an isomorphism between $\bar{\mathscr{F}} \mid T_k^2$ and $\bar{\mathscr{F}}_0^{(k)}$ for $k=1, 2, \cdots, s$.

Let

$$g_t^{(k)}: T_k^2 \to T_k^2, \quad 0 \leqq t \leqq 1,$$
$$g_0^{(k)} = \text{identity}, \quad g_1^{(k)} = g^{(k)}$$

be the isotopy for $k=1, 2, \cdots, s$. Let c_k: $T^2 \times I \to E$ be a sufficiently thin collar of T_k^2 in E such that $\pi(c_k(\{y\} \times I)) = \pi(c_k(y, 0))$ $(y \in T^2)$ for $k=1, 2, \cdots, s$. Then, by realizing the isotopy $g_t^{(k)}$ $(0 \leqq t \leqq 1)$ in the collar $c_k(T^2 \times I)$ $(k=1, 2, \cdots, s)$, we obtain a C^r isotopy

$$\hat{g}_t: E \to E, \quad 0 \leqq t \leqq 1$$

having the following properties:

(i) $\hat{g}_0$ is the identity map.

(ii) $\hat{g}_1 \mid T_k^2 = g^{(k)}$, $k=1, 2, \cdots, s$.

(iii) Let $(\hat{g}_t)_* \mathscr{F}' = \{\hat{g}_t(L'); L' \in \mathscr{F}'\}$ be a codimension one C^r foliation of E $(0 \leqq t \leqq 1)$. Then $(\hat{g}_t)_* \mathscr{F}'$ is transverse to $\mathscr{F}_\pi^\varepsilon$ for all t.

Thus we have the following proposition:

Proposition 5.3. *Let E, $\mathscr{F}_\pi^\varepsilon$, T_k^2 and $\mathscr{F}'$ be as above. Then there exists a C^r diffeomorphism $\hat{g}$: $E \to E$ isotopic to the identity map such that, for the codimension one C^r foliation $\hat{g}_* \mathscr{F}' = \{\hat{g}(L'); L' \in \mathscr{F}'\}$, one dimensional C^r foliation $(\mathscr{F}_\pi^\varepsilon \cap \hat{g}_* \mathscr{F}') \mid T_k^2$ of T_k^2 is normalized for $k=1, 2, \cdots, s$.*

§ 6. Compactification of vector fields on noncompact leaves of $\mathscr{F}_\pi^\varepsilon$

Let $\pi: E_1 \to S^1$ be a C^∞ fibering with one punctured surface of genus g, $\Sigma_g(1) = \Sigma_g - \text{Int}\, D^2$ as fibre and an orientable 3-dimensional manifold E_1 as total space, and let $\mathscr{F}_\pi$ be a codimension one C^∞ foliation of E_1 with $\varepsilon(1)=1$ as in Section 4.

Suppose that $\mathscr{F}'$ is a transversely orientable codimension one C^r foliation of E_1 ($r \geqq 2$) transverse to $\mathscr{F}_\pi$. Let X be a non-singular C^{r-1} vector field on $\mathscr{F}_\pi$ such that the orbits of X form the one dimensional C^r foliation $\bar{\mathscr{F}} = \mathscr{F}_\pi \cap \mathscr{F}'$ as in Section 5.

Furthermore let V be a non-singular C^{r-1} vector field on E_1 as (5.2) and let $\varphi(t, z)$ denote the integral curve of V through $z \in E_1$.

Now let $f\colon E_1 \to \boldsymbol{R}$ be a C^∞ function such that

$$\begin{aligned} f(z) &> 0 \qquad \text{if } z \in \operatorname{Int} E_1, \\ f(z) &= 0 \qquad \text{if } z \in \partial E_1. \end{aligned}$$

The existence of such a function f is obvious. Define a non-singular C^{r-1} vector field X_f on Int E_1 by

$$X_f(z) = f(z) X(z) \qquad (z \in \operatorname{Int} E_1).$$

Then two vector fields $X \mid \operatorname{Int} E_1$ and X_f have obviously the same orbits.

Let

$$\eta\colon \operatorname{Int} E_1 \to \operatorname{Int} E_1$$

be a natural C^r diffeomorphism which maps each noncompact leaf L_θ of $\mathscr{F}_\pi$ onto Int $\pi^{-1}(\theta)$ for $\theta \in S^1$, where L_θ is the leaf obtained from Int $\pi^{-1}(\theta)$ by the turbulization.

We compactify Int E_1 by adding a circle $S^1_\infty = \{p_\infty(\theta);\ \theta \in S^1\}$ so that Int $\pi^{-1}(\theta) \cup p_\infty(\theta)$ is the one point compactification of Int $\pi^{-1}(\theta)$ for each $\theta \in S^1$, and denote by $\hat{E}_1$ the closed 3-dimensional C^∞ manifold thus obtained:

$$\hat{E}_1 = \operatorname{Int} E_1 \cup S^1_\infty.$$

Denote the orientable surface Int $\pi^{-1}(\theta) \cup p_\infty(\theta)$ of genus g by $(\Sigma_g)_\theta$. Then by defining the map $\hat{\pi}\colon \hat{E}_1 \to S^1$ by

$$\hat{\pi}((\Sigma_g)_\theta) = \theta,$$

$\hat{E}$ is the total space of a C^∞ fibering over S^1 with Σ_g as fibre and $\hat{\pi}$ as projection.

Let us define a homeomorphism

$$\hat{\phi}\colon \Sigma_g \to \Sigma_g$$

by

$$\hat{\phi}(z) = \begin{cases} \eta \circ \Phi_{0,1} \circ \eta^{-1}(z) & z \in \operatorname{Int} \pi^{-1}(1), \\ p_\infty(0) = p_\infty(1) & z = p_\infty(1), \end{cases}$$

where $\Phi_{0,1}$ is the homeomorphism defined in Section 5 and we consider as $\Sigma_g = \mathrm{Int}\ \pi^{-1}(1) \cup p_\infty(1) = \mathrm{Int}\ \pi^{-1}(0) \cup p_\infty(0)$, $\eta^{-1}(z) \in L_1$, $\Phi_{0,1} \circ \eta^{-1}(z) \in L_0 = L_1$. Then $\hat{E}$ is the quotient space obtained from the product space $I \times \Sigma_g$ by identifying $(0, y)$ and $(1, \hat{\phi}(y))$ for $y \in \Sigma_g$.

Let $\phi\colon \Sigma_g(1) \to \Sigma_g(1)$ be the C^∞ diffeomorphism used to construct E_1 in Section 4. Then the following proposition is obvious:

Proposition 6.1. *Let ϕ and $\hat{\phi}$ be as above. Then the following diagram commutes*:

$$\begin{array}{ccc} H_1(\Sigma_g(1)) & \xrightarrow{\phi_*} & H_1(\Sigma_g(1)) \\ \cong\downarrow & & \cong\downarrow \\ H_1(\Sigma_g) & \xrightarrow{\hat{\phi}_*} & H_1(\Sigma_g). \end{array}$$

Let us define a continuous vector field $\hat{X}$ on $\hat{E}_1$ by

$$\hat{X}(z) = \begin{cases} d\eta(X_f(\eta^{-1}(z))) & z \in \hat{E}_1 - S^1_\infty, \\ 0 & z \in S^1_\infty. \end{cases}$$

Then the restriction $\hat{X} | (\hat{E}_1 - S^1_\infty)$ is a non-singular C^{r-1} vector field and the restriction $\hat{X} | (\Sigma_g)_\theta$ is a continuous tangent vector field on $(\Sigma_g)_\theta$ for $\theta \in S^1$. We denote by $\mathscr{F}_\theta$ the codimension one foliation of $(\Sigma_g)_\theta$ with a singularity $p_\infty(\theta)$ formed by the orbits of $\hat{X} | (\Sigma_g)_\theta$. Thus $\mathscr{F}_\theta | ((\Sigma_g)_\theta - p_\infty(\theta))$ is a codimension one C^{r-1} foliation. Making use of the C^r diffeomorphism $\Phi_{\theta',\theta}\colon L_\theta \to L_{\theta'}$ defined in Section 5, we define a homeomorphism

$$\hat{\Phi}_{\theta',\theta}\colon (\Sigma_g)_\theta \to (\Sigma_g)_{\theta'} \quad (0 \leqq \theta' < \theta \leqq 1)$$

by

$$\hat{\Phi}_{\theta',\theta}(z) = \begin{cases} \eta \circ \Phi_{\theta',\theta} \circ \eta^{-1}(z) & z \in (\Sigma_g)_\theta - p_\infty(\theta), \\ p_\infty(\theta') & z = p_\infty(\theta). \end{cases}$$

It is obvious that $\hat{\Phi}_{\theta',\theta}$ maps each leaf of $\mathscr{F}_\theta$ onto a leaf of $\mathscr{F}_{\theta'}$.

In the following we study the property of the vector field $\hat{X}_\theta = \hat{X} | (\Sigma_g)_\theta$ on $(\Sigma_g)_\theta$ around the singular point $p_\infty(\theta)$. We take a simple closed curve C on ∂E_1 so that $\pi(C) = 0 \in S^1$.

Let $\boldsymbol{R}_-$ denote the interval $(-\infty, 0]$ and let

$$\tilde{\pi}_0\colon \boldsymbol{R}_- \times S^1 \to \partial E_1$$

be a submersion such that

$$\pi(\tilde{\pi}_0(\{u\} \times S^1)) = u \bmod 1 \quad (u \in \boldsymbol{R}_-),$$
$$\tilde{\pi}_0(\{0\} \times S^1) = C.$$

Let $\tilde{X}$ be the non-singular C^{r-1} tangent vector field on $\boldsymbol{R}_- \times S^1$ such that

$$d\tilde{\pi}_0(\tilde{X}(u, y)) = X(\tilde{\pi}_0(u, y)) \quad u \in \boldsymbol{R}_-, \ y \in S^1.$$

We can take a simple closed curve C_1 on L_1 $(1 \in S^1)$ so that C_1 is very close to ∂E_1 and the integral curves $\varphi(t, z)$ intersect C_1 for any $z \in C$. Let D_1 denote the compact subset of L_1 bounded by C_1 and let $W_1 = L_1 - \operatorname{Int} D_1$. Then W_1 is contained in a thin collar of ∂E_1 in E_1.

It is easy to see that there exists a C^r diffeomorphism

$$\Phi_1 \colon \boldsymbol{R}_- \times S^1 \to W_1$$

such that $\Phi_1(\{0\} \times S^1) = C_1$ and $\Phi_1(u, y) \in (\bigcup_{t>0} \varphi(t, \tilde{\pi}_0(u, y))) \cap L_1$.

Let $D_- = (\boldsymbol{R}_- \times S^1) \cup \{p_\infty\}$ denote the one point compactification of $\boldsymbol{R}_- \times S^1$. D_- is homeomorphic to the 2-disk. Let $\tilde{X}_\infty$ be a continuous vector field on D_- defined by

$$\tilde{X}_\infty(z) = \begin{cases} \dfrac{1}{1+u^2} \tilde{X}(u, y) & z = (u, y) \in \boldsymbol{R}_- \times S^1, \\ 0 & z = p_\infty. \end{cases}$$

The non-singular C^{r-1} vector field $X|\partial E_1$ has at least one closed orbit such that the homology class represented by it is $a\alpha + b\beta$, $a \neq 0$ by Proposition 4.3. Thus the codimension one C^r foliation $\mathscr{F}|\partial E_1$ is decomposed into plus Reeb components $\mathscr{F}|K_1^{(+)}, \mathscr{F}|K_2^{(+)}, \cdots, \mathscr{F}|K_p^{(+)}$, minus Reeb components $\mathscr{F}|K_1^{(-)}, \mathscr{F}|K_2^{(-)}, \cdots, \mathscr{F}|K_q^{(-)}$ and the union of foliated I-bundles which is the restriction of $\mathscr{F}$ onto the closure of $\partial E_1 - \bigcup_{i=1}^{p} K_i^{(+)} - \bigcup_{i=1}^{q} K_i^{(-)}$ as in Section 3.

Let $\tilde{K}_i^{(+)}$ $(i = 1, 2, \cdots, p)$ and $\tilde{K}_i^{(-)}$ $(i = 1, 2, \cdots, q)$ be subsets of D_- defined by

$$\tilde{K}_i^{(+)} = \tilde{\pi}_0^{-1}(K_i^{(+)}) \quad (i = 1, 2, \cdots, p),$$
$$\tilde{K}_i^{(-)} = \tilde{\pi}_0^{-1}(K_i^{(-)}) \quad (i = 1, 2, \cdots, q).$$

Then the following proposition is a direct consequence of properties of plus and minus Reeb components (Fig. 16):

Proposition 6.2. *The orbits of $\tilde{X}_\infty$ on D_- are as follows:*

(i) *If z is a point of* $\operatorname{Int} \tilde{K}_i^{(+)}$ $(i = 1, 2, \cdots, p)$ *near p_∞, then the α-limit set and the ω-limit set of z are both p_∞, or one of the α-limit set and the ω-limit set of z is $\{p_\infty\}$ and the orbit $\psi(t, z)$ of $\tilde{X}_\infty$ through z goes out D_- for $t \to -\infty$ or $t \to \infty$.*

(ii) *If z is a point of* Int $\widetilde{K}_i^{(-)}$ $(i=1, 2, \cdots, q)$, *then the orbit* $\psi(t, z)$ *of* $\widetilde{X}_\infty$ *through z goes out* D_-.

(iii) *If z is a point of* $D_- - \bigcup_{i=1}^p \mathrm{Int}\, \widetilde{K}_i^{(+)} - \bigcup_{i=1}^q \mathrm{Int}\, \widetilde{K}_i^{(-)} - \{p_\infty\}$ *near* p_∞, *then one of the* α*-limit set and the* ω*-limit set of z is* $\{p_\infty\}$ *and the orbit* $\psi(t, z)$ *of* $\widetilde{X}_\infty$ *through z goes out* D_- *for* $t\to\infty$ (*resp.* $t\to-\infty$) *if* p_∞ *is the* α*-limit set* (*resp.* ω*-limit set*) *of z.*

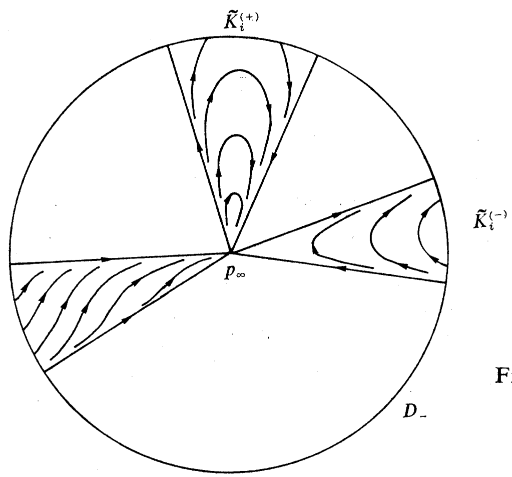


Fig. 16.

Define a map

$$\hat{\Phi}_\theta\colon D_-\to(\Sigma_g)_\theta \quad (0\leqq\theta\leqq 1)$$

by

$$\hat{\Phi}_\theta(z)=\begin{cases}\hat{\Phi}_{\theta,1}\circ\eta\circ\Phi_1(z) & z\in \boldsymbol{R}_-\times S^1,\\ p_\infty(\theta) & z=p_\infty.\end{cases}$$

Then it is obvious that $\hat{\Phi}_\theta\colon D_-\to\hat{\Phi}_\theta(D_-)$ is a homeomorphism and

$$\hat{\Phi}_\theta\,|\,(\boldsymbol{R}_-\times S^1)\colon \boldsymbol{R}_-\times S^1\to(\Sigma_g)_\theta$$

maps each orbit of $\widetilde{X}$ into an orbit of $\hat{X}\,|\,(\Sigma_g)_\theta$. Thus we have the following proposition.

Proposition 6.3. *$\hat{\Phi}_\theta$ maps orbits of $\widetilde{X}_\infty$ around p_∞ to orbits of $\hat{X}\,|\,(\Sigma_g)_\theta$ around $p_\infty(\theta)$ isomorphically for* $0\leqq\theta\leqq 1$.

We remark that the consideration on E_1 in this section can be naturally generalized to E as in Section 4.

Let z_0 be a point of D_- which is contained in $\partial\tilde{K}_i^{(\pm)}$ and is near p_∞. Thus one of the α-limit set and the ω-limit set of z_0 with respect to $\tilde{X}_\infty$ is $\{p_\infty\}$. Let l_0 be the simple curve of D_- with end points z_0 and p_∞ consisting of a part of an orbit of $\tilde{X}_\infty$ through z_0 and p_∞. Then the following proposition important to prove Theorem C in Section 8 holds:

Proposition 6.4. *If* $|a|=1$, *then for subsets* $\hat{\Phi}_1(l_0)$ *and* $\hat{\Phi}_0(l_0)$ *of* $(\Sigma_g)_1=(\Sigma_g)_0$, *we have*

$$\hat{\Phi}_0(l_0)\supset\hat{\Phi}_1(l_0).$$

Proof. Since $z_0\in\partial\tilde{K}_i^{(\pm)}$, we have $\tilde{\pi}_0(z_0)\in\partial K_i^{(\pm)}$. Thus $\tilde{\pi}_0(z_0)$ is a point of a compact leaf of $\mathscr{F}$, say $\tilde{\pi}_0(z_0)\in\bar{L}_c$.

Since $\hat{\Phi}_\theta\colon D_-\to(\Sigma_g)_\theta$ maps orbits of $\tilde{X}_\infty$ to orbits of $\hat{X}|(\Sigma_g)_\theta$, the subsets $\hat{\Phi}_1(l_0)-\{p_\infty(1)\}$ and $\hat{\Phi}_0(l_0)-\{p_\infty(1)\}$ are contained in orbits of $\hat{X}|(\Sigma_g)_1$, say $\hat{\Phi}_1(l_0)-\{p_\infty(1)\}\subset\hat{L}_1$, $\hat{\Phi}_0(l_0)-\{p_\infty(1)\}\subset\hat{L}_2$. Denote $z_0=(u,y)\in\boldsymbol{R}_-\times S^1$. Then, since $\bar{L}_c$ is a simple closed curve and $|a|=1$, we have $(u-1,y)\in l_0$. Let $\bar{l}$ denote the simple curve contained in l_0 whose end points are $z_0=(u,y)$ and $(u-1,y)$. Then, as is easily verified, $\hat{\Phi}_0(\bar{l})$ is contained in $\hat{\Phi}_0(l_0)-\{p_\infty(1)\}$ and that $\hat{\Phi}_0(u-1,y)=\hat{\Phi}_1(z_0)$. Thus this proposition is proved.

§ 7. Vector fields on the torus with one singular point and asymptotic homology classes

In this section let X be a continuous vector field on the torus T^2 with possibly one singular point such that X is C^r for regular points $(r\geqq 1)$ and let $\varphi(t,z)$ denote the orbit of X through $z\in T^2$.

For a point z_0 of T^2, we can classify the ω-limit set of z_0 as follows:

(7.1) (i) The ω-limit set of z_0 consists of one singular point.

(ii) The orbit through z_0 is periodic and, thus, the ω-limit set of z_0 is a closed orbit.

(iii) The orbit through z_0 is not periodic and the ω-limit set of z_0 contains a regular point.

Now suppose that the ω-limit set of z_0 is as (iii) above. Then, since a regular point has a local section through itself, there exists a simple closed C^r curve C_0 transverse to X which intersects the positive semi-orbit $\{\varphi(t;z_0),\ t\geqq 0\}$ through z_0. By the assumption that the singular point of X is at most one, the homology class $[C_0]$ represented by C_0 is non-zero.

In case the positive semi-orbit through z_0 intersects C_0 at only finite points, the ω-limit set of z_0 is contained in T^2-C_0, and thus, by applying the standard arguments of dynamical systems on the 2-sphere, it follows that the ω-limit set of z_0 is a closed orbit or a union of countable orbits whose α-limit sets and ω-limit sets are the singular point.

Let $\tilde{\pi}$: $\boldsymbol{R}^2 \to T^2$ be a universal covering as in Section 1 and let $\tilde{X}$ be a continuous vector field on $\boldsymbol{R}^2$ such that

$$\tilde{\pi}_*(\tilde{X}) = X.$$

Furthermore let $\bar{\alpha}$ and $\bar{\beta}$ denote the homology classes represented by the images of the x-axis and the y-axis by $\tilde{\pi}$ respectively.

Let z_0 be a point of T^2 such that the ω-limit set of z_0 is not a singular point, and let $\hat{z}_0 \in \boldsymbol{R}^2$ be a lift of z_0, i.e. $\tilde{\pi}(\hat{z}_0) = z_0$. Let $\tilde{\varphi}(t, \hat{z}_0)$ denote the orbit of $\tilde{X}$ through $\hat{z}_0$ and let $\tilde{\varphi}(t, \hat{z}_0) = (\bar{x}(t), \bar{y}(t))$ be the coordinates with respect to the x-axis and the y-axis of $\boldsymbol{R}^2$. Then the following lemma holds:

Lemma 7.2. $$\lim_{t \to \infty} (\bar{x}(t) : \bar{y}(t)) = \hat{a} : \hat{b},$$

where $\hat{a}$ and $\hat{b}$ are real numbers and the pair $(\hat{a}, \hat{b})$ is uniquely determined up to positive multiples, that is to say, an equivalence class of the pairs of real numbers such that at least one of them is non-zero by the relation $(\hat{a}, \hat{b}) \sim (\lambda\hat{a}, \lambda\hat{b})$ for $\lambda > 0$ is uniquely determined.

Proof. (I) First suppose that the orbit through z_0 is not closed and the ω-limit set of z_0 is a closed orbit, say $C^{(\omega)}$. Let p_0 be a point of $C^{(\omega)}$ and let l_0 denote a local section through p_0. We denote the set of intersection points of the positive semi-orbit $\{\varphi(t, z_0);\ t \geqq 0\}$ with l_0 by v_i $(i = 0, 1, 2, \cdots)$, where

$$v_i = \varphi(t_i, z_0), \qquad i = 0, 1, 2, \cdots,$$
$$0 \leqq t_0 < t_1 < t_2 < \cdots.$$

Let $C^{(i)}$ be a simple closed C^0 curve consisting of $\{\varphi(t, z_0);\ t_i \leqq t \leqq t_{i+1}\}$ and the subset of l_0 bounded by v_i and v_{i+1}. Then, since the singular point of X is at most one, the homology class $[C^{(i)}]$ represented by $C^{(i)}$ is non-zero and, for sufficiently large i, $[C^{(i)}] = [C^{(\omega)}]$, where $[C^{(\omega)}]$ is the homology class represented by the ω-limit set $C^{(\omega)}$ of z_0. This implies that at least one of $\lim_{t \to \infty} \bar{x}(t)$ and $\lim_{t \to \infty} \bar{y}(t)$ is $\pm\infty$. We let $[C^{(\omega)}] = a'\bar{\alpha} + b'\bar{\beta}$. Then, for a constant r, the inequalities

$$a'j - r \leqq \bar{x}(t_{i+j}) - \bar{x}(t_i) \leqq a'j + r$$
$$b'j - r \leqq \bar{y}(t_{i+j}) - \bar{y}(t_i) \leqq b'j + r$$

hold for $j = 0, 1, 2, \cdots$ and sufficiently large i. Thus we have

$$\lim_{t\to\infty}(\bar{x}(t):\bar{y}(t))=a':b'.$$

(II) In case the orbit through z_0 is closed, the lemma is obvious.

(III) In case the ω-limit set of z_0 is a union of countable orbits whose α-limit sets and ω-limit sets are the singular point, this lemma is proved by a similar argument as in (I), since only a finite number of the closures of orbits contained in the ω-limit set of z_0 are not homologous to zero.

(IV) Let C_0 be as above and suppose that $\{\varphi(t, z_0); t\geqq 0\}$ intersects C_0 at infinite points $u_0, u_1, u_2, \cdots$, where

$$u_i=\varphi(t_i', z_0) \quad i=0, 1, 2, \cdots,$$
$$0\leqq t_0'<t_1'<t_2'<\cdots.$$

Let $[C_0]=c'\bar{\alpha}+d'\bar{\beta}$ and let $c'\bar{\alpha}+d'\bar{\beta}$ and $e'\bar{\alpha}+f'\bar{\beta}$ be generators of $H_1(T^2)$. We take a C^r curve $\tilde{C}_0$ in $\boldsymbol{R}^2$ which is a lift of C_0, i.e. $\bar{\pi}(\tilde{C}_0)=C_0$, and $\tilde{\varphi}(t_0', \hat{z}_0)\in\tilde{C}_0$. Let

$$\tilde{C}_i=\{(x+e'i, y+f'i); (x, y)\in\tilde{C}_0\} \quad (i=\pm 1, \pm 2, \cdots),$$

then $\tilde{C}_i$ is a lift of C_0. We may suppose

$$\tilde{\varphi}(t_i', \hat{z}_0)\in\tilde{C}_i.$$

First we assume that $\tilde{C}_i$ $(i=0, \pm 1, \pm 2, \cdots)$ are parallel lines in $\boldsymbol{R}^2$. Let $\tilde{u}_i=\tilde{\varphi}(t_i', \hat{z}_0)$ $(i=0, 1, 2, \cdots)$ and let $\tilde{u}_i'$ and $\tilde{u}_i''$ be two points of $\tilde{C}_i$ such that $\bar{\pi}(\tilde{u}_i')=\bar{\pi}(\tilde{u}_i'')=\bar{\pi}(\tilde{u}_0)$, $|\tilde{u}_i'-\tilde{u}_i''|=\sqrt{(c')^2+(d')^2}$ and that $\tilde{u}_i$ lies between $\tilde{u}_i'$ and $\tilde{u}_i''$ for $i=1, 2, 3, 4, \cdots$ (Fig. 17).

Let l_i' and l_i'' denote half lines starting from $\tilde{u}_0$ and through $\tilde{u}_i'$ and $\tilde{u}_i''$ respectively. Let us consider curves

$$F_i=\{\tilde{\varphi}(t, \hat{z}_0); t_0'\leqq t\leqq t_i'\}=\{\tilde{\varphi}(t, \tilde{u}_0); 0\leqq t\leqq t_i'-t_0'\},$$
$$F_i'=\{\tilde{\varphi}(t, \tilde{u}_i'); 0\leqq t\leqq t_i'-t_0'\},$$
$$F_i''=\{\tilde{\varphi}(t, \tilde{u}_i''); 0\leqq t\leqq t_i'-t_0'\}.$$

F_i' and F_i'' are parallel displacements of F_i, and it follows from the definition of $\tilde{u}_i'$ and $\tilde{u}_i''$ that $\{\tilde{\varphi}(t, \hat{z}_0); t_i'\leqq t\leqq t_{2i}'\}$ lies between F_i' and F_i''. It is obvious that $\tilde{\varphi}(t_i'-t_0', \tilde{u}_i')$ and $\tilde{\varphi}(t_i'-t_0', \tilde{u}_i'')$ lie in the domain between l_i' and l_i''. Thus $\tilde{u}_{2i}=\tilde{\varphi}(t_{2i}', \hat{z}_0)$ lies in this domain. Therefore, $\tilde{\varphi}(t_{mi}', \hat{z}_0)$ $(m=1, 2, 3, \cdots)$ lie in the domain between l_i' and l_i'' for $i=1, 2, 3, \cdots$. This implies that $\lim_{t\to\infty}(\bar{x}(t):\bar{y}(t))$ exists.

Next we consider the general case. As is well known, there exists a C^r diffeomorphism

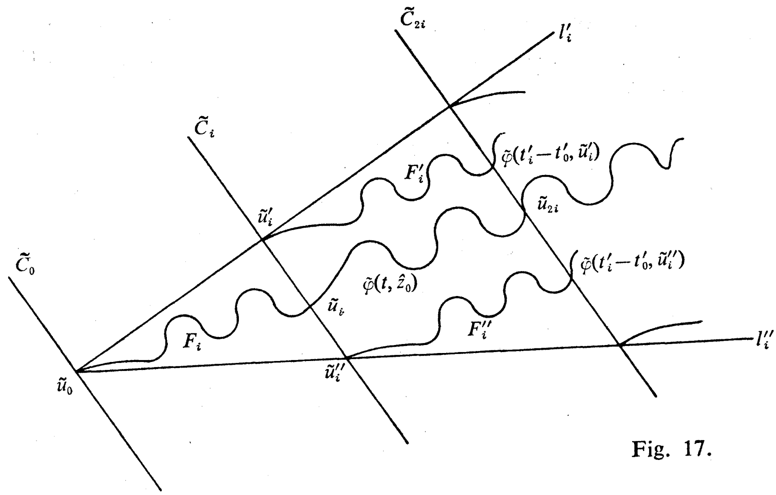


Fig. 17.

$$g\colon T^2 \to T^2$$

isotopic to the identity such that $g(C_0)$ is the image of a line of $\boldsymbol{R}^2$ by $\tilde{\pi}$. Let $\tilde{g}\colon \boldsymbol{R}^2 \to \boldsymbol{R}^2$ be a lift of g. Then, by the result above, $\lim_{t\to\infty}(\bar{x}(t); \bar{y}(t)) = \lim_{t\to\infty}(\tilde{g}(\bar{x}(t))\colon \tilde{g}(\bar{y}(t)))$ exists. Thus this lemma is proved.

For a positive semi-orbit $\{\varphi(t, z_0);\ t \geqq 0\}$ through z_0 whose ω-limit set is not the singular point, the homology class $\hat{a}\bar{\alpha} + \hat{b}\bar{\beta}$ of $H_1(T^2; \boldsymbol{R})$ is called the *asymptotic homology class* of it and is denoted by $A^+(z_0)$, where $\hat{a}$, $\hat{b}$ are real numbers in Lemma 7.2. Asymptotic homology classes are determined up to positive multiples. As is easily verified, $A^+(z_0)$ is independent of the choice of z_0. For a closed orbit, the asymptotic homology class is one represented by the closed orbit. The asymptotic homology classes can be similarly defined for negative semi-orbits $\{\varphi(t, z_0);\ t \leqq 0\}$ of z_0 whose α-limit set is not the singular point, which is denoted by $A^-(z_0)$.

The above definition is closely related to the rotation number in Nishimori [3, Section 10]. For definition and properties of asymptotic homology classes (cycles) in more general setting and references of them, see Yano [7].

§ 8. Proof of Theorem C

In this section we let E_1 be an orientable 3-dimensional C^∞ mainfold, $\pi\colon E_1 \to S^1$ a C^∞ fibering over S^1 with the one punctured torus $\Sigma_1(1) = T^2 - \mathrm{Int}\, D^2$ as fibre and $\mathscr{F}_\pi$ a codimension one C^∞ foliation of E_1 as in Sections 4 and 6. The fibering π is constructed by an orientation preserving C^∞ diffeomorphism $\phi\colon T^2 - \mathrm{Int}\, D^2 \to T^2 - \mathrm{Int}\, D^2$ and π is determined uniquely by the diffeomorphism up to isotopy. Furthermore, as is well known, the isotopy class of an orientation preserving diffeomorphism ϕ is determined by the homomorphism

$$\phi_*: H_1(T^2-\mathrm{Int}\, D^2)\to H_1(T^2-\mathrm{Int}\, D^2)$$

induced by ϕ in this case. Let $\{\xi, \mu\}$ be a set of generators of $H_1(T^2-\mathrm{Int}\, D^2)$. Then ϕ_* is expressed by a 2×2 matrix $\begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} \in SL(2; Z)$.

Lemma 8.1. *Let $\tau(\mathscr{F}_\pi)$ be the tangent 2-plane bundle of $\mathscr{F}_\pi$. Then $\tau(\mathscr{F}_\pi)$ is trivial and, thus, $\mathscr{F}_\pi$ admits transverse 2-plane fields.*

Proof. Let us consider a C^∞ bundle π_0: $E_0\to S^1$ over S^1 with an orientable closed 3-dimensional C^∞ manifold E_0 as total space and the torus as fibre. We may assume that the monodromy map ϕ: $T^2\to T^2$ associated to this bundle is linear with respect to the universal covering space of T^2. This implies that a non-singular linear tangent vector field given on a fibre can be extended to a non-singular vector field tangent to fibres of the bundle. Therefore the 2-plane bundle τ^2 over E_0 tangent to fibres is trivial.

Suppose that $E_0\supset E_1$ and $\pi_0|E_1=\pi$. Then, for a collar $c(T^2\times I)$ of ∂E_1 in E_1, two 2-plane bundles $\tau^2|(E_1-\mathrm{Int}\, c(T^2\times I))$ and $\tau(\mathscr{F}_\pi)|(E_1-\mathrm{Int}\, c(T^2\times I))$ are isomorphic. Since $E_1-\mathrm{Int}\, c(T^2\times I)$ is a deformation retract of E_1 and $\tau^2|(E_1-\mathrm{Int}\, c(T^2\times I))$ is trivial, $\tau(\mathscr{F}_\pi)$ is trivial. Thus this lemma is proved.

Suppose that $\mathscr{F}'$ is a transversely orientable codimension one C^r foliation of E_1 transverse to $\mathscr{F}_\pi$, where $r\geqq 2$. We let $\bar{\mathscr{F}}$, X, a, p, q, $\Phi_{\theta',\theta}$, $\hat{X}$, $\hat{E}$, $\hat{\Phi}_{\theta',\theta}$, $\hat{\Phi}_\theta$, D_- and $\tilde{X}_\infty$ etc. be as in Section 6.

Lemma 8.2. $|a|=1, \quad p-q=-2.$

Proof. By Propositions 4.3 and 5.1, we have

$$|a|(p-q)=-2.$$

Since $\mathscr{F}'$ is transversely orientable, it follows by the same argument used in the proof of Proposition 3.6 that $p-q$ is an even integer. Thus this lemma is proved.

Let $\hat{L}_1, \hat{L}_2, \cdots, \hat{L}_n$ be closed orbits of the C^{r-1} vector field $X|\partial E_1$ on ∂E_1 such that

(i) $\hat{L}_j$ is a connected component of $\partial K_i^{(\pm)}$ for some i,

(ii) $\hat{L}_j$ $(j=1, 2, \cdots, n')$ (resp. $(j=n'+1, n'+2, \cdots, n)$) are towards the minus (resp. plus) direction of S^1.

Let $z^{(j)}$ be a point of $\tilde{\pi}_0^{-1}(\hat{L}_j) \subset D_-$ situated near p_∞ ($j=1, 2, \cdots, n$). Let us consider the continuous vector field $\hat{X}|(\Sigma_1)_1$ with one singular point $p_\infty(1)$ as in Section 6 and orbits $\hat{\varphi}(t, \hat{\Phi}_1(z^{(j)}))$ of $\hat{X}|(\Sigma_1)_1$ through $\hat{\Phi}_1(z^{(j)}) \in (\Sigma_1)_1$ for $j=1, 2, \cdots, n$. It is obvious that the ω-limit set (resp. α-limit set) of $\hat{\varphi}(t, \hat{\Phi}_0(z^{(j)}))$ is $\{p_\infty(1)\}$ for $j=1, 2, \cdots, n'$ (resp. $j=n'+1, n'+2, \cdots, n$).

Lemma 8.3. *One of the following* (a), (b) *holds for* $j=1, 2, \cdots, n$:

(a) *There exists at least one orbit $\hat{\varphi}(t, \hat{\Phi}_1(z^{(j)}))$ such that the α-limit set and the ω-limit set of it are both $\{p_\infty(1)\}$ and that the homology class $[\hat{C}_j]$ represented by a simple closed curve $\hat{C}_j$ formed by the orbit $\bigcup_{-\infty<t<\infty} \hat{\varphi}(t, \hat{\Phi}_1(z^{(j)}))$ and $p_\infty(1)$ is not zero.*

(b) *There exists at least one orbit $\hat{\varphi}(t, \hat{\Phi}_1(z^{(j)}))$ such that one of the α-limit set and the ω-limit set is not $\{p_\infty(1)\}$.*

Proof. Assume that the case (a) does not occur. Then, for any orbit $\hat{\varphi}(t, \hat{\Phi}_1(z^{(j)}))$ whose α-limit set and ω-limit set are $\{p_\infty(1)\}$, the simple closed curve $\hat{C}_j$ consisting of the orbit and $p_\infty(1)$ is homologous to zero. Thus $\hat{C}_j$ bounds a 2-disk in $(\Sigma_1)_1$, say $\hat{D}_j$. Let $\hat{K}_i^{(\pm)} = \hat{\Phi}_1(\tilde{K}_i^{(\pm)})$. Let $N_j^{(+)}$ (resp. $N_j^{(-)}$) be the subset of $\{1, 2, \cdots, p\}$ (resp. $\{1, 2, \cdots, q\}$) such that $\hat{K}_i^{(+)} \subset \hat{D}_j$ (resp. $\hat{K}_i^{(-)} \subset \hat{D}_j$) if and only if $i \in N_j^{(+)}$ (resp. $i \in N_j^{(-)}$). Let $p^{(j)}$ (resp. $q^{(j)}$) denote the number of the elements of $N_j^{(+)}$ (resp. $N_j^{(-)}$). Then, by considering the vector field $\hat{X}|\hat{D}_j$, we have

$$p^{(j)} - q^{(j)} = 1.$$

Now let $\hat{C}_{j_1}, \hat{C}_{j_2}, \cdots, \hat{C}_{j_m}$ be the set of the simple closed curves with the property as above and let $\hat{D}_{j_1}, \hat{D}_{j_2}, \cdots, \hat{D}_{j_m}$ be 2-disks in $(\Sigma_1)_1$ such that $\partial \hat{D}_{j_i} = \hat{C}_{j_i}$ ($i=1, 2, \cdots, m$) as above. Here we take $\hat{C}_{j_1}, \hat{C}_{j_2}, \cdots, \hat{C}_{j_m}$ so that $\hat{D}_{j_i} \cap \hat{D}_{j_{i'}} = \{p_\infty(1)\}$ if $i \neq i'$. Since $p-q=-2$ by Lemma 8.2 and $p^{(j_i)} - q^{(j_i)} = 1$ for $i=1, 2, \cdots, m$, there exist at least $m+2$ of $\hat{K}_i^{(-)}$ ($i=1, 2, \cdots, q$) such that $\hat{K}_i^{(-)}$ is not contained in $\bigcup_{i=1}^m \hat{D}_{j_i}$. As is easily verified, it follows from this consideration that one of $\hat{\Phi}_1(z^{(j)})$ ($j=1, 2, \cdots, n$), say $\hat{\Phi}_1(z^{(h)})$, is not contained in $\bigcup_{i=1}^m \hat{D}_{j_i}$. Thus, by the assumption, one of the α-limit set and the ω-limit set of $\hat{\Phi}_1(z^{(h)})$ is not $\{p_\infty(1)\}$. Thus this lemma is proved.

Now we prove the following theorem which is the "only if" part of Theorem C in Section 0, making use of asymptotic homology classes of orbits defined in Section 7.

Theorem 8.4. *Let E_1, $\mathscr{F}_\pi$ and ϕ_* be as above. If there exists a codimension one C^r foliation $\mathscr{F}'$ of E_1 ($r \geqq 2$) transverse to $\mathscr{F}_\pi$, then the trace of ϕ_* is $\geqq 2$.*

Proof. First suppose that the case (a) of Lemma 8.3 occurs. Then, by Proposition 6.4 and Lemma 8.2, the orbit $\hat{\varphi}(t, \hat{\Phi}_0(z^{(j)}))$ of $\hat{X}|(\Sigma_1)_0$ is mapped onto the orbit $\hat{\varphi}(t, \hat{\Phi}_1(z^{(j)}))$ of $\hat{X}|(\Sigma_1)_1$ by $\hat{\phi}$. Therefore the homology class $[\hat{C}^{(j)}]=a\xi+b\mu$ ($a, b \in \boldsymbol{Z}$) is invariant under ϕ_*:

$$\begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}=\begin{pmatrix} a \\ b \end{pmatrix}.$$

This shows that $\det \begin{pmatrix} c_{11}-1 & c_{12} \\ c_{21} & c_{22}-1 \end{pmatrix}=0$. Thus we have

$$\text{Trace}\ \phi_*=c_{11}+c_{22}=2.$$

Next suppose that the case (b) of Lemma 8.3 occurs. Then, by Proposition 6.4 and Lemma 8.2, the orbit $\hat{\varphi}(t, \hat{\Phi}_0(z^{(j)}))$ of $\hat{X}|(\Sigma_1)_0$ is mapped onto the orbit $\hat{\varphi}(t, \hat{\Phi}_1(z^{(j)}))$ of $\hat{X}|(\Sigma_1)_1$ by $\hat{\phi}$. This implies that the ω-limit set and the α-limit set of $\hat{\varphi}(t, \hat{\Phi}_0(z^{(j)}))$ are mapped onto the ω-limit set and the α-limit set of $\hat{\varphi}(t, \hat{\Phi}_1(z^{(j)}))$ by $\hat{\phi}$ respectively. We may assume that the ω-limit set of $z^{(j)}$ is not $\{p_\infty(1)\}$.

In case the ω-limit set of $\hat{\varphi}(t, \hat{\Phi}_0(z^{(j)}))$ is a closed orbit, say C_ω, the homology class represented by C_ω is not homologous to zero by the reason that C_ω cannot bound a 2-disk in $(\Sigma_1)_1$. Since the homology class $[C_\omega]$ is invariant under ϕ_*, we have Trace $\phi_*=2$ as above.

Even in case the ω limit set of $\hat{\varphi}(t, \hat{\Phi}_0(z^{(j)}))$ is not a closed orbit, the asymptotic homology class $A^+(z^{(j)})$ of $\hat{\Phi}_0(z^{(j)})$ can be defined as in Section 7.

Since $A^+(z^{(j)})=\hat{a}\xi+\hat{b}\mu$ ($\hat{a}, \hat{b} \in \boldsymbol{R}$) is invariant under ϕ_* up to a positive multiple $\lambda>0$, we have

$$\begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}\begin{pmatrix} \hat{a} \\ \hat{b} \end{pmatrix}=\lambda\begin{pmatrix} \hat{a} \\ \hat{b} \end{pmatrix}.$$

It follows from $\det \begin{pmatrix} c_{11}-\lambda & c_{12} \\ c_{21} & c_{22}-\lambda \end{pmatrix}=0$ that

$$\text{Trace}\ \phi_*=c_{11}+c_{22}\geqq 2.$$

Thus this theorem is proved.

The "if" part of Theorem C can be proved as follows. For details, see Nishimori [3, Section 11]. Let $\hat{E}_1$ be a torus bundle over S^1 such that the trace of $\phi_*: H_1(T^2)\to H_1(T^2)$ is $\geqq 2$. Let λ_1 and λ_2 be the real proper values of ϕ_*. Then we can define a C^∞ vector field $\hat{X}$ on $\hat{E}_1$ having the following properties:

(i) The vectors of $\hat{X}$ are tangent to each fibre of $\hat{\pi}$: $\hat{E} \to S^1$.

(ii) The orbits of the restriction of $\hat{X}$ to each fibre $(\Sigma_1)_\theta = \hat{\pi}^{-1}(\theta)$ consist of the images of lines of $\boldsymbol{R}^2$ with the direction of the eigenvector of λ_1 by the projection $\boldsymbol{R}^2 \to T^2$ and one singular point.

As we constructed the vector field $\hat{X}$ on $\hat{E}_1$ from the vector field X on E_1, we can conversely construct a C^∞ vector field X on E_1 from $\hat{X}$ above. Then the vector field X on E_1 thus obtained is transversely integrable.

Remark 8.5. If the foliation $\mathscr{F}'$ of Theorem 8.4 does not contain any non-proper leaf, then, since the orbit $\hat{\varphi}(t, \hat{\Phi}_0(z^{(j)}))$ in the proof of Theorem 8.4 is proper, we have Trace $\phi_* = 2$.

§ 9. Cutting down of ends of noncompact leaves of $\mathscr{F}_\pi^\varepsilon$ and bifurcation of leaves of foliations of punctured surfaces

Let E and $\pi: E \to S^1$ be an orientable 3-dimensional C^∞ manifold and a C^∞ fibering over the circle with fibre $\Sigma_g(m)$ and let $\mathscr{F}_\pi^\varepsilon$ be a C^∞ foliation of E as in Section 4. For simplicity we assume that the C^∞ diffeomorphism ϕ: $\Sigma_g(m) \to \Sigma_g(m)$ associated to π as in Section 4, maps each connected component of $\partial\Sigma_g(m)$ onto itself. Thus ∂E consists of m copies of the torus $T_1^2, T_2^2, \cdots, T_m^2$.

Let $\mathscr{F}'$ be a transversely orientable codimension one C^r foliation of E transverse to $\mathscr{F}_\pi^\varepsilon$ $(r \geqq 2)$ and let $\bar{\mathscr{F}} = \mathscr{F}_\pi^\varepsilon \cap \mathscr{F}$ as in Section 5. We assume that each codimension one C^r foliation $\bar{\mathscr{F}} \mid T_k^2$ of T_k^2 is normalized for $k = 1, 2, \cdots, m$, by taking $(\hat{g})_* \mathscr{F}'$ instead of $\mathscr{F}'$ making use of $\hat{g}$ of Proposition 5.3, if necessary.

Let $c^{(k)}$: $T^2 \times I \to E$, $c^{(k)}(T^2 \times \{0\}) \subset \partial E$ be a sufficiently thin collar of T_k^2 in E such that $\pi(c^{(k)}(\{y\} \times I)) = \pi(c^{(k)}(y, 0))$ $(y \in T^2)$ for $k = 1, 2, \cdots, m$ and let

$$A = E - \bigcup_{k=1}^{m} c^{(k)}(T^2 \times [0, 1)),$$

$$T_k' = c^{(k)}(T^2 \times \{1\}) \quad k = 1, 2, \cdots, m,$$

$$\partial A = \bigcup_{k=1}^{m} T_k'.$$

For a noncompact leaf L_θ of $\mathscr{F}_\pi^\varepsilon$ $(\theta \in S^1)$, we denote

$$A_\theta = A \cap L_\theta \quad (\theta \in S^1).$$

Then we have $A = \bigcup_{\theta \in S^1} A_\theta$ and A_θ is obviously diffeomorphic to $\Sigma_g(m)$, and furthermore, we have a C^∞ fibering $\bar{\pi}$: $A \to S^1$ with $\Sigma_g(m)$ as fibre by defining $\pi(A_\theta)=\theta$. Denote $S_{k,\theta}=A_\theta \cap T'_k$. Then we have

$$T'_k=\bigcup_{\theta\in S^1} S_{k,\theta} \quad k=1, 2, \cdots, m.$$

Let $\bar{\mathscr{F}}_k=\{T'_k\cap L';\ L'\in \mathscr{F}'\}$. Since leaves of $\mathscr{F}$ are transverse to T^2_k, $\bar{\mathscr{F}}_k$ is a codimension one C^r foliation of T'_k $(k=1, 2, \cdots, m)$.

Let V be the non-singular C^{r-1} vector field on E as in (5.2). Then the orbits of V give a C^r diffeomorphism $T^2_k \to T'_k$ which is denoted by μ_k for $k=1, 2, \cdots, m$. The C^r diffeomorphism μ_k is obviously an isomorphism between $\mathscr{F}\,|\,T^2_k$ and $\bar{\mathscr{F}}_k$.

Now we assume that each $\mathscr{F}\,|\,T^2_k$ has at least one compact leaf, say $L^{(k)}_{comp}$, such that

$$[L^{(k)}_{comp}]=a_k\alpha_k+b_k\beta_k, \quad a_k\neq 0$$

for $k=1, 2, \cdots, m$, where $[L^{(k)}_{comp}]$ is the homology class of $H_1(T^2_k)$ represented by $L^{(k)}_{comp}$ and α_k and β_k are generators of $H_1(T^2_k)$ such that $\pi_*(\beta_k)=0$. Then $\mathscr{F}\,|\,T^2_k$ has plus Reeb components $\mathscr{F}\,|\,K^{(+)}_{k,i}$ $(i=1, 2, \cdots, p_k)$ and minus Reeb components $\mathscr{F}\,|\,K^{(-)}_{k,i}$ $(i=1, 2, \cdots, q_k)$.

Since $\mathscr{F}\,|\,T^2_k$ is normalized, $\bar{\mathscr{F}}_k$ has the following properties (Fig. 18):

(i) $\bar{\mathscr{F}}_k$ has plus Reeb components $\bar{\mathscr{F}}_k\,|\,\bar{K}^{(+)}_{k,i}$ $(i=1, 2, \cdots, p_k)$ and minus Reeb components $\bar{\mathscr{F}}_k\,|\,\bar{K}^{(-)}_{k,i}$ $(i=1, 2, \cdots, q_k)$, where $\bar{K}^{(+)}_{k,i}=\mu_k(K^{(+)}_{k,i})$ and $\bar{K}^{(-)}_{k,i}=\mu_k(K^{(-)}_{k,i})$.

(ii) For each θ, the simple closed curve $S_{k,\theta}$ is transverse to leaves of $\bar{\mathscr{F}}_k$ except $|a_k|(p_k+q_k)$ points

$$\bar{z}_{k,i,j,\theta}\ (i=1, 2, \cdots, p_k;\, j=1, 2, \cdots, |a_k|),$$
$$\bar{z}'_{k,i,j,\theta}\ (i=1, 2, \cdots, q_k;\, j=1, 2, \cdots, |a_k|),$$

such that

$$\bar{z}_{k,i,j,\theta}\in \operatorname{Int} \bar{K}^{(+)}_{k,i} \quad (j=1, 2, \cdots, |a_k|),$$
$$\bar{z}'_{k,i,j,\theta}\in \operatorname{Int} \bar{K}^{(-)}_{k,i} \quad (j=1, 2, \cdots, |a_k|).$$

(iii) A leaf of $\bar{\mathscr{F}}_k$ is tangent to $S_{k,\theta}$ at $\bar{z}_{k,i,j,\theta}$(resp. $\bar{z}'_{k,i,j,\theta}$) from the minus side (resp. plus side) of $S_{k,\theta}$ with respect to the orientation of S^1.

Let

$$Q_{k,i}=\bigcup_{\substack{\theta\in S^1\\ j=1,2,\cdots,|a_k|}} \bar{z}_{k,i,j,\theta}, \qquad Q'_{k,i}=\bigcup_{\substack{\theta\in S^1\\ j=1,2,\cdots,|a_k|}} \bar{z}'_{k,i,j,\theta}.$$

Then $Q_{k,i}$ and $Q'_{k,i}$ are simple closed C^r curves in $\operatorname{Int} \bar{K}^{(+)}_{k,i}$ and $\operatorname{Int} \bar{K}^{(-)}_{k,i}$ respectively.

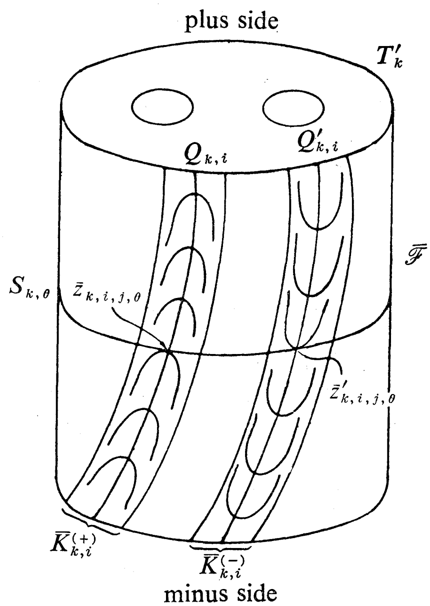


Fig. 18.

Let us consider the restriction $\bar{\mathscr{F}}\,|\,A_\theta$ of $\bar{\mathscr{F}}$ to A_θ. It is obvious that $\bar{\mathscr{F}}\,|\,A_\theta$ consists of C^r simple curves of A_θ and $\sum_{k=1}^m |a_k| p_k$ points $\bar{z}_{k,i,j,\theta}$ for k with $\varepsilon(k)=1$ and $\sum_{k=1}^m |a_k| q_k$ points $\bar{z}'_{k,i,j,\theta}$ for k with $\varepsilon(k)=-1$ such that there exist two simple curves in $\bar{\mathscr{F}}\,|\,A_\theta$ having a common point $\bar{z}'_{k,i,j,\theta}$ for each $\bar{z}'_{k,i,j,\theta}$ with $\varepsilon(k)=1$ and $\bar{z}_{k,i,j,\theta}$ for each $\bar{z}_{k,i,j,\theta}$ with $\varepsilon(k)=-1$, where we understand that $\bar{L} \in \bar{\mathscr{F}}\,|\,A_\theta$ is simple if $\bar{L} \cap \operatorname{Int} A_\theta$ is connected (Fig. 19). Thus $\bar{\mathscr{F}}\,|\,(A_\theta - \bigcup_{k,i,j} \bar{z}_{k,i,j,\theta} - \bigcup_{k,i,j} \bar{z}'_{k,i,j,\theta})$ is a codimension one C^r foliation.

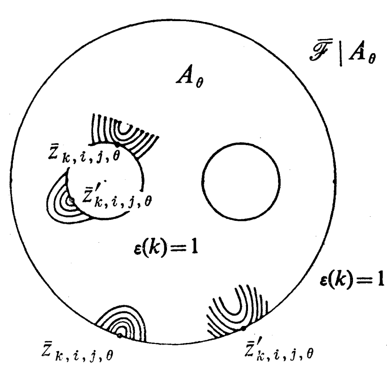


Fig. 19.

The simple curves of $\mathscr{F}|A_\theta$ form a family of concentric half circles around $\bar{z}_{k,i,j,\theta}$ (resp. $\bar{z}'_{k,i,j,\theta}$) if $\varepsilon(k)=1$ (resp. $\varepsilon(k)=-1$), and the simple curves of $\mathscr{F}|A_\theta$ form an upper part of conforcal parabolas around $\bar{z}'_{k,i,j,\theta}$ (resp. $\bar{z}_{k,i,j,\theta}$) if $\varepsilon(k)=1$ (resp. $\varepsilon(k)=-1$) (Fig. 19).

In this sense, the point $\bar{z}_{k,i,j,\theta}$ (resp. $\bar{z}'_{k,i,j,\theta}$) is said to be a *plus singular point* of $\mathscr{F}|A_\theta$ if $\varepsilon(k)=1$ (resp. $\varepsilon(k)=-1$) and the point $\bar{z}'_{k,i,j,\theta}$ (resp. $\bar{z}_{k,i,j,\theta}$) is said to be a *minus singular point* of $\mathscr{F}|A_\theta$ if $\varepsilon(k)=1$ (resp. $\varepsilon(k)=-1$).

Now let W be a C^r vector field on A satisfying the following conditions (see [6, Section 3]). The existence of such W is obvious.

(9.1) (i) W is tangent to leaves of $\mathscr{F}'$.

(ii) Let $\bar{Q}$ (resp. $\bar{Q}'$) denote the union of $Q_{k,i}$ ($k=1, 2, \cdots, m$, $i=1, 2, \cdots, p_k$) (resp. $Q'_{k,i}$ ($k=1, 2, \cdots, m$; $i=1, 2, \cdots, q_k$)). Then

$$W(z)=0 \qquad \text{if } z \in \bar{Q} \cup \bar{Q}',$$
$$W(z)\neq 0 \qquad \text{if } z \in A-\bar{Q}-\bar{Q}'.$$

That is, the singular set of W is $\bar{Q} \cup \bar{Q}'$.

(iii) $W|\partial A$ is tangent to ∂A.

(iv) For $z \in A_\theta-(A_\theta \cap (\bar{Q} \cup \bar{Q}'))$, $W(z)$ is transverse to A_θ and directs to the positive direction of S^1.

(v) The vector field W near a singular point $z \in \bar{Q} \cup \bar{Q}'$ is as follows (Fig. 20):

(a) In case $z \in Q_{k,i}$ with $\varepsilon(k)=1$, the closure of the union of orbits of W whose ω-limit sets are $\{z\}$ forms a half elliptic paraboloid with z as the maximal point in a neighborhood of z.

(b) In case $z \in Q'_{k,i}$ with $\varepsilon(k)=1$, there exist exactly two orbits of W with $\{z\}$ as the α-limit set, and exactly one orbit of W with $\{z\}$ as the ω-limit set. They are contained in T'_k and Int A near z respectively.

(c) In case $z \in Q_{k,i}$ with $\varepsilon(k)=-1$, there exist exactly two orbits of W with $\{z\}$ as the ω-limit set, and exactly one orbit of W with $\{z\}$ as the α-limit set. They are contained in T'_k and Int A near z respectively.

(d) In case $z \in Q'_{k,i}$ with $\varepsilon(k)=-1$, the closure of the union of orbits of W whose α-limit sets are $\{z\}$ forms a half elliptic paraboloid with z as the minimal point in a neighborhood of z.

A point z of $Q_{k,i}$ is said to be an *attracting point* (resp. a *joining point*) of W in case $\varepsilon(k)=1$ (resp. $\varepsilon(k)=-1$), and a point z of $Q'_{k,i}$ is said to be a *branching point* (resp. *repelling point*) of W in case $\varepsilon(k)=1$ (resp. $\varepsilon(k)=-1$).

Let $\bar{\varphi}(t, z)$ denote the orbit of W through $z \in A$. In the following we denote $A_{\theta,\varepsilon}=\bigcup_{\theta \leq \theta' < \theta+\varepsilon} A_{\theta'}$.

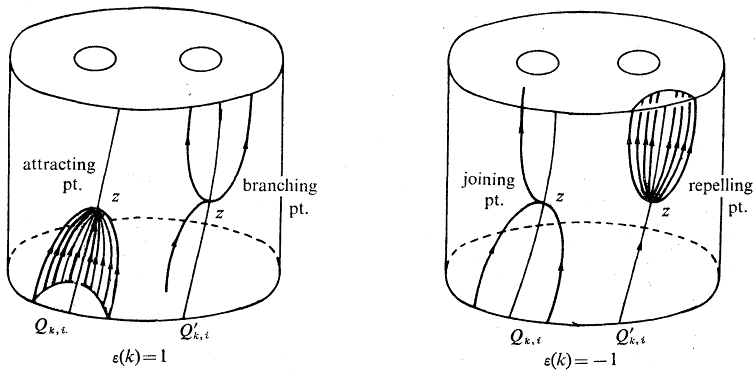


Fig. 20.

It follows from the conditions (iv), (v) of W that there exists a sufficiently small real number $\varepsilon>0$ such that there is no orbit of $W|A_{\theta,\varepsilon}$ whose α-limit set and ω-limit set belong both to $A_{\theta,\varepsilon}$ except singular points for any $\theta \in S^1$.

For a point z of A_θ, we define a subset $\bar{\varphi}[z]$ of $A_{\theta,\varepsilon}$ as follows (Fig. 21):

(i) In case z is not a singular point of W and $\bar{\varphi}(t, z)$ goes through $A_{\theta+\varepsilon}$ for $t\geqq 0$, we define

$$\bar{\varphi}[z]=\{\bar{\varphi}(t, z);\ 0\leqq t<t_\varepsilon\},$$

where we denote by t_ε the least positive real number such that $\bar{\varphi}(t_\varepsilon, z) \in A_{\theta+\varepsilon}$.

(ii) In case z is not a singular point of W and $\bar{\varphi}(t, z)$ approaches to an attracting point z_ω of W for $t>0$ satisfying $\overline{\{\bar{\varphi}(t, z);\ 0\leqq t\}} \subset A_{\theta,\varepsilon}$, we define

$$\bar{\varphi}[z]=\{\bar{\varphi}(t, z);\ 0\leqq t<\infty\} \cup \{z_\omega\}.$$

(iii) In case z is not a singular point of W and $\bar{\varphi}(t, z)$ approaches to a branching point or a joining point of W, say z_ω, for $t>0$ satisfying $\overline{\{\bar{\varphi}(t, z);\ 0\leqq t\}}\subset A_{\theta,\varepsilon}$, we define

$$\begin{aligned}\bar{\varphi}[z]=&\{\bar{\varphi}(t, z);\ 0\leqq t<\infty\} \cup \{z_\omega\}\\ &\cup \{z';\ \lim_{t\to-\infty} \bar{\varphi}(t, z')=z_\omega,\ \bigcup_{-\infty<t\leqq 0} \bar{\varphi}(t, z')\subset A_{\theta,\varepsilon}\}.\end{aligned}$$

(iv) In case z is an attracting point of W, we define

$$\bar{\varphi}[z]=\{z\}.$$

(v) In case z is a branching point or a joining point or a repelling point, we define

$$\bar{\varphi}[z]=\{z\}\cup\{z';\ \lim_{t\to-\infty}\bar{\varphi}(t, z')=z,\ \bigcup_{-\infty<t\leqq 0}\bar{\varphi}(t, z')\subset A_{\theta,\varepsilon}\}.$$

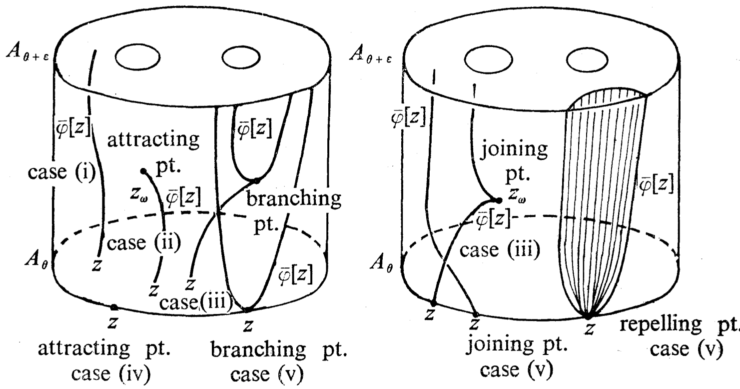


Fig. 21.

Now, for a point z of A_θ and $0\leqq s<\varepsilon$, we define a subset $\Psi_{\theta+s,\theta}(z)$ of $A_{\theta+s}$ (possibly $\Psi_{\theta+s,\theta}(z)=\phi$) by

$$\Psi_{\theta+s,\theta}(z)=\bar{\varphi}[z]\cap A_{\theta+s}.$$

Furthermore, for a subset G of A_θ, we define

$$\Psi_{\theta+s,\theta}(G)=\bigcup_{z\in G}\Psi_{\theta+s,\theta}(z).$$

Then $\Psi_{\theta+s,\theta}\colon \mathscr{P}(A_\theta)\to\mathscr{P}(A_{\theta+s})$ is a correspondence, where $\mathscr{P}(A_\theta)$, $\mathscr{P}(A_{\theta+s})$ denote the families of subsets of A_θ and $A_{\theta+s}$ respectively.

Let θ, θ' be real numbers such that $\theta\leqq\theta'$. We take a sequence of real numbers $\theta_0, \theta_1, \theta_2, \cdots, \theta_n$ so that

$$\theta=\theta_0<\theta_1<\theta_2<\cdots<\theta_n=\theta',$$
$$|\theta_i-\theta_{i-1}|<\varepsilon \qquad i=1, 2, \cdots, n.$$

Recall that θ, θ' and θ_i represent points of S^1 (see Section 1). We define a correspondence

$$\Psi_{\theta',\theta}\colon \mathscr{P}(A_\theta)\to\mathscr{P}(A_{\theta'})$$

by

$$\Psi_{\theta',\theta}(G)=\Psi_{\theta',\theta_{n-1}}\circ\Psi_{\theta_{n-1},\theta_{n-2}}\circ\cdots\circ\Psi_{\theta_1,\theta}(G) \quad (G\subset A_\theta).$$

Then $\Psi_{\theta',\theta}$ is independent of the choice of a sequence as above.

Let us consider the image $\Psi_{\theta',\theta}(\bar{L})$ of a simple curve $\bar{L}$ in $\mathscr{F}|A_\theta$ by $\Psi_{\theta',\theta}$ for $\theta<\theta'$. The bifurcation phenomena occur for $\{\Psi_{\theta',\theta}(\bar{L});\ \theta\leqq\theta'\}$ when θ' varies from θ to ∞. Although plural bifurcations of types (III), (IV) below may occur complexly at the same A_{θ_1}, we restrict here our attention to the case where a bifurcation occurs at one point of A_{θ_1} for simplicity. (Fig. 22).

(9.2) (I) If $\Psi_{\theta',\theta}(\bar{L})$ does not contain any singular point of W for $\theta\leqq\theta'\leqq\theta_2$, then $\Psi_{\theta',\theta}(\bar{L})$ is a simple curve of $\mathscr{F}|A_{\theta'}$ and $\Psi_{\theta',\theta}\colon \bar{L}\to\Psi_{\theta',\theta}(\bar{L})$ is a C^r diffeomorphism for $\theta\leqq\theta'\leqq\theta_2$.

(II) If $\Psi_{\theta',\theta}(\bar{L})$ does not contain any singular point of W for $\theta\leqq\theta'\leqq\theta_2$ except exactly one attracting point $z_1\in\Psi_{\theta_1,\theta}(\bar{L})$ of W, then we have

$$\Psi_{\theta_1,\theta}(\bar{L})=\{z_1\},\qquad \Psi_{\theta',\theta}(\bar{L})=\phi \text{ for } \theta_1<\theta'.$$

(III) If $\Psi_{\theta',\theta}(\bar{L})$ does not contain any singular point of W for $\theta\leqq\theta'\leqq\theta_2$ except exactly one branching point $z_1\in\Psi_{\theta_1,\theta}(\bar{L})$ of W, then $\Psi_{\theta',\theta}(\bar{L})$ consists of two simple curves $\bar{L}'_{\theta'}$, $\bar{L}''_{\theta'}$ of $\mathscr{F}|A_{\theta'}$ for $\theta_1<\theta'\leqq\theta_2$ such that $\bar{L}'_{\theta'}$ and $\bar{L}''_{\theta'}$ have one of two points $\Psi_{\theta',\theta_1}(z_1)$ as one of end points respectively.

(IV) If $\Psi_{\theta',\theta}(\bar{L})$ does not contain any singular point of W for $\theta\leqq\theta'\leqq\theta_2$ except exactly one joining point $z_1\in\Psi_{\theta_1,\theta}(\bar{L})$ of W, then there exist two simple curves $\bar{L}'$, $\bar{L}''$ of $\mathscr{F}|A_{\theta_1}$ such that $\bar{L}'\cap\bar{L}''=\{z_1\}$, $\Psi_{\theta_1,\theta}(\bar{L})=\bar{L}'$ and that the union of $\Psi_{\theta',\theta_1}(\bar{L}')$ and $\Psi_{\theta',\theta_1}(\bar{L}'')$ forms a simple curve of $\mathscr{F}|A_{\theta'}$ for $\theta_1<\theta'\leqq\theta_2$.

(V) If $z_1\in A_\theta$ is a repelling point of W and $\Psi_{\theta',\theta}(z_1)$ does not contain any singular point of W for $\theta\leqq\theta'\leqq\theta_2$ except z_1, then $\Psi_{\theta',\theta}(z_1)$ is a simple curve of $\mathscr{F}|A_{\theta'}$ for $\theta<\theta'\leqq\theta_2$.

Remark 9.3. The bifurcation occurs when $\Phi_{\theta',\theta}(\bar{L})$ meets one of $Q_{k,i}$ and $Q'_{k,i}$. And if $\Psi_{\theta'_0,\theta}(\bar{L})\cap Q_{k,i}\neq\phi$ (resp. $\Psi_{\theta'_0,\theta}(\bar{L})\cap Q'_{k,i}\neq\phi$), then $\Psi_{\theta',\theta}(\bar{L})$ does not meet with $Q_{k,i}$ (resp. $Q'_{k,i}$) for $0<|\theta'-\theta'_0|<\varepsilon'$, where $\varepsilon'>0$ is a sufficiently small real number. Thus the number of bifurcation occurs when θ' varies from θ to θ'' is finite, where $\theta''<\infty$.

The following proposition shows the bifurcation of simple curves in $\mathscr{F}|A_\theta$ in case $\varepsilon(k)=1$ $(k=1, 2, \cdots, m)$.

Proposition 9.4. *Let $\mathscr{F}|A_\theta$, $\bar{K}^{(+)}_{k,i}$, $\bar{K}^{(-)}_{k,i}$ and $\Psi_{\theta',\theta}$ etc. be as above. We assume that $\varepsilon(k)=1$ for $k=1, 2, \cdots, m$.*

(i) *Let $\bar{L}$ be a simple curve in $\mathscr{F}|A_\theta$. Then there exists θ' $(\theta<\theta')$ such that $\Psi_{\theta',\theta}(\bar{L})$ is an attracting point of W if and only if the two end points of $\bar{L}$ belong both to the same* Int $\bar{K}^{(+)}_{k,i}$.

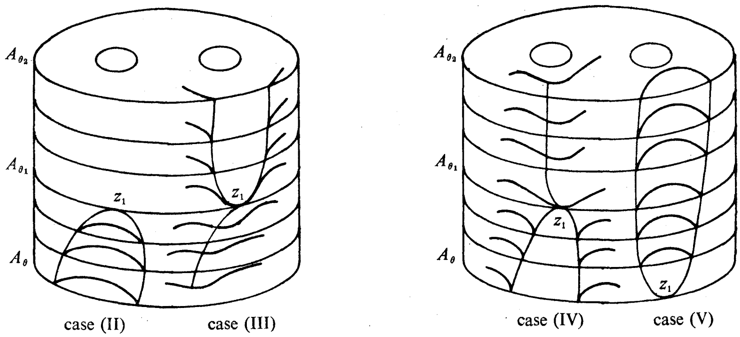


Fig. 22.

(ii) *Let $\bar{L}$ be a simple curve in $\mathscr{F}|A_\theta$. Then $\Psi_{\theta',\theta}(\bar{L})$ consists of a finite number of simple curves in $\mathscr{F}|A_{\theta'}$, say $\bar{L}^{(1)}, \bar{L}^{(2)}, \cdots, \bar{L}^{(r)}$, such that one of the end points of $\bar{L}^{(u)}$ and one of the end points of $\bar{L}^{(u+1)}$ belong to the same $\bar{K}^{(-)}_{k,i}$ for $u=1, 2, \cdots, r-1$.*

Proof. If $\Psi_{\theta'',\theta}(\bar{L})$ contains a branching point for $\theta \leqq \theta'' < \theta'$, then $\Psi_{\theta'',\theta}(\bar{L})$ consists of simple curves of $\mathscr{F}|A_{\theta''}$ such that one of the end points of each simple curve belongs to some $\bar{K}^{(-)}_{k,i}$. This implies that $\Psi_{\theta',\theta}(\bar{L})$ cannot contain an attracting point of W. Therefore the conclusion of (i) is a direct consequence of the definition of the attracting point.

The conclusion of (ii) is a direct consequence of the definition of the branching point.

§ 10. Bifurcation of leaves of foliations of two punctured 2-disk. Proof of Theorem B

Let $\Sigma_0(3)$ denote the 3 punctured 2-sphere, that is, the 2 punctured 2-disk, and let

$$\partial\Sigma_0(3) = S^1_0 \cup S^1_1 \cup S^1_2.$$

We specify an orientation on $\Sigma_0(3)$ and give the boundary orientation on S^1_k $(k=0, 1, 2)$. Recall that S^1 is always oriented. Let α_k be the homology classes of $H_1(S^1 \times S^1_k)$ represented by $S^1 \times \{*\}$ for $k=0, 1, 2$, and let β_0 (resp. β_k $(k=1, 2)$) be the homology class of $H_1(S^1 \times S^1_0)$ (resp. $H_1(S^1 \times S^1_k)$) represented by $-(\{**\} \times S^1_0)$ (resp. $\{**\} \times S^1_k$).

Let $\pi: S^1 \times \Sigma_0(3) \to S^1$ be the projection onto the first factor. Let $T^2_k = S^1 \times S^1_k$ and $\varepsilon(k)=1$ for $k=0, 1, 2$, and let $\mathscr{F}^\varepsilon_\pi$ be the codimension one C^∞ foliation of $S^1 \times \Sigma_0(3)$ as in Section 4.

Let us consider the tangent bundle $\tau(\mathscr{F}^\varepsilon_\pi)$. Since each noncompact leaf L_θ is diffeomorphic to Int $\Sigma_0(3)$, there exists a natural framing of the tangent bundle $\tau(L_\theta)$ of L_θ considering $L_\theta = \text{Int}\,\Sigma_0(3)$ is a subset of $\boldsymbol{R}^2$. Thus $\tau(\mathscr{F}^{\varepsilon}_{\pi})\,|\,(S^1 \times \text{Int}\,\Sigma_0(3))$ is trivial. Furthermore the framing of $\tau(L_\theta)$ induces framings of $\tau(S^1 \times S^1_k)$ $(k=0, 1, 2)$. Thus $\tau(\mathscr{F}^{\varepsilon}_{\pi})$ is trivial, which implies that $\mathscr{F}^{\varepsilon}_{\pi}$ admits transverse 2-plane fields.

The foliation $\mathscr{F}^{\varepsilon}_{\pi}$ admits a transverse codimension one C^∞ foliation $\mathscr{F}'$ as follows (Fig. 23). We divide $\Sigma_0(3)$ into 10 pieces B_i $(i=1, 2, \cdots, 10)$ as in Fig. 23, and we give the plus half Reeb foliation of $S^1 \times D^2_+$ (Section 5) for $S^1 \times B_i$ $(i=1, 2, 3, 4)$ and the *TS* component of $S^1 \times H$ (Section 5) for $S^1 \times B_i$ $(i=5, 6, 7, 8, 9, 10)$. The codimension one C^∞ foliation of $S^1 \times \Sigma_0(3)$ obtained as the union of them is transverse to $\mathscr{F}^{\varepsilon}_{\pi}$. Fig. 31, (a) shows $\bar{\mathscr{F}}\,|\,A_\theta$ for $\bar{\mathscr{F}} = \mathscr{F}^{\varepsilon}_{\pi} \Cap \mathscr{F}'$ and A_θ as in Section 9.

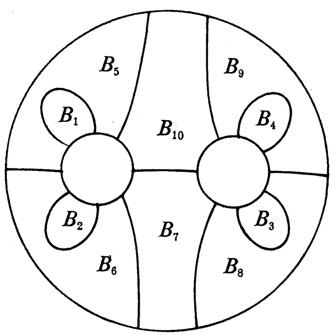


Fig. 23.

Now we have the following theorem:

Theorem 10.1. *Let* $\pi\colon S^1 \times \Sigma_0(3) \to S^1$ *and* $\mathscr{F}^{\varepsilon}_{\pi}$ *be as above, and let* $\mathscr{F}'$ *be a transversely orientable codimension one* C^r *foliation* $(r \geqq 2)$ *of* $S^1 \times \Sigma_0(3)$ *transverse to* $\mathscr{F}^{\varepsilon}_{\pi}$. *Suppose that the one dimensional* C^r *foliation* $\bar{\mathscr{F}} = \mathscr{F}^{\varepsilon}_{\pi} \Cap \mathscr{F}'$ *formed by the intersection of leaves of* $\mathscr{F}^{\varepsilon}_{\pi}$ *and* $\mathscr{F}'$ *satisfies the following assumptions*:

(i) $\bar{\mathscr{F}}\,|\,(S^1 \times S^1_k)$ *has at least one compact leaf for* $k=1, 2$.

(ii) *The homology class of* $H_1(S^1 \times S^1_k)$ *represented by a compact leaf of* $\bar{\mathscr{F}}\,|\,(S^1 \times S^1_k)$ *is* $a_k\alpha_k + b_k\beta_k$, $a_k \neq 0$, *for* $k=1, 2$.

(iii) *It holds that*

$$|a_1|(p_1 - q_1) = |a_2|(p_2 - q_2) = 2,$$

where p_k *and* q_k *denote the numbers of plus Reeb components and minus Reeb components of* $\bar{\mathscr{F}}\,|\,(S^1 \times S^1_k)$ *respectively for* $k=1, 2$.

Then $\bar{\mathscr{F}}\,|\,(S^1 \times S^1_0)$ *has at least one compact leaf and the homology class of* $H_1(S^1 \times S^1_0)$ *represented by a compact leaf of* $\bar{\mathscr{F}}\,|\,(S^1 \times S^1_0)$ *is* $\pm\alpha_0$.

Proof. (Step 1) Let a_0, b_0, p_0 and q_0 be integers defined for $\mathscr{F}|(S^1\times S^1_0)$ as in Section 4. That is, if $\mathscr{F}|(S^1\times S^1_0)$ does not have any compact leaf, then $a_0=b_0=p_0=q_0=0$, and if $\mathscr{F}|(S^1\times S^1_0)$ has a compact leaf, then the homology class of $H_1(S^1\times S^1_0)$ represented by the compact leaf is $\pm(a_0\alpha_0+b_0\beta_0)$ and the numbers of plus Reeb components and minus Reeb components of $\mathscr{F}|(S^1\times S^1_0)$ are p_0 and q_0 respectively. By Propositions 4.2, 5.1 and the assumption (iii), it holds that

$$|a_0|(p_0-q_0)=-6.$$

Therefore, since p_0-q_0 is even as was shown in Proposition 3.4, it follows that $\mathscr{F}|(S^1\times S^1_0)$ has at least one compact leaf and that

$$|a_0|=1,\quad p_0-q_0=-6 \quad \text{or} \quad |a_0|=3,\quad p_0-q_0=-2.$$

Making use of Proposition 5.3, we may assume that $\mathscr{F}|(S^1\times S^1_i)$ $(i=0, 1, 2)$ are normalized. Let A be the closed subset of $S^1\times\Sigma_0(3)$ obtained by cutting down the ends of noncompact leaves of $\mathscr{F}^{\varepsilon}_{\pi}$ as in Section 9, and let $A=\bigcup_{\theta\in S^1}A_\theta$, $A_\theta=A\cap L_\theta$ be as in Section 9, where L_θ $(\theta\in S^1)$ are noncompact leaves of $\mathscr{F}^{\varepsilon}_{\pi}$.

(Step 2) The restriction $\mathscr{F}|A_0$ of $\mathscr{F}$ to A_0 $(0\in S^1)$ is a codimension one C^r foliation with $|a_0|(p_0+q_0)+|a_1|(p_1+q_1)+|a_2|(p_2+q_2)$ singular points in ∂A_0 as was shown in Section 8. (Fig. 31, (a) shows an example of $\mathscr{F}|A_0$.)

Let $\partial A=T'_0\cup T'_1\cup T'_2$ as in Section 9, where T'_k is a torus imbedded in $S^1\times\Sigma_0(3)$ which bounds a collar of $S^1\times S^1_k$ $(k=0, 1, 2)$. Let $\mathscr{F}_k=$ {connected components of $T'_k\cap L'$; $L'\in\mathscr{F}'$} as in Section 9. Then $\mathscr{F}_k$ is a codimension one C^r foliation of T'_k having plus Reeb components $\mathscr{F}_k|\bar{K}^{(+)}_{k,i}$ $(i=1, 2, \cdots, p_k)$ and minus Reeb components $\mathscr{F}_k|\bar{K}^{(-)}_{k,i}$$(i=1, 2, \cdots, q_k)$ for $k=0, 1, 2$, as was observed in Section 9. Let

$$S^1_{k,0}=A_0\cap T'_k \quad (k=0, 1, 2).$$

Then we have

$$\partial A_0=S^1_{0,0}\cup S^1_{1,0}\cup S^1_{2,0}.$$

The intersection $\partial A_0\cap\bar{K}^{(+)}_{k,i}$ (resp. $\partial A_0\cap\bar{K}^{(-)}_{k,i}$) consists of $|a_k|$ connected components, say $\bar{K}^{(+)}_{k,i,j,0}$ $(j=1, 2, \cdots, |a_k|)$ (resp. $\bar{K}^{(-)}_{k,i,j,0}$ $(j=1, 2, \cdots, |a_k|)$).

The numbers of the plus singular points and the minus singular points of $\mathscr{F}|A_0$ in $S^1_{0,0}$ are $|a_0|p_0$ and $|a_0|q_0$ respectively. Since $|a_0|q_0-|a_0|p_0=6$ as was observed in Step 1, there exists a connected arc $\bar{l}_1$ in $S^1_{0,0}$ having minus singular points z_-, z'_- as its end points such that any other singular point does not belong to $\bar{l}_1$ (Fig. 24).

(Step 3) Let $z_1 \in \bar{l}_1 \cap \partial \bar{K}^{(-)}_{0,i,j,0}$, where z_- or z'_- is belonging to $\bar{K}^{(-)}_{0,i,j,0}$. Let $\bar{L}_1$ be the simple curve of $\mathscr{F} \mid A_0$ containing z_1.

If $\bar{L}_1 \cap \partial A_0 = \{z_1\}$, then there exists a simple closed curve in $\mathscr{F} \mid A_0$ by the Poincaré-Bendixson theorem, which is a contradiction as is easily shown by an argument on the Euler number and singular points using the assumption (iii). Thus we have

$$\bar{L}_1 \cap \partial A_0 = \{z_1, z'_1\}.$$

Now if $z'_1 \in S^1_{0,0}$, then $A_0 - \bar{L}_1$ consists of two connected components, say G_1 and G'_1, and one of the following two cases occurs (Fig. 24):

(a) Both $S^1_{1,0}$ and $S^1_{2,0}$ are contained in one of $\bar{G}_1$ and $\bar{G}'_1$ (Fig. 24, (a)).

(b) $S^1_{1,0}$ and $S^1_{2,0}$ are separated by $\bar{L}_1$ (Fig. 24, (b)).

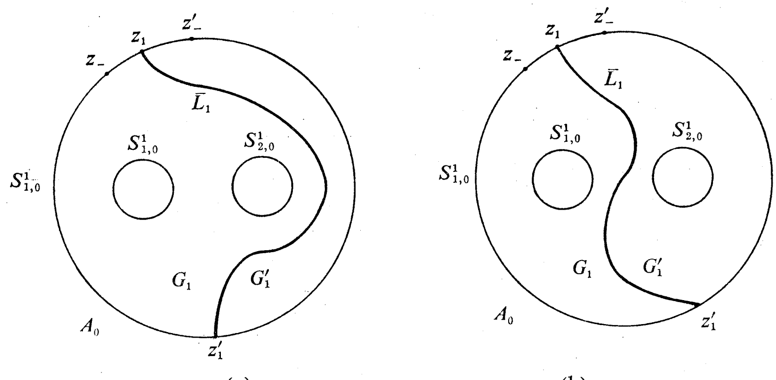


Fig. 24.

Suppose that the case (a) (resp. (b)) above occurs. We let $S^1_{1,0} \cup S^1_{2,0} \subset \bar{G}_1$ (resp. $S^1_{1,0} \subset \bar{G}_1$). Thus $\bar{G}_1$ is homeomorphic to the 2 punctured 2-disk (resp. one punctured 2-disk), and $\bar{G}'_1$ is homeomorphic to the 2-disk (resp. one punctured 2-disk).

Let us consider the number of singular points in $\bar{G}_1 \cap S^1_{0,0}$ and $\bar{G}'_1 \cap S^1_{0,0}$. In case z'_1 is a minus singular point, we understand that z'_1 is a singular point of $\bar{G}_1 \cap S^1_{0,0}$ (resp. $\bar{G}'_1 \cap S^1_{0,0}$) and not a singular point of $\bar{G}'_1 \cap S^1_{0,0}$ (resp. $\bar{G}_1 \cap S^1_{0,0}$) if z'_1 is a cusp of $\partial \bar{G}'_1$ (resp. $\partial \bar{G}_1$). Let $\bar{p}_1$ and $\bar{q}_1$ (resp. $\bar{p}'_1$ and $\bar{q}'_1$) denote the numbers of the plus singular points and the minus singular points in $\bar{G}_1 \cap S^1_{0,0}$ (resp. $\bar{G}'_1 \cap S^1_{0,0}$).

Let $\bar{G}_1 \cup \bar{G}_1$ (resp. $\bar{G}'_1 \cup \bar{G}'_1$) be the double of $\bar{G}_1$ (resp. $\bar{G}'_1$) obtained from two copies of $\bar{G}_1$ (resp. $\bar{G}'_1$) by identifying two copies of $\bar{G}_1 \cap \partial A_0$ (resp. $\bar{G}'_1 \cap \partial A_0$). The double of $\mathscr{F} \mid \bar{G}_1$ (resp. $\mathscr{F} \mid \bar{G}'_1$) defines a codimension one C^0 foliation of $\bar{G}_1 \cup \bar{G}_1$ (resp. $\bar{G}'_1 \cup \bar{G}'_1$) with $\bar{p}_1 + p_1 + p_2$ or $\bar{p}_1 + p_1$ (resp. $\bar{p}'_1$ or $\bar{p}'_1 + p_2$) plus singular points and $\bar{q}_1 + q_1 + q_2$ or $\bar{q}_1 + q_1$ (resp. $\bar{q}'_1$

or $\bar{q}_1'+q_2$) minus singular points according to the case (a) or (b). Here, in case z_1' is a minus singular point, we understand that z_1' is a minus singular point of $\bar{G}_1\cup\bar{G}_1$ (resp. $\bar{G}_1'\cup\bar{G}_1'$) and not a minus singular point of $\bar{G}_1'\cup\bar{G}_1'$ (resp. $\bar{G}_1\cup\bar{G}_1$) if z_1' is a cusp of $\partial\bar{G}_1'$ (resp. $\partial\bar{G}_1$).

$\bar{G}_1\cup\bar{G}_1$ is homeomorphic to $\Sigma_2(1)$ (resp. $\Sigma_1(1)$) and $\bar{G}_1'\cup\bar{G}_1'$ is homeomorphic to D^2 (resp. $\Sigma_1(1)$) in the case (a) (resp. (b)). Thus, by the assumption (iii), we have

$$\bar{p}_1-\bar{q}_1=-7, \qquad \bar{p}_1'-\bar{q}_1'=1$$
$$(\text{resp. } \bar{p}_1-\bar{q}_1=-3,\ \bar{p}_1'-\bar{q}_1'=-3).$$

It follows from $\bar{p}_1'-\bar{q}_1'=1$ (resp. $\bar{p}_1'-\bar{q}_1'=-3$) that $G_1'\cap S_{0,0}^1$ must contain at least one plus singular point (resp. three minus singular points). This implies that $\bar{L}$ separates z_- and z_-'. Thus we have

$$\bar{q}_1<|a_0|q_0.$$

By the equation $\bar{p}_1-\bar{q}_1=-7$ (resp. $\bar{p}_1-\bar{q}_1=-3$), there exists a connected arc $\bar{l}_2$ in $G_1\cap S_{0,0}^1$ having minus singular points as its end points such that any other singular point does not belong to $\bar{l}_2$. Let $z_2\in \bar{l}_2\cap\partial\bar{K}_{0,i,j,0}^{(-)}$ for some $\bar{K}_{0,i,j,0}^{(-)}$, and let $\bar{L}_2$ be the simple curve of $\mathscr{F}|A_0$ containing z_2. Then, by the same reason as above, we have

$$\bar{L}_2\cap(S_{0,0}^1\cap\bar{G}_1)=\{z_2, z_2'\}.$$

If $z_2'\in S_{0,0}^1$, then $\bar{L}_2$ divides G_1 into two connected components, say G_2 and G_2'. We let $S_{1,0}^1\subset\bar{G}_2$.

Let $\bar{p}_2$ and $\bar{q}_2$ (resp. $\bar{p}_2'$ and $\bar{q}_2'$) denote the numbers of plus singular points and minus singular points in $\bar{G}_2\cap S_{0,0}^1$ (resp. $\bar{G}_2'\cap S_{0,0}^1$). Then, by the same argument as above, we have

$$\bar{p}_2-\bar{q}_2\leqq-4, \qquad \bar{q}_2<\bar{q}_1.$$

Furthermore, the number of connected components of $G_2\cap S_{0,0}^1$ is at most two. Thus there exists a connected arc $\bar{l}_3$ in $G_2\cap S_{0,0}^1$ having minus singular points as its end points such that any other singular point does not belong to $\bar{l}_3$. We can take $z_3\in\bar{l}_3\cap\partial\bar{K}_{0,i,j,0}^{(-)}$ for some $\bar{K}_{0,i,j,0}^{(-)}$ and repeat the process as above. Therefore, we can finally find a point z_0 of $S_{0,0}^1$ with the following property:

(10.2) Let $\bar{L}_0$ be the simple curve of $\mathscr{F}|A_0$ containing z_0. Then another end point of $\bar{L}_0$ belongs to $S_{1,0}^1$ or $S_{2,0}^1$.

(Step 4) In the following we assume that the homology class of $H_1(S^1\times S_0^1)$ represented by a compact leaf of $\mathscr{F}|(S^1\times S_0^1)$ is $a_0\alpha_0+b_0\beta_0$, $b_0\neq 0$, and will show that the assumption $b_0\neq 0$ yields a contradiction.

Now suppose that $|a_0|=1$, $p_0-q_0=-6$. Let z_0 be a point of $S_{0,0}^1$

satisfying the condition (10.2), and let $\bar{L}_0$ denote the simple curve of $\mathscr{F}\,|\,A_0$ containing z_0.

Suppose that another end point z_0' of $\bar{L}_0$ belongs to $S^1_{1,0}$. Let us consider the image of $\bar{L}_0$ by Ψ_s when parameter s varies from 0 to 1 as in Section 9. Since $|a_0|=1$, we have

$$\Psi_1(z_0)=z_0.$$

The assumption $b_0\neq 0$ implies that bifurcation occurs for Ψ_s $(0\leqq s\leqq 1)$. By the uniqueness of the simple curve in $\mathscr{F}\,|\,A_0=\mathscr{F}\,|\,A_1$ containing z_0, it follows that $\Psi_1(\bar{L}_0)$ contains $\bar{L}_0$, and thus, by Proposition 9.4, the point z_0' belongs to one of $\bar{K}^{(-)}_{1,i,j,0}$, say $z_0'\in K^{(-)}_{1,i_0,j_0,0}$, and $\Psi_1(\bar{L}_0)$ consists of l simple curves in $\mathscr{F}\,|\,A_0=\mathscr{F}\,|\,A_1$, say

$$\bar{L}_0,\ \bar{L}_1,\ \bar{L}_2,\ \cdots,\ \bar{L}_{l-1}$$

such that, letting z_s and z_s' be end points of $\bar{L}_s$ $(s=0, 1, 2, \cdots, l-1)$, two points z_{s-1}' and z_s belong to the interior of one of $\bar{K}^{(-)}_{k,i,j,0}$ for $s=1, 2, \cdots, l-1$. (Fig. 25).

Let C_s be the simple arc connecting z_{s-1}' and z_s in such a $\bar{K}^{(-)}_{k,i,j,0}$. Then the union of $\bar{L}_0$, C_1, $\bar{L}_1$, $\cdots$, C_{l-1}, $\bar{L}_{l-1}$ forms a C^0 curve C in $A_0=A_1$ connecting z_0 and $\Psi_1(z_0')$ (Fig. 25).

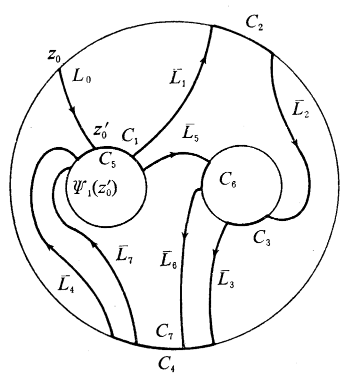


Fig. 25.

Let $\{C_{s_1}, C_{s_2}, \cdots, C_{s_u}\}$ be the subset of $\{C_s;\, s=1, 2, \cdots, l-1\}$ such that

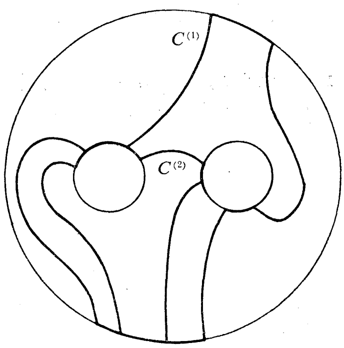
$$C_{s_i}\subset \bar{K}^{(-)}_{1,i_0,j_0,0}, \qquad i=1, 2, \cdots, u,$$
$$1=s_1<s_2<\cdots<s_u,$$

and let $C^{(i)}$ (resp. $C^{(u)}$) denote a closed C^0 curve in A_0 obtained as the union of $\bar{L}_{s_i}$, C_{s_i+1}, $\bar{L}_{s_i+1}$, $\cdots$, $\bar{L}_{s_{i+1}-1}$ and an arc in $\bar{K}^{(-)}_{1,i_0,j_0,0}$ with end points z_{s_i}, $z'_{s_{i+1}-1}$ for $i=1, 2, \cdots, u-1$ (resp. as the union of $\bar{L}_{s_u}$, $\bar{C}_{s_u+1}$, $\cdots$, $\bar{L}_{l-1}$ and an arc in $\bar{K}^{(-)}_{1,i_0,j_0,0}$ with end points z_{s_u} and $z'_0=z'_{l-1}$). (Fig. 26, (a)).

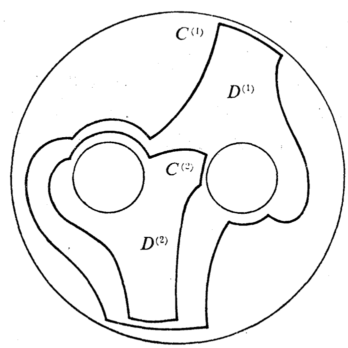
By pushing each C_s and arc contained in $\bar{K}^{(-)}_{1,i_0,j_0,0}$ slightly into the interior of A_0, we can make $C^{(i)}$ a simple closed C^0 curve in Int A_0 for $i=1, 2, \cdots, u$ (Fig. 26, (b)). By straightening the corner, we may suppose that each $C^{(i)}$ is a simple closed C^r curve.

Let us suppose that A_0 is a subset of $\boldsymbol{R}^2$ in the natural manner, and let $D^{(i)}$ denote the closed set of $\boldsymbol{R}^2$ bounded by $C^{(i)}$. $D^{(i)}$ is diffeomorphic to the 2-disk (Fig. 26, (b)). Then, since $b_0\neq 0$, at least one of $D^{(i)}$ $(i=1, 2, \cdots, u)$ contains $S^1_{2,0}$, say $D^{(i')}\supset S^1_{2,0}$.

Let Y_0 be a non-singular C^r vector field on A_0 whose orbits are $\mathscr{F}|A_0$. Then, by changing Y_0 in a small neighborhood of $C^{(i')}$ near C_s, we obtain a non-singular C^r vector field Y'_0 such that $C^{(i')}$ is a closed orbit of Y'_0 (Fig. 27).



(a)



(b)

Fig. 26.

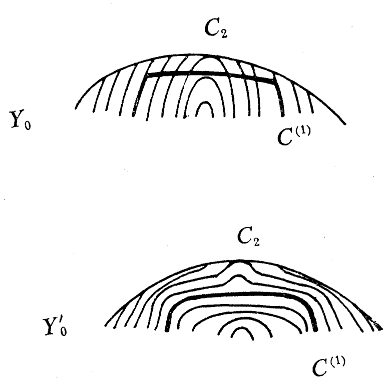


Fig. 27.

Let $\Sigma = D^{(i')} \cap A_0$ and let $\Sigma \cup \Sigma$ be the double of Σ obtained from two copies of Σ by identifying their boundaries. Then $\Sigma \cup \Sigma$ is diffeomorphic to the torus or the orientable closed surface Σ_2 of genus 2 according to $D^{(i')} \not\supset S^1_{1,0}$ or $D^{(i')} \supset S^1_{1,0}$. On the other hand the double of $Y'_0 | \Sigma$ defines a C^0 vector field with p_1 (resp. $p_1 + p_2$) plus singular points and q_1 (resp. $q_1 + q_2$) minus singular points if $D^{(i')} \not\supset S^1_{1,0}$ (resp. $D^{(i')} \supset S^1_{1,0}$). This is a contradiction.

In case $|a_0| = 3$, $p_0 - q_0 = -2$, it can be proved by considering Ψ_s $(0 \leqq s \leqq 3)$ that a contradiction also occurs.

The above results imply that the assumption $b_0 \neq 0$ yields a contradiction. Thus this theorem is proved.

Now we prove Theorem B in Section 0. Let $\mathscr{F}_0$ be as in Theorem B, that is, $\mathscr{F}_0$ is the union of the codimension one C^∞ foliations $\mathscr{F}^\varepsilon_\pi$ of $S^1 \times \Sigma_0(3)$, $\mathscr{F}^{(+)}_R$ of $S^1 \times D^2_1$, $\mathscr{F}^{(+)}_R$ of $S^1 \times D^2_2$ and $\mathscr{F}^{(+)}_R$ of $D^2 \times S^1$, and let $\mathscr{F}'$ be a codimension one C^r foliation $(r \geqq 2)$ of S^3 transverse to $\mathscr{F}_0$, where $\Sigma_0(3) = D^2 - \mathrm{Int}\, D^2_1 - \mathrm{Int}\, D^2_2$.

Let $\bar{\mathscr{F}} = \mathscr{F}_0 \Cap \mathscr{F}'$. Consider $\mathscr{F}' | \Sigma_0(3)$ and $\bar{\mathscr{F}} | \Sigma_0(3)$. Then, by Proposition 3.6, $\bar{\mathscr{F}} | (S^1 \times \partial D^2_1)$ and $\bar{\mathscr{F}} | (S^1 \times \partial D^2_2)$ have compact leaves such that homology classes represented by the compact leaves are $a_1\alpha_1 + b_1\beta_1$ $(a_1 \neq 0)$ and $a_2\alpha_2 + b_2\beta_2$ $(a_2 \neq 0)$ respectively, and $|a_1|(p_1 - q_1) = 2$, $|a_2|(p_2 - q_2) = 2$. Thus the assumptions of Theorem 10.1 are satisfied for $\mathscr{F}^\varepsilon_\pi$. Therefore, by Theorem 10.1, $\bar{\mathscr{F}} | (S^1 \times \partial D^2)$ has a compact leaf such that the homology class represented by it is $\pm \alpha_0$.

On the other hand, by the consideration on $\bar{\mathscr{F}} | (D^2 \times S^1)$, the homology class represented by a compact leaf of $\bar{\mathscr{F}} | (\partial D^2 \times S^1)$ is $\pm \beta_0 + a'_0\alpha_0$. This is a contradiction. Thus Theorem B is proved.

The most results on existence problem in Nishimori [2] can be proved by the arguments as in Sections 9 and 10.

§ 11. Proof of Theorem D

Let $h_n \colon S^1 \times D^2 \to S^1 \times D^2$ denote the C^∞ diffeomorphism defined by

$$h_n(e^{2\pi i x}, re^{2\pi i y}) = (e^{2\pi i x}, re^{2\pi i (y + nx)}), \quad (0 \leqq x \leqq 1, 0 \leqq y \leqq 1).$$

Then, for the 2 punctured 2-sphere $\Sigma_0(3) = D^2 - \mathrm{Int}\, D^2_1 - \mathrm{Int}\, D^2_2$, we have a decomposition of the solid torus $S^1 \times D^2$ as follows:

$$S^1 \times D^2 = h_n(S^1 \times \Sigma_0(3)) \cup h_n(S^1 \times D^2_1) \cup h_n(S^1 \times D^2_2).$$

For the codimension one C^∞ foliation $\mathscr{F}^\varepsilon_\pi$ of $S^1 \times \Sigma_0(3)$ as in Section 10, we denote $\mathscr{F}^{(n)}_0 = \{h_n(L);\, L \in \mathscr{F}^\varepsilon_\pi\}$. Then the union of $\mathscr{F}^{(n)}_0$ and two plus Reeb foliations $\mathscr{F}^{(+)}_R$ of $h_n(S^1 \times D^2_2)$ and $h_n(S^1 \times D^2_2)$ determines a codimension one C^∞ foliation of $S^1 \times D^2$, which is denoted by $\mathscr{F}_1^{(n)}$. Furthermore, let $\mathscr{F}^{(n)}$ denote the codimension one C^∞ foliation of $S^1 \times D^2 = (S^1 \times \Sigma_0(3))' \cup (S^1 \times D_1^2)' \cup (S^1 \times D_2^2)'$ consisting of codimension one C^∞ foliations $\mathscr{F}_\pi^\varepsilon$ of $(S^1 \times \Sigma_0(3))'$ as in Section 10, $\mathscr{F}_1^{(n)}$ of $(S^1 \times D_1^2)'$ as above and the plus Reeb foliation $\mathscr{F}_R^{(+)}$ of $(S^1 \times D_2^2)'$ (Fig. 28).

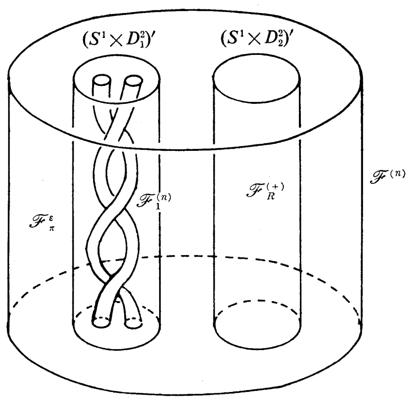


Fig. 28.

We have the following proposition:

Proposition 11.1. *Let $\mathscr{F}^{(n)}$ be the codimension one C^∞ foliation of the solid torus $S^1 \times D^2$ as above. Then, if $n \neq 0$, there does not exist any codimension one C^r foliation ($r \geq 2$) of $S^1 \times D^2$ transverse to $\mathscr{F}^{(n)}$.*

Proof. Suppose that $n \neq 0$ and there exists a transversely orientable codimension one C^r foliation $\mathscr{F}'$ of $S^1 \times D^2$ transverse to $\mathscr{F}^{(n)}$. Denote $\bar{\mathscr{F}} = \mathscr{F}^{(n)} \pitchfork \mathscr{F}'$. Then $\mathscr{F}' | (S^1 \times D_1^2)'$ is a codimension one C^r foliation transverse to $\mathscr{F}_1^{(n)}$, and thus, $\mathscr{F}'' = \{h_n^{-1}(L'); L' \in \mathscr{F}' | (S^1 \times D_1^2)'\}$ is a codimension one C^r foliation of $S^1 \times D^2$ transverse to the codimension one C^r foliation of $S^1 \times D^2$ consisting of codimension one foliations $\mathscr{F}_\pi^\varepsilon$ of $\Sigma_0(3)$ and two plus Reeb foliations of $S^1 \times D_1^2$ and $S^1 \times D_2^2$. Thus it follows from Theorem 10.1 (cf. Proof of Theorem B) that $\mathscr{F}_\pi^\varepsilon \pitchfork \mathscr{F}'' | (S^1 \times D^2)$ has at least one compact leaf and that the homology class of $H_1(S^1 \times \partial D^2)$ represented by a compact leaf is $\pm\alpha$, the homology class represented by a longitude. This implies that $\bar{\mathscr{F}} | \partial(S^1 \times D_1^2)'$ has at least one compact leaf and the homology class represented by a compact leaf is $\pm\alpha_1 \pm n\beta_1$, where α_1 and β_1 are homology classes as in Section 10.

Next let us consider $\bar{\mathscr{F}} | (S^1 \times \Sigma_0(3))'$. Let α_k, β_k, a_k, b_k ($k = 0, 1, 2$) be as in Section 10, and let p_k and q_k be the numbers of plus Reeb components and minus Reeb components of $\bar{\mathscr{F}} | (S^1 \times S_k^1)'$ for $k = 0, 1, 2$.

Then, as was used in Theorem 10.1, we have

$$|a_1|(p_1-q_1)=-6, \qquad |a_2|(p_2-q_2)=2.$$

Therefore, by Proposition 4.2, we have

$$|a_0|(p_0-q_0)=2.$$

Thus the assumptions (i), (ii), (iii) of Theorem 10.1 are satisfied for $(S^1\times S^1_0)'$ and $(S^1\times S^1_2)'$, and it follows from Theorem 10.1 that the homology class represented by a compact leaf of $\mathscr{F}\,|\,\partial(S^1\times D^2_1)'$ should be $\pm\alpha_1$. This is a contradiction. In case $\mathscr{F}'$ is not transversely orientable, by considering the double covering of $S^1\times D^2$, the same arguments work. Thus this proposition is proved.

Now we prove Theorem D in Section 0. Let M be a 3-dimensional C^∞ manifold. Then, for an imbedding $g\colon S^1\times D^2\to M$, there exists a codimension one C^∞ foliation $\hat{\mathscr{F}}$ of $M-\operatorname{Int} g(S^1\times D^2)$ with $g(S^1\times\partial D^2)$ as a compact leaf. Let $\mathscr{F}^{(n)}$ be the codimension one C^∞ foliation of $S^1\times D^2$ as in Proposition 11.1, and let $\mathscr{F}$ be a codimension one C^∞ foliation of M consisting of $\hat{\mathscr{F}}$ and $g_*\mathscr{F}^{(n)}=\{g(L);\, L\in\mathscr{F}^{(n)}\}$ $(n\neq 0)$. Then $\mathscr{F}$ does not admit any transverse codimension one C^r foliation $(r\geqq 2)$ by Proposition 11.1. Thus Theorem D is proved.

§ 12. Proof of Theorem A

Let k be a non-trivial fibred knot in the 3-sphere and let $N(k)$ be a tubular neighborhood of k. Let $\pi\colon E_1\to S^1$ be a C^∞ fibering over the circle with $\Sigma_g(1)$ as fibre, where $E_1=S^3-\operatorname{Int} N(k)$ and $\Sigma_g(1)$ is the one punctured surface of genus g $(g\geqq 1)$. Thus we have

$$S^3=N(k)\cup E_1, \qquad N(k)=S^1\times D^2.$$

We specify orientations on $S^1\times\{*\}$ and $\{**\}\times\partial D^2$ for $*\in\partial D^2$, $**\in S^1$. By the natural identification of $\{**\}\times\partial D^2$ with the base space S^1 of π, an orientation on the base space S^1 is specified. Let α and β be generators of $H_1(\partial N(k))$ represented by the longitude and the meridian with orientations as above.

Let $\mathscr{F}$ denote the codimension one C^∞ foliation of S^3 which is the union of the plus Reeb foliation $\mathscr{F}^{(+)}_R$ of $N(k)=S^1\times D^2$ and the codimension one C^∞ foliation $\mathscr{F}_\pi$ of E_1 as in Section 4:

$$\mathscr{F}=\mathscr{F}^{(+)}_R\cup\mathscr{F}_\pi.$$

Thus $\mathscr{F}$ has a unique compact leaf $\partial N(k)=T^2$

Suppose that there exists a codimension one C^r foliation $\mathscr{F}'$ of S^3 transverse to $\mathscr{F}$ $(r\geqq 2)$. Obviously $\mathscr{F}'$ is transversely orientable. As is well-known, $\mathscr{F}'$ has a Reeb component by Novikov's result [4], that is, there exists a subset N' of S^3 diffeomorphic to $S^1\times D^2$ such that $\mathscr{F}'|N'$ is a Reeb foliation of N'. We specify an orientation of $S^1\times\{*'\}$ $(*'\in\partial D^2)$ so that $\mathscr{F}'|N'$ is a plus C^r Reeb foliation of N'. Let α' and β' be generators of $H_1(\partial N')$ represented by $S^1\times\{*'\}$ and $\{**'\}\times\partial D^2$.

We let

$$\bar{\mathscr{F}}=\mathscr{F}\cap\mathscr{F}'.$$

Then, by Proposition 5.1, the following lemma holds:

Lemma 12.1. (I) (i) *$\bar{\mathscr{F}}|\partial N(k)$ has a compact leaf.*

(ii) *The homology class of $H_1(\partial N(k))$ represented by a compact leaf of $\bar{\mathscr{F}}|\partial N(k)$ is $\pm(\alpha+b\beta)$, where $|b|=2g-1$.*

(iii) *There exist closed subsets $K_1, K_2, \cdots, K_{q+2}, K_1', K_2', \cdots, K_q'$ of $\partial N(k)=\partial E_1$ such that $\bar{\mathscr{F}}|K_i$ is a plus Reeb component with respect to $\mathscr{F}_R^{(+)}$ and a minus Reeb component with respect to $\mathscr{F}_\pi$ for $i=1, 2, \cdots, q+2$, and that $\bar{\mathscr{F}}|K_i'$ is a minus Reeb component with respect to $\mathscr{F}_R^{(+)}$ and a plus Reeb component with respect to $\mathscr{F}_\pi$ for $i=1, 2, \cdots, q$.*

(II) (i) *$\bar{\mathscr{F}}|\partial N'$ has a compact leaf.*

(ii) *The homology class of $H_1(\partial N')$ represented by a compact leaf of $\bar{\mathscr{F}}|\partial N'$ is $\pm(\alpha'+b'\beta)$.*

(iii) *$\bar{\mathscr{F}}|\partial N'$ has plus Reeb components $\bar{\mathscr{F}}|K_1^{(+)}, \bar{\mathscr{F}}|K_2^{(+)}, \cdots, \bar{\mathscr{F}}|K_{q'+2}^{(+)}$ and minus Reeb components $\bar{\mathscr{F}}|K_1^{(-)}, \bar{\mathscr{F}}|K_2^{(-)}, \cdots, \bar{\mathscr{F}}|K_{q'}^{(-)}$ with respect to $\mathscr{F}'|N'$.*

Proof. (I) (i), (II) (i), (ii) and (iii) are direct consequences of Proposition 5.1. The homology class of $H_1(\partial N(k))$ represented by a compact leaf of $\bar{\mathscr{F}}|\partial N(k)$ is $\pm\alpha+b\beta$ by Proposition 5.1. Let p and q (resp. $\bar{p}$ and $\bar{q}$) be the numbers of plus Reeb components and minus Reeb components of $\bar{\mathscr{F}}|\partial N(k)=\bar{\mathscr{F}}|\partial E_1$ with respect to $\mathscr{F}_R^{(+)}$ (resp. $\mathscr{F}_\pi$). Then, by Proposition 5.1, we have

$$p-q=2, \qquad |b|(\bar{p}-\bar{q})=2(1-2g).$$

The conclusions of (I) (ii) and (iii) follow from these equations. Thus this lemma is proved.

Lemma 12.2. *$N(k)\cap N'\neq\phi$, $E_1\cap N'\neq\phi$.*

Proof. If $E_1\cap N'=\phi$, then N' is contained in Int $N(k)$. This implies that each leaf of $\bar{\mathscr{F}}|\partial N'$ is compact, which contradicts Lemma

12.1, (II), (iii). If $N(k)\cap N'=\phi$, then N' is contained in Int E_1. This implies that each leaf of $\mathscr{F}\mid\partial N'$ is compact, which contradicts also Lemma 12.1, (II), (iii). Thus this lemma is proved.

Lemma 12.3. *Let K be a connected component of $\partial N(k)\cap N'$. Then $\mathscr{F}\mid K$ is a plus or a minus Reeb component with respect to $\mathscr{F}_R^{(+)}$.*

Proof. Obviously ∂K consists of two compact leaves of $\mathscr{F}\mid\partial N(k)$, say $\bar{L}$ and $\bar{L}'$. Each leaf of $\mathscr{F}\mid\text{Int } K$ is an intersection of $\partial N(k)$ and a noncompact leaf of $\mathscr{F}'\mid\text{Int } N'$. If there exists a compact leaf in $\mathscr{F}'\mid\text{Int } N'$, then there exists a compact leaf in $\mathscr{F}\mid L'$ for a noncompact leaf L' of $\mathscr{F}'\mid\text{Int } N'$. Since L' is diffeomorphic to $\boldsymbol{R}^2$, this is a contradiction. Therefore there does not exist any compact leaf in $\mathscr{F}\mid\text{Int } K$. Furthermore, by considering the holonomy of $\mathscr{F}'$ with respect to $\partial N'$, it follows that the holonomy of $\mathscr{F}\mid K$ with respect to $\bar{L}$ and $\bar{L}'$ having orientations induced from $S^1\times\{*\}$ are both contracting or expanding. Thus this lemma is proved.

Lemma 12.4. *Let B be a connected component of $N(k)\cap N'$. Then $B\cap\partial N(k)$ is connected and $\mathscr{F}\mid(B\cap\partial N(k))$ is a plus Reeb component with respect to $\mathscr{F}_R^{(+)}$.*

Proof. Suppose that $B\cap\partial N(k)$ consists of connected components $\bar{K}_1, \bar{K}_2, \cdots, \bar{K}_m$. By considering the holonomy of $\mathscr{F}'$ with respect to $\partial N'\cap N(k)$, it follows that all of $\mathscr{F}\mid\bar{K}_1, \mathscr{F}\mid\bar{K}_2, \cdots, \mathscr{F}\mid\bar{K}_m$ are plus Reeb components or minus Reeb components with respect to $\mathscr{F}_R^{(+)}$.

Let L be a noncompact leaf of $\mathscr{F}_R^{(+)}$. Then $L\cap B$ has m ends corresponding to $\bar{K}_i$ $(i=1, 2, \cdots, m)$ and $\mathscr{F}\mid(L\cap B)$ is a codimension one C^r foliation of $L\cap B$. Let Σ be a polygon obtained from $L\cap B$ by cutting down the ends of $L\cap B$ as in the proofs of Propositions 3.6 and 4.2. Then $\mathscr{F}\mid\Sigma$ is a codimension one C^r foliation of Σ with m singular points in $\partial\Sigma$. Let $\Sigma\cup\Sigma$ be the double of Σ obtained from two copies of Σ by identifying the two copies of the closure of $\partial\Sigma-(\partial\Sigma\cap\partial B)$. The double of $\mathscr{F}\mid\Sigma$ determines a codimension one C^r foliation of the double $\Sigma\cup\Sigma$ with m singular points. The indices of these singular points are all 1 or all -1 according to $\mathscr{F}\mid\bar{K}_i$ $(i=1, 2, \cdots, m)$ are plus Reeb components or minus Reeb components. $\Sigma\cup\Sigma$ is the m punctured 2-sphere. Therefore, by considering the Euler number of $\Sigma\cup\Sigma$, we have

$$2-m=\pm m.$$

This implies that $m=1$ and $\mathscr{F}\mid\bar{K}_1$ is a plus Reeb component. Thus this lemma is proved.

Let us consider $N' \cap E_1$. By Lemma 12.4, $N' \cap E_1$ is connected. Let $\hat{K}_1, \hat{K}_2, \cdots, \hat{K}_n$ be connected components of $N' \cap \partial E_1$. Then, by Lemma 12.1, (I), (iii) and Lemma 12.4, $\bar{\mathscr{F}} | \hat{K}_i$ $(i=1, 2, \cdots, n)$ are minus Reeb components with respect to $\mathscr{F}_\pi$.

Let L' be a noncompact leaf of $\mathscr{F}_\pi$. Then $L' \cap N'$ has n ends corresponding to $\hat{K}_i$ $(i=1, 2, \cdots, n)$ and $\bar{\mathscr{F}} | (L' \cap N')$ is a codimension one C^r foliation of $L' \cap N'$. Let Σ' be a polygon obtained from $L' \cap N'$ by cutting down the ends of $L' \cap N'$ as in the proofs of Propositions 3.5 and 4.2. Then $\bar{\mathscr{F}} | \Sigma'$ is a codimension one C^r foliation of Σ' with n singular points in $\partial \Sigma'$. Let $\Sigma' \cup \Sigma'$ be the double of Σ' obtained from two copies of Σ' by identifying the two copies of the closure of $\partial \Sigma' - (\partial \Sigma' \cap \partial N')$. The double of $\bar{\mathscr{F}} | \Sigma'$ determines a codimension one C^r foliation of the double $\Sigma' \cup \Sigma'$ with n singular points. The indices of these singular points are all -1, since $\bar{\mathscr{F}} | \hat{K}_i$ $(i=1, 2, \cdots, n)$ are minus Reeb components. $\Sigma' \cup \Sigma'$ is an n punctured surface of genus $2g'$ for some $g' \leqq g$. Thus, by considering the Euler number of $\Sigma' \cup \Sigma'$, we have

$$2-4g'-n=-n.$$

This is a contradiction. Therefore there does not exist a codimension one C^r foliation transverse to $\mathscr{F}$. Thus Theorem A in Section 0 is proved.

Remark 12.5. In case k is a trefoil knot, the homomorphism ϕ_* induced from the monodromy map of the fibering $\pi: E_1 \to S^1$ is given by the matrix $\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$. Thus Theorem C implies Theorem A in case k is a trefoil knot. On the contrary, in case k is a figure eight knot, the homomorphism ϕ_* is given by $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$. Thus, by Theorem C, there exists a transversely orientable codimension one C^r foliation of E_1 transverse to $\mathscr{F}_\pi$ in this case. However, Theorem A shows that this codimension one foliation cannot be extended to a codimension one C^r foliation of S^3 transverse to $\mathscr{F}$ in Theorem A.

§ 13. Some examples of vector fields on foliations

In this section we study vector fields on codimension one C^r foliations of 3-dimensional C^∞ manifolds which are not transversely integrable $(r \geqq 1)$.

Let E be a compact connected orientable 3-dimensional C^∞ manifold with boundary and let $\pi: E \to S^1$ be a C^∞ fibering over S^1 with fibre $\Sigma_g(m)$ $(m \geqq 1)$. We fix a Riemannian metric on E. Let $\mathscr{F}_\pi^\varepsilon$ be a codimension one C^∞ foliation of E as in Section 4 and let L be a noncompact leaf of $\mathscr{F}^{\varepsilon}_{\pi}$.

Let $\partial E=\bigcup_{k=1}^{s} T_k^2$ as in Section 4 and let $c^{(k)}: T^2\times I\to E$, $c^{(k)}(T^2\times\{0\})\subset\partial E$ be a sufficiently thin collar of T_k^2 in E such that $\pi(c^{(k)}(\{y\}\times I))=\pi(c^{(k)}(y, 0))$ for $k=1, 2, \cdots, s$. Let $p^{(k)}: c^{(k)}(T^2\times[0, 1))\to T^2$ be the projection defined by $p^{(k)}(y, t)=y$. Let x be a point of T_k^2 and let $x_1, x_2, \cdots, x_n, \cdots$ be points of $c^{(k)}(\{x\}\times[0, 1))\cap L$ such that

$$\lim_{n\to\infty} x_n=x.$$

Obviously this implies that

$$\lim_{n\to\infty} p^{(k)}(x_n)=x,$$

and that there exist sufficiently small neighborhoods U_x of x in T_k^2 and U_{x_n} of x_n in L $(n=1, 2, \cdots)$ such that $p^{(k)}|U_{x_n}: U_{x_n}\to U_x$ is a C^∞ diffeomorphism. A C^r tangent vector field Y on L is said to be *convergent* if, for any point $x\in T_k^2$ $(k=1, 2, \cdots, s)$ and any choice of x_n and U_{x_n} $(n=1, 2, \cdots)$, the sequence of vector fields $dp^{(k)}(Y|U_{x_n})$ $(n=1, 2, \cdots)$ converges to a C^r vector fields of U_x in the C^r topology. A convergent C^r tangent vector field Y on L is said to be *non-singular* if there exists a positive real number $\varepsilon>0$ such that $|Y(x)|>\varepsilon$ $(x\in L)$.

Let Y be a convergent C^r tangent vector field on L. Then we can define a vector field $\bar{Y}$ on ∂E by

$$\bar{Y}(z)=\lim_{n\to\infty} dp^{(k)}(Y(x_n)) \qquad (z\in T_k^2),$$

where $\{x_n\}$ is a sequence of points of L such that $\lim_{n\to\infty} x_n=z$. As is easily verified, $\bar{Y}$ is a C^r tangent vector field on ∂E. The vector field $\bar{Y}$ on ∂E defined as above is called the *limit vector field* of Y and is denoted by $\lim Y$. In case Y is non-singular convergent, then $\lim Y$ is non-singular.

We have the following propositions:

Proposition 13.1. *Let $\mathscr{F}_R^{(+)}$ be the plus Reeb foliation of the solid torus $S^1\times D^2$ and let L be a noncompact leaf of $\mathscr{F}_R^{(+)}$. Suppose that Y is a non-singular convergent C^r tangent vector field on L $(r\geqq 1)$. Then the limit vector field* $\lim Y$ *on $S^1\times\partial D^2$ has the following properties:*

(i) $\lim Y$ *has at least one closed orbit.*

(ii) *Let L_{comp} be a closed orbit of* $\lim Y$ *and $[L_{comp}]$ be the homology class of $H_1(S^1\times\partial D^2)$ represented by L_{comp}. Then it holds that*

$$[L_{comp}]=a\alpha+b\beta, \quad a=\pm 1,$$

where α *and* β *are generators of* $H_1(S^1\times\partial D^2)$ *as in Section* 2.

(iii) *Let* p *and* q *be the numbers of the plus and the minus Reeb components in the codimension one* C^r *foliation of* $S^1\times\partial D^2$ *formed by the orbits of* $\lim Y$. *Then it holds that*

$$p-q=2.$$

The proof of Proposition 13.1 is the same as that of Proposition 3.6.

Proposition 13.2. *Let* E *and* $\mathscr{F}^{\varepsilon}_{\pi}$ *be as above, and let* L *be a noncompact leaf of* $\mathscr{F}^{\varepsilon}_{\pi}$. *Suppose that* Y *is a non-singular convergent* C^r *tangent vector field on* L $(r\geq 1)$. *Let* $\bar{\mathscr{F}}$ *denote the codimension one* C^r *foliation of* ∂E *formed by the orbits of the limit vector field* $\lim Y$ *of* Y, *and let* a_k, b_k, c_k, p_k *and* q_k *be as in Section* 4. *Then the equation of Proposition* 4.2 *holds.*

The proof of Proposition 13.2 is the same as that of Proposition 4.2.

Proposition 13.3. *Let* $\mathscr{F}^{(+)}_R$, L *and* Y *be as in Proposition* 13.1. *Suppose that there exists a* C^r *vector field* Y_1 *on a neighborhood* U *of* $S^1\times\partial D^2$ *in* $S^1\times D^2$ *tangent to* $\mathscr{F}^{(+)}_R$ *such that* $Y_1|(L\cap U)=Y|(L\cap U)$. *Then there exists a non-singular* C^r *vector field* $\hat{Y}$ *on* $\mathscr{F}^{(+)}_R$ *such that* $\hat{Y}|L=Y$.

Proof. Let $\tau(S^1\times D^2)$ denote the tangent bundle of $S^1\times D^2$ and let τ_1 denote the 2-plane bundle over $S^1\times D^2$ which is a subbundle of $\tau(S^1\times D^2)$ consisting of vectors tangent to leaves of $\mathscr{F}^{(+)}_R$. Let G be a subset of $\{*\}\times D^2$ such that G is diffeomorphic to $\partial D^2\times I$ and is contained in U and that one of the connected components of ∂G is a submanifold of L and the other is a submanifold of $S^1\times\partial D^2$. (Fig. 29). Let Σ denote the compact subset of L bounded by $G\cap L$. We denote by W the compact 3-dimensional manifold obtained by cutting $S^1\times D^2$ at $\Sigma\cup G$ (Fig. 29). Since W is homeomorphic to the 3-disk, the 2-plane bundle τ_1' over W obtained from τ_1 is trivial. Thus, by making use of a trivialization of τ_1', the union of the vector fields $Y|\Sigma$ and $Y_1|(G\cup S^1\times\partial D^2)$ defines a continuous map

$$\partial W\to \boldsymbol{R}^2-\{0\}.$$

Since $\pi_2(\boldsymbol{R}^2-\{0\})=0$, this map can be extended over W. This implies the existence of $\hat{Y}$ as in this proposition.

We remark that, for given two non-singular convergent C^r tangent vector fields Y on L and Y' on L' such that $\lim Y=\lim Y'$, a result similar to Proposition 13.3 holds, where L and L' are noncompact leaves of $\mathscr{F}^{(+)}_R$.

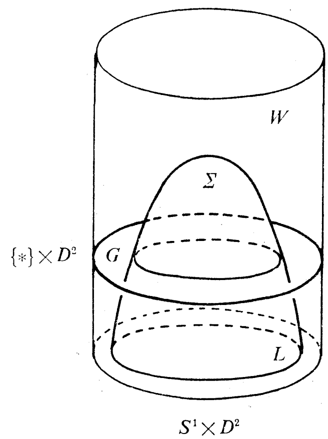


Fig. 29.

Proposition 13.4. *Let $\Sigma_0(3)$, $S^1\times S^1_k$, α_k, β_k $(k=0, 1, 2)$, $\pi: S^1\times\Sigma_0(3)\to S^1$ and $\mathscr{F}^\varepsilon_\pi$ be as in Section* 10, *and let L be a noncompact leaf of $\mathscr{F}^\varepsilon_\pi$. Suppose that Y is a non-singular convergent C^r tangent vector field $(r\geqq 1)$ on L satisfying the following conditions* (i), (ii), (iii):

(i) *There exists a C^r vector field Y_1 on a neighborhood U of ∂E in E tangent to $\mathscr{F}^\varepsilon_\pi$ such that $Y_1|(L\cap U)=Y|(L\cap U)$.*

(ii) $\lim Y|(S^1\times S^1_k)$ *has at least one closed orbit for $k=0, 1, 2$.*

(iii) *Let the homology class of $H_1(S^1\times S^1_k)$ represented by a compact leaf of* $\lim Y|(S^1\times S^1_k)$ *be $a_k\alpha_k+b_k\beta_k$ and let p_k and q_k be the numbers of plus and minus Reeb components of the codimension one C^r foliation of $S^1\times S^1_k$ formed by the orbits of* $\lim Y|(S^1\times S^1_k)$ *for $k=0, 1, 2$; then it holds that $a_k=1$ and $p_k-q_k=2$ for $k=1, 2$.*

Then there exists a non-singular C^r vector field $\hat{Y}$ tangent to $\mathscr{F}^\varepsilon_\pi$ such that $\hat{Y}|L=Y$ if and only if $b_1=3b_0$, $b_2=3b_0$.

Proof. By Proposition 13.2, we have $|a_0|(p_0-q_0)=-6$. Let $\overline{y_0y_1}$ and $\overline{y_0'y_2}$ be two straight lines in $\Sigma_0(3)$ such that $y_0, y_0'\in S^1_0$, $y_1\in S^1_1$, $y_2\in S^1_2$ and $\overline{y_0y_1}\cap\overline{y_0'y_2}=\phi$ (Fig. 30).

First suppose that there exists a C^r vector field $\hat{Y}$ as above. As was mentioned in Section 10, the natural framing of the tangent bundle $\tau(L)$ of a noncompact leaf L of $\mathscr{F}^\varepsilon_\pi$ gives a trivialization of $\tau(\mathscr{F}^\varepsilon_\pi)$. The framing as above induces the framings $\{\partial/\partial\theta, \partial/\partial\theta_0\}$ of $\tau(S^1\times S^1_0)$ on $S^1\times\{y_0\}$ and $\{-(\partial/\partial\theta), \partial/\partial\theta_1\}$ of $\tau(S^1\times S^1_1)$ on $S^1\times\{y_1\}$, where $\partial/\partial\theta$, $\partial/\partial\theta_0$ and $\partial/\partial\theta_1$ are unit tangent vectors of S^1, S^1_0 and S^1_1 with orientations as in Section 10 respectively (Fig. 30).

The vector field $\hat{Y}|(S^1\times\overline{y_0y_1})$ defines a continuous map

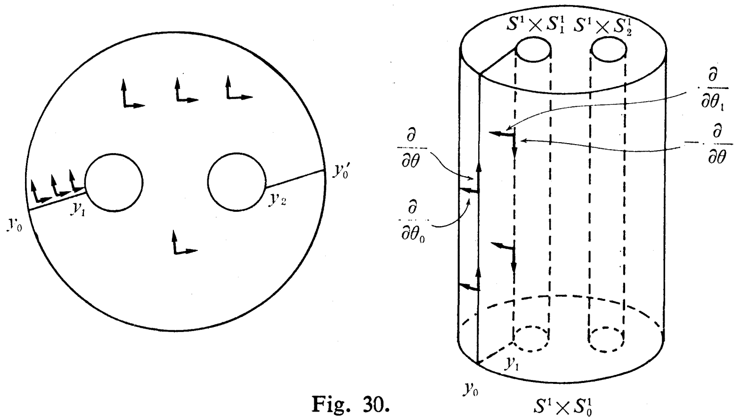


Fig. 30.

$$f_{\hat{Y}}: S^1\times \overline{y_0y_1}\to S^1$$

by

$$f_{\hat{Y}}(x, s)=(u/\sqrt{u^2+v^2}, v/\sqrt{u^2+v^2}),$$

where u and v are components of $\hat{Y}(x, s)$ with respect to the framing as above at $(x, s)\in S^1\times\overline{y_0y_1}$. Since $a_1=1$ and $p_1-q_1=2$, the degree of the map $f_{\hat{Y}}|(S^1\times\{y_1\})$: $S^1\times\{y_1\}\to S^1$ is $-b_1$. On the other hand, since $|a_0|(p_0-q_0)=-6$, the degree of the map $f_{\hat{Y}}|(S^1\times\{y_0\})$: $S^1\times\{y_0\}\to S^1$ is $-3b_0$. Obviously $f_{\hat{Y}}|(S^1\times\{y_1\})$ and $f_{\hat{Y}}|(S^1\times\{y_0\})$ are homotopic. Thus we have $b_1=3b_0$. Similarly we have $b_2=3b_0$.

Conversely if it holds that $b_1=3b_0$ and $b_2=3b_0$, then, by making use of the argument as above, the tangent vector field $(Y\cup Y_1)|((S^1\times\overline{y_0y_1})\cap U')$ can be extended to a non-singular C^r tangent vector field of $\tau(\mathscr{F}^{\varepsilon}_{\pi})$ on $S^1\times\overline{y_0y_1}$, where U' is a suitably chosen neighborhood of $(S^1\times\{y_0\})\cup(S^1\times\{y_1\})$ such that $U'\subset U$. We can make a similar extension for $S^1\times\overline{y_0'y_2}$.

The 3-dimensional C^∞ manifold with corner obtained from $S^1\times\Sigma_0(3)$ by cutting along $S^1\times\overline{y_0y_1}$ and $S^1\times\overline{y_0'y_2}$ is homeomorphic to the solid torus. Therefore, by the same argument as in the proof of Proposition 13.3, we have a vector field $\hat{Y}$ with desired properties. Thus this proposition is proved.

Now we show an example of vector fields as in Proposition 13.4. Let Y be a C^∞ tangent vector field on a noncompact leaf L_0 of $\mathscr{F}^{\varepsilon}_{\pi}$ shown by the vector field on $A_0=L_0\cap A$ as in Fig. 31, (a) with the limit vector field lim Y as in Fig. 31, (b), where A_0 is as in Section 9. Remark that a_k, b_k, p_k and q_k in Proposition 13.4 are as follows:

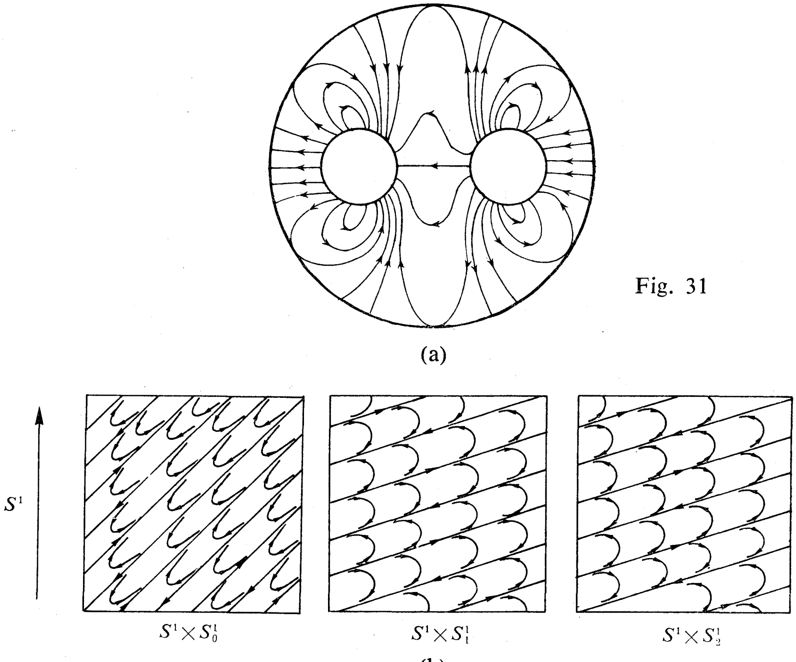


Fig. 31

$$a_0=1,\quad b_0=1,\quad a_1=a_2=1,\quad b_1=b_2=3,$$
$$p_0=0,\quad q_0=6,\quad p_1=p_2=2,\quad q_1=q_2=0.$$

Then Y can be extended to a C^∞ vector field $\hat{Y}$ on $S^1\times\Sigma_0(3)$ tangent to $\mathcal{F}^\varepsilon_\pi$ as is shown in Fig. 32. This construction is due to Koichi Yano.

The following proposition is obvious (cf. [6, Theorem 7]).

Proposition 13.5. *Let $\mathcal{F}$ be a codimension one C^r foliation of a 3-dimensional C^∞ manifold M ($r\geq 2$). Then a codimension one C^r foliation $\mathcal{F}'$ of M is transverse to $\mathcal{F}$ if and only if there exists a non-singular C^{r-1} vector field X on M tangent to $\mathcal{F}$ such that X is transverse to each leaf of $\mathcal{F}'$.*

Thus, by Theorem 10.1, there does not exist any codimension one C^r foliation of $S^1\times\Sigma_0(3)$ to which C^∞ vector field $\hat{Y}$ of Fig. 32 is transverse. Making use of the following proposition, this fact can be shown directly as below.

Let Y be a non-singular C^r vector field ($r\geq 1$) on a 3-dimensional C^∞ manifold M and let C be a simple closed C^r curve on M. If $|\sin\theta_x|<\varepsilon$ holds for the angle θ_x formed by $Y(x)$ and the tangent vector of C at x for any point $x\in C$, the simple curve C is called an *ε-closed orbit* of Y.

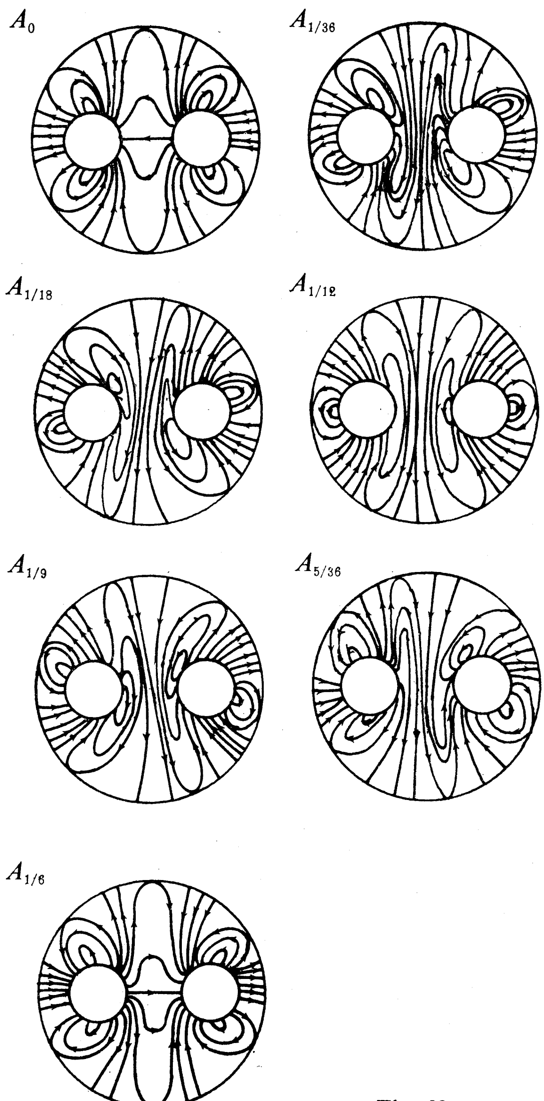


Fig. 32.

Proposition 13.6. *Let Y be a non-singular C^r vector field ($r \geqq 2$) on a compact 3-dimensional C^∞ manifold such that $Y|\partial M$ is tangent to ∂M if $\partial M \neq \phi$ and let $\{C_\sigma\}(\sigma \in \Sigma)$ be the set of all the closed orbits of Y. Suppose that, for given $\varepsilon > 0$, there exists always an ε-closed orbit $C(\varepsilon)$ such that $C(\varepsilon)$ is null homotopic in $M - \bigcup_{\sigma \in \Sigma} C_\sigma$. Then there does not exist any codimension one C^r foliation of M transverse to the one dimensional foliation $\bar{\mathscr{F}}$ of M formed by the orbits of Y.*

Proof. Assume that there exists a codimension one C^r foliation $\mathscr{F}$ of M transverse to $\bar{\mathscr{F}}$. Then, for a sufficiently small $\varepsilon > 0$, an ε closed orbit $C(\varepsilon)$ of Y as above is transverse to $\mathscr{F}$. $C(\varepsilon)$ bounds a 2-disk immersed in $M - \bigcup_{\sigma \in \Sigma} C_\sigma$. Therefore, by Novikov's result [4], there exists $S^1 \times D^2$ imbedded in M such that $\{*\} \times D^2$ is contained in the immersed 2-disk and $\mathscr{F}|(S^1 \times D^2)$ is the Reeb foliation. Since $Y|(S^1 \times D^2)$ is transverse to $\mathscr{F}|(S^1 \times D^2)$, there exists a closed orbit of $Y|(S^1 \times D^2)$ which intersects the immersed 2-disk. This is a contradiction. Thus this proposition is proved.

The original type of this proposition is due to Yano [8].

For the C^∞ vector field of Fig. 32, an ε-closed orbit $C(\varepsilon)$ of Y satisfying the conditions in Proposition 13.6 exists as is shown in Fig. 33. (Fig. 33 shows a part of a covering of A.) This proves the statement above.

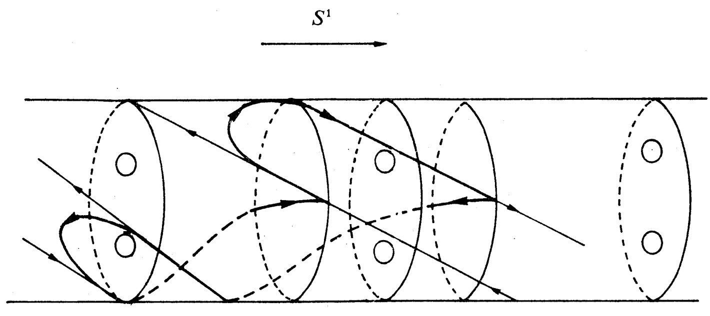
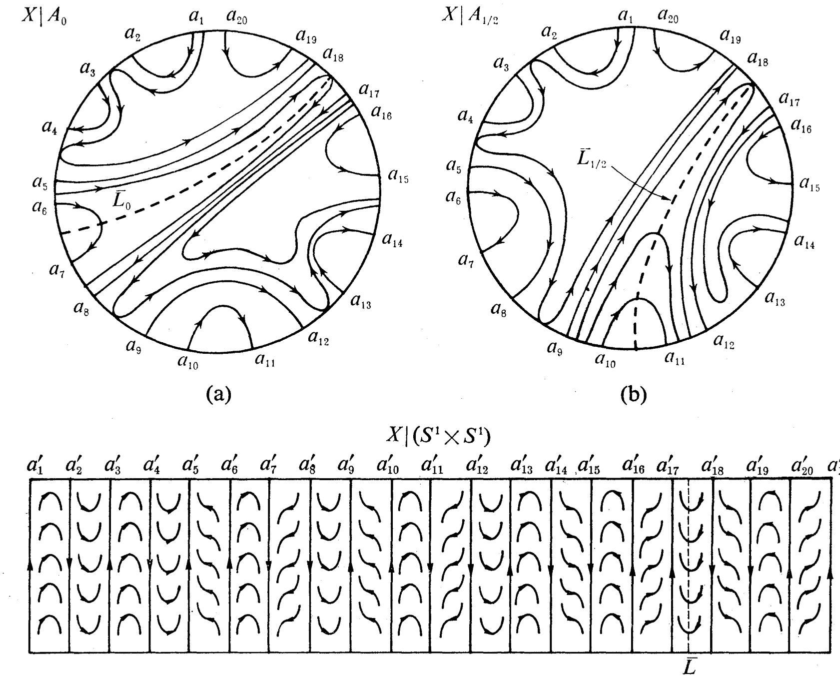


Fig. 33.

In the following we show the existence of a non-singular C^∞ vector field X on the plus Reeb foliation $\mathscr{F}_R^{(+)}$ of the solid torus $S^1 \times D^2$ such that there does not exist any codimension one C^r foliation ($r \geqq 2$) of $S^1 \times D^2$ transverse to X. Let X be a C^∞ vector field on $\mathscr{F}_R^{(+)}$ such that $X|A_0$, $X|A_{1/2}$ and $X|(S^1 \times S^1)$ are as in Fig. 34. This construction is also due to Koichi Yano. By the remark after Proposition 13.3, such a C^∞ vector field X exists. Suppose that there exists a codimension one C^r foliation



(c)

Fig. 34.

$(r \geqq 2)$ transverse to X, say $\mathscr{F}'$. Let $\bar{\mathscr{F}} = \mathscr{F}_R^{(+)} \pitchfork \mathscr{F}'$. Then $\bar{\mathscr{F}} | (S^1 \times D^2)$ has a compact leaf $\bar{L}$ as in Fig. 34, (c). Consider the leaf L' of $\mathscr{F}'$ containing $\bar{L}$ and the simple curves $\bar{L}_0 = L' \cap A_0$ in A_0 and $\bar{L}_{1/2} = L' \cap A_{1/2}$ in $A_{1/2}$ (Fig. 34, (a), (b)). $\bar{L}_0$ shows that

$$L' \cap (S^1 \times \partial D^2) - \bar{L} \subset S^1 \times [a_5, a_8]$$

and, on the other hand, $\bar{L}_{1/2}$ shows that

$$L' \cap (S^1 \times \partial D^2) - \bar{L} \subset S^1 \times [a_9, a_{12}].$$

However the leaf $L' \cap (S^1 \times \partial D^2) - \bar{L}$ of $\bar{\mathscr{F}}$ transverse to $X | (S^1 \times \partial D^2)$ cannot satisfy the above implications. Thus $\mathscr{F}'$ as above does not exist.

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