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Discontinuous Invariants of Foliations

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§ 1. Introduction

Let $\mathscr{F}_c$ be the linear foliation on the torus $T^2=\boldsymbol{R}^2/\boldsymbol{Z}^2$, whose leaves consist of parallel translations of the line $y=cx$ $(c\in\boldsymbol{R}\cup\infty)$. $\{\mathscr{F}_c\}_c$ should be considered as a C^∞ family of codimension one foliations on T^2 in any sense. However as is well known the global geometric property of $\mathscr{F}_c$ changes *discontinuously* with respect to the parameter c. Namely if $c\in\boldsymbol{Q}\cup\infty$, then all leaves of $\mathscr{F}_c$ are closed, while if $c\notin\boldsymbol{Q}\cup\infty$, then all leaves of $\mathscr{F}_c$ are dense in T^2. One way to express this phenomenon by numerical invariants would be as follows.

$\mathscr{F}_c$ is defined by a non singular closed 1-form ω and we have the corresponding cohomology class $[\omega]\in H^1(T^2;\boldsymbol{R})$. This cohomology class is well defined up to non-zero scalar and a particular choice corresponds to defining a transverse orientation and a transverse invariant Riemannian metric. Now consider the question whether ω bounds as a non singular closed 1-form, namely whether there is a compact 3-manifold W with boundary T^2 which has a non singular closed 1-form $\tilde{\omega}$ such that $\tilde{\omega}$ restricts to the given ω on the boundary. It is easy to see that this is the case if and only if $c\in\boldsymbol{Q}\cup\infty$. Now write $[\omega]=a[dx]+b[dy]$, where $[dx]$ and $[dy]\in H^1(T^2;\boldsymbol{Z})$ form the standard basis $(c=-(a/b))$. Consider $[\omega]_2=a\wedge b\in\Lambda^2_{\boldsymbol{Q}}(\boldsymbol{R})$, where $\Lambda^2_{\boldsymbol{Q}}(\boldsymbol{R})$ denotes the 2-fold exterior power of $\boldsymbol{R}$ over $\boldsymbol{Q}$. Then it is easy to see that $[\omega]_2$ does not depend on the choice of the basis of $H^1(T^2;\boldsymbol{Z})$ and we can say that ω bounds if and only if $[\omega]_2=0$. We will think of $[\omega]_2$ as a kind of characteristic number which detects the discontinuous phenomenon described above.

It turns out that this kind of phenomenon arises whenever we are given *real* cohomology classes. Now there is a theory of characteristic classes of foliations and one distinctive feature of them is that they are in general cohomology classes which have values essentially in the *reals*. This reflects on the fact that sometimes they can vary continuously on a C^∞ family of foliations. Now the purpose of the present paper is to show that the same reason gives rise to *discontinuous invariants of folia-*

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tions (usually in higher degrees than the ones where continuous variations occur), which might detect some discontinuous phenomena of foliations. More concretely in Section 2 we shall give a typical example of such invariants which is associated with the Godbillon-Vey class for codimension one foliations. Then in Sections 3 and 4, we shall give the general definition. At present we have only a few examples of foliations with non-trivial discontinuous invariants. Sections 5–7 are devoted to them, among which Theorem 7.1 would be most interesting.

§ 2. A homomorphism $\mathscr{F}\Omega_{3k,1}\to\Lambda_{\boldsymbol{Q}}^k(\boldsymbol{R})$

Let $\mathscr{F}_i$ be a codimension q foliation on a closed oriented n-dimensional manifold M_i $(i=1, 2)$. Two foliated manifolds $(M_1, \mathscr{F}_1)$ and $(M_2, \mathscr{F}_2)$ are said to be *foliated cobordant* if there is a compact oriented $(n+1)$-manifold W with a codimension q foliation $\tilde{\mathscr{F}}$ on it such that

(i) $\tilde{\mathscr{F}}$ is transverse to ∂W

(ii) $\partial(W, \tilde{\mathscr{F}})=(M_2, \mathscr{F}_2)+(-M_1, \mathscr{F}_1)$

where $-M_1$ denotes M_1 with the opposite orientation. This defines an equivalence relation on the set of all diffeomorphism classes of closed oriented codimension q foliated n-manifolds and we have the quotient set $\mathscr{F}\Omega_{n,q}$. The disjoint union induces a structure of an abelian group on $\mathscr{F}\Omega_{n,q}$ and we call it the foliated cobordism group of codimension q foliated n-manifolds.

Let $\Lambda_{\boldsymbol{Q}}^k(\boldsymbol{R})$ be the k-fold exterior power of $\boldsymbol{R}$ over $\boldsymbol{Q}$. The purpose of this section is to define a homomorphism

$$\mathrm{GV}_k\colon \mathscr{F}\Omega_{3k,1}\longrightarrow\Lambda_{\boldsymbol{Q}}^k(\boldsymbol{R})$$

for all $k=1, 2, \cdots$. As stated in the Introduction, these are typical examples of our discontinuous invariants for $k>1$. Let $\mathscr{F}$ be a codimension one foliation on a closed oriented manifold M. Then there is defined the Godbilon-Vey class $\mathrm{gv}(\mathscr{F})\in H^3(M; \boldsymbol{R})$ ([GV]). If $\dim M=3$, then we obtain the so called Godbillon-Vey number

$$\mathrm{GV}(M, \mathscr{F})=\langle[M], \mathrm{gv}(\mathscr{F})\rangle$$

where $[M]$ denotes the fundamental cycle of M. It is well known that this number depends only on the foliated cobordism calss of $(M, \mathscr{F})$ and we have a homomorphism

$$\mathrm{GV}\colon \mathscr{F}\Omega_{3,1}\longrightarrow\boldsymbol{R}.$$

Of course we set $\mathrm{GV}_1=\mathrm{GV}$. Now we proceed to define GV_k for $k>1$.

So assume that $\dim M = 3k$. Let $\{x_1, \cdots, x_m\}$ be a basis of $H^3(M; \boldsymbol{Q})$. Then we have

$$\mathrm{gv}(\mathscr{F}) = a_1 x_1 + \cdots + a_m x_m$$

for some $a_i \in \boldsymbol{R}$. We define

$$\mathrm{GV}_k(M, \mathscr{F}) = \sum_{i_1 < \cdots < i_k} \langle [M], x_{i_1} x_{i_2} \cdots x_{i_k} \rangle a_{i_1} \wedge a_{i_2} \wedge \cdots \wedge a_{i_k} \in \Lambda^k_{\boldsymbol{Q}}(\boldsymbol{R}).$$

Proposition 2.1. *The above expression is well defined. Namely it does not depend on the choice of the basis* $\{x_1, \cdots, x_m\}$.

Proof. Let $\{y_1, \cdots, y_m\}$ be another basis of $H^3(M; \boldsymbol{Q})$. Then there is a matrix $C = (c_{ij}) \in GL(m, \boldsymbol{Q})$ such that

$$y_i = \sum c_{ij} x_j.$$

If we write $C^{-1} = (\bar{c}_{ij})$, then

$$x_i = \sum \bar{c}_{ij} y_j.$$

Let $I = \{i_1, \cdots, i_k\}$ $(1 \leqq i_1 < \cdots < i_k \leqq m)$ and $J = \{j_1, \cdots, j_k\}$ $(1 \leqq j_1 < \cdots < j_k \leqq m)$ be multi-indices. We write $c(I, J)$ for the minor determinant of C of degree k corresponding to (I, J) and similarly for C^{-1}. Now let

$$\mathrm{gv}(\mathscr{F}) = b_1 y_1 + \cdots + b_m y_m$$

so that

$$a_i = \sum c_{ji} b_j.$$

We have

$$x_{i_1} \cdots x_{i_k} = \sum_{J = \{j_1, \cdots, j_k\}} \bar{c}(I, J) y_{j_1} \cdots y_{j_k}, \quad \text{and}$$
$$a_{i_1} \wedge \cdots \wedge a_{i_k} = \sum_{J' = \{j'_1, \cdots, j'_k\}} c(J', I) b_{j'_1} \wedge \cdots \wedge b_{j'_k}.$$

Hence

$$\begin{aligned}
&\sum_{I = \{i_1, \cdots, i_k\}} \langle [M], x_{i_1} \cdots x_{i_k} \rangle a_{i_1} \wedge \cdots \wedge a_{i_k} \\
&\quad = \sum_I \{\langle [M], \sum_J \bar{c}(I, J) y_{j_1} \cdots y_{j_k} \rangle \sum_{J'} c(J', I) b_{j'_1} \wedge \cdots \wedge b_{j'_k}\} \\
&\quad = \sum_J \{\sum_{J'} \sum_I c(J', I) \bar{c}(I, J) \langle [M], y_{j_1} \cdots y_{j_k} \rangle b_{j'_1} \wedge \cdots \wedge b_{j'_k}\} \\
&\quad = \sum_J \langle [M], y_{j_1} \cdots y_{j_k} \rangle b_{j_1} \wedge \cdots \wedge b_{j_k}
\end{aligned}$$

because $c(J', I)\bar{c}(I, J)=\delta_{J'J}$ by the Laplace's expansion theorem. This completes the proof.

Proposition 2.2. $\mathrm{GV}_k(M, \mathscr{F})$ *depends only on the foliated cobordism class of* $(M, \mathscr{F})$.

Proof. It is enough to prove that if $\partial(W, \tilde{\mathscr{F}})=(M, \mathscr{F})$, then $\mathrm{GV}_k(M, \mathscr{F})=0$. Let $i\colon M=\partial W\to W$ be the inclusion and let $i^*\colon H^3(W; \boldsymbol{Q})\to H^3(M; \boldsymbol{Q})$ be the induced homomorphism. Let $\{x_1, \cdots, x_s\}$ be a basis of $\mathrm{Im}\, i^*\subset H^3(M; \boldsymbol{Q})$ and choose $y_i \in H^3(W; \boldsymbol{Q})$ $(i=1, \cdots, s)$ such that $i^*(y_i)=x_i$. Let $\{y_1, \cdots, y_s, y_{s+1}, \cdots, y_{s+t}\}$ be a basis of $H^3(W; \boldsymbol{Q})$ such that $y_{s+1}, \cdots, y_{s+t} \in \mathrm{Ker}\, i^*$. Now write

$$\mathrm{gv}(\tilde{\mathscr{F}})=a_1y_1+\cdots+a_{s+t}y_{s+t}.$$

Then by the naturality of the Godbillon-Vey class, we have

$$\mathrm{gv}(\mathscr{F})=i^*\,\mathrm{gv}(\tilde{\mathscr{F}})=a_1x_1+\cdots+a_sx_s.$$

Then

$$\begin{aligned}
\mathrm{GV}_k(M, \mathscr{F})&=\sum_{i_1<\cdots<i_k}\langle[M], x_{i_1}\cdots x_{i_k}\rangle a_{i_1}\wedge\cdots\wedge a_{i_k}\\
&=\sum_{i_1<\cdots<i_k}\langle[M], i^*y_{i_1}\cdots i^*y_{i_k}\rangle a_{i_1}\wedge\cdots\wedge a_{i_k}\\
&=\sum_{i_1<\cdots<i_k}\langle i_*[M], y_{i_1}\cdots y_{i_k}\rangle a_{i_1}\wedge\cdots\wedge a_{i_k}\\
&=0.
\end{aligned}$$

Combining Propositions 2.1 and 2.2, we obtain the desired homomorphism $\mathrm{GV}_k\colon \mathscr{F}\Omega_{3k,1}\to\Lambda^k_{\boldsymbol{Q}}(\boldsymbol{R})$. It is clear from the definition that GV_k factors through $H_{3k}(B\Gamma_1; \boldsymbol{Z})$, where $B\Gamma_1$ is the classifying space for codimension one Haefliger structures (or Γ_1-structures) ([Ha]). Namely there is a similar homomorphism $\mathrm{GV}_k\colon H_{3k}(B\Gamma_1; \boldsymbol{Z})\to\Lambda^k_{\boldsymbol{Q}}(\boldsymbol{R})$ (we use the same letter), making the following diagram commutative

$$\begin{tikzcd}
\mathscr{F}\Omega_{3k,1} \arrow[rr, "\mathrm{GV}_k"] \arrow[dr, "\pi"'] & & \Lambda^k_{\boldsymbol{Q}}(\boldsymbol{R}) \\
& H_{3k}(B\Gamma_1; \boldsymbol{Z}) \arrow[ur, "\mathrm{GV}_k"'] &
\end{tikzcd}$$

where π is the natural map. Thurston [Th 1] has proved that GV_1 is a surjection. We propose

Conjecture 2.3. *The homomorphisms*

$$\mathrm{GV}_k\colon \mathscr{F}\Omega_{3k,1} \longrightarrow \Lambda^k_Q(\boldsymbol{R})$$

are surjective for all k.

The difficulty of the problem to prove or disprove the above Conjecture increases as k becomes larger. The case $k=2$ is already extremely difficult. In fact even the non-triviality of GV_2 is unknown. The following problem is related to this.

Problem 2.4. *For any given* $a, b \in \boldsymbol{R}$, *construct a codimension one foliation* $\mathscr{F}$ *on* $S^3 \times S^3$ *such that* $\mathrm{gv}(\mathscr{F})=(a, b) \in \boldsymbol{R}\oplus\boldsymbol{R} \cong H^3(S^3\times S^3; \boldsymbol{R})$.

Let $\mathscr{F}_t$ $(t \in \boldsymbol{R})$ be one of Thurston's foliations on S^3 such that $\mathrm{GV}(\mathscr{F}_t)=t$ ([Th 1]) and let $\gamma_t \in \pi_3(B\Gamma_1)$ be the corresponding element. We have a Γ_1-structure $\gamma_a \vee \gamma_b$ on $S^3 \vee S^3$. The obstruction to extend this structure to whole of $S^3 \times S^3$ is represented by the Whitehead product $[\gamma_a, \gamma_b] \in \pi_5(B\Gamma_1)$ and if we can prove that it vanishes, then we could obtain a solution to Problem 2.4. If a and b are linearly dependent over $\boldsymbol{Q}$, then $[\gamma_a, \gamma_b]=0$ because Tsuboi [Ts] has proved that Thurston's examples define a direct summand $\boldsymbol{R} \subset \pi_3(B\Gamma_1)$. However if a and b are linearly independent over $\boldsymbol{Q}$, then we cannot say anything because at present nothing is known about the group $\pi_5(B\Gamma_1)$. We shall consider this problem again in Section 6 from a different point of view.

§ 3. Homology of $K(\boldsymbol{R}, q)$

In the preceding section we have observed that the Godbillon-Vey class for codimension one foliations, which is a *real* cohomology class of degree 3, gives rise to various foliated cobordism invariants with values in $\Lambda^k_Q(\boldsymbol{R})$ $(k=2, 3, \cdots)$ which are vector spaces over the *rationals* rather than the reals. In this section we examine homotopy theoretic background of this phenomenon. Thus let X be a reasonable topological space (e.g. a CW complex) and let $\alpha \in H^q(X; \boldsymbol{R})$ be a cohomology class. Then there is defined the corresponding continuous map $\alpha\colon X \to K(\boldsymbol{R}, q)$ (we use the same letter) such that $\alpha^*(\iota)=\alpha$, where $\iota \in H^q(K(\boldsymbol{R}, q); \boldsymbol{R})$ is the fundamental cohomology class of the Eilenberg-MacLane space $K(\boldsymbol{R}, q)$, namely it corresponds to the id: $\boldsymbol{R} \to \boldsymbol{R}$ under the isomorphism $H^q(K(\boldsymbol{R}, q); \boldsymbol{R}) \cong \mathrm{Hom}_{\boldsymbol{Z}}(\boldsymbol{R}, \boldsymbol{R})$. We would like to identify the induced homomorphism $\alpha_*\colon H_*(X; \boldsymbol{Z}) \to H_*(K(\boldsymbol{R}, q); \boldsymbol{Z})$ (see Theorem 3.5). To do so we first recall the homology of $K(\boldsymbol{R}, q)$, which should be well-known (see [Ro] for the case $q=1$). Let $S^k_Q(\boldsymbol{R})$ be the k-fold symmetric power of $\boldsymbol{R}$ over $\boldsymbol{Q}$. For a real number $a \in \boldsymbol{R}$, we write $\hat{a}$ for the corresponding element of $S^1_Q(\boldsymbol{R}) \cong \boldsymbol{R}$.

Proposition 3.1. (i) *If q is odd, then we have*

$$H_*(K(\boldsymbol{R}, q); \boldsymbol{Z}) = \begin{cases} \boldsymbol{Z} & * = 0 \\ \Lambda^k_{\boldsymbol{Q}}(\boldsymbol{R}) & * = kq \\ 0 & \text{otherwise.} \end{cases}$$

(ii) *If q is even, then we have*

$$H_*(K(\boldsymbol{R}, q); \boldsymbol{Z}) = \begin{cases} \boldsymbol{Z} & * = 0 \\ S^k_{\boldsymbol{Q}}(\boldsymbol{R}) & * = kq \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Let $\{a_i; i \in I\}$ be a basis of $\boldsymbol{R}$ as a vector space over $\boldsymbol{Q}$. For a finite subset F of I, let $\boldsymbol{Q}_F$ be the vector subspace of $\boldsymbol{R}$ generated by a_i $(i \in F)$. Then clearly $\boldsymbol{R} = \varinjlim_F \boldsymbol{Q}_F$ and so

$$K(\boldsymbol{R}, q) = \varinjlim_F K(\boldsymbol{Q}_F, q).$$

The homology of $K(\boldsymbol{Q}, q)$ is well-known:

If q is odd,

$$H_*(K(\boldsymbol{Q}, q); \boldsymbol{Z}) = \begin{cases} \boldsymbol{Z} & * = 0 \\ \boldsymbol{Q} & * = q \\ 0 & \text{otherwise.} \end{cases}$$

If q is even,

$$H_*(K(\boldsymbol{Q}, q); \boldsymbol{Z}) = \begin{cases} \boldsymbol{Z} & * = 0 \\ \boldsymbol{Q} & * = kq \\ 0 & \text{otherwise.} \end{cases}$$

From this we can calculate $H_*(K(\boldsymbol{Q}_F, q); \boldsymbol{Z})$ using the theorem of Künneth and we obtain the desired result because

$$H_*(K(\boldsymbol{R}, q); \boldsymbol{Z}) = \varinjlim_F H_*(K(\boldsymbol{Q}_F, q); \boldsymbol{Z}).$$

Here we look into the above isomorphism more closely. Let

$$\mu: K(\boldsymbol{R}, q) \times K(\boldsymbol{R}, q) \longrightarrow K(\boldsymbol{R}, q)$$

be the map defining the natural H-space structure on $K(\boldsymbol{R}, q)$. Namely it is characterized by the property $\mu^*(\iota) = \iota \times 1 + 1 \times \iota$. (If one worries about the topology of $K(\boldsymbol{R}, q) \times K(\boldsymbol{R}, q)$, one can consider everything at the levels of $K(\boldsymbol{Q}_F, q)$'s, because $K(\boldsymbol{Q}_F, q)$ can be assumed to be a countable CW complex so that $K(\boldsymbol{Q}_F, q)\times K(\boldsymbol{Q}_F, q)$ is also a CW complex. This remark applies also to the proof of the next Proposition.) This induces the Pontrjagin product on $H_*(K(\boldsymbol{R}, q); \boldsymbol{Z})$: if $u \in H_p(K(\boldsymbol{R}, q); \boldsymbol{Z})$ and $v \in H_{p'}(K(\boldsymbol{R}, q); \boldsymbol{Z})$, then $u*v \in H_{p+p'}(K(\boldsymbol{R}, q); \boldsymbol{Z})$ is defined by $u*v=\mu_*(u\times v)$, where $u\times v$ is the cross product of u and v. For a real number $a_i \in \boldsymbol{R}$ $(i \in I)$, let $u_i \in H_q(K(\boldsymbol{R}, q); \boldsymbol{Z})$ be the corresponding homology class. Then we can say the following

With respect to the Pontrjagin product, $H_*(K(\boldsymbol{R}, q); \boldsymbol{Z})$ is a free graded commutative algebra over $\boldsymbol{Q}$ generated by the elements u_i $(i \in I)$.

The isomorphisms in the statement of Proposition 3.1 are given by the correspondences

$$u_{i_1}*\cdots*u_{i_k}\longrightarrow a_{i_1}\wedge\cdots\wedge a_{i_k} \quad (q\colon \text{odd})$$
$$u_{i_1}*\cdots*u_{i_k}\longrightarrow \hat{a}_{i_1}\cdots\hat{a}_{i_k} \quad (q\colon \text{even}).$$

Let d: $K(\boldsymbol{R}, q)\to K(\boldsymbol{R}, q)\times K(\boldsymbol{R}, q)$ be the diagonal map.

Proposition 3.2.

(i) $\mu_*((u_{i_1}*\cdots*u_{i_k})\times(u_{j_1}*\cdots*u_{j_l}))=u_{i_1}*\cdots*u_{i_k}*u_{j_1}*\cdots*u_{j_l}.$

(ii) $d_*(u_{i_1}*\cdots*u_{i_k})$

$$=\begin{cases}\sum_S \operatorname{sgn} S(u_{i_{s(1)}}*\cdots*u_{i_{s(l)}})\times(u_{i_{t(1)}}*\cdots*u_{i_{t(m)}}) & (q\colon \text{odd})\\ \sum_S (u_{i_{s(1)}}*\cdots*u_{i_{s(l)}})\times(u_{i_{t(1)}}*\cdots*u_{i_{t(m)}}) & (q\colon \text{even}),\end{cases}$$

where $S=\{s(1), \cdots, s(l)\}$, $1\leqq s(1)<\cdots<s(l)\leqq k$, $S\cup\{t(1), \cdots, t(m)\}=\{1, \cdots, k\}$ $(m=k-l)$, $1\leqq t(1)<\cdots<t(m)\leqq k$ *and*

$$\operatorname{sgn} S=\operatorname{sgn}\begin{pmatrix}1\cdots\cdots\cdots\cdots k\\ s(1)\cdots s(l)t(1)\cdots t(m)\end{pmatrix}.$$

Proof. (i) is clear. We prove (ii) by the induction on k. If $k=1$, clearly we have $d_*(u_i)=u_i\times 1+1\times u_i$. Next consider the following commutative diagram:

$$\begin{array}{ccc}
K(\boldsymbol{R}, q)\times K(\boldsymbol{R}, q) & \xrightarrow{\quad\mu\quad} & K(\boldsymbol{R}, q)\\
\downarrow{\scriptstyle d\times d} & & \\
K(\boldsymbol{R}, q)\times K(\boldsymbol{R}, q)\times K(\boldsymbol{R}, q)\times K(\boldsymbol{R}, q) & & \downarrow{\scriptstyle d}\\
\downarrow{\scriptstyle 1\times T\times 1} & & \\
K(\boldsymbol{R}, q)\times K(\boldsymbol{R}, q)\times K(\boldsymbol{R}, q)\times K(\boldsymbol{R}, q) & \xrightarrow{\mu\times\mu} & K(\boldsymbol{R}, q)\times K(\boldsymbol{R}, q)
\end{array}$$

where $T\colon K(\boldsymbol{R}, q)\times K(\boldsymbol{R}, q)\to K(\boldsymbol{R}, q)\times K(\boldsymbol{R}, q)$ is defined by $T(s, t)=(t, s)$ $(s, t\in K(\boldsymbol{R}, q))$. Now we assume q is odd and compute

$$\begin{aligned}
&d_*(u_{i_1}*\cdots *u_{i_{k+1}})\\
&\quad=d_*\mu_*((u_{i_1}*\cdots *u_{i_k})\times u_{i_{k+1}})\\
&\quad=(\mu\times\mu)_*(1\times T\times 1)_*(d\times d)_*((u_{i_1}*\cdots *u_{i_k})\times u_{i_{k+1}})\\
&\quad=(\mu\times\mu)_*(1\times T\times 1)_*(\sum_S \operatorname{sgn} S(u_{i_{s(1)}}*\cdots *u_{i_{s(l)}})\times(u_{i_{t(1)}}*\cdots *u_{i_{t(m)}})\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\times(u_{i_{k+1}}\times 1+1\times u_{i_{k+1}}))\\
&\quad=(\mu\times\mu)_*(\sum_S (-1)^m \operatorname{sgn} S(u_{i_{s(1)}}*\cdots *u_{i_{s(l)}})\\
&\qquad\qquad\qquad\qquad\qquad\qquad\times u_{i_{k+1}}\times(u_{i_{t(1)}}*\cdots *u_{i_{t(m)}})\times 1\\
&\qquad+\sum_S \operatorname{sgn} S(u_{i_{s(1)}}*\cdots *u_{i_{s(l)}})\times 1\times(u_{i_{t(1)}}*\cdots *u_{i_{t(m)}})\times u_{i_{k+1}})\\
&\quad=\sum_S (-1)^m \operatorname{sgn} S(u_{i_{s(1)}}*\cdots *u_{i_{s(l)}}*u_{i_{k+1}})\times(u_{i_{t(1)}}*\cdots *u_{i_{t(m)}})\\
&\qquad+\sum_S \operatorname{sgn} S(u_{i_{s(1)}}*\cdots *u_{i_{s(l)}})\times(u_{i_{t(1)}}*\cdots *u_{i_{t(m)}}*u_{i_{k+1}}).
\end{aligned}$$

This proves the assertion (ii) for q odd. The case q: even is easier (just forget the sgn in the above computation).

Let $f_i\colon \boldsymbol{R}\to\boldsymbol{R}$ $(i=1,\cdots,k)$ be an *additive* (in general discontinuous) homomorphism. We consider f_i as an element of

$$H^q(K(\boldsymbol{R}, q); \boldsymbol{R})\cong \operatorname{Hom}_Z(\boldsymbol{R}, \boldsymbol{R}).$$

Proposition 3.3.

$$\langle u_{i_1}*\cdots *u_{i_k}, f_1\cdots f_k\rangle=\begin{cases}\sum\limits_{\sigma\in S_k} \operatorname{sgn}\sigma f_1(a_{i_{\sigma(1)}})\cdots f_k(a_{i_{\sigma(k)}}) & (q\text{: odd})\\ \sum\limits_{\sigma\in S_k} f_1(a_{i_{\sigma(1)}})\cdots f_k(a_{i_{\sigma(k)}}) & (q\text{: even}),\end{cases}$$

where S_k is the k-th symmetric group.

Proof. We assume q is odd and use the induction on k. If $k=1$, then the assertion is clear. Now

$$\begin{aligned}
&\langle u_{i_1}*\cdots *u_{i_k}, f_1\cdots f_k\rangle\\
&\quad=\langle u_{i_1}*\cdots *u_{i_k}, d^*(f_1\cdots f_{k-1}\times f_k)\rangle\\
&\quad=\langle d_*(u_{i_1}*\cdots *u_{i_k}), f_1\cdots f_{k-1}\times f_k\rangle\\
&\quad=\sum_{l=1}^{k} (-1)^{k-l}\langle u_{i_1}*\cdots u_{i_{l-1}}*u_{i_{l+1}}*\cdots u_{i_k}\times u_{i_l}, f_1\cdots f_{k-1}\times f_k\rangle\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\text{(Proposition 3.2, (ii))}\\
&\quad=\sum (-1)^{k-l}\langle u_{i_1}*\cdots u_{i_{l-1}}*u_{i_{l+1}}*\cdots *u_{i_k}, f_1\cdots f_{k-1}\rangle\langle u_{i_l}, f_k\rangle
\end{aligned}$$

$$=(\sum(-1)^{k-l}\sum_{\tau\in S'}\operatorname{sgn}\tau f_1(a_{i_{\tau(1)}})\cdots f_{k-1}(a_{i_{\tau(k)}}))f_k(a_{i_l})$$
$$=\sum_{\sigma\in S_k}\operatorname{sgn}\sigma f_1(a_{i_{\sigma(1)}})\cdots f_k(a_{i_{\sigma(k)}})$$

where S' is the permutation group of $\{1,\cdots,l-1,l+1,\cdots,k\}$. The case q: even is similar.

Now for any $i\in I$, let $\iota_i\in H^q(K(\boldsymbol{R},q);\boldsymbol{R})\cong\operatorname{Hom}_{\boldsymbol{Z}}(\boldsymbol{R},\boldsymbol{R})$ be the discontinuous cohomology class defined by $\iota_i(a_i)=1$, $\iota_i(a_j)=0$ $(j\neq i)$. We have $\iota_i(u_j)=\delta_{ij}$. In view of Proposition 3.3, we have

Proposition 3.4. *We have*

$$\langle u_{i_1}*\cdots*u_{i_k},\iota_{j_1}\cdots\iota_{j_k}\rangle\begin{cases}=0 & \textit{if } \{i_1,\cdots,i_k\}\neq\{j_1,\cdots,j_k\}\\ \neq 0 & \textit{if } \{i_1,\cdots,i_k\}=\{j_1,\cdots,j_k\}.\end{cases}$$

Therefore the cohomology classes $\iota_{j_1}\cdots\iota_{j_k}$ detects all the elements of $H_{kq}(K(\boldsymbol{R},q);\boldsymbol{Z})$.

Theorem 3.5. *Let X be a reasonable topological space* (e.g. *a finite CW complex*) *and let $\alpha\in H^q(X;\boldsymbol{R})$. We use the same letter α for the corresponding map $X\to K(\boldsymbol{R},q)$. Let $\{x_1,\cdots,x_m\}$ be a basis of $H^q(X;\boldsymbol{Q})$ and write*

$$\alpha=a_1x_1+\cdots+a_mx_m.$$

Let $u\in H_{kq}(X;\boldsymbol{Z})$ be an element. Then we have

$$\alpha_*(u)=\begin{cases}\displaystyle\sum_{i_1<\cdots<i_k}\langle u,x_{i_1}\cdots x_{i_k}\rangle a_{i_1}\wedge\cdots\wedge a_{i_k}\in\Lambda^k_{\boldsymbol{Q}}(\boldsymbol{R})\cong H_{kq}(K(\boldsymbol{R},q);\boldsymbol{Z}) & (q\text{: odd})\\ \displaystyle\frac{1}{k!}\sum_{i_1,\cdots,i_k}\langle u,x_{i_1}\cdots x_{i_k}\rangle\hat{a}_{i_1}\cdots\hat{a}_{i_k}\in S^k_{\boldsymbol{Q}}(\boldsymbol{R})\cong H_{kq}(K(\boldsymbol{R},q);\boldsymbol{Z}) & (q\text{: even}).\end{cases}$$

Proof. Let $u_i\in H_q(X;\boldsymbol{Q})$ $(i=1,\cdots,m)$ be elements such that

$$\langle u_i,x_j\rangle=\delta_{ij}.$$

First we claim that

$$\alpha_*(u_i)=a_i\in\boldsymbol{R}\cong H_q(K(\boldsymbol{R},q);\boldsymbol{Z}).$$

This follows because

$$\langle\alpha_*(u_i),\iota\rangle=\langle u_i,\alpha^*(\iota)\rangle=\langle u_i,\alpha\rangle=a_i.$$

Next let $f\in H^q(K(\boldsymbol{R},q);\boldsymbol{R})\cong\operatorname{Hom}_{\boldsymbol{Z}}(\boldsymbol{R},\boldsymbol{R})$ be an element. Then we have

$$\begin{aligned}\langle u_i, \alpha^*(f)\rangle &= \langle \alpha_*(u_i), f\rangle \\ &= \langle a_i, f\rangle \\ &= f(a_i).\end{aligned}$$

Hence we obtain

$$\alpha^*(f) = f(a_1)x_1 + \cdots + f(a_m)x_m.$$

Now let $f_1, \cdots, f_k \in H^q(K(\boldsymbol{R}, q); \boldsymbol{R})$. Assuming that q is odd, we have

$$\begin{aligned}&\langle \alpha_*(u), f_1\cdots f_k\rangle \\ &\quad = \langle u, \alpha^*(f_1)\cdots\alpha^*(f_k)\rangle \\ &\quad = \langle u, (f_1(a_1)x_1 + \cdots + f_1(a_m)x_m)\cdots(f_k(a_1)x_1 + \cdots + f_k(a_m)x_m)\rangle \\ &\quad = \langle u, \sum_{i_1,\cdots,i_k} f_1(a_{i_1}) \cdots f_k(a_{i_k})x_{i_1}\cdots x_{i_k}\rangle \\ &\quad = \sum_{i_1<\cdots<i_k}\sum_{\sigma\in S_k} \operatorname{sgn}\sigma f_1(a_{i_{\sigma(1)}})\cdots f_k(a_{i_{\sigma(k)}})\langle u, x_{i_1}\cdots x_{i_k}\rangle \\ &\quad = \sum_{i_1<\cdots<i_k} \langle u_{i_1}*\cdots*u_{i_k}, f_1\cdots f_k\rangle\langle u, x_{i_1}\cdots x_{i_k}\rangle \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(see Proposition 3.3)} \\ &\quad = \sum_{i_1<\cdots<i_k} \langle a_{i_1}\wedge\cdots\wedge a_{i_k}, f_1\cdots f_k\rangle\langle u, x_{i_1}\cdots x_{i_k}\rangle.\end{aligned}$$

Since elements of $H_{kq}(K(\boldsymbol{R}, q); \boldsymbol{Z})$ can be detected by cohomology classes of the form $f_1\cdots f_k \in H^{kq}(K(\boldsymbol{R}, q); \boldsymbol{R})$ (see Proposition 3.4), we obtain

$$\alpha_*(u) = \sum_{i_1<\cdots<i_k} \langle u, x_{i_1}\cdots x_{i_k}\rangle a_{i_1}\wedge\cdots\wedge a_{i_k}.$$

The case when q is even is similar and omitted.

In this way, any real cohomology class $\alpha \in H^q(X; \boldsymbol{R})$ gives rise to a homomorphism $\alpha_*: H_{kq}(X; \boldsymbol{Z}) \to \Lambda^k_{\boldsymbol{Q}}(\boldsymbol{R})$ (or $S^k_{\boldsymbol{Q}}(\boldsymbol{R})$) according as q is odd (or even), which we would like to call *discontinuous invariants arising from* α for $k>1$. If α is a rational cohomology class, then clearly all the discontinuous invariants vanish. Also it is clear that the homomorphisms $\mathrm{GV}_k: H_{3k}(B\Gamma_1; \boldsymbol{Z}) \to \Lambda^k_{\boldsymbol{Q}}(\boldsymbol{R})$ defined in Section 2 are nothing but the discontinuous invariants arising from the Godbillon-Vey class $\mathrm{gv} \in H^3(B\Gamma_1; \boldsymbol{R})$.

§ 4. Discontinuous invariants —general framework—

In the preceding section, we have observed that any real cohomology class of a topological space induces discontinuous invariants, which are homomorphisms from the integral homology groups of the space to various vector spaces over $\boldsymbol{Q}$ such as $\Lambda^k_{\boldsymbol{Q}}(\boldsymbol{R})$ or $S^k_{\boldsymbol{Q}}(\boldsymbol{R})$. In this section we generalize this procedure. More precisely we consider systems of real cohomology classes instead of just one cohomology class and give the general definition of our discontinuous invariants. This is done in the framework of the theory of Sullivan ([Su], in particular § 8). Thus let $\mathscr{A}$ be a differential graded algebra (d.g.a.) over $\boldsymbol{R}$. An $\mathscr{A}$-differential system on a space X is a d.g.a. map $\mathscr{A}\to A^*(X)$, where $A^*(X)$ is the set of all C^∞ forms on X. Here X is either a C^∞ manifold, $A^*(X)$ being its de Rham complex or else X is a simplicial complex, $A^*(X)$ being its Sullivan-de Rham complex, namely the set of all "compatible C^∞ forms" on X. In any case we have the de Rham theorem

$$H^*(A^*(X))\cong H^*(X;\boldsymbol{R}).$$

Now let $\mathscr{A}\to A^*(X)$ be an $\mathscr{A}$-differential system on X. Then we have a homomorphism

$$H^*(\mathscr{A})\longrightarrow H^*(X;\boldsymbol{R})$$

which induces a system of real cohomology classes of X. There is a universal space $B\mathscr{A}$, called the spatial realization of $\mathscr{A}$, which is defined to be the geometric realization of the simplicial set whose n-simplices consist of all $\mathscr{A}$-differential systems on the standard n-simplex Δ^n and the face, degeneracy operators are defined by restrictions and pull backs of forms. An $\mathscr{A}$-differential system on X naturally defines a classifying map $X\to B\mathscr{A}$ and conversely any map $X\to B\mathscr{A}$ induces an $\mathscr{A}$-differential system on X well defined up to homotopy. Here two $\mathscr{A}$-differential systems on X are said to be homotopic if there is an $\mathscr{A}$-differential system on $X\times I$ such that the restrictions to $X\times\{0\}$, $X\times\{1\}$ are the given ones. Then we have a bijection

$$\begin{matrix}\text{the set of homotopy}\\ \text{classes of } \mathscr{A}\text{-differential}\\ \text{systems on } X\end{matrix} \approx [X, B\mathscr{A}]$$

for any reasonable X. Now one has a natural d.g.a. map $\mathscr{A}\to A^*(B\mathscr{A})$ and for an $\mathscr{A}$-differential system on X, the following diagram clearly commutes

$$\begin{tikzcd} H^*(\mathscr{A}) \arrow[r] \arrow[dr] & H^*(B\mathscr{A};\boldsymbol{R}) \arrow[d] \\ & H^*(X;\boldsymbol{R}) \end{tikzcd}$$

where the vertical homomorphism is induced by the classifying map $X\to B\mathscr{A}$. The classifying map also induces homomorphisms

$$H_k(X;\boldsymbol{Z})\longrightarrow H_k(B\mathscr{A};\boldsymbol{Z})\quad (k=1,2,\cdots)$$

which we call *discontinuous invariants* arising from the given $\mathscr{A}$-differential system on X. The point here is that the natural map $\mathscr{A}\to A^*(B\mathscr{A})$ does not induce isomorphism on cohomology. In fact the image of the homomorphism $H^*(\mathscr{A})\to H^*(B\mathscr{A};\boldsymbol{R})$ consists of those cohomology classes which can be represented by cocycles on $B\mathscr{A}$ which are continuous with respect to the coarse topology of $B\mathscr{A}$ which is induced from the natural topology on C^∞ forms. Thus to detect homology classes of $B\mathscr{A}$, we have to consider discontinuous cohomology classes also. This motivates the naming of our invariants.

Now we apply the above construction to the case of characteristic classes of foliations. Thus let $\mathscr{F}$ be a codimension q foliation on a C^∞ manifold M. Assume that the normal bundle of $\mathscr{F}$ is trivialized. Then $\mathscr{F}$ is defined by certain 1-forms ω^i $(i=1, \cdots, q)$. The integrability condition implies that there are 1-forms ω^i_j such that

$$d\omega^i+\sum\omega^i_j\wedge\omega^j=0.$$

If we differentiate the above equation, we see that there are 1-forms ω^i_{jk} such that

$$d\omega^i_k+\sum\omega^i_{jk}\wedge\omega^j+\sum\omega^i_j\wedge\omega^j_k=0.$$

One can continue this procedure indefinitely to obtain a system of 1-forms $\{\omega^i;\omega^i_j;\omega^i_{jk};\omega^i_{jkl};\cdots\}$ on M. It turns out that this defines an $\mathfrak{a}_q$-structure on M. Here $\mathfrak{a}_q$ denotes the topological Lie algebra of formal vector fields on $\boldsymbol{R}^q$ and an $\mathfrak{a}_q$-structure on M means an $A^*_c(\mathfrak{a}_q)$-differential system on it, where $A^*_c(\mathfrak{a}_q)$ is the d.g.a. consisting of all continuous cochains on $\mathfrak{a}_q$ (see [B 3]). This construction is natural so that there is defined a map

$$B\bar{\Gamma}_q\longrightarrow B\mathfrak{a}_q$$

where $B\bar{\Gamma}_q$ is the classifying space for codimension q Haefliger structures with trivial normal bundles ([Ha]) and $B\mathfrak{a}_q$ denotes the spatial realization of $A^*_c(\mathfrak{a}_q)$. Sullivan [Su] asks whether this map is a homotopy equivalence or not.

The cohomology of $A^*_c(\mathfrak{a}_q)$ was determined by Gel'fand-Fuks [GF] and can be described as follows. Define a d.g.a. W_q as

$$W_q=E(h_1, \cdots, h_q)\otimes\hat{\boldsymbol{R}}[c_1, \cdots, c_q]$$
$$dh_i=c_i,\ \deg h_i=2i-1$$

where E denotes the exterior algebra and $\hat{\boldsymbol{R}}$ denotes the polynomial algebra truncated by the ideal consisting of elements of $\deg>2q$. Then there is a d.g.a. map

$$W_q \longrightarrow A_c^*(\mathfrak{a}_q)$$

inducing an isomorphism on cohomology. Thus we have a homomorphism

$$H^*(W_q) \longrightarrow H^*(B\bar{\Gamma}_q; \boldsymbol{R})$$

which defines the so-called characteristic classes of foliations (see [BH] for example). There is a nice topological model for the d.g.a. W_q which is defined as follows. Let BU_q be the classifying space for the unitary group U_q and let $\pi: EU_q \to BU_q$ be the universal U_q bundle over BU_q. Let $BU_q^{(2q)}$ be the $2q$-skeleton of BU_q (with respect to the natural cell structure of BU_q) and set $Y_q = \pi^{-1}(BU_q^{(2q)})$. Then there is a d.g.a. map $W_q \to A^*(Y_q)$ which induces an isomorphism on cohomology.

Now in general let X be a simply connected finite simplicial complex and let $\mathcal{M}_X$ be the minimal model of $A^*(X)$ in the sense of Sullivan [Su]. We call $B\mathcal{M}_X$ the *real type* of X and denote it by $X_{\boldsymbol{R}}$. The homotopy group of $X_{\boldsymbol{R}}$ is isomorphic to $\pi_*(X) \otimes \boldsymbol{R}$ and the integral homology groups of $X_{\boldsymbol{R}}$ are vector spaces over $\boldsymbol{Q}$ (see [Su]). For example it is easy to see that $S_{\boldsymbol{R}}^{2q+1}$ is a $K(\boldsymbol{R}, 2q+1)$ and hence we know $H_*(S_{\boldsymbol{R}}^{2q+1}; \boldsymbol{Z})$ by Proposition 3.1. However the computation of $H_*(X; \boldsymbol{Z})$ is in general very difficult. Even the case when $X = S^{2q}$ seems to be non-trivial:

Problem 4.1. *Compute* $H_*(S_{\boldsymbol{R}}^{2q}; \boldsymbol{Z})$.

$S_{\boldsymbol{R}}^{2q}$ is the classifying space for real cohomology classes $\alpha \in H^{2q}(\ ; \boldsymbol{R})$ such that $\alpha^2 = 0$.

Now clearly $B\mathfrak{a}_q$ is homotopy equivalent to $(Y_q)_{\boldsymbol{R}}$ and hence we have a map

$$B\bar{\Gamma}_q \longrightarrow (Y_q)_{\boldsymbol{R}}.$$

This induces homomorphisms

$$H_k(B\bar{\Gamma}_q; \boldsymbol{Z}) \longrightarrow H_k((Y_q)_{\boldsymbol{R}}; \boldsymbol{Z}) \quad (k = 1, 2, \cdots)$$

and we call them *discontinuous invariants* of *foliations*. If $q = 1$, then Y_1 has the homotopy type of S^3 and hence $(Y_1)_{\boldsymbol{R}}$ is a $K(\boldsymbol{R}, 3)$. In this case the discontinuous invariants defined above coincide with the homomorphisms $\mathrm{GV}_k: H_{3k}(B\bar{\Gamma}_1; \boldsymbol{Z}) \longrightarrow \Lambda_{\boldsymbol{Q}}^k(\boldsymbol{R}) \cong H_{3k}((Y_1)_{\boldsymbol{R}}; \boldsymbol{Z})$ defined in Section 2. In general it is known that Y_q has the homotopy type of bouquet of spheres and in accordance with that Hurder [Hu] has constructed many non-trivial foliations on spheres. However at present no example of C^∞

foliations is known which is detected by essentially discontinuous invariants.

Problem 4.2. *Calculate* $H_*((Y_q)_{\boldsymbol{R}}; \boldsymbol{Z})$.

Problem 4.3. *Prove the non-triviality of discontinuous invariants of foliations.*

It is clear that the above considerations can be applied to various geometric structures other than the foliations. It is enough to assume only that there is defined a system of C^∞ forms (or real cohomology classes) in a functorial manner. For example we can consider foliated M-bundles or flat G-bundles where G is a Lie group. Here is a sample problem for the latter case.

Problem 4.4. *Let* $v \in H^3(SL_2\boldsymbol{C}; \boldsymbol{R})$ *be the "volume class" (cf.* § 7). *Is the homomorphism*

$$H_6(SL_2\boldsymbol{C}; \boldsymbol{Z}) \longrightarrow \Lambda^2_{\boldsymbol{Q}}(\boldsymbol{R})$$

induced by v *non-trivial? (Here we understand* $SL_2\boldsymbol{C}$ *as a discrete group.)*

§ 5. Examples

(I) The volume class of Riemannian foliations

Let $\mathscr{F}$ be a transversely oriented codimension q Riemannian foliation on a C^∞ manifold M. Roughly speaking there is defined a metric on the normal direction to the leaves of $\mathscr{F}$ which is invariant by the action of the holonomy pseudo-group of $\mathscr{F}$. (See [LP] [P] for more precise definitions). In particular we have the volume class $v(\mathscr{F}) \in H^q(M; \boldsymbol{R})$. If we denote $BR\Gamma_q^+$ for the classifying space for codimension q transversely oriented Riemannian Haefliger structures, then we have the universal volume class $v \in H^q(BR\Gamma_q^+; \boldsymbol{R})$. Clearly we have $v^2=0$. Therefore by Theorem 3.5 and the results in Section 4, we have homomorphisms

$$V_k\colon H_{kq}(BR\Gamma_q^+; \boldsymbol{Z}) \longrightarrow \begin{cases} \Lambda^k_{\boldsymbol{Q}}(\boldsymbol{R}) & (q\colon \text{odd}) \\ H_{kq}(S^q_{\boldsymbol{R}}; \boldsymbol{Z}) & (q\colon \text{even}). \end{cases}$$

Pasternack [P] has proved that $BR\Gamma_1^+$ is a $K(\boldsymbol{R}, 1)$ so that V_k is an isomorphism for all k.

Conjecture 5.1. *The homomorphisms* V_k $(k=1, 2, \cdots)$ *are surjective for any* q.

Let $L: \boldsymbol{R}^{kq} \to \boldsymbol{R}^q$ be a linear map of maximal rank. Then it induces a codimension q Riemannian foliation on $\boldsymbol{R}^{kq}$, which is invariant under parallel translations of $\boldsymbol{R}^{kq}$. Hence taking the quotients of it with respect to various lattices of $\boldsymbol{R}^{kq}$ of maximal rank, we obtain codimension q Riemannian foliations on T^{kq}. It is likely that these examples are enough to prove Conjecture 5.1 at least for the case when q is odd.

There is defined the notion of characteristic classes of Riemannian foliations ([LP] [Mor 1]).

Problem 5.2. *Prove the non-triviality of discontinuous invariants of Riemannian foliations.*

(II) Foliated S^1-bundles

Let $\mathrm{Diff}_+ S^1$ be the topological group of all orientation preserving C^∞ diffeomorphisms of S^1 and let $\widetilde{\mathrm{Diff}}_+ S^1$ be its universal covering group. We denote $\mathrm{Diff}_+^\delta S^1$, $\widetilde{\mathrm{Diff}}_+^\delta S^1$ for the same groups but with the discrete topologies. The Godbillon-Vey class for codimension one foliations gives rise to cohomology classes

$$\alpha \in H^2(B\mathrm{Diff}_+^\delta S^1; \boldsymbol{R})$$
$$\alpha \in H^2(B\widetilde{\mathrm{Diff}}_+^\delta S^1; \boldsymbol{R})$$
$$\beta \in H^3(B\widetilde{\mathrm{Diff}}_+^\delta S^1; \boldsymbol{R})$$

(see [Mor 2] for more details). By the procedure given in Section 3, these cohomology classes induce discontinuous invariants. To study the non-triviality of them, we first consider the group $\mathrm{Diff}_K \boldsymbol{R}$ consisting of all C^∞ diffeomorphisms of $\boldsymbol{R}$ with compact supports. If we embed $\boldsymbol{R}$ in S^1 as an oriented open interval, then $\mathrm{Diff}_K \boldsymbol{R}$ is a subgroup of $\mathrm{Diff}_+ S^1$. Hence by restriction we have

$$\alpha \in H^2(B\mathrm{Diff}_K^\delta \boldsymbol{R}; \boldsymbol{R}).$$

It is easy to see that this cohomology class is well defined independent of the choice of the embedding $\boldsymbol{R} \subset S^1$. Now for any $k \in \boldsymbol{N}$, choose k mutually disjoint oriented open intervals U_i $(i=1, \cdots, k)$ of $\boldsymbol{R}$. These define an injective homomorphism $\mathrm{Diff}_K \boldsymbol{R} \times \cdots \times \mathrm{Diff}_K \boldsymbol{R} \to \mathrm{Diff}_K \boldsymbol{R}$ and hence a map

$$j\colon \overbrace{B\mathrm{Diff}_K^\delta \boldsymbol{R} \times \cdots \times B\mathrm{Diff}_K^\delta \boldsymbol{R}}^{k \text{ times}} \longrightarrow B\mathrm{Diff}_K^\delta \boldsymbol{R}.$$

Let $\alpha\colon B\mathrm{Diff}_K^\delta \boldsymbol{R} \to K(\boldsymbol{R}, 2)$ be the map defined by the cohomology class α. Then it is easy to see that the following diagram is homotopy commutative:

$$\begin{array}{ccc} B\mathrm{Diff}^{\delta}_{K}\boldsymbol{R}\times\cdots\times B\mathrm{Diff}^{\delta}_{K}\boldsymbol{R} & \xrightarrow{j} & B\mathrm{Diff}^{\delta}_{K}\boldsymbol{R} \\ \downarrow{\scriptstyle \alpha\times\cdots\times\alpha} & & \downarrow{\scriptstyle \alpha} \\ K(\boldsymbol{R},2)\times\cdots\times K(\boldsymbol{R},2) & \xrightarrow{\lambda} & K(\boldsymbol{R},2) \end{array}$$

where λ is a map characterized by

$$\lambda^*(\iota)=\sum_{i=1}^{k} 1\times\cdots\times\overset{i}{\iota}\times\cdots\times 1.$$

Now it can be shown that the induced homomorphism

$$\lambda_*\colon H_{2k}(K(\boldsymbol{R},2)\times\cdots\times K(\boldsymbol{R},2);\boldsymbol{Z})\longrightarrow H_{2k}(K(\boldsymbol{R},2);\boldsymbol{Z})$$

is surjective (cf. Proposition 3.2, (i)). From this and the fact that

$$\alpha_*\colon H_2(B\mathrm{Diff}^{\delta}_{K}\boldsymbol{R};\boldsymbol{Z})\longrightarrow\boldsymbol{R}$$

is a surjection (see [Th 1] [Ma] and Theorem 6.5 below) we conclude

Proposition 5.3. *The homomorphisms*

$$\alpha_*\colon H_{2k}(B\mathrm{Diff}^{\delta}_{K}\boldsymbol{R};\boldsymbol{Z})\longrightarrow S^k_{\boldsymbol{Q}}(\boldsymbol{R})\quad(k=1,2,\cdots)$$

are all surjective.

Corollary 5.4. *There are surjections*

$$\begin{aligned} &H_{2k}(B\mathrm{Diff}^{\delta}_{+}S^1;\boldsymbol{Z})\longrightarrow S^k_{\boldsymbol{Q}}(\boldsymbol{R})\longrightarrow 0\\ &H_{2k+1}(B\widetilde{\mathrm{Diff}}^{\delta}_{+}S^1;\boldsymbol{Z})\longrightarrow S^{k-1}_{\boldsymbol{Q}}(\boldsymbol{R})\otimes\boldsymbol{R}\longrightarrow 0\quad(k=1,2,\cdots)\\ &H_{2k}(B\widetilde{\mathrm{Diff}}^{\delta}_{+}S^1;\boldsymbol{Z})\longrightarrow S^k_{\boldsymbol{Q}}(\boldsymbol{R})\longrightarrow 0. \end{aligned}$$

Proof. The first and the third homomorphisms are discontinuous invariants arising from the cohomology class α and their surjectivity follows directly from Proposition 5.3. The second surjection is induced from the homomorphism

$$H_{2k+1}(B\mathrm{Diff}^{\delta}_{+}S^1;\boldsymbol{Z})\longrightarrow H_{2k-2}(K(\boldsymbol{R},2);\boldsymbol{Z})\otimes_{\boldsymbol{Z}}H_3(K(\boldsymbol{R},3);\boldsymbol{Z})$$

which is defined by the discontinuous invariants arising from α and β.

§ 6. The Godbillon-Vey class (real case)

In Section 2 we have defined a homomorphism

$$\mathrm{GV}_2\colon H_6(B\Gamma_1;\boldsymbol{Z})\longrightarrow\Lambda^2_{\boldsymbol{Q}}(\boldsymbol{R})$$

and proposed the problem to prove the surjectivity of it. In this section we relate it with another problem about the homology of $B\mathrm{Diff}_K^\delta \boldsymbol{R}$. Recall that we denote $\gamma_t \in \pi_3(B\bar{\Gamma}_1)=\pi_3(B\Gamma_1)$ for the element corresponding to Thurston's codimension one foliation $\mathscr{F}_t$ on S^3 such that $\mathrm{gv}(\mathscr{F}_t)=t \in H^3(S^3;\boldsymbol{R})\cong\boldsymbol{R}$. We would like to know whether the Whitehead product $[\gamma_a, \gamma_b]$ vanishes in $\pi_5(B\bar{\Gamma}_1)$ or not when a and b are linearly independent over $\boldsymbol{Q}$.

Now in general let X be a simply connected topological space. We have a natural isomorphism $\pi_{p+1}(X)\cong\pi_p(\Omega X)$ for any p. For an element $\alpha \in \pi_{p+1}(X)$, we write $\bar{\alpha} \in \pi_p(\Omega X)$ for the corresponding element under the above isomorphism. Let $\mu\colon \Omega X\times\Omega X\to\Omega X$ be the map defined by the composition of loops. This induces the Pontrjagin product on the homology $H_*(\Omega X;\boldsymbol{Z})$: if $u \in H_p(\Omega X;\boldsymbol{Z})$, $v \in H_q(\Omega X;\boldsymbol{Z})$, then a homology class $u*v \in H_{p+q}(\Omega X;\boldsymbol{Z})$ is defined to be $u*v=\mu_*(u\times v)$, where $u\times v$ is the cross product of u and v. Let $\xi\colon \pi_p(\Omega X)\to H_p(\Omega X;\boldsymbol{Z})$ be the Hurewicz homomorphism. The following is well known.

Proposition 6.1. *For any simply connected topological space X, the kernel of the Hurewicz homomorphism $\xi\colon \pi_p(\Omega X)\to H_p(\Omega X;\boldsymbol{Z})$ is a torsion group for any p.*

There is a close relation between the Whitehead products on the homotopy group of a simply connected space and the Pontrjagin products on the homology group of its loop space as the following theorem indicates.

Theorem 6.2 (Samelson [Sa]). *Let X be a simply connected topological space and let $\alpha \in \pi_{p+1}(X)$, $\beta \in \pi_{q+1}(X)$ so that we have $[\alpha, \beta] \in \pi_{p+q+1}(X)$. Let $\bar{\alpha} \in \pi_p(\Omega X)$, $\bar{\beta} \in \pi_q(\Omega X)$, $\overline{[\alpha, \beta]} \in \pi_{p+q}(\Omega X)$ be the corresponding elements. Then we have*

$$\xi(\overline{[\alpha, \beta]})=(-1)^p(\xi(\bar{\alpha})*\xi(\bar{\beta})-(-1)^{pq}\xi(\bar{\beta})*\xi(\bar{\alpha})).$$

Combining Proposition 6.1 and Theorem 6.2, we obtain

Proposition 6.3. *Let X be a simply connected topological space. If the Pontrjagin products on $H_*(\Omega X;\boldsymbol{Q})$ is graded commutative, then all the Whitehead products in $\pi_*(X)$ have finite orders.*

Now we go back to our problem. We are concerned with the element $[\gamma_a, \gamma_b] \in \pi_5(B\bar{\Gamma}_1)$ or equivalently the corresponding element $\overline{[\gamma_a, \gamma_b]} \in \pi_4(\Omega B\bar{\Gamma}_1)$ (recall that $B\bar{\Gamma}_1$ is simply connected [H]). By Proposition 6.1, $[\gamma_a, \gamma_b]$ has finite order if and only if $\xi(\overline{[\gamma_a, \gamma_b]})$ has finite order in

$H_4(\Omega B\bar{\Gamma}_1; \boldsymbol{Z})$. By Theorem 6.2, we have

$$\xi(\overline{[\gamma_a, \gamma_b]}) = \xi(\bar{\gamma}_a) * \xi(\bar{\gamma}_b) - \xi(\bar{\gamma}_b) * \xi(\bar{\gamma}_a).$$

Hence we obtain

Proposition 6.4. *$[\gamma_a, \gamma_b]$ has finite order if and only if*

$$\xi(\bar{\gamma}_a) * \xi(\bar{\gamma}_b) = \xi(\bar{\gamma}_b) * \xi(\bar{\gamma}_a) \quad \textit{up to torsion.}$$

At this place let us recall the result of Mather [Ma] relating the homology of $B\bar{\Gamma}_1$ to that of $B\mathrm{Diff}_K^\delta \boldsymbol{R}$. Let h: $B\mathrm{Diff}_K^\delta \boldsymbol{R} \times \boldsymbol{R} \to B\bar{\Gamma}_1$ be the natural map classifying the universal codimension one foliation on $B\mathrm{Diff}_K^\delta \boldsymbol{R} \times \boldsymbol{R}$. Since this foliation is trivial in a neighborhood of the infinity of the $\boldsymbol{R}$-factor, the map h has an adjoint map H: $B\mathrm{Diff}_K^\delta \boldsymbol{R} \to \Omega B\bar{\Gamma}_1$.

Theorem 6.5 (Mather [Ma]). *The map H: $B\mathrm{Diff}_K^\delta \boldsymbol{R} \to \Omega B\bar{\Gamma}_1$ induces an isomorphism on the integral homology.*

Now let μ: $\mathrm{Diff}_K^\delta \boldsymbol{R} \times \mathrm{Diff}_K^\delta \boldsymbol{R} \to \mathrm{Diff}_K^\delta \boldsymbol{R}$ be the homomorphism defined by identifying $\boldsymbol{R}$ on the first factor with $(-\infty, 0)$ by the map say $-\exp -t$ and $\boldsymbol{R}$ on the second factor with $(0, \infty)$ by the map say $\exp t$. The homomorphism μ induces a product on the homology of $B\mathrm{Diff}_K^\delta \boldsymbol{R}$, which we denote by the letter $*$. Thus if

$$u \in H_p(B\mathrm{Diff}_K^\delta \boldsymbol{R}; \boldsymbol{Z}) \quad \text{and} \quad v \in H_q(B\mathrm{Diff}_K^\delta \boldsymbol{R}; \boldsymbol{Z}),$$

then

$$u * v = \mu_*(u \times v) \in H_{p+q}(B\mathrm{Diff}_K^\delta \boldsymbol{R}; \boldsymbol{Z}).$$

It is easy to see that the above $*$-product does not depend on the particular choices of identifications of $\boldsymbol{R}$ with $(-\infty, 0)$ and $(0, \infty)$. It is clear that the following diagram is homotopy commutative:

$$\begin{tikzcd} B\mathrm{Diff}_K^\delta \boldsymbol{R} \times B\mathrm{Diff}_K^\delta \boldsymbol{R} \arrow[r, "\mu"] \arrow[d, "H \times H"] & B\mathrm{Diff}_K^\delta \boldsymbol{R} \arrow[d, "H"] \\ \Omega B\bar{\Gamma}_1 \times \Omega B\bar{\Gamma}_1 \arrow[r, "\mu"] & \Omega B\bar{\Gamma}_1. \end{tikzcd}$$

From this follows

Proposition 6.6. *The $*$-product on $H_*(B\mathrm{Diff}_K^\delta \boldsymbol{R}; \boldsymbol{Z})$ corresponds to the Pontrjagin product on $H_*(\Omega B\bar{\Gamma}_1; \boldsymbol{Z})$ under the isomorphism of Theorem 6.5.*

Now we propose

Conjecture 6.7. *The $*$-product on $H_*(B\mathrm{Diff}_K^\delta \boldsymbol{R}; \boldsymbol{Z})$ is graded commutative at least up to torsion.*

Here it is amazing to observe the following. Let $\iota\colon \boldsymbol{R}\to S^1$ be any embedding. This induces a homomorphism $\mathrm{Diff}_K\boldsymbol{R}\to\mathrm{Diff}_+S^1$ and hence a map $i\colon B\mathrm{Diff}_K^\delta\boldsymbol{R}\to B\mathrm{Diff}_+^\delta S^1$. Then we have

$$\iota_*(u*v-(-1)^{pq}v*u)=0$$

for any $u\in H_p(B\mathrm{Diff}_K^\delta\boldsymbol{R}; \boldsymbol{Z})$ and $v\in H_q(B\mathrm{Diff}_K^\delta\boldsymbol{R}; \boldsymbol{Z})$. This follows from the fact that on S^1 we can go from $+\infty$ to $-\infty$ without passing through $\boldsymbol{R}$ and the fact that the inner automorphisms of a group induce the identity on homology. In view of Propositions 6.4 and 6.6, we obtain

Proposition 6.8. *If the $*$-product on $H_*(B\mathrm{Diff}_K^\delta\boldsymbol{R}; \boldsymbol{Q})$ is (graded) commutative for $*=2$, then the homomorphism $\mathrm{GV}_2\colon H_6(B\bar{\Gamma}_1; \boldsymbol{Z})\to\Lambda^2_{\boldsymbol{Q}}(\boldsymbol{R})$ is non-trivial. In fact its cokernel is a torsion group.*

In this way, we have reduced a problem of homotopy groups of $B\bar{\Gamma}_1$ to that of homology groups of $B\mathrm{Diff}_K^\delta\boldsymbol{R}$. However this latter problem seems to be still extremely difficult. One reason for that can be explained as follows. There is a result of Gel'fand-Feigin-Fuks [GFF] about a curious phenomenon on the characteristic classes of family of foliations. Let us consider one special case. So let $\mathscr{F}_t$ ($t\in\boldsymbol{R}$) be a C^∞ one-parameter family of codimension one foliations on a manifold M. We may assume that $\mathscr{F}_t$ is defined by a 1-form ω_t which depends smoothly on t. By the integrability condition, there is a 1-form η_t on M such that

$$d\omega_t=\eta_t\wedge\omega_t.$$

We may assume that η_t also depends smoothly on t. Now the cohomology class represented by the closed form $\eta_t\wedge d\eta_t$ is the Godbillon-Vey class $\mathrm{gv}(\mathscr{F}_t)\in H^3(M; \boldsymbol{R})$. Since $\mathscr{F}_t$ depends smoothly on t, we can differentiate $\mathrm{gv}(\mathscr{F}_t)$ with respect to t to obtain a cohomology class $\mathrm{gv}'(\mathscr{F}_t)|_{t=0}\in H^3(M; \boldsymbol{R})$. Now the result of Gel'fand-Feigin-Fuks in this particular case claims

$$\mathrm{gv}(\mathscr{F}_0)\,\mathrm{gv}'(\mathscr{F}_t)|_{t=0}=0.$$

This can be easily proved as follows. This cohomology class is represented by

$$\begin{aligned}\eta_0\wedge d\eta_0\wedge(\eta_t\wedge d\eta_t)'|_{t=0}&=\eta_0\wedge d\eta_0\wedge\{\eta_0'\wedge d\eta_0+\eta_0\wedge d\eta_0'\}\\&=0\end{aligned}$$

because we have $(d\eta_0)^2=0$. From this follows, for example, that if $\mathscr{F}_t$ ($t \in \boldsymbol{R}$) is a C^∞ one parameter family of codimension one foliations on $S^3 \times S^3$ and if we write $\mathrm{gv}(\mathscr{F}_t)=(a_t, b_t) \in \boldsymbol{R}^2 \cong H^3(S^3 \times S^3; \boldsymbol{R})$, then the ratios a_t/b_t are independent of t. It follows from this fact that even if Conjecture 6.7 were true, it cannot be proved by a construction on elements of $\mathrm{Diff}_K^\delta \boldsymbol{R}$ which varies continuously with respect to the C^∞ topology. By the same reason we can say that even if $B\bar{\Gamma}_1$ were *a* $K(\boldsymbol{R}, 3)$, the corresponding H-space structure on $B\bar{\Gamma}_1$ cannot be continuous with respect to the coarse topology of it (for the coarse topology see [Mos]).

In any case there are two extremal candidates for the space $B\bar{\Gamma}_1$, the Eilenberg-MacLane space $K(\boldsymbol{R}, 3)$ and the Moore space $M(\boldsymbol{R}, 3)$. It would be too early to say something about the homotopy type of $B\bar{\Gamma}_1$, but the result of the next section (§ 7) seems to support the first candidate.

§ 7. The Godbillon-Vey class (complex case)

There is a theory of characteristic classes for holomorphic foliations which is analogous to the C^∞ case (see [B1]). In particular if we denote $B\bar{\Gamma}_q\boldsymbol{C}$ for the classifying space of codimension q holomorphic Haefliger structures with trivial normal bundles, then there is defined a map

$$H^*(W_q \otimes_{\boldsymbol{R}} \boldsymbol{C}) \longrightarrow H^*(B\bar{\Gamma}_q\boldsymbol{C}; \boldsymbol{C})$$

or at the space level we have a map

$$B\bar{\Gamma}_q\boldsymbol{C} \longrightarrow (Y_q)_{\boldsymbol{C}}$$

where $(Y_q)_{\boldsymbol{C}}$ is the *complex type* of the space Y_q which is defined by using complex valued C^∞ forms. As before this induces discontinuous invariants of holomorphic foliations. In particular, we have homomorphisms

$$\mathrm{GV}_k^{\boldsymbol{C}}\colon H_{3k}(B\bar{\Gamma}_1\boldsymbol{C}; \boldsymbol{Z}) \longrightarrow \Lambda_{\boldsymbol{Q}}^k(\boldsymbol{C}) \quad (k=1, 2, \cdots)$$

which should be considered as complex analogue of the homomorphisms GV_k defined in Section 2. Namely they are induced by the Godbillon-Vey class $\mathrm{gv} \in H^3(B\bar{\Gamma}_1\boldsymbol{C}; \boldsymbol{C})$ which is now a complex valued cohomology class of degree 3 defined for any codimension one holomorphic foliation with trivial normal bundle.

Bott [B2] has proved that $\mathrm{GV}_1^{\boldsymbol{C}}$ is a surjection. In fact for non zero complex numbers α, β, let $\mathscr{F}(\alpha, \beta)$ be the codimension one holomorphic foliation on $\boldsymbol{C}^2-\boldsymbol{0}$ defined by the holomorphic 1-form $\alpha z_2 dz_1 + \beta z_1 dz_2$. Then Bott shows

$$\langle [S^3], \mathrm{gv}(\mathscr{F}(\alpha, \beta))\rangle = 4\pi^2\left(\frac{\beta}{\alpha}+\frac{\alpha}{\beta}-2\right)$$

which can be assumed to take any complex value by suitable choices of α and β.

Now we would like to show the non-triviality of the homomorphism GV_2^C. For that observe that the Hopf fibration $S^3 \to CP^1$ defines a codimension one holomorphic foliation on S^3, by taking the fibres as leaves, whose Godbillon-Vey number is equal to $-16\pi^2$ because we can take $\alpha/\beta = -1$. To prove the non-triviality of GV_2^C it is enough to show the existence of a certain non-trivial family of the Hopf fibrations along a closed 3-manifold. There exists such a thing. Namely let M^3 be a closed orientable hyperbolic 3-manifold (see [Th 2]) and let T_1M be its unit tangent bundle. Then there is a codimension two foliation on T_1M, called the Anosov foliation, whose leaves are transverse to the fibres of the projection $T_1M \to M$. The total holonomy group of this foliated S^2-bundle lies in PSL_2C which acts on $S^2 = CP^1$ holomorphically. Hence this foliation can be considered as a codimension one holomorphic foliation. Since M is parallelizable, the total holonomy group lifts to SL_2C (the unique obstruction to lift a homomorphism $\pi_1(M) \to PSL_2C$ to SL_2C is the second Stiefel-Whitney class which is zero in this case). This implies that we can take an S^1-bundle E_M over T_1M which is a sort of family of Hopf fibrations over M. E_M has a codimension one holomorphic foliation $\mathscr{F}_M$ whose restriction to any fibre is isomorphic to the one induced from the Hopf fibration. We will compute the discontinuous invariant $\mathrm{GV}_2^C(E_M, \mathscr{F}_M) \in \Lambda_Q^2(C)$ of this foliation. Let $v(M)$ be the volume of M and let $\eta(M)$ be the η-invariant of M (see [APS]).

Theorem 7.1. *Let M be a closed oriented hyperbolic 3-manifold. Then we have*

$$\mathrm{GV}_2^C(E_M, \mathscr{F}_M) = 192\pi^2 \wedge \pi^2\eta(M) - 64\pi^2 \wedge iv(M) \in \Lambda_Q^2(C).$$

Corollary 7.2. *The homomorphism* $\mathrm{GV}_2^C\colon H_6(B\bar{\Gamma}_1C; Z) \to \Lambda_Q^2(C)$ *is non-trivial.*

Proof of Theorem 7.1. First of all we take a second way of viewing our foliation $\mathscr{F}_M$ which is convenient for computation. Let H^3 be the 3-dimensional hyperbolic space. Then the group of isometries of it is PSL_2C and we have a fibration

$$PSU_2 \longrightarrow PSL_2C \xrightarrow{p} H^3$$

which can be seen as the principal $SO(3)$-bundle associated to the tangent bundle of H^3. We have also a fibration

$$SU_2 \xrightarrow{i} SL_2\boldsymbol{C} \xrightarrow{p} H^3$$

which should be considered as the "fibre-wise Hopf fibration" over the unit tangent bundle T_1H^3. Now the fundamental group $\pi_1(M)$ acts on these fibrations freely and if we take the quotient, then we obtain the tangent orthonormal frame bundle of M

$$SO(3) \longrightarrow F_M \longrightarrow M$$

and the fibration

$$S^3 \longrightarrow E_M \longrightarrow M.$$

Now our foliation $\mathscr{F}_M$ on E_M is the quotient by $\pi_1(M)$ of the codimension one holomorphic foliation $\mathscr{F}$ on $SL_2\boldsymbol{C}$ defined as follows. Let

$$X_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad X_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

be a basis over $\boldsymbol{C}$ of the Lie algebra $\mathfrak{sl}_2\boldsymbol{C}$ and let ω_i ($i=0, 1, 2$) be the dual basis. We consider ω_i as a left invariant complex valued 1-form on $PSL_2\boldsymbol{C}$ and $SL_2\boldsymbol{C}$. It is easy to see that

$$\begin{aligned} d\omega_0 &= -\omega_1 \wedge \omega_2 \\ d\omega_1 &= -2\omega_0 \wedge \omega_1 \\ d\omega_2 &= 2\omega_0 \wedge \omega_2. \end{aligned}$$

Now $\mathscr{F}$ is defined by the 1-form ω_1. Therefore the associated Godbillon-Vey form is $-4\omega_0\omega_1\omega_2$. If we write

$$\omega_0\omega_1\omega_2 = \phi + i\psi$$

for some real forms ϕ, ψ, then computation shows that

(i) On $PSL_2\boldsymbol{C}$, ϕ is $\pi^2 Q$ where Q is the Chern-Simons form corresponding to the 1-st Pontrjagin class,

(ii) $\psi = -p^*v +$ exact form, where v is the volume form of H^3 and $p\colon PSL_2\boldsymbol{C}$ (or $SL_2\boldsymbol{C}$)$\to H^3$ is the projection

(see [Y]). Moreover the restriction of $\omega_0\omega_1\omega_2$ to the fibre SU_2 equals $2\times$volume-form of SU_2. Now choose a cross-section $s\colon M \to E_M$ (recall that M is parallelizable so that $E_M \cong M \times S^3$). This induces a cross-section

$\bar{s}\colon M \to F_M$. Now let z and w be the homology classes of E_M represented by the fibre S^3 and $s(M)$ respectively. Then we have

$$\begin{aligned}\langle z, \mathrm{gv}(\mathscr{F}_M)\rangle &= \int_{SU_2} i^*(-4\omega_0\omega_1\omega_2)\\ &= \int_{SU_2} -8 \text{ volume form of } SU_2\\ &= -16\pi^2\end{aligned}$$

because $v(S^3)=2\pi^2$. Observe that the above computation coincides with Bott's one for the Hopf fibration. Next we have

$$\begin{aligned}\langle w, \mathrm{gv}(\mathscr{F}_M)\rangle &= \int_{\bar{s}(M)} -4(\phi+i\psi)\\ &= -4\pi^2 \int_{\bar{s}(M)} Q + 4i \int_M v\\ &= -4\pi^2(3\eta(M)-3\delta(M, \bar{s}))+4iv(M)\end{aligned}$$

because according to Yoshida [Y], we have

$$\int_{\bar{s}(M)} Q = 3\eta(M) - 3\delta(M, \bar{s})$$

for any closed oriented Riemannian 3-manifold M with an orthonormal framing $\bar{s}$, where $\delta(M, \bar{s})$ denotes the Hirzebruch's invariant of the framed manifold $(M, \bar{s})$. If we denote $x, y \in H^3(E_M; Z)$ for the cohomology classes dual to z, w, then the above computation implies

$$\mathrm{gv}(E_M, \mathscr{F}_M) = -16^2\pi x + \{-4\pi^2(3\eta(M)-3\delta(M, \bar{s}))+4iv(M)\}y.$$

Hence if we orient E_M by requiring $xy=1 \in H^6(E_M, Z)$, then we have

$$\mathrm{GV}_2^C(E_M, \mathscr{F}_M) = 192\pi^2 \wedge \pi^2\eta(M) - 64\pi^2 \wedge iv(M).$$

Here we have used the fact that $3\delta(M, \bar{s})$ is an integer. This completes the proof.

Since there are only countably many isometry classes of hyperbolic 3-manifolds, the set of values of the above examples is very small in $\Lambda^2_Q(C)$.

Problem 7.3. *Prove that the homomorphism* $\mathrm{GV}_2^C\colon H_6(B\bar{\Gamma}_1 C; Z) \to \Lambda^2_Q(C)$ *is surjective.*

Remark 7.4. Observe that the non-triviality of the above examples can be detected by the cohomology class $\mathrm{gv}\overline{\mathrm{gv}} \in H^6(B\bar{\Gamma}_1 C; C)$. From this point of view the above computation can be generalized to the homogeneous codimension q holomorphic foliation on $SL_{q+1}C$ which is given similarly as above (the case $q=1$) and also to the foliations constructed by Rasmussen [Ra]. In this way we obtain many non-triviality results for the discontinuous invariants of $B\bar{\Gamma}_q C$. However we omit the details.

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