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On Decompositions and Approximations of Foliated Manifolds

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Introduction

Let $(M, \mathscr{F})$ be a foliated manifold. By this, unless otherwise specified, we mean that M is a compact C^∞-manifold and $\mathscr{F}$ is a transversely oriented, codimension one, C^∞-foliation of M. Recall that a leaf F is *resilient* if F accumulates to itself by a contracting holonomy. A resilient leaf exhibits rather bizarre behaviour and does not submit to concrete qualitative study. In this paper, we observe that a foliation without resilient leaves is decomposed into three types of compact foliated submanifolds (*units*) each of which is an immersed image of a foliated interval bundle or a without holonomy foliation (Theorem 1). We call it an *NT-decomposition.*

Among foliations without resilient leaves, *PA-foliations* and *foliations of finite type* are simple ones. A foliation is said to be PA if each leaf has polynomial growth and the germinal holonomy group of each leaf is abelian, and a foliation is said to be of finite type if it is a finite union, along proper leaves, of open, connected saturated subsets without holonomy. These foliations are characterized by the property that they have some good decompositions (Proposition (3.2.2) and Theorem 2). A theorem of Mizutani states that a PA-foliation is cobordant to a union of foliated S^1-bundles over tori [Mi].

PA-foliations and foliations of finite type constitute important subspaces of the space of foliations without resilient leaves of a given manifold. We study how large these subspaces are; that is, we study when a foliation is approximated by PA-foliations or finite type foliations. By using NT-decompositions, we show that an "almost PA" foliation of a 3-manifold is C^∞-approximated by PA-foliations (Theorem 3). In general, however, C^∞-approximations seem to be hopeless. At the expense of differentiability, we can prove that a foliation without resilient leaves is $\mathscr{D}$-approximated by PA or finite type foliations of class $\mathscr{D}$ (Theorem 4).

We plan to show that the Godbillon-Vey class is defined for foliations

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of class $\mathscr{D}$, and the above theorem implies Duminy's theorem [Du 1, 2]: the vanishing of the Godbillon-Vey classes of foliations without resilient leaves.

The paper is organized as follows. In Section 1, we review the theory of levels and local minimal sets developed by Cantwell-Conlon [C-C 1] and Hector. In Section 2, we recall the tameness of totally proper leaves from [Tsuc 2]. In Section 3, we define NT-decompositions and prove Theorem 1. The proof is straightforward from the arguments in Section 2. Also, we study PA-foliations and foliations of finite type. In Section 4, we deal with the problem of C^∞-approximations by PA or finite type foliations. In Section 5, we introduce the class of foliations of class $\mathscr{D}$, and prove the $\mathscr{D}$-approximation theorem.

For a foliated manifold $(M, \mathscr{F})$, we fix a 1-dimensional foliation $\mathscr{L}$ transverse to $\mathscr{F}$, and assume each holonomy of $\mathscr{F}$ is defined with respect to $\mathscr{L}$.

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§ 1. Preliminaries

In this section we review some terminology and facts about foliations without resilient leaves. We refer the reader to [C-C 1] for a comprehensive exposition.

(1.1) Open saturated sets. Let U be an open, connected $\mathscr{F}$-saturated subset of M. Let $\hat{U}$ be the *Dippolito completion* of U (see [Di]); that is, $\hat{U}$ is the completion of U with respect to a Riemannian metric induced from M. Then $\hat{U}$ is a manifold and the inclusion $i: U \to M$ extend naturally to an immersion $\hat{i}: \hat{U} \to M$. Foliations $\hat{\mathscr{F}}$ and $\hat{\mathscr{L}}$ are induced by $\mathscr{F}$ and $\mathscr{L}$. The foliation $\hat{\mathscr{F}}$ is tangent to $\partial\hat{U}$ and $\partial\hat{U}$ is a union of finitely many leaves of $\hat{\mathscr{F}}$. There is a *Dippolito decomposition* $\hat{U}=K \cup \hat{U}_1 \cup \cdots \cup \hat{U}_q$. Here K is a compact manifold and each $\hat{U}_i$ is diffeomorphic to $B_i \times [0, 1]$, where $B_i \subset \partial\hat{U}$ is a non-compact connected submanifold and each $\{x\} \times [0, 1]$, $x \in B_i$, is a leaf of $\hat{\mathscr{L}}$. Thus $\hat{\mathscr{F}}_{|\hat{U}_i}$ is a foliated $[0, 1]$-bundle over B_i. Fixing an identification of $[0, 1]$ with $\{x_0\} \times [0, 1]$, $x_0 \in B_i$, one obtains the total holonomy homomorphism $q: \pi_1(B_i, x_0) \to \mathrm{Diff}[0, 1]$ and the total holonomy group $G = \mathrm{Image}(q)$.

The manifold K is called the *nucleus* of $\hat{U}$ and each $\hat{U}_i$ is called an *arm*.

Definition (1.1.1). If the nucleus $K \subset \hat{U}$ can be chosen so that, in each arm $\hat{U}_i = B_i \times [0, 1]$, $\hat{\mathscr{F}}$ restricts to the product foliation, then U is said to be *trivial at infinity*.

(1.2) Levels and local minimal sets. A subset X of M is a *local minimal set* if there is an open $\mathscr{F}$-saturated set U and $X \subset U$ is a minimal set of the foliation $\mathscr{F}_{|U}$.

These sets are of three types;

(a) every proper leaf is a local minimal set;

(b) an open $\mathscr{F}$-saturated set $U \subset M$, in which each leaf of $\mathscr{F}_{|U}$ is dense in U, is said to be an *open local minimal set* or a local minimal set of *locally dense type*;

(c) a local minimal set of neither type (a) nor type (b) is said to be of *exceptional type.*

There is a *level filtration* $\{M_k\}$ of M which is defined by the following;

(a) $M_{-1} = \emptyset$;

(b) $M_{k+1} = M_k \cup$ {all minimal sets of $M - M_k$};

(c) $M_\infty = \bigcup_{k \geqq 0} M_k$.

Then each M_k is a closed $\mathscr{F}$-saturated subset and M_∞ is dense in M. A local minimal set X, and each of its leaves, is said to be at *level* k if $X \subset M_k - M_{k-1}$. A leaf F is said to be *at infinite level* if $F \not\subset M_\infty$. The *height* $h(\mathscr{F})$ of $\mathscr{F}$ is defined to be $h(\mathscr{F}) = \sup\{k;\ M_k \neq \emptyset\}$.

Theorem (1.2.1) ([C-C1; Lemma (5.3)]). *There is an integer $p(\mathscr{F})$ such that, for each $p \geq p(\mathscr{F})$, each connected component of $M - M_p$ is a foliated $\mathring{I}$-bundle.*

A one dimensional submanifold T of M is called a *sufficient transversal* if T is transverse to $\mathscr{F}$ and each leaf of $\mathscr{F}$ meets the interior of T. Choose a sufficient transversal T which is a finite union of compact subarcs of the leaves of $\mathscr{L}$ and choose a Riemannian metric of M. For a foliated $\mathring{I}$-bundle $U \subset M$, we define the *total width* $\delta(U)$ of U by $\delta(U)$ = length of $T \cap U$. Since M_∞ is dense in M, we get the following.

Corollary (1.2.2). *Given $\varepsilon > 0$, there is an integer p_ε such that each connected component of $M - M_{p_\varepsilon}$ is a foliated $\mathring{I}$-bundle of total width $< \varepsilon$.*

(1.3) Resilient leaves. A leaf F is *resilient* if there exist elements f, g of the holonomy pseudogroup of $\mathscr{F}$ and a point $x \in F \cap \mathrm{dom}(f) \cap \mathrm{dom}(g)$ such that $g(x) = y \neq x$ and $\lim_{n \to \infty} f^n(y) = x$. A resilient leaf is non-proper, at a finite level, and has exponential growth. It is easy to see that an open local minimal set U contains a resilient leaf unless $\mathscr{F}_{|U}$ is without holonomy. On the other hand, a local exceptional minimal set contains a resilient leaf by a generalized version of Sacksteder's theorem [C-C 1]. Thus we find the following.

Proposition (1.3.1). *Assume $\mathcal{F}$ has no resilient leaves. Then each local minimal set is either a proper leaf or an open local minimal set without holonomy. The set M_∞ is a disjoint union of proper leaves and countably many open, connected $\mathcal{F}$-saturated subsets without holonomy.*

Definition (1.3.2) (see [Di]). A proper leaf F is said to be *semi-stable* on the positive side if it has arbitrarily thin, $\mathcal{F}$-saturated, one-sided tubular neighbourhoods on the positive side. Such a neighbourhood is called a *semi-stable collar* on the positive side of F. A proper leaf is said to be *stable* on the positive side if there is a trivially foliated semi-stable collar on the positive side of F.

Definition (1.3.3). A proper leaf F is said to be *unbounded* on the positive side if there is a leaf F' which accumulates to F from the positive side. A one-sided tubular neighbourhood $F\times[0, 1]$ of the positive side of F with $F\times\{0\}=F$ is called an *unbounded collar* if each leaf of $\mathcal{F}_{|F\times(0,1]|}$ contains $F=F\times\{0\}$ in its limit set. A proper leaf F is said to be *contracting* on the positive side if the holomony group of the positive side of F contains a contracting element.

Obviously, an unbounded side of a leaf has an unbounded collar, and a contracting side of a leaf is unbounded.

Lemma (1.3.4). *Let F be a proper leaf of a foliation without resilient leaves. If the positive side of F is unbounded, then F is contracting on the positive side.*

Proof. If the leaf F' in (1.3.3) is chosen to be totally proper, then F' accumulates to F in a staircase (see § 2) and F is contracting. Otherwise all leaves in $F\times(0, 1]$ are contained in an open local minimal set, and the holonomy group on the positive side of F is fixed point free (see (1.4)). Thus F is contracting. q.e.d.

Proposition (1.3.5). *Let $\mathcal{F}$ be a foliation without resilient leaves and F a proper leaf. Then the positive side of F is either semi-stable or contracting.*

Proof. In general, it is known that a proper side of a leaf is either semi-stable or unbounded [Di]. Proposition follows from the above lemma. q.e.d.

Corollary (1.3.6). *Reeb stability for proper leaves* (*see* [In]) *holds for foliations without resilient leaves.*

(1.4) Open saturated sets without holonomy. In this subsection we assume that $\mathscr{F}$ has no resilient leaves. Let U be an open, connected $\mathscr{F}$-saturated set without holonomy. Let L be a leaf of $\hat{\mathscr{L}}$ (L is either a closed interval, a half open interval, an open interval or a circle). Since L is oriented, for points x, y of L, we can and do use an interval notation $[x, y]$. Let x_0 be a point of L. One can define the *Novikov transformation* $\hat{q}$: $\pi_1(\hat{U}, x_0) \to \mathrm{Diff}(L)$ as follows (see e.g. [Tsuc 1, § 5]). Let α be an element of $\pi_1(\hat{U}, x_0)$, $c: (S^1, 0) \to (\hat{U}, x_0)$ a representative of α and x a point of L. Consider the loop $\sigma = [x, x_0]*c*[x_0, x]$ based at x. It is seen that σ is homotopic relative to $\{x\}$ to a loop of the form $\tau_1 * \tau_2$ where τ_1 is contained in L and τ_2 is contained in the leaf of $\hat{\mathscr{F}}$ through x. We define $\hat{q}$ by $\hat{q}(\alpha)(x) =$ the initial point of τ_1. One can prove the following.

Proposition (1.4.1). *The map $\hat{q}$ is a well-defined homomorphism, the image* $\mathrm{Im}(\hat{q})$ *of $\hat{q}$ acts freely on* $\mathrm{Int}(L)$ *and is abelian. Each leaf of $\mathscr{F}_{|U}$ is closed in U if* $\mathrm{rank}(\mathrm{Im}(\hat{q})) \leqq 1$. *Otherwise, each leaf of $\mathscr{F}_{|U}$ is dense in U and U is an open local minimal set.*

Since the image of $\hat{q}$ is abelian, it factors as follows:

$$\hat{q}:\ \pi_1(\hat{U}, x_0) \longrightarrow H_1(\hat{U}; Z) \xrightarrow{\ q\ } \mathrm{Diff}(L).$$

Let F be a $\hat{\mathscr{F}}$-leaf in $\partial \hat{U}$, $x_0 \in F$, and assume L is the $\hat{\mathscr{L}}$-leaf through x_0. Let $\mathrm{hol}_F^+: \pi_1(F, x_0) \to G$ be the holonomy map of the leaf F where G is the group of germs at x_0 of local diffeomorphisms of (L, x_0). From (1.4.1), one can easily get the following.

Proposition (1.4.2). *The map* hol_F^+ *lifts canonically to a homomorphism* $\widehat{\mathrm{hol}_F^+}: \pi_1(F, x_0) \to \mathrm{Diff}(L, x_0)$ *and factors through q in the following diagram;*

$$\begin{tikzcd}
\pi_1(F, x_0) \arrow[r, "\mathrm{hol}_F^+"] \arrow[d] \arrow[dr, "\widehat{\mathrm{hol}_F^+}"] & G \\
\pi_1(\hat{U}, x) \arrow[r, "\hat{q}"] \arrow[d] & \mathrm{Diff}(L, x_0) \arrow[u] \\
H_1(\hat{U}; Z) \arrow[ur, "q"'] &
\end{tikzcd}$$

where vertical arrows are natural homomorphisms.

§ 2. Totally proper leaves and staircases

(2.1) A leaf F is *totally proper* if each leaf contained in the limit set of F is proper [C-C 1]. It is known that a totally proper leaf spirals on leaves in its limit set very finely [C-C 1], [Tsuc 2]. There are some ways to describe the situation. Here we prefer to use the notion of staircases [N], [Tsuc 2], since it is closely related with our definition of decompositions. The material of this section is a summary of [Tsuc 2].

We need some notions. Let K be a connected compact manifold and let N be a closed, transversely oriented, codimension one submanifold of the interior of K which does not separate K. Let $C(K, N)$ denote the compact manifold with boundary which is obtained from $K-N$ by attaching two copies N_1 and N_2 of N as boundary, where the transverse orientation is inward (resp. outward) pointing on N_1 (resp. N_2). Let ι: $N_2 \rightarrow N_1$ be the identity map. Let f: $[0, \delta_1] \rightarrow [0, \delta_2]$, $\delta_2 = f(\delta_1) < \delta_1$, be a contracting diffeomorphism. We denote by $X(K, N, f)$ the manifold with corner which is the quotient space of $C(K, N) \times [0, \delta_1]$ by the equivalence relation $\sim$ which is defined by $(\iota(x), t) \sim (x, f(t))$ for $t \in [0, \delta_1]$ and $x \in N_2$. Let $\mathscr{F}(K, N, f)$ denote the foliation of $X(K, N, f)$ induced from the product foliation $\{C(K, N) \times \{t\}\}$, $t \in [0, \delta_1]$, of $C(K, N) \times [0, \delta_1]$. Finally $\mathscr{L}(K, N, f)$ denotes the one dimensional foliation of $X(K, N, f)$ which is induced from the foliation $\{\{x\} \times [0, \delta_1]\}$, $x \in C(K, N)$, of $C(K, N) \times [0, \delta_1]$.

Definition (2.1.1). Let $(S, \mathscr{F}_S)$ be a compact foliated manifold. We say $(S, \mathscr{F}_S)$ is a *staircase* if there are K, N, f as above and a diffeomorphism h from $X(K, N, f)$ to S which sends the leaf of $\mathscr{F}(K, N, f)$ through $N_1 \times \{\delta_1\}$ to a leaf of $\mathscr{F}_S$, and $\mathscr{L}(K, N, f)$ to the one-dimensional foliation $\mathscr{L}$ transverse to $\mathscr{F}_S$. If the diffeomorphism h can be chosen to be foliation-preserving, we call $(S, \mathscr{F}_S)$ a *regular staircase*.

We call $C(S) = h(C(K,N) \times \{\delta_1\})$, $F(S) = h(C(K, N) \times \{0\})$, $W(S) = h(N_2 \times [\delta_2, \delta_1])$ and $D(S) = h(\partial K \times [0, \delta_1])$, the *ceiling*, the *floor*, the *wall* and the *door* of $(S, \mathscr{F}_S)$ respectively. And we call f the *slope* of the staircase.

Let $(S, \mathscr{F}_S)$ be a staircase which is the image of the composed map

$$h: C(K, N) \times [0, \delta_1] \rightarrow X(K, N, f) \rightarrow S.$$

The induced foliation $h^*(\mathscr{F}_S)$ of $C(K, N) \times [0, \delta_1]$ is transverse to the fibres $\{x\} \times [0, \delta_1]$, $x \in C(K, N)$. So $h^*(\mathscr{F}_S)$ is a foliated interval bundle and is determined by the total holonomy map q: $\pi_1(C(K, N)) \rightarrow \mathrm{Diff}[0, \delta_1]$. We call q (resp. the image of q) the *reduced total holonomy map* (resp. the *reduced total holonomy group*) of $(S, \mathscr{F}_S)$. The staircase $(S, \mathscr{F}_S)$ may be viewed as an immersed image of the foliated interval bundle $(C(K, N) \times [0, \delta_1], h^*(\mathscr{F}_S))$. Now let $(M, \mathscr{F})$ be a foliated manifold, $(S, \mathscr{F}_S)$ a staircase and ϕ a foliation preserving imbedding of $(S, \mathscr{F}_S)$ into $(M, \mathscr{F})$. By abuse of language, we often identify $(S, \mathscr{F}_S)$ and its image $(\phi(S), \phi(\mathscr{F}_S))$ in $(M, \mathscr{F})$. Let F be a leaf of $\mathscr{F}$ which intersects the staircase S.

Definition (2.1.2). We say F is *well-behaved* in S if each connected component of $F \cap S$ is closed in the interior of S.

The following lemma is easy to prove (see [Tsuc 2]).

Lemma (2.1.3). *A leaf F is well-behaved in S if and only if each element of the reduced holonomy group of S leaves the points of $F \cap \{x_0\} \times [0, \delta_1]$ pointwise fixed, where x_0 is a base point of N_1.*

Let $\mathfrak{S}$ be a finite family of staircases of $(M, \mathscr{F})$ satisfying the following conditions.

(A1) The interiors $\mathrm{Int}(S)$ with $S \in \mathfrak{S}$ are disjoint.

(A2) The walls $W(S)$ with $S \in \mathfrak{S}$ are disjoint.

(A3) For each $S \in \mathfrak{S}$, the door $D(S)$ of S is contained in the union $\cup \{W(S'); S' \in \mathfrak{S}\}$.

For two staircases $S, S' \in \mathfrak{S}$, we denote $S < S'$ if $D(S') \cap W(S) \neq \emptyset$, We also denote by the same symbol $<$ the relation in $\mathfrak{S}$ which is generated by the above relation $<$. For $S \in \mathfrak{S}$, we define $B(S) = \cup \{S' \in \mathfrak{S}; S' \leqq S\}$. If $X \subset Y \subset M$, the *$\mathscr{F}$-saturation* $\mathrm{Sat}_Y(X)$ of X in Y is the set of points y of Y such that the leaf F_y of the restricted foliation $\mathscr{F}_{|Y}$ through y intersects X.

Definition (2.1.4). We say $\mathfrak{S}$ is an *admissible family* of staircases if $\mathfrak{S}$ satisfies the above three conditions (A1)-(A3) and the followings.

(A4) The relation $<$ is a partial order of $\mathfrak{S}$.

(A5) For each $S \in \mathfrak{S}$, the saturations $\mathrm{Sat}_{B(S)}(C(S))$ and $\mathrm{Sat}_{B(S)}(F(S))$ are well-behaved in each staircase $S' < S$.

Definition (2.1.5). A leaf F is *tame* if there is an admissible family $\mathfrak{S}$ of staircases satisfying the following conditions.

(T 1) F is well-behaved in each staircase S of $\mathfrak{S}$.

(T 2) The set $F - \cup \{S; S \in \mathfrak{S}\}$ is relatively compact in F.

In this case we say F is tame in $\mathfrak{S}$ or $\mathfrak{S}$ tames F.

The following term "*thinning*", which was introduced by Nishimori [N], is useful afterwards.

Definition (2.1.6). Let $(S, \mathscr{F}_S)$ be a staircase which is the image of $C(K, N) \times [0, \delta_1]$. Let n be a non-negative integer. The n-thinning $(S^{(n)}, \mathscr{F}_{S^{(n)}})$ of $(S, \mathscr{F}_S)$ is the staircase which is the image of $C(K, N) \times [0, f^n(\delta_1)]$, where f is the slope of S.

Let $\mathfrak{S}$ be an admissible family of staircases and α a non-negative integer valued function on $\mathfrak{S}$. Then there exist uniquely an admissible family $\mathfrak{S}^{(\alpha)}$ of staircases and a bijection $j^{(\alpha)}: \mathfrak{S} \to \mathfrak{S}^{(\alpha)}$ such that $j^{(\alpha)}(S) \cap S$ is the $\alpha(S)$-thinning of S for each $S \in \mathfrak{S}$ (see [N]).

Definition (2.1.7). The admissible family $\mathfrak{S}^{(\alpha)}$ as above is called the α-thinning of $\mathfrak{S}$.

Lemma (2.1.8) (Nishimori [N; Proposition 7]). *Let $\mathfrak{S}$ be an admissible family of staircases. Let K be a compact subset of M such that $K \cap F^*(S) = \emptyset$ for each $S \in \mathfrak{S}$, where $F^*(S)$ is the leaf of $\mathcal{F}$ through $F(S)$. Then there is a non-negative integer valued function α on $\mathfrak{S}$ such that $K \cap \bigcup\{j^{(\alpha)}(S); S \in \mathfrak{S}\} = \emptyset$.*

Now we can state the main result of [Tsuc 2].

Theorem (2.1.9). *Let $C = \bigcup_{i=1}^{k} F_i$ be a closed saturated subset of $(M, \mathcal{F})$ consisting of finitely many leaves. Then there is an admissible family $\mathfrak{S}$ of staircases which satisfies the following conditions.*

(1) *For each $S \in \mathfrak{S}$, the floor $F(S)$ and the ceiling $C(S)$ are contained in C.*

(2) *Each $F_i \subset C$ is tame in $\mathfrak{S}$.*

(2.2) Scaffoldings. Let C be a closed subset of M consisting of finitely many leaves of $\mathcal{F}$. Of course, each leaf in C is totally proper.

Definition (2.2.1). We say C is a *scaffolding* of $\mathcal{F}$ if the following condition is satisfied: Let U be a connected component of $M - C$ (these components are finite in number since C consists of finitely many leaves). Then one of the following two cases occurs.

(A) The restricted foliation $\mathcal{F}_{|U}$ is without holonomy.

(B) In the Dippolito completion $\hat{U}$ of U, each leaf of the induced one-dimensional foliation $\hat{\mathcal{L}}$ is diffeomorphic to the unit interval I. In other words, the induced foliation $\hat{\mathcal{F}}$ is a foliated I-bundle.

We say U is a *type* (A) *component* if $\mathcal{F}_{|U}$ is without holonomy. Otherwise, U is said to be a *type* (B) *component.*

Proposition (2.2.2). *Let $(M, \mathcal{F})$ be a closed foliated manifold without resilient leaves. Let C_{-1} be a closed subset of M consisting of finitely many leaves of $\mathcal{F}$, and let ε be a positive real number. Then there is a scaffolding $C \supset C_{-1}$ which satisfies the following condition; for each type* (B) *component U of $M - C$, the total width $\delta(U)$ of U is smaller than ε.*

Proof. Inductively, we define an increasing sequence of subsets $C_0 \subset C_1 \subset \cdots \subset C_p \subset \cdots \subset M$ which satisfies the following conditions.

(1) C_p is a closed $\mathscr{F}$-saturated subsets consisting of finitely many leaves at level $\leqq p$.

(2) Each connected component U of $M-C_p$ is one of the following three types;

(A) $\mathscr{F}_{|U}$ is without holonomy;

(B) U is a foliated $\mathring{I}$-bundle of total width $<\varepsilon$;

(C) U is neither type (A) nor type (B) and U is a connected component of $M-M_p$.

From the conditions, M_p is contained in the union $C_p \cup$ type (A) components $\cup$ type (B) components. So if $p \geqq p_\varepsilon$ (see (1.2.2)), then $C=C_{-1} \cup C_p$ is a desired scaffolding.

First we define C_0. Let T_0 be the union of all compact leaves of $\mathscr{F}$, and T_0' be the set of compact leaves which are semi-stable on the positive or negative side. T_0' is a compact subset of M. For each leaf $K \subset T_0'$, we choose a possibly one-sided collar of K in M as follows. If K is semi-stable on the positive side (resp. negative side) and is contracting on the negative side (resp. positive side), we choose a semistable one-sided collar $K \times [0, 1]$ of total width $<\varepsilon$ on the positive (resp. negative) side of K. If K is semi-stable on both sides, we choose a neighbourhood $K \times [-1, 1]$ of total width $<\varepsilon$ with $K \times \{0\} = K$, and $K \times [0, 1]$ and $K \times [-1, 0]$ being semi-stable one-sided collars of K. Since T_0' is compact, there is a finite subcover $\cup K_i \times [0, 1] \cup \cup K_j \times [-1, 1]$ of the above covering. Let C_0 be the compact $\mathscr{F}$-saturated set consisting of $K_i \times \{0\}$, $K_i \times \{1\}$, $K_j \times \{-1\}$, $K_j \times \{1\}$ and all other compact leaves K contained in $M - \cup K_i \times [0, 1] \cup \cup K_j \times [-1, 1]$. Then C_0 satisfies the conditions (1), (2) with $p=0$.

Assume that C_p with the conditions (1) and (2) is defined. We define C_{p+1}. Let V_j $(j=1, \cdots, k)$ be the connected components of type (C) of $M-C_p$, and let $V=\bigcup_{j=1}^k V_j$. Then each V_j contains a totally proper leaf at level $p+1$. Let $T_{p+1}(V)$ be the set of all such leaves. Then $T_{p+1}(V)$ is a closed $\mathscr{F}$-saturated subset of V. Let $T_{p+1}'(V)$ be the set of semi-stable proper leaves in $T_{p+1}(V)$. For each $F_i \in T_{p+1}'(V)$, there is a semi-stable, possibly one-sided, collar $F_i \times [0, 1]$, $F_i \times [-1, 0]$ or $F_i \times [-1, 1]$ of F as above of total width $<\varepsilon$.

Assertion. *There is a finite subcover of the above covering.*

Proof. Let $\hat{V} = \hat{V}_1 \cup \cdots \cup \hat{V}_k$ be the Dippolito completion of V. For each j, fix a nucleus K_j of V_j. For each leaf $F \subset \partial \hat{V}_j$, there is an unbounded collar $N(F)$ of F in V_j. Put $N_j = \cup \{N(F);\ F \subset \partial V_j\}$, and put $K = \bigcup_{j=1}^k \{K_j - N_j\}$. Then K is a compact subset of V, and it follows that $T_{p+1}'(V) \cap K$ is compact. Choose a finite subcover $\mathscr{U}_K$ by $F_i \times [0, 1] \cap K$, $F_i \times [-1, 0] \cap K$ or $F_i \times [-1, 1] \cap K$ of K. Then the $\mathscr{F}$-saturation $\mathscr{U}$ of

$\mathscr{U}_K$ is a finite covering of $T'_{p+1}(V)$ by semistable collars.

Let C_{p+1} be the union of C_p, $\{\hat{i}(\partial\hat{U});\ U \in \mathscr{U}\}$ and all other totally proper leaves at level $p+1$ that are contained in $V - \bigcup\{\hat{i}(\partial\hat{U});\ U \in \mathscr{U}\}$. Then C_{p+1} satisfies the conditions (1) and (2). q.e.d.

§ 3. NT-decomposition

(3.1) Units and decompositions. Let M be a compact connected manifold possibly with corner and $\mathscr{F}$ a codimension one foliation of M. We assume that the boundary ∂M of M is divided by the corner into two parts; the *tangent boundary* $\partial_{\tan}M$ which is tangent to $\mathscr{F}$ and the *transverse boundary* $\partial_{\mathrm{tr}}M$ which is transverse to $\mathscr{F}$. Such a foliated manifold $(M, \mathscr{F})$ will be called a *unit*. We always choose a one-dimensional foliation $\mathscr{L}$ transverse to $\mathscr{F}$ so that it is tangent to $\partial_{\mathrm{tr}}M$. A nucleus of the Dippolito decomposition of an open $\mathscr{F}$-saturated set and a staircase are important examples of units.

Definition (3.1.1). Let M be a closed manifold of dimension n, and $\mathscr{F}$ a codimension one foliation of M. A pair (Δ, ϕ), where $\Delta = \{(M_i, \mathscr{F}_i);\ i=1, \cdots, m\}$ is a finite family of n-dimensional units and ϕ is a foliation preserving immersion from the disjoint union $\bigsqcup_{i=1}^{m}(M_i, \mathscr{F}_i)$ to $(M, \mathscr{F})$, is called a *decomposition* of $(M, \mathscr{F})$ if the following conditions are satisfied;

(D 1) for each i, $\phi|_{\mathrm{Int}\,(M_i)}$ is an imbedding,

(D 2) if $i \neq j$, then $\phi\,(\mathrm{Int}\,(M_i)) \cap \phi\,(\mathrm{Int}\,(M_j)) = \emptyset$, and

(D 3) $\bigcup_{i=1}^{m} \phi(M_i) = M$.

As in [N] and [Tsuc 3], we use three types of units. One of those is a staircase.

Definition (3.1.2). A unit (M, F) is said to be a *room* if it admits a structure of a foliated I-bundle with fibres being leaves of $\mathscr{L}$. If the total holonomy group of $(M, \mathscr{F})$ is abelian, then we say that $(M, \mathscr{F})$ is an *abelian room.*

Definition (3.1.3). A unit $(M, \mathscr{F})$ is said to be a *hall* if each corner of M is convex (see [Tsuc 3]) and each interior leaf has trivial holonomy.

If $(M, \mathscr{F})$ is a room or a hall, $D(M) = \partial_{\mathrm{tr}}M$ is called the *door* of $(M, \mathscr{F})$.

Definition (3.1.4). Let $(M, \mathscr{F})$ be a closed foliated manifold. A decomposition $(\Delta = \{(M_i, \mathscr{F}_i);\ i=1, \cdots, m\}, \phi)$ of $(M, \mathscr{F})$ is called an *NT-decomposition* if the following conditions are satisfied;

(NT 1) each unit $(M, \mathscr{F}_i)$ is either a staircase, a room or a hall;

(NT 2) for each i, and for each connected component D of the door of $(M_i, \mathscr{F}_i)$, there is a staircase $(M_j, \mathscr{F}_j)$ of Δ such that $\phi(D)$ is contained in $\phi(W(M_j))$, where $W(M_j)$ is the wall of M_j;

(NT 3) let $\mathfrak{S}(\Delta)$ be the set of staircases of Δ, then $\mathfrak{S}(\Delta)$ is admissible;

(NT 4) for each unit $(M_i, \mathscr{F}_i)$ in Δ, the leaves of $\mathscr{F}$ through $\phi(\partial_{\tan} M_i)$ are tame in $\mathfrak{S}(\Delta)$; and

(NT 5) for each hall $(M_i, \mathscr{F}_i)$ of Δ, the $\mathscr{F}$-saturation Sat $(\phi\,(\mathrm{Int}(M_i))$ is without holonomy.

Remark (3.1.5). (1) Again, by abuse of language, we often identify a unit $(M_i, \mathscr{F}_i)$ and its immersed image in M. In that context, $\mathrm{Int}\,(M_i)$ denotes the set $\phi(\mathrm{Int}\,(M_i))$.

(2) For each unit $(M_i, \mathscr{F}_i)$ of an NT-decomposition, we use the term *total holonomy* for the pseudogroup generated by the slope and the reduced total holonomy group in the case of a staircase, the total holonomy group in the case of a room and the image of the Novikov transformation along loops in $\mathrm{Int}\,(M_i)$ in the case of a hall respectively.

(3) The notions of rooms and halls are not mutually exclusive. When definiteness is needed, we call each unit $(M_i, \mathscr{F}_i)$ a hall if the saturation $\mathrm{Sat}\,(\mathrm{Int}\,(M_i))$ is without holonomy.

(4) Let $(M_i, \mathscr{F}_i)$ be a room or hall, and let U be the saturation $\mathrm{Sat}_M\,(\mathrm{Int}\,(M_i))$ of $\mathrm{Int}\,(M_i)$. Then the decomposition $\hat{U}=\widehat{\mathrm{Int}\,(M_i)} \cup \bigcup\{\widehat{U \cap S};\ S \in \mathfrak{S}(\Delta)\}$ gives a Dippolito decomposition of $\hat{U}$.

Let (Δ, ϕ) be an NT-decomposition. There are two natural ways to modify the decomposition. We explain them by figures. First we can modify Δ by *thinning* the set $\mathfrak{S}(\Delta)$ of staircases of Δ. See Figure 1.

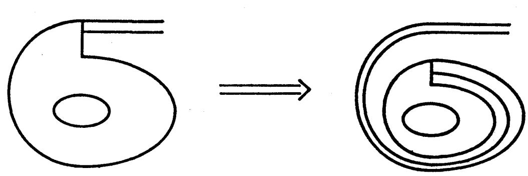

Fig. 1.

Secondly, let S be a staircase of Δ and $(M_i, \mathscr{F}_i)$ a unit such that $D(M_i) \cap W(S) \neq \emptyset$. Let F be a leaf through a connected component of $\partial_{\tan} M_i$.

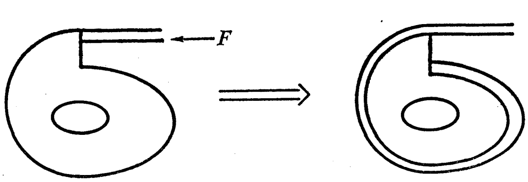

Fig. 2.

We can consider a decomposition obtained by *changing the ceiling of S to F*. See Figure 2.

The notions of NT-decompositions and scaffoldings are closely related. Let $(\varDelta, \phi)$ be an NT-decomposition. From the conditions (NT 2), (NT 4) and (NT 5), the saturation C of $\bigcup_i \{\partial_{\tan} M_i; M_i \in \varDelta\}$ is seen to be a scaffolding of M. We call C the *scaffolding associated with the decomposition.*

Conversely, from a scaffolding, one can canonically construct an NT-decomposition.

Theorem (3.1.6). *Let C be a scaffolding. Then there exists an NT-decomposition $(\varDelta, \phi)$ such that the scaffolding associated with $(\varDelta, \phi)$ coincides with C.*

Proof. Let $\mathfrak{S}$ be an admissible family of staircases which tames each leaf of C (see (2.1.9)). By changing the ceilings of staircases of $\mathfrak{S}$ and ignoring unnecessary staircases, we may assume that for each $S \in \mathfrak{S}$, the ceiling $C(S)$ and the floor $F(S)$ of S are contained in C. Let U_i be a connected component of $M-C-\bigcup\{S; S \in \mathfrak{S}\}$. Then U_i is the interior of a compact manifold with corner M_i. Let $\mathscr{F}_i$ be the induced foliation of M_i, Then $(M_i, \mathscr{F}_i)$ is a room or a hall from the definition of a scaffolding and the choice of $\mathfrak{S}$. There is a natural foliation preserving immersion $\phi_i: (M_i, \mathscr{F}_i) \to (M, \mathscr{F})$. Put $\varDelta = \mathfrak{S} \cup \{(M_i, \mathscr{F}_i)\}$ and $\phi = \{\mathrm{id}_S; S \in \mathfrak{S}\} \cup \{\phi_i\}$. Then $(\varDelta, \phi)$ gives an NT-decomposition of $(M, \mathscr{F})$.

q.e.d.

From (2.2.2) and (3.1.6) we get the following.

Theorem 1. *Let $(M, \mathscr{F})$ be a closed foliated manifold without resilient leaves. Then $(M, \mathscr{F})$ admits an NT-decomposition.*

Examples. (1) An NT-decomposition by regular staircases and abelian rooms whose total holonomy is cyclic is an *SRH-decomposition* of Nishimori [N]. A foliation admits an SRH-decomposition if and only if it is of finite depth (that is, each leaf is proper and there is a finite upper bound to the levels of leaves), and the holonomy group of each leaf is abelian.

(2) An NT-decomposition by abelian rooms and halls is a Hector-Imanishi decomposition of an almost without holonomy foliation (see e.g. [Im] and [M-M-T]).

Generalizing the above two classes of foliations, we consider PA-foliations and foliations of finite type in the next section.

(3.2) Foliations of finite type and PA-foliations.

Definition (3.2.1). A foliation $\mathscr{F}$ is said to be of *finite type* if there is a scaffolding C such that each connected component U of $M-C$ is of type (A).

The following proposition is easy to prove.

Proposition (3.2.2). *The following three conditions are equivalent.*

(1) *$(M, \mathscr{F})$ is of finite type.*

(2) *$(M, \mathscr{F})$ has an NT-decomposition into halls and staircases.*

(3) *All leaves of $\mathscr{F}$ except finitely many proper leaves have trivial holonomy.*

Definition (3.2.3). A foliation $\mathscr{F}$ is said to be *PA* if each leaf of $\mathscr{F}$ has polynomial growth and the holonomy group of each leaf is abelian.

In [Tsuc 3], an NT-decomposition by regular staircases, abelian rooms and halls was called a *Nishimori decomposition.* One of the main theorems of [Tsuc 3] is the following.

Theorem 2. *A foliation $\mathscr{F}$ is PA if and only if it admits a Nishimori decomposition.*

A foliation of finite type can have a leaf of non-polynomial growth and a leaf with non-abelian holonomy group. These phenomena result from the existence of open local minimal sets which are not trivial at infinity. A PA-foliation is a disjoint union of countably many open local minimal sets which are trivial at infinity and totally proper leaves at some bounded levels. A PA-foliation of finite type is a disjoint union of finitely many open local minimal sets which are trivial at infinity and totally proper leaves.

Proposition (3.2.4). *A PA-foliation $(M, \mathscr{F})$ is C^∞-approximated by PA-foliations of finite type.*

Let G be a finitely generated abelian subgroup of Diff$[0, 1]$. We say G is of finite type if the number of connected components of $[0, 1]-\mathrm{Fix}\, G$ is finite. The proposition follows from the following.

Lemma (3.2.5). *Let G be a finitely generated abelian subgroup of* Diff$[0, 1]$. *Then G is approximated by groups of finite type in the following sense: There are homomorphisms ϕ_n: $G \to$* Diff$[0, 1]$ $(n=1, 2. \cdots)$, *such that*

(1) *$\phi_n(G)$ is of finite type,*

(2) *for each $g \in G$, $\phi_n(g) \to g$, as $n \to \infty$ in the C^∞-topology and*

(3) *for each $g \in G$ and sufficiently large n, we have $j_0^\infty(\phi_n(g)) = j_0^\infty(g)$ and $j_1^\infty(\phi_n(g)) = j_1^\infty(g)$.*

Proof of (3.2.4) *from* (3.2.5). Let (Δ, ϕ) be a Nishimori decomposition of $(M, \mathcal{F})$. We alter $\mathcal{F}$ in abelian rooms of Δ. Let $(R, \mathcal{F}_R)$ be an abelian room with $R = B \times [0, 1]$, $q: \pi_1(B) \to \mathrm{Diff}[0, 1]$ its total holonomy homomorphism and $G = \mathrm{Im}(q)$. By (3.2.5), there are homomorphisms $\phi_n: G \to \mathrm{Diff}[0, 1]$ which approximate G. Let $\mathcal{F}_R(n)$ be the foliation of $R = B \times [0, 1]$ defined by the total holonomy homomorphism $\phi_n \circ q$. Then $\mathcal{F}_R(n) \to \mathcal{F}_R$ as $n \to \infty$, and the foliations $\{\mathcal{F}_R(n);\ R$ is a room of $\Delta\} \cup \{\mathcal{F}_H;\ H$ is a hall of $\Delta\} \cup \{\mathcal{F}_S:\ S \in \mathfrak{S}(\Delta)\}$ fit together to a C^∞-foliation $\mathcal{F}(n)$ of finite type of M for sufficiently large n. We have got a desired approximation. q.e.d.

A finitely generated abelian subgroup G of Diff [a, b] is said to be *plain* if the derived set $(\mathrm{Fix}\, G)'$ of $\mathrm{Fix}\, G$ is $\{a\}$ or $\{b\}$.

Assertion. *To prove* (3.2.5), *we may assume that G is plain.*

Proof. Let G be as in (3.2.5). Given $n > 0$, there is a sequence $0 = x_0 < x_1 < \cdots < x_m = 1$ of points of $\mathrm{Fix}\, G$ with the following properties;

(1) $\{x_1, x_2, \cdots, x_{m-1}\} \subset (\mathrm{Fix}\, G)'$ and

(2) if $x_{i+1} - x_i > 1/n$, then $(x_i, x_{i+1}) \cap (\mathrm{Fix}\, G)' = \emptyset$.

Let ϕ_n; $G \to \mathrm{Diff}[0, 1]$ be the homomorphism defined by;

$$\phi_n(g)_{|[x_i, x_{i+1}]} = \begin{cases} \mathrm{id}_{|[x_i, x_{i+1}]} & \text{if} \quad x_{i+1} - x_i \leqq 1/n \quad \text{and} \\ g_{|[x_i, x_{i+1}]} & \text{if} \quad x_{i+1} - x_i > 1/n. \end{cases}$$

Then $\{\phi_n(G)\}$ converges to G in the sense of (3.2.5). And each $\phi_n(G)_{|[x_i, x_{i+1}]}$ satisfies one of the following four conditions:

(i) $\phi_n(G)_{|[x_i, x_{i+1}]}$ is trivial.

(ii) $(\mathrm{Fix}\,(\phi_n(G)_{|[x_i, x_{i+1}]}))' = \emptyset$.

(iii) $\phi_n(G)_{|[x_i, x_{i+1}]}$ is plain.

(iv) There is $x \in (x_i, x_{i+1})$ such that $\phi_n(G)_{|[x_i, x]}$ and $\phi_n(G)_{|[x, x_{i+1}]}$ are plain. q.e.d.

To prove (3.2.5), we use the "flattening homomorphism" introduced by Tsuboi and Muller ([Tsub 2], [Mu]). Let h be a homeomorphism of [0, 1] which satisfies the following conditions.

(1) $h_{|(0,1]}$ is a C^∞-diffeomorphism.

(2) $h(x) = \begin{cases} \exp(-1/x), & \text{near} \quad x = 0 \\ x \qquad , & \text{near} \quad x = 1. \end{cases}$

Let Ψ: Diff [0, 1]$\to$Homeo [0, 1] be the homomorphism defined by $\Psi(g)=h^{-2}\circ g\circ h^2$. We use the following notations; for a C^∞-function k on [0, 1], $\delta^r(k)$ = the r-th derivative of k,

$$|k|_r=\sup_{0\leqq x\leqq 1}|\delta^r(k)(x)| \quad \text{and} \quad \|k\|_r=\max\{|k|_i;\ i=1,\cdots,r\}.$$

By some calculations, one can see that the homomorphism Ψ enjoys the following properties.

(*) The image of Ψ is contained in the group $\mathrm{Diff}_\infty[0, 1]$ of C^∞-diffeomorphisms of [0, 1] which are infinitely tangent to the identity at 0.

(**) For each $r\geqq 0$, there is a constant K_r such that $|\Psi(g)-\mathrm{id}|_r\leqq K_r\cdot\|g-\mathrm{id}\|_{r+2}$.

Proof of (3.2.5). We may assume that $(\mathrm{Fix}\,G)'=\{0\}$ and $\mathrm{Fix}\,G=\{0<\cdots<x_n<\cdots<x_1<x_0=1\}$. For $m, n\in N$, we define a homomorphism $\phi_{m,n}$; $G\to$Diff [0,1] by

$$\Phi_{m,n}(g)(x)=\begin{cases} g(x) & ,\quad \text{if } x\geqq x_m,\\ A_{m,n}^{-1}\circ(\Psi(A_{m,n}\circ g\circ A_{m,n}^{-1}))\circ A_{m,n}(x), & \quad \text{if } x_{m+n}\leqq x\leqq x_m,\\ x & ,\quad \text{if } x\leqq x_{m+n},\end{cases}$$

where $A_{m,n}$ is the affine homeomorphism from $[x_{m+n}, x_m]$ to [0, 1].

Then $\Phi_{m,n}(g)$ is a C^∞-diffeomorphism of [0, 1] from (*), and $\Phi_{m,n}(G)$ is of finite type. We show that $\{\Phi_{m,n}(G)\}$ accumulates to G. For each $r\geqq 2$ and each $g\in G$, we get the following inequality from (**) and the fact that g is infinitely tangent to the identity at 0,

$$\sup_{x_{m+n}\leqq x\leqq x_m}|\delta^r\Phi_{m,n}(g)(x)|\leqq(x_m/x_m-x_{m+n})^r\cdot K_r\cdot\sup_{0\leqq x\leqq x_m}|\delta^{r+2}g(x)|.$$

For each m, choose $n(m)$ such that $x_m/x_m-x_{m+n(m)}\leqq 2$. Then we have

$$\sup_{x_{m+n(m)}\leqq x\leqq x_m}|\delta^r\Phi_{m,n(m)}(g)(x)\leqq 2^r\cdot K_r\cdot\operatorname{Sup}_{s\leqq x\leqq x_m}|\delta^{r+2}g(x)|.$$

From this inequality, it is easy to see that $\Phi_{m,n(m)}(g)\to g$ in the C^∞-topology. q.e.d.

(3.3) GV-decompositions and the localization of the Godbillon-Vey class. A unit $(M, \mathscr{F})$ is called a *GV-unit* if $\mathscr{F}$ is trivial near the transverse boundary $\partial_{\mathrm{tr}}M$. For such a unit, we can define the Godbillon-Vey class $\mathrm{gv}(\mathscr{F})$ which is an element of $H^3(M, \partial M)$ [Tsuc 3]. We say $(M, \mathscr{F})$ is an *OGV-unit* if the class $\mathrm{gv}(\mathscr{F})$ is zero. An NT-decomposition is said to be a *GV-decomposition* (resp. *OGV-decomposition*) if each unit is a GV-unit (resp. OGV-unit).

In [Tsuc 3], we have proved the following propositions.

Proposition (3.3.1). *The Godbillon-Vey class of a closed foliated manifold* $(M, \mathscr{F})$ *is zero if it admits an OGV-decomposition.*

Proposition (3.3.2). *Regular staircases, GV-abelian rooms and GV-halls are OGV-units.*

These imply, from Theorem 2, that the Godbillon-Vey class of a PA-foliation is zero. Furthermore, using the results of Duminy and Sergiescu [D-S], one can prove that GV-staircases and GV-rooms are OGV if they contain no resilient leaves. We then get the following.

Proposition (3.3.3). If $(M, \mathscr{F})$ *admits a GV-decomposition each of whose unit contains no resilient leaves, then the Godbillon-Vey class* gv $(\mathscr{F})$ *is zero.*

§ 4. C^∞-approximations by PA-foliations of finite type

In this section we consider when a foliation is C^∞-approximated by PA-foliations of finite type.

(4.1) Let $(S, \mathscr{F}_S)$ be a staircase in $(M, \mathscr{F})$ which is the image of the composed map $h\circ\pi$: $C(K, N)\times[0, \delta]\xrightarrow{\pi}S\xrightarrow{h}M$. We say $(S, \mathscr{F}_S)$ is *trivial on the wall* if $\mathscr{F}_{S|\pi(N_2\times[0,\delta])}$ is trivial. And we say $(S, \mathscr{F}_S)$ is *flat on the ceiling* if each holonomy along the leaf $h\circ\pi(C(K, N)\times\{\delta, f(\delta), \cdots, f^n(\delta), \cdots\})$ is infinitely tangent to the identity, where f is the slope of S.

Let (Δ, ϕ) be an NT-decomposition and let R be a room in Δ. We say R is *globally abelian* if the saturation Sat (Int (R)) of the interior of R is a foliated $\mathring{I}$-bundle of abelian total holonomy.

Proposition (4.1.1). *Let* $(M, \mathscr{F})$ *be a closed foliated manifold which admits an NT-decomposition* (Δ, ϕ) *satisfying the following conditions*;

(1) *each room in* Δ *is globally abelian*;

(2) *each staircase in* Δ *is trivial on the wall and flat on the ceiling.*

Then $\mathscr{F}$ *is* C^∞*-approximated by PA-foliations of finite type.*

Proof. It is sufficient to approximate $\mathscr{F}$ by PA-foliations (see (3.2.4)). We alter $\mathscr{F}$ in each staircase $(S, \mathscr{F}_S)$ of Δ. Let $(S, \mathscr{F}_S)$ be the image of $C(K, N)\times[0, \delta]$, f its slope and q: $\pi_1(C(K, N))\to\mathrm{Diff}\,[0, \delta]$ its reduced total holonomy homomorphism. For $n\in N$, we define a staircase $(S_{(n)}, \mathscr{F}_{S_{(n)}})$ as follows. The underlying manifold $S_{(n)}$ is the same as S and the reduced total holonomy homomorphism $q_{(n)}\colon \pi_1(C(K, N))\to \mathrm{Diff}\,[0, \delta]$ of $\mathscr{F}_{S_{(n)}}$ is given by;

$$q_{(n)}(\alpha)(x)=x \qquad \text{if } x \in [0, f^n(\delta)],$$

and

$$q_{(n)}(\alpha)(x)=q(\alpha)(x) \qquad \text{if } x \in [f^n(\delta), \delta],$$

for each $\alpha \in \pi_1(C(K, N))$.

Then $\mathscr{F}_{S_{(n)}}$ is well-defined since $(S, \mathscr{F}_S)$ is trivial on the wall, and is smooth since S is flat on the ceiling. Evidently, $\mathscr{F}_{S_{(n)}}\to\mathscr{F}_S$ as $n\to\infty$. The n-thinning of $(S_{(n)}, \mathscr{F}_{S_{(n)}})$ is a regular staircase. Gathering these foliations, we get a smooth PA-foliation of M since each $S \in \varDelta$ is trivial on the wall. These foliations approach to $\mathscr{F}$ in the C^∞-topology. q.e.d.

(4.2) Inflexible staircase. We consider when a foliation has an NT-decomposition each of whose staircase is flat on the ceiling.

First we recall a theorem due to Sternberg, Takens and Sergeraert (see e.g., [Tsub 1; (3.5)]).

Theorem (4.2.1). *Let g be a C^∞-diffeomorphism of* $[0, 1]$ *such that* $\mathrm{Fix}\,(g)=\{0, 1\}$, *and g' be a diffeomorphism of* $[0, 1]$ *which commutes with g. Then either*

(a) *there are coprime integers m, n and a C^∞-diffeomorphism h of* $[0, 1]$ *such that $g=h^m$ and $g'=h^n$, or*

(b) *there are real numbers s, t and a vector field ξ, C^1 on* $[0, 1]$ *and C^∞ on* $(0, 1)$, *such that g and g' are the time s and the time t map of ξ respectively.*

In the latter case, if $j_0^\infty(g)\neq j_0^\infty$ (id), *then ξ is of class C^∞ at* 0.

In the case (a), we write $g'=g^{n/m}$, and in the case (b), we write $g'=g^{t/s}$.

Let $(\varDelta, \phi)$ be an NT-decomposition and let $\mathfrak{S}(\varDelta)$ be the set of staircases of $\varDelta$. Let S be an element of $\mathfrak{S}(\varDelta)$ which is not maximal with respect to the order $<$ in $\mathfrak{S}(\varDelta)$. So there is $S_1 \in \mathfrak{S}(\varDelta)$ such that $W(S)\cap D(S_1)\neq\emptyset$. Then , by changing the ceiling of S, one can modify S to S' so that the ceiling of S' is contained in the saturation of the floor of S_1 (see Figure 3).

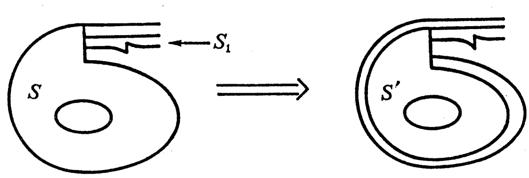

Fig. 3.

Modifying each such staircase in this way, we obtain an NT-decomposition Δ' with the following extra condition:

(NT-6) The ceiling of each staircase S in Δ' which is not maximal in $\mathfrak{S}(\Delta')$ is contained in the saturation of a floor of some staircase in Δ'.

From now on, NT-decompositions are assumed to satisfy the above additional condition.

Definition (4.2.2). Let $(S, \mathscr{F}_S)$ be a staircase, f its slope and $G \subset \mathrm{Diff}[0, \delta]$ its reduced total holonomy group. We say $(S, \mathscr{F}_S)$ is *inflexible* if there is $g \in G$ and a finite sequence of points $f(\delta)=t_0<t_1<\cdots<t_l=\delta$ which satisfy the following conditions:

(1) $\mathrm{Fix}(g) \cap [f(\delta), \delta]=\{t_0, t_1, \cdots, t_l\}$.

(2) $j^\infty_{t_i}(g) \neq j^\infty_{t_i}(\mathrm{id})$, for each i.

(3) There is a real number $\alpha \in (0, 1)$ such that $f^{-n} \circ g \circ f^n = g^{\alpha^n}$.

Lemma (4.2.3). Let $(S, \mathscr{F}_S)$ be an inflexible staircase without resilient leaves. Let f, G, g, $\{t_i\}$ and α be as above. Then we have the following properties:

(1) Let $k=\min\{i; j^i_{t_0}(g) \neq j^i_{t_0}(\mathrm{id})\}$, then

$$\alpha=\frac{\log \delta g(t_0)}{\log \delta g(t_l)} \qquad \text{if} \quad k=1, \text{ and}$$

$$\alpha=\frac{\delta^k g(t_0)}{\delta^k g(t_l)}(\delta f(t_l))^{k-1} \qquad \text{if} \quad k \geqq 2.$$

(2) The leaves of $\mathscr{F}_S$ through $\{t_i\}$ are tame in S, and all other leaves are dense in S.

(3) For each $h \in G$, there is $\beta \in \boldsymbol{R}$ such that $h=g^\beta$.

Proof. The first assertion is easily obtained by calculating the derivative of the equality $f^{-1} \circ g \circ f = g^\alpha$ at t_l. The second assertion follows from the fact $g^{\alpha^n} \to \mathrm{id}$ as $n \to \infty$. From this, each connected component of $\mathrm{Int}(S) - \bigcup$ {leaves through $\{t_i\}$} is an open local minimal set. It follows from (1.3.1) and (1.4) that the reduced total holonomy group is abelian. The third assertion then follows from (4.2.1). q.e.d.

Lemma (4.2.4). *Each non-proper leaf in an inflexible staircase has exponential growth.*

Proof. We use the notations of (4.2.2). Let F be a non-proper leaf. Choose a point $x \in [f(\delta), \delta] \cap F$. Let Γ_n be the set of points of $[0, \delta]$ which are mapped from x by words of length $\leqq n$ in f and g. Let γ_n be the cardinality of Γ_n. Then γ_n is dominated by the growth function of F (see e.g. [Tsuc 1]).

Taking α^n instead of α if necessary, we may assume that $\alpha<1/3$. The set Γ_{2n} contains the following points; $g^{\varepsilon_n}\circ f\circ\cdots\circ g^{\varepsilon_1}\circ f(x)$ where $\varepsilon_i=\pm 1$. Since $\alpha<1/3$ these points are easily seen to be distinct. So $\gamma(2n)\geqq 2^n$, and γ has exponential growth. q.e.d.

Lemma (4.2.5). *Let (Δ, ϕ) be an NT-decomposition of a closed foliated manifold $(M, \mathscr{F})$ without resilient leaves. Assume that each staircase of Δ is not inflexible. Then we can modify (Δ, ϕ) to a decomposition (Δ', ϕ') in which each staircase is flat on the ceiling.*

Proof. Let $\mathfrak{S}(\Delta)$ be the set of staircases of Δ. From the conditions (NT 3) and (NT 6), each staircase which is not maximal in $\mathfrak{S}(\Delta)$ is flat on the ceiling. Let S be a staircase which is maximal in $\mathfrak{S}(\Delta)$. We modify Δ so that the staircase corresponding to S is flat on the ceiling. There are two cases.

Case (1). There is a room $(R, \mathscr{F}_R)$ in Δ with $D(R)\cap W(S)\neq\emptyset$ and the saturation U of Int (R) has a leaf with non-trivial holonomy. Let F be a leaf through a connected component of $\partial_{\tan} R$. Then either F is semi-stable on the side of R or there is a totally proper leaf in U whose limit set contains F. In the first case, in the decomposition Δ' obtained by changing the ceiling of S to F, the staircase S' corresponding to S is flat on the ceiling. See Figure 4.

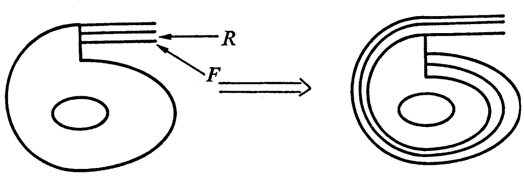

Fig. 4.

In the second case, from (2.1.9), the contracting holonomy of F is compactly supported in the side of R. That is, there is a compact subset K of F such that each holonomy along a loop in $F-K$ on the side of R is not contracting. From (2.1.8), there is $\alpha\colon \mathfrak{S}(\Delta)\to N$ such that the α-thinning $\mathfrak{S}^{(\alpha)}(\Delta)$ of $\mathfrak{S}(\Delta)$ does not meet K. Let Δ' be the decomposition obtained from Δ by α-thinning $\mathfrak{S}(\Delta)$ and changing the ceiling of $S^{\alpha(S)}$ to F. Then the staircase S' of Δ' corresponding to S is flat on the ceiling.

Case (2). Each unit adjacent to $W(S)$ is a hall. Assume that a hall $(H, \mathscr{F}_H)$ adjacent to $W(S)$ satisfies one of the following two conditions;

(1) a leaf F through a connected component of $\partial_{\tan} H$ with $F\cap S\neq\emptyset$ has flat holonomy, or

(2) the holonomy of a leal F through a connected component of

$\partial_{\tan} H$ with $F \cap S \neq \emptyset$ is compactly supported.

Then by thinning $\mathfrak{S}(\Delta)$ and changing the ceiling as in case (1), we get a desired decomposition.

Otherwise, there are $f(\delta)=t_0<t_1<\cdots<t_l=\delta$ and $g \in G$ such that $g(t_i)=t_i$ and $j_{t_i}^{k_i}(g) \neq j_{t_i}^{k_i}(\mathrm{id})$ for each i and for some k_i, where f is the slope of S and G is the reduced total holonomy group of S. Since each unit adjacent to $W(S)$ is a hall, the diffeomorphisms $f^{-n} \circ g \circ f^n{}_{|[f(\delta),\delta]}$ and $g_{|[f(\delta),\delta]}$ are commutative. From (4.2.1), there is $\alpha_n \in R$ such that $f^{-n} \circ g \circ f^n{}_{|[f(\delta),\delta]}=g^{\alpha_n}_{|[f(\delta),\delta]}$. Considering the derivative of g at $f^n(\delta)$, we get $\alpha_n=\alpha_1^n$ and $f^{-n} \circ g \circ f^n{}_{|[0,\delta]}=g^{\alpha_1^n}_{|[0,\delta]}$. Since g is C^2, we have $0<\alpha_1<1$ by a version of Kopell's lemma. So the staircase S is inflexible. This contradicts our assumption. q.e.d.

From (4.2.4) and (4.2.5), we get the following.

Proposition (4.2.6). *If $(M, \mathscr{F})$ has no exponential leaves, then $(M, \mathscr{F})$ has an NT-decomposition which is flat on the ceiling of each staircase.*

(4.3) 3-dimensional case. Let (Δ, ϕ) be an NT-decomposition of $(M, \mathscr{F})$ and let $(M_i, \mathscr{F}_i)$ be a room or a hall of Δ. Let U_i be the $\mathscr{F}$-saturation of $\mathrm{Int}(M_i)$. We say U_i is *trivial on the walls* if for each staircase $S=h(C(K, N)\times[0, \delta]) \in \mathfrak{S}(\Delta)$, the foliation $\mathscr{F}_{|U_i \cap N \times [0,\delta]}$ is trivial. If the dimension of M is three, we get the following.

Proposition (4.3.1) (see [C-C 3; (2.3)]). *Let (Δ, ϕ) be an NT-decomposition of a foliated 3-manifold $(M, \mathscr{F})$. Then each hall and each globally abelian room of Δ are trivial on the walls.*

Proof. Let $(M_i, \mathscr{F}_i)$ be a hall and U be the saturation of $\mathrm{Int}(M_i)$. For each staircase $S \in \mathfrak{S}(\Delta)$, which is the image of $C(K, N)\times[0, \delta]$, we show that $\mathscr{F}_{|U \cap N \times [0,\delta]}$ is trivial, by induction on the level of the leaf through the floor $F(S)$ of S. Assume $F(S)$ is at level 0. And let f be the slope of S. Since $\dim M=3$, the manifold N is diffeomorphic to a circle. Let g be the element of the reduced holonomy group of S defined by N. Let (a, b) be a connected component of $U \cap [f(\delta), \delta]$, where we identify a fibre of $C(K, N)\times[0, \delta]$ with the interval $[0, \delta]$. Then the intersection of $[0, \delta]$ and the connected component of $U \cap S$ containing (a, b) is the disjoint union $(a, b) \cup (f(a), f(b)) \cup \cdots \cup (f^n(a), f^n(b)) \cup \cdots$. We prove that $g_{|(f^n(a), f^n(b))}=\mathrm{id}$ for each n. Let $N_n=N\times\{f^n(\delta)\}$ be the lift of N, and $g_n\colon (f^n(a), f^n(b)) \to (f^n(a), f^n(b))$ be the corresponding holonomy of U. The map g_n is a diffeomorphism of $(f^n(a), f^n(b))$ by the arguments in (1.4). Since N_n and N_1 are homologous in the leaf $F=\mathrm{Sat}_S(\{a\})$, they define the same holonomy of U (see (1.4.2)). In other words, $f^{-n} \circ g_n \circ f^n{}_{|(a,b)}=g_{|(a,b)}$. But g_n is the restriction of g on $(f^n(a), f^n(b))$. So we get $f^{-n} \circ g \circ f^n{}_{|(a,b)} = g_{|(a,b)}$ for each n. A version of Kopell's lemma implies that such a diffeomorphism g can not be of class C^2 unless $g_{|(f^n(a), f^n(b))}$ is the identity map. Thus U is trivial on the wall of S. By induction, we can prove that U is trivial on the wall of each staircase of $\mathfrak{S}(\Delta)$.

The case of an abelian room is the same and the proof is omitted.

q.e.d.

From (4.1.1), (4.2.4) and (4.3.1), we get the following.

Theorem 3. *Let $(M, \mathscr{F})$ be a closed foliated 3-manifold without exponential leaves. Assume that $(M, \mathscr{F})$ admits an NT-decomposition each of whose room is globally abelian. Then $\mathscr{F}$ is C^∞-approximated by PA-foliations of finite type. In particular, finite type foliations of 3-manifolds are C^∞-approximated by PA-foliations of finite type.*

§ 5. Foliations of class $\mathscr{D}$ and $\mathscr{D}$-approximations

(5.1) Foliations of class $\mathscr{D}$. In [D-S], Duminy and Sergiescu introduced an important subgroup of (local) homeomorphisms of I (or $\boldsymbol{R}$). For a map k: $\boldsymbol{R} \to \boldsymbol{R}$, let $\delta^R k$ denote the right derivative of k and Var (k) denotes the total variation of k. Let f be an orientation preserving (local) homeomorphism of $\boldsymbol{R}$ (or I) with compact support which is smooth (C^2) except at countably many points. We say f is of class $\mathscr{D}$ if Var $(\log \delta^R(f))$ is finite. Let $\mathscr{D}$ be the group of all such homeomorphisms. For $f \in \mathscr{D}$, $\log \delta^R f$ has the right derivative a.e., and $\delta^R \log \delta^R f$ is intergrable (L^1). And we define the $\mathscr{D}$-norm $|f|_{\mathscr{D}}$ of f by $|f|_{\mathscr{D}} = |\log \delta^R f|_\infty + |\delta^R \log \delta^R f|_1$ where $|\ \ |_\infty$ (resp. $|\ \ |_1$) denotes the essential sup. norm (resp. L^1-norm).

Now let $\mathscr{F}$ be a C^0-foliation with C^∞-leaves of a compact n-manifold M.

Definition (5.1.1). We say $\mathscr{F}$ is of class $\mathscr{D}$, if there is an open covering $\{U_i\}$ of M and homeomorphisms ϕ_i: $U_i \to \mathring{D}^{n-1} \times \mathring{D}^1$ which satisfy the following conditions.

(1) For each i and $t \in \mathring{D}^1$, the map $\phi_i^{-1}{}_{|\mathring{D}^{n-1} \times \{t\}}$: $\mathring{D}^{n-1} \times \{t\} \to M$ is smooth.

(2) If $U_i \cap U_j \neq \emptyset$, then the map $\phi_i \circ \phi_j^{-1}$ is locally of the form $(x, y) \to (g(x, y), h(y))$, where $x, g(x, y) \in \mathring{D}^{n-1}$, $y, h(y) \in \mathring{D}^1$ and $y \to h(y)$ belongs to the class $\mathscr{D}$.

Note that C^2-foliations are of class $\mathscr{D}$ and that the holonomy pseudogroup of a foliation of class $\mathscr{D}$ is contained in $\mathscr{D}$. Kopell's lemma, Denjoy's inequality etc., are true for local $\mathscr{D}$-homeomorphisms. Thus the theory of levels (1.2) holds for foliations of class $\mathscr{D}$ (see [Tsuc 4]).

We want to define a topology on the space of foliations of class $\mathscr{D}$ by saying that two $\mathscr{D}$-foliations $\mathscr{F}$ and $\mathscr{F}'$ are near if each corresponding elements of the holonomy pseudogroups of $\mathscr{F}$ and $\mathscr{F}'$ are $\mathscr{D}$-near. A general definition seems to be cumbersome, so we adopt the following definition.

Definition (5.1.2). Let $\mathscr{F}$ be a foliation of class $\mathscr{D}$ of a compact manifold M. Assume $(M, \mathscr{F})$ has an NT-decomposition $(\varDelta, \phi)$ whose scaffolding is C. Let $\mathscr{F}_n$, $n=1, 2, \cdots$ be a sequence of foliations of class $\mathscr{D}$ of M. We say $\{\mathscr{F}_n\}$ accumulates to $\mathscr{F}$ along C (or $(\varDelta, \phi)$) if the following conditions are satisfied.

(1) The scaffolding C is a scaffolding of $\mathscr{F}_n$, for each n, also the underlying spaces M_i of $(\varDelta, \phi)$ give an NT-decomposition of the foliated manifold $(M, \mathscr{F}_n)$.

(2) For each unit $(M_i, \mathscr{F}_i)$ of $\varDelta$, the foliated manifold

$$(M_i, \phi^{-1}(\mathscr{F}_{n|\phi(M_i)}))=(M_i, \mathscr{F}_i^n)$$

is a same type of unit as $(M_i, \mathscr{F}_i)$.

(3) For each unit $(M_i, \mathscr{F}_i^n)$, the total holonomy group of $(M_i, \mathscr{F}_i^n)$ converges to that of $(M_i, \mathscr{F}_i)$ in the $\mathscr{D}$-topology.

If $\mathscr{F}$ is accumulated by $\{\mathscr{F}_n\}$ along some scaffolding, we say $\mathscr{F}$ is $\mathscr{D}$-approximated by $\{\mathscr{F}_n\}$.

One can define the GV-invariant of a foliation of class $\mathscr{D}$, and one can prove that the invariant is continuous with respect to the above convergence.

Duminy and Sergiescu [D-S] proved that if $(M, \mathscr{F})$ is a foliated I-bundle over a compact manifold B without resilient leaves, then $(M, \mathscr{F})$ is $\mathscr{D}$-approximated (along the trivial scaffolding $B\times\{0\}\cup B\times\{1\}$), by finite-type foliations of class $\mathscr{D}$. We prove that a corresponding theorem is true in the general case.

(5.2) $\mathscr{D}$-approximations of foliations without resilient leaves. In this section, we prove the following.

Theorem 4. *Let $(M, \mathscr{F})$ be a closed foliated manifold. Assume that $\mathscr{F}$ has no resilient leaves. Let C be a scaffolding of $\mathscr{F}$. Then $\mathscr{F}$ is $\mathscr{D}$-approximated along C by finite type foliations $\{\mathscr{F}_n\}$, $n=1, 2, \cdots$, of class $\mathscr{D}$, which satisfy the following conditions; there is a scaffolding $C_n\supset C$ of $\mathscr{F}_n$ such that for each connected component U of $M-C_n$, $\mathscr{F}_{n|U}$ is smooth and is without holonomy.*

Proof. We use the method of Duminy and Sergiescu (see [D-S] and [Tsuc 4]). For $\lambda>0$, let $h_\lambda \in \mathrm{Diff}(I)$ be the diffeomorphism

$$h_\lambda(x)=\frac{x}{(1-\lambda)x+\lambda}.$$

Choose an NT-decomposition $(\varDelta, \phi)$ associated with the scaffolding C. For each staircase $S=\pi$ $(C(K, N)\times I)$ (or a room $R=B\times I$), fix parametrizations of $\{b\}\times I$, $b\in N$ (or $\{b\}\times I$, $b\in B$). We call such a subset a *pillar* of $(\varDelta, \phi)$.

Let n be a positive integer. From (2.2.2), there is a scaffolding $C_n\supset C$, such that the total width of each type (B) component of $M-C_n$ is smaller than $1/n$. We alter $\mathscr{F}$ in each type (B) component of $M-C_n$ by h_λ. Let I be a connected component of the pillar of $(\varDelta, \phi)$ which is contained in a staircase S or a room R, and let (a, b) be a connected component of the intersection of a type (B) component of $M-C_n$ with I. We define a homomorphism π_n from the total holonomy group G of S (or R) to the pseudogroup of local homeomorphisms Loc $\mathscr{D}(I)$ of I of class $\mathscr{D}$ as follows. For $x\in(a,b)$ and $g\in G$, we define $\pi_n(g)(x)=(g(b)-g(a))h_\lambda((x-a)/(b-a))+g(a)$, where $\lambda=\sqrt{\delta g(b)/\delta g(a)}$. For x which is not contained in a type (B) component of $M-C_n$, we set $\pi_n(g)(x)=g(x)$. It is seen that π_n is a homomorphism into Loc $\mathscr{D}(I)$ and the foliation defined by π_n in each unit gathers compatibly to a finite type foliation $\mathscr{F}_n$ of class $\mathscr{D}$.

For each total holonomy group G of a staircase or a room of $\varDelta$, choose a finite symmetric generating set Γ. Let

$$K=\sup_{g\in\Gamma}\frac{\max_{x\in I}|\delta^2 g(x)|}{\min_{x\in I}\delta g(x)}.$$

Then it is seen that for each $g\in\Gamma$,

$$|\log\delta g-\log\delta^R\pi_n(g)|_\infty+|\delta\log\delta g-\delta^R\log\delta^R\pi_n(g)|_\infty\leqq 2(K+K^2)\,(1/n).$$

(see [Tsuc 4]). Thus the foliations $\{\mathscr{F}_n\}$ accumulate to $\mathscr{F}$ along C. q.e.d.

Theorem (5.2.1). *Let $(M^3, \mathscr{F})$ be a closed foliated 3-manifold. Assume $\mathscr{F}$ has no resilient leaves. Then $\mathscr{F}$ is $\mathscr{D}$-approximated by finite type, PA-foliations of class $\mathscr{D}$.*

Proof. This follows from Theorem 4 and Theorem 3, since the existence of an inflexible staircase makes no trouble when we are considering $\mathscr{D}$-approximations. q.e.d.

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