A COLLECTION OF PAPERS IN MATHEMATICS AND RELATED SCIENCES,
A FESTSCHRIFT IN HONOUR OF THE LATE GALAYE DIA
EDITORS : PROFS HAMET SEYDI, GANE SAMB LO, ABOUBAKARY DIAKHABY
SPAS-EDS (SAINT-LOUIS, CALGARY, 2018), WWW.STATPAS.ORG/SPASEDS/
IN EUCLID (WWW.PROJECTEUCLID.ORG)
DOI : 10.16929/SBS/2018.100

CHAPTER 19

Homogenization of semilinear periodic parabolic PDEs with Reflecting boundary condition, *by A. Diedhiou, I. Sane, A. Coulibaly*

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Abstract. We establish homogenization results for semilinear partial differential equations (PDEs) on the half space according to the reflecting Neumann boundary condition and highly oscillating coefficients. Our method is based on the martingale property of the stochastic exponential and Girsanov transformation.

Keywords. Homogenization; viscosity solutions; stochastic differential equation; backward stochastic differential equation.
AMS 2010 Mathematics Subject Classification. 60H30; 60G20; 35B27.

Cite the chapter as :
Diedhiou A., Sane I. and Coulibaly A. (2018). Homogenization of semilinear periodic parabolic PDEs with Reflecting boundary condition. In *A Collection of Papers in Mathematics and Related Sciences, a festschrift in honour of the late Galaye Dia* (Editors : Seydi H., Lo G.S. and Diakhaby A.). Spas Editions, Euclid Series Book, (Doi : 10.16929/sbs/2018.100. pp. 343 –362. Doi : 10.16929/sbs/2018.100-04-02

1. Introduction

Let $\boldsymbol{D} = \left\{\left(x^1, x^2, \ldots, x^d\right) \in \mathbb{R}^d : x^1 > 0\right\}$, $\boldsymbol{\partial D} = \left\{\left(x^1, x^2, \ldots, x^d\right) \in \mathbb{R}^d : x^1 = 0\right\}$ and let

$$L_\varepsilon^{\delta_t} = \frac{1}{2}\sum_{i,j}^{d} \alpha_{i,j}\left(\frac{x}{\varepsilon}\right)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d}\left\{\frac{1}{\varepsilon}\beta_i\left(\frac{x}{\varepsilon}\right) + c_i\left(\frac{x}{\varepsilon}\right) + \tau\left(\frac{x}{\varepsilon}\right)\cdot\delta_t\right\}\frac{\partial}{\partial x_i}, \tag{1.1}$$

$$\Gamma_\varepsilon := \sum_{i=1}^{d}\gamma_i\left(\frac{x}{\varepsilon}\right)\frac{\partial}{\partial x_i} \tag{1.2}$$

be given (with $\epsilon > 0$). The problem of constructing diffusion equipped with boundary conditions has been discussed by many authors (see, for example Stroock and Varadhan (1971) ,Andreson (1976a), Anderson (1976b)). It well know that such diffusions do exist in the case that all the coefficients are smooth and functions periodic in each variable, that β satisfies certain centering conditions, and that $\gamma_1 = 1$, the first component of the vector-valued γ (precise assumptions on the coefficient are stated in Section 2). Tanaka (1984) gave a formulation of the boundary value problem associated with such diffusions for the case $\delta_t \equiv 0$. Using the formulation Tanaka (1984), Ouknine and Pardoux (2002) investigated the homogenization of semilinear PDEs with nonlinear Neumann boundary condition, periodic coefficients and highly oscillating drift and nonlinear term. The purpose of the present note is to study a similar problem in the following sense :

$$\begin{cases} \dfrac{\partial u^\varepsilon}{\partial t}(t,x) = L_\varepsilon^{\delta_t} u^\varepsilon(t,x) + f\left(\dfrac{x}{\varepsilon}, u^\varepsilon(t,x)\right) \\ \qquad\qquad + \dfrac{1}{\varepsilon} e\left(\dfrac{x}{\varepsilon}, u^\varepsilon(t,x)\right), \quad x \in \boldsymbol{D},\ 0 < t \\ \Gamma_\varepsilon u^\varepsilon(t,x) + h\left(\dfrac{x}{\varepsilon}, u^\varepsilon(t,x)\right) = 0, \qquad x \in \partial\boldsymbol{D},\ 0 \le t \\ u^\varepsilon(0,x) = g(x), \qquad x \in \boldsymbol{D}. \end{cases} \tag{1.3}$$

Here e, f, g and h are subject to suitable conditions.

The approach developed in this work is the *method of transformation of drift* to the study of the diffusion in terms of martingale problem according to *Girsanov Formula*. The present note requires for its development the systematic generalization of certain results contained in the papers of Ouknine and Pardoux (2002). When the details of a particular proof do not appreciably differ from the presentation of Ouknine and Pardoux (2002) and do not affect the clarity of our development, we will feel free to refer the reader to the original proofs in Ouknine and Pardoux (2002) and simply describe the significant differences which lead to our generalizations.

This note is organized as follows. In Section 2, we deal with the formulation of the problem and state theorem of (Backward) stochastic differential equations (BSDE) SDE and weak convergence. In Section 3, we deal with the homogenization of equation (1.3).

2. Preliminaries

2.1. Statement of the problem - SDE and weak convergence.

The following hypotheses are required :

- **(H.1)** The matrix-value $\alpha(x) = (\alpha_{ij}(x))$ is uniformly non-degenerate for all x, and can be factored as $\alpha := \tau\tau^*$ where * denotes the transpose.
- **(H.2)** The maps $\tau : \mathbb{R}^d \longrightarrow \mathbb{R}^{d\times d}$, $\beta : \mathbb{R}^d \longrightarrow \mathbb{R}^d$, $c : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ and $\gamma : \left(\boldsymbol{\partial D} \approx \mathbb{R}^{d-1}\right) \longrightarrow \mathbb{R}^d$ are smooth and periodic of period 1 in each variable. In particular, we can choose the direction of reflection γ such that

$$\gamma^1(x) := 1.$$

- **(H.3)** We also choose $\delta_t : \mathbb{R}_+ \longrightarrow \mathbb{R}^d$ such that it is bounded.

The differential operator $L_\epsilon^{\delta_t}$ (1.1) inside $\boldsymbol{D}$ along with the boundary condition $\Gamma_\varepsilon u = 0$ on $\boldsymbol{\partial D}$ determine a unique diffusion process X^ε in $\overline{\boldsymbol{D}}$ which we call the $\left(L_\varepsilon^{\delta_t}, \Gamma_\varepsilon\right)$-diffusion.

Given a d-dimensional Brownian motion $\left\{W_t, t \geq 0\right\}$ defined on a completed probability space $\left(\Omega, \mathcal{F}, \mathbb{P}\right)$, such a process X^ε with the local time ϕ^ε of $X^{1,\varepsilon}$ on $\boldsymbol{\partial D}$, satisfies the reflected SDE :

(2.1)
$$\begin{cases} dX_t^\varepsilon = \tau\left(\frac{X_t^\varepsilon}{\varepsilon}\right)dB_t + \tau\left(\frac{X_t}{\varepsilon}\right)\cdot\delta_t dt + \frac{1}{\varepsilon}\beta\left(\frac{X_t^\varepsilon}{\varepsilon}\right)dt \\ \qquad\qquad + c\left(\frac{X_t^\varepsilon}{\varepsilon}\right)dt + \gamma\left(\frac{X_t^\varepsilon}{\varepsilon}\right)d\varphi_t^\varepsilon, \quad 0 < t, \\ X_t^{1,\varepsilon} \geq 0,\ \varphi^\varepsilon \text{ is continuous and increasing and} \\ \int_0^t X_s^{1,\varepsilon} d\varphi_s^\varepsilon = 0, \quad 0 < t, \\ X_0^\varepsilon = x. \end{cases}$$

Next we let

(2.2)
$$\widehat{B}_t = B_t + \int_0^t \delta_r dr.$$

By (**H.3**), it follows from Girsanov's theorem that there exists a new probability measure $\mathbb{P}^{\delta_t}$ equivalent to $\mathbb{P}$, defined as

(2.3)
$$\frac{d\mathbb{P}^{\delta_t}}{d\mathbb{P}}|_{\mathcal{F}_t} = \exp\left(-\int_0^t \delta_r dB_r - \frac{1}{2}\int_0^t \|\delta_r\|^2 dr\right),$$

under which $\widehat{B}_t$ is a standard Brownian motion. This implies that $\left(X^\varepsilon, \widehat{B}_t\right)$ is a solution on the probability space $\left(\Omega, \mathcal{F}, \mathbb{P}^{\delta_t}\right)$ of the following SDE with the reference family $(\mathcal{F}_t)_{t\geq 0}$:

(2.4)
$$\begin{cases} dX_t^\varepsilon = \tau\left(\frac{X_t^\varepsilon}{\varepsilon}\right)d\widehat{B}_t + \frac{1}{\varepsilon}\beta\left(\frac{X_t^\varepsilon}{\varepsilon}\right)dt \\ \qquad\qquad + c\left(\frac{X_t^\varepsilon}{\varepsilon}\right)dt + \gamma\left(\frac{X_t^\varepsilon}{\varepsilon}\right)d\varphi_t^\varepsilon, \quad 0 < t, \\ X_t^{1,\varepsilon} \geq 0,\ \varphi^\varepsilon \text{ is continuous and increasing and} \\ \int_0^t X_s^{1,\varepsilon} d\varphi_s^\varepsilon = 0, \quad 0 < t, \\ X_0^\varepsilon = x. \end{cases}$$

Throughout the paper, we denote by $\widehat{\mathbb{E}}$ the expectation operator associate with $\mathbb{P}^\delta$.

Now we set

$$L := \frac{1}{2}\sum_{i,j}^{d}\alpha_{i,j}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d}\beta_i \frac{\partial}{\partial x_i}, \quad x \in \mathbb{R}^d, \tag{2.5}$$

and let $\overline{X}$ denote the unique diffusion process with values in the d-dimensional torus $\mathbb{T}^d$, whose generator is the operator L. Then it well know that $\overline{X}$ is ergodic. We denote by m its unique invariant measure. In order for the process X^ε on $\left(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P}^{\delta_t}\right)$ to have a limit in law as $\varepsilon \longrightarrow 0$, we need that the following be in force :

(H.4) centering condition : $\displaystyle\int_{\mathbb{T}^d}\beta(x)m(dx) = 0.$

First, we check that under **(H.4)** there exists a unique periodic solution $\widehat{\beta}$ of $L\widehat{\beta} = -\beta$ with zero integral against the measure m. Such solution is given by $\widehat{\beta}(x) := \displaystyle\int_0^{+\infty}\widehat{\mathbb{E}}_x\left[\beta\left(\overline{X}_t\right)\right]dt$ where, under $\mathbb{P}^\delta$ with the reference family $(\mathcal{F}_t)_{t\geq 0}$, $\overline{X}$ starts from x.

Now we put

$$\begin{aligned}
c_0 &= \int_{\mathbb{T}^d}\left(I + \nabla\widehat{\beta}\right)(x)c(x)m(dx),\\
\alpha_0 &= \int_{\mathbb{T}^d}\left(I + \nabla\widehat{\beta}\right)(x)\alpha(x)\left(I + \nabla\widehat{\beta}\right)^*(x)m(dx),\\
L_0 &= \frac{1}{2}\sum_{i,j}^{d}\alpha_0^{i,j}\partial_i\partial_j + \sum_{i=1}^{d}c_0^i\partial_i.
\end{aligned}$$

Write $v = \mathcal{H}\varphi$ for the solution v of :

$$\begin{cases} Lv = 0 & \text{in } \boldsymbol{D}\\ v = \varphi & \text{on } \partial\boldsymbol{D}. \end{cases} \tag{2.6}$$

Then $\mathcal{H}$ sends functions defined on $\partial\boldsymbol{D}$ to functions defined on $\overline{\boldsymbol{D}}$, while $\Gamma\mathcal{H}$ sends functions defined on $\partial\boldsymbol{D}$ to functions on $\partial\boldsymbol{D}$, where
$\Gamma := \displaystyle\sum_{i=1}^{d}\gamma_i(x)\partial_i$. It well know there exists a unique Markov process on $\partial\boldsymbol{D}$ with generator $\Gamma\mathcal{H}$. By the periodicity assumption, this process induces

a Markov process on $\mathbb{T}^{d-1}$; let $\widetilde{m}$ be the unique invariant measure of the induced Markov process, and let us set

$$\gamma_0 := \int_{\mathbb{T}^{d-1}} \left(I + \nabla\widehat{\beta}\right)\gamma(x)\widetilde{m}(dx) \quad \text{and} \quad \Gamma_0 := \sum_{i=1}^{d}\gamma_0\partial_i.$$

Let us introduce the process $\hat{X}_t^\varepsilon$ defined as:

$$\begin{aligned}\hat{X}_t^\varepsilon &= X_t^\varepsilon + \varepsilon\left[\hat{\beta}\left(\frac{X_t}{\varepsilon}\right) - \hat{\beta}\left(\frac{x}{\varepsilon}\right)\right]\\ &= x + \int_0^t \left(I + \nabla\hat{\beta}\right)\tau\left(\frac{X_s^\varepsilon}{\varepsilon}\right)d\widehat{B}_s + \int_0^t \left(I + \nabla\hat{\beta}\right)c\left(\frac{X_s^\varepsilon}{\varepsilon}\right)ds\\ &\quad + \int_0^t \left(I + \nabla\hat{\beta}\right)\gamma_\varepsilon\left(\frac{X_s^\varepsilon}{\varepsilon}\right)d\varphi_s^\varepsilon.\end{aligned} \tag{2.7}$$

Let us write (2.7) in coordinate form :

$$\begin{cases}\hat{X}_t^{1,\varepsilon}\\ = x_1 + \int_0^t \left(1 + \nabla\hat{\beta}_1\right)\tau_1\left(\frac{X_s^\varepsilon}{\varepsilon}\right)d\widehat{B}_s^1\\ + \int_0^t \left(1 + \nabla\hat{\beta}_1\right)c_1\left(\frac{X_s^\varepsilon}{\varepsilon}\right)ds\\ + \int_0^t \left(1 + \nabla\hat{\beta}_1\right)\left(\frac{X_s^\varepsilon}{\varepsilon}\right)d\varphi_s^\varepsilon\end{cases}. \tag{2.8}$$

and (continued)

$$\begin{cases}\hat{X}_t^{j,\varepsilon}\\ x_j + \int_0^t \left(1 + \nabla\hat{\beta}_j\right)\tau_j\left(\frac{X_s^\varepsilon}{\varepsilon}\right)d\widehat{B}_s^j\\ + \int_0^t \left(1 + \nabla\hat{\beta}_j\right)c_j\left(\frac{X_s^\varepsilon}{\varepsilon}\right)ds\\ \text{for all } j = 2, \dots, d.\end{cases}.$$

Then there exists a bounded and smooth solution η of the PDE with Neumann-type boundary condition :

$$
(2.9) \qquad \begin{cases} L\eta = 0 & \text{in } \boldsymbol{D} \\ \gamma \cdot \nabla\eta = \left(1 + \nabla\hat{\beta}_1\right) - \int_{\mathbb{T}^{d-1}} \left(1 + \nabla\hat{\beta}_1\right)(x)\widetilde{m}(dx) & \text{on } \boldsymbol{\partial D}. \end{cases}
$$

Taking such a solution η, we have by Itô :

$$
\begin{aligned}
& \varepsilon\left[\eta\left(\frac{X_t^\varepsilon}{\varepsilon}\right) - \eta\left(\frac{x}{\varepsilon}\right)\right] \\
(2.10) \qquad & = \int_0^t \nabla\eta\tau\left(\frac{X_s^\varepsilon}{\varepsilon}\right) d\widehat{B}_s + \int_0^t \nabla\eta c\left(\frac{X_s^\varepsilon}{\varepsilon}\right) ds \\
& + \int_0^t \left(1 + \nabla\hat{\beta}_1\right)\left(\frac{X_s^\varepsilon}{\varepsilon}\right) d\varphi_s^\varepsilon - \varphi_t^\varepsilon \int_{\mathbb{T}^{d-1}} \left(1 + \nabla\hat{\beta}_1\right)(x)\widetilde{m}(dx).
\end{aligned}
$$

Putting (2.10) into the first component of (2.8) we have

$$
\begin{aligned}
\hat{X}_t^{1,\varepsilon} &= x_1 + \int_0^t \left(1 + \nabla\hat{\beta}_1\right)\tau_1\left(\frac{X_s^\varepsilon}{\varepsilon}\right) d\widehat{B}_s^1 + \int_0^t \left(1 + \nabla\hat{\beta}_1\right) c_1\left(\frac{X_s^\varepsilon}{\varepsilon}\right) ds \\
&+ \varphi_t^\varepsilon \int_{\mathbb{T}^{d-1}} \left(1 + \nabla\hat{\beta}_1\right)(x)\widetilde{m}(dx) \\
& \underbrace{- \int_0^t \nabla\eta\tau\left(\frac{X_s^\varepsilon}{\varepsilon}\right) d\widehat{B}_s - \int_0^t \nabla\eta c\left(\frac{X_s^\varepsilon}{\varepsilon}\right) ds + \varepsilon\left[\eta\left(\frac{X_t}{\varepsilon}\right) - \eta\left(\frac{x}{\varepsilon}\right)\right]}_{A_\varepsilon(t)}.
\end{aligned}
$$

From (**H.3**) and (2.10) using lemma 6.3 of Tanaka (1984) we can see that the term:

$$
(2.11) \qquad \lim_{\varepsilon\to 0} \widehat{\mathbb{E}}_x \left\{ \max_{0\le t\le T} |A_\varepsilon(t)| \right\} = 0.
$$

Before proceeding, we introduce some definition :

And finally, we have according to Theorem 2.1 in Ouknine and Pardoux (2002):

THEOREM 54. *Assume that the assumptions* **(H.1)** *to* **(H.4)** *hold true. Then the* $\left(L_{\varepsilon}^{\delta_t}, \Gamma_{\varepsilon}\right)$*-reflected diffusion process* X^{ε} *on* $\left(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P}^{\delta_t}\right)$ *converges in law to the* (L_0, Γ_0)*-reflected diffusion process* X *as* $\varepsilon \downarrow 0$*. Moreover,*

$$\left(X^{\varepsilon}, \widehat{M}_t^{X^{\varepsilon}}, \varphi^{\varepsilon}\right) \Longrightarrow \left(X, M^X, \varphi\right),$$

where $\widehat{M}^{X^{\varepsilon}} := \int_0^t \left(I + \nabla\widehat{\beta}\right) \tau \left(\frac{X_s^{\varepsilon}}{\varepsilon}\right) d\widehat{B}_s$ *and with*

- M^X *is the martingale part of* X*;*
- φ *(resp.* φ^{ε}*) is the local time of* X^1 *(resp.* $X^{1,\varepsilon}$*).*

Thus, it easily follows a from result in Tanaka (1984).

LEMMA 32. *Under the assumptions of* (54) *for any* $p > 1$ *,*

$$\sup_{\varepsilon} \widehat{\mathbb{E}}_x \left(|X_t^{\varepsilon}|^p + \varphi_t^{\varepsilon} \right) < \infty .$$

2.2. BSDE and weak convergence.

Here, we require that :

- $e : \mathbb{R}^d \times \mathbb{R} \longrightarrow \mathbb{R}$ is a measurable mapping, which is periodic, of period one in each direction in the first argument, continuous in the second argument uniformly with respect to the first, and satisfies :

$$\int_{\mathbb{T}^d} e(x,y) m(dx) = 0, \qquad \forall y \in \mathbb{R} \tag{2.12}$$

  Suppose e be twice continuously differentiable in y, uniformly with respect to x, and there exists a constant K such that :

$$|e(x,y)| + \left|\frac{\partial}{\partial y} e(x,y)\right| + \left|\frac{\partial^2}{\partial y^2} e(x,y)\right| \leq K, \quad \forall x \in \mathbb{R}^d,\ y \in \mathbb{R} \tag{2.13}$$

- $f : \mathbb{R}^d \times \mathbb{R} \longrightarrow \mathbb{R}$, $g : \mathbb{R}^d \longrightarrow \mathbb{R}$ and $h : \mathbb{R}^d \times \mathbb{R} \longrightarrow \mathbb{R}$ are sufficiently smooth functions.
  Equivalently the coefficients can be seen as periodic functions with respect to the first variable with period one in each direction on $\mathbb{R}^d$ which are such that for some $c > 0$, $p > 0$, $\mu \in \mathbb{R}$, $\beta < 0$, and all $x \in \mathbb{R}^d$, $y, y' \in \mathbb{R}$:

(H.5) $|g(x)| \leq c\,(1 + |x|^p)$,
(H.6) $|f(x,y)| \leq c\left(1 + |y|^2\right)$,
(H.7) $(y - y')\,[f(x,y) - f\,(x,y')] \leq \mu\,|y - y'|^2$,

(H.8) $(y-y')\left[h(x,y)-h(x,y')\right]\leq\beta\left|y-y'\right|^2$,
(H.9) $|h(x,y)|\leq c\left(1+|y|^2\right)$.

Throughout this note, the triple $\left(X^\varepsilon, M_t^{X^\varepsilon}, \varphi^\varepsilon\right)_{\varepsilon>0}$ is the one which appears in the statement of Theorem 54. We now consider a type of BSDE which has been introduced in Pardoux and Zhang (1998).

Let $\left\{(Y_s^\varepsilon, Z_s^\varepsilon); 0\leq s\leq T\right\}$ be the solution of the following BSDE :
for $(t,x)\in[0,T]\times\overline{\boldsymbol{D}}$,

$$\begin{aligned} Y_s^\varepsilon =& g\left(X_t^\varepsilon\right)+\int_s^t f\left(\frac{X_r^\epsilon}{\varepsilon}, Y_r^\varepsilon\right)dr+\frac{1}{\varepsilon}\int_s^t e\left(\frac{X_r^\varepsilon}{\varepsilon}, Y_r^\epsilon\right)dr \\ &+\int_s^t h\left(\frac{X_r^\varepsilon}{\varepsilon}, Y_r^\varepsilon\right)d\varphi_r^\varepsilon-\int_s^t Z_r^\varepsilon dB_r-\int_s^t \delta_r\left(Z_r^\varepsilon\right)^* dr. \end{aligned} \tag{2.14}$$

From (2.2) we have :

$$\begin{aligned} Y_s^\varepsilon =& g\left(X_t^\varepsilon\right)+\int_s^t f\left(\frac{X_r^\epsilon}{\varepsilon}, Y_r^\varepsilon\right)dr+\frac{1}{\varepsilon}\int_s^t e\left(\frac{X_r^\varepsilon}{\varepsilon}, Y_r^\epsilon\right)dr \\ &+\int_s^t h\left(\frac{X_r^\varepsilon}{\varepsilon}, Y_r^\varepsilon\right)d\varphi_r^\varepsilon-\int_s^t Z_r^\varepsilon d\widehat{B}_r. \end{aligned} \tag{2.15}$$

For each fixed $y\in\mathbb{R}$, let set $\hat{e}$ be the solution of the Poisson equation :

$$L\hat{e}(x,y)+e(x,y)=0,\ x\in\mathbb{T}^d,\ y\in\mathbb{R}. \tag{2.16}$$

More precisely by (2.12), $\hat{e}$ is centered with respect to the invariant measure m and is given by the formula:

$$\hat{e}(x,y)=\int_0^\infty \widehat{\mathbb{E}}_x e\left(\overline{X}_t, y\right)dt. \tag{2.17}$$

Note that $\hat{e}\in C^{0,2}\left(\mathbb{T}^d,\mathbb{R}\right)$ (see Pardoux and Veretennikov (2001)) and

$$\hat{e}(.,y),\ \frac{\partial}{\partial y}\hat{e}(.,y),\ \frac{\partial^2}{\partial y^2}\hat{e}(.,y)\in W^{2,p}\left(\mathbb{T}^d\right),$$

and for any $p\geq 1$ there exists K' such that for all $y\in\mathbb{R}$

$$\|\hat{e}(.,y)\|_{W^{2,p}\left(\mathbb{T}^d\right)}+\left\|\frac{\partial}{\partial y}\hat{e}(.,y)\right\|_{W^{2,p}\left(\mathbb{T}^d\right)}+\left\|\frac{\partial^2}{\partial y^2}\hat{e}(.,y)\right\|_{W^{2,p}\left(\mathbb{T}^d\right)}\leq K'. \tag{2.18}$$

In the same way we define $\Big\{(Y_s, Z_s)\,; 0 \leq s \leq T\Big\}$ as the unique solution of the BSDE :

$$Y_s = g\left(X_t\right) + \int_s^t f_0\left(Y_r\right) dr + \int_s^t h_0\left(Y_r\right) d\varphi_r - \int_s^t Z_r d\widehat{B}_r, \tag{2.19}$$

where

$$f_0(y) = \int_{\mathbb{T}^d} \left(f + \left[\Big\langle \frac{\partial \hat{e}}{\partial x}, c(x) \Big\rangle - \left(\frac{\partial \hat{e}}{\partial y} \times e \right) + \frac{\partial^2 \hat{e}}{\partial x \partial y} \alpha(x) \Big(\frac{\partial \hat{e}}{\partial x} \Big)^* \right] \right) (x,y) m(dx),$$

$$h_0(y) = \int_{\mathbb{T}^{d-1}} \left(h(x,y) + \left\langle \frac{\partial \hat{e}}{\partial x}(x,y), \gamma(x) \right\rangle \right) \widetilde{m}(dx).$$

Choose Z_t^ε such that $Z_t^\varepsilon := U_t^\varepsilon \tau\left(X_t^\varepsilon/\varepsilon\right)$ and we introduce the notation,

$$M_{s:}^\varepsilon = \int_0^s U_r^\varepsilon dM_r^{X^\varepsilon} \quad \text{and} \quad M_s := \int_0^s U_r dM_r^X, \quad 0 \leqslant s \leqslant t.$$

So we consider the quintuple $\Big(X, M^X, \varphi, Y, M\Big)$ (*resp.* $\Big(X^\varepsilon, M^{X^\varepsilon}, \varphi^\varepsilon, Y^\varepsilon, M^\varepsilon\Big)$) as a random element of the space $C\left([0,t], \mathbb{R}^{2d+1}\right) \times D\left([0,t], \mathbb{R}^2\right)$, where we equip the first factor with the sup-norm topology, and the second factor with the S-topology of Jakubowski (1997).

THEOREM 55. *Assume the conditions* **(H.1)**–**(H.9)** *hold true. On the space* $C\Big([0,t], \mathbb{R}^{2d+1}\Big) \times D\Big([0,t], \mathbb{R}^2\Big)$ *equipped with the sup-norm topology in the first factor and the* S*-topology of Jakubowski in the second, we have*

$$\left(X^\varepsilon, M^{X^\varepsilon}, \varphi^\varepsilon, Y^\varepsilon, M^\varepsilon\right) \Rightarrow \left(X, M^X, \varphi, Y, M\right).$$

Moreover, $Y_0^\varepsilon \longrightarrow Y_0$ *in* $\mathbb{R}$.

Proof of Theorem 55. Let us proceed step by step.

$*$ *Step 1: Transformation of the BSDE.*
Set $\check{X}_t^\varepsilon := \dfrac{1}{\varepsilon} X_t^\varepsilon$ and consider the following process

$$\widetilde{Y}_s^\varepsilon := Y_s^\varepsilon + \varepsilon \Big(\hat{e}(\check{X}_t^\varepsilon, Y_t^\varepsilon) - \hat{e}(\check{X}_s^\varepsilon, Y_s^\varepsilon) \Big).$$

By applying Itô's formula, we have :

$$\begin{aligned}\widetilde{Y}_s^\varepsilon =& g(X_t^\varepsilon) + \int_s^t \left(\frac{\partial \hat{e}}{\partial x}\cdot c - \frac{\partial \hat{e}}{\partial y}e\right)(\check{X}_r^\varepsilon, Y_r^\varepsilon)dr + \int_s^t \frac{\partial \hat{e}}{\partial x}(\check{X}_r^\varepsilon, Y_r^\varepsilon)\gamma\left(\check{X}_r^\varepsilon\right)d\varphi_r^\varepsilon \\ &+ \int_s^t \left(1 - \varepsilon\frac{\partial \hat{e}}{\partial y}(\check{X}_r^\varepsilon, Y_r^\varepsilon)\right)\left(h(\check{X}_r^\varepsilon, Y_r^\varepsilon)d\varphi_r^\varepsilon + f(\check{X}_r^\varepsilon, Y_r^\varepsilon)dr\right) \\ &+ \int_s^t \frac{\partial^2 \hat{e}}{\partial x \partial y}(\check{X}_r^\varepsilon, Y_r^\varepsilon)\tau\left(\check{X}_r^\varepsilon\right)(Z_r^\varepsilon)^* dr + \int_s^t \left(\frac{\partial \hat{e}}{\partial x}(\check{X}_r^\varepsilon, Y_r^\varepsilon)\tau\left(\check{X}_r^\varepsilon\right) - Z_r^\varepsilon\right)d\widehat{B}_r \\ &+ \varepsilon\int_s^t \frac{\partial \hat{e}}{\partial y}(\check{X}_r^\varepsilon, Y_r^\varepsilon)Z_r^\varepsilon d\widehat{B}_r + \frac{\varepsilon}{2}\int_s^t \frac{\partial^2 \hat{e}}{\partial y^2}(\check{X}_r^\varepsilon, Y_r^\varepsilon)|Z_r^\varepsilon|^2 dr.\end{aligned}$$

Let us set

$$\widetilde{Z}_s^\varepsilon := Z_s^\varepsilon - \frac{\partial \hat{e}}{\partial x}(\check{X}_s^\varepsilon, Y_s^\epsilon)\tau\left(\check{X}_s^\varepsilon\right),\ 0 \leqslant s \leqslant t,$$

and note that the difference between $\widetilde{Z}_s^\epsilon$ and Z_s^ϵ is a uniformly bounded process.
Thus we have :

$$\begin{aligned}\widetilde{Y}_s^\varepsilon =& g(X_t^\varepsilon) + \int_s^t \left[\frac{\partial \hat{e}}{\partial x}\cdot c - \frac{\partial \hat{e}}{\partial y}e + \frac{\partial^2 \hat{e}}{\partial x \partial y}\alpha\left(\frac{\partial \hat{e}}{\partial x}\right)^*\right](\check{X}_r^\varepsilon, Y_r^\varepsilon)dr \\ &+ \int_s^t \frac{\partial \hat{e}}{\partial x}(\check{X}_r^\varepsilon, Y_r^\varepsilon)\gamma\left(\check{X}_r^\varepsilon\right)d\varphi_r^\varepsilon + \frac{\varepsilon}{2}\int_s^t \frac{\partial^2 \hat{e}}{\partial y^2}(\check{X}_r^\varepsilon, Y_r^\varepsilon)|Z_r^\varepsilon|^2 dr \\ &+ \int_s^t \left(1 - \varepsilon\frac{\partial \hat{e}}{\partial y}(\check{X}_r^\varepsilon, Y_r^\varepsilon)\right)\left(h(\check{X}_r^\varepsilon, Y_r^\varepsilon)d\varphi_r^\varepsilon + f(\check{X}_r^\varepsilon, Y_r^\varepsilon)dr\right) \\ &+ \varepsilon\int_s^t \frac{\partial \hat{e}}{\partial y}(\check{X}_r^\varepsilon, Y_r^\varepsilon)Z_r^\varepsilon d\widehat{B}_r - \int_s^t \widetilde{Z}_r^\varepsilon\left(d\widehat{B}_r - \tau^*(\check{X}_r^\varepsilon)\frac{\partial^2 \hat{e}}{\partial x \partial y}(\check{X}_r^\varepsilon, Y_r^\varepsilon)dr\right).\end{aligned}$$

We next define

$$\widetilde{B}_s := \widehat{B}_s - \int_0^s \tau^*(\check{X}_r^\varepsilon)\frac{\partial^2 \hat{e}}{\partial x \partial y}(\check{X}_r^\epsilon, Y_r^\epsilon)dr.$$

Thanks to Girsanov's theorem, it well know there exists a new probability measure $\widetilde{\mathbb{P}}$ equivalent to $\mathbb{P}^\delta$ under which $\widetilde{B}_s$ is a Brownian motion. Denote by $\widetilde{\mathbb{E}}$ the expectation operator associated to $\widetilde{\mathbb{P}}$.

Let $\widetilde{X}_s^\varepsilon := X_s^\varepsilon + \varepsilon\left[\widehat{\beta}(\check{X}_s^\epsilon) - \widehat{\beta}(x/\varepsilon)\right]$. In order to simplify the further exposition, we have (with the notations $c_\epsilon = c(\check{X}_s^\varepsilon)$, and similarly for $(I + \nabla\hat{\beta}), \alpha, \tau, \gamma$)

$$d\widetilde{X}_s^\epsilon = \left(I + \nabla\widehat{\beta}\right)_\varepsilon \left(\tau_\varepsilon d\widetilde{B}_s + c_\epsilon ds + \alpha_\varepsilon \frac{\partial^2 \hat{e}}{\partial x \partial y}(\check{X}_s^\epsilon, Y_s^\epsilon)ds + \gamma_\epsilon d\varphi_s^\epsilon\right), \quad \widetilde{X}_0^\epsilon = x$$

and

$$\begin{aligned}\widetilde{Y}_s^\varepsilon \;&=\; g(X_t^\varepsilon) + \int_s^t \left[\frac{\partial \hat{e}}{\partial x}\cdot c_\varepsilon - \frac{\partial \hat{e}}{\partial y} e + \frac{\partial^2 \hat{e}}{\partial x \partial y}\alpha_\varepsilon \left(\frac{\partial \hat{e}}{\partial x}\right)^*\right](\check{X}_r^\varepsilon, Y_r^\varepsilon)dr \quad (CC)\\ &+\; \int_s^t \left(\left(1 - \varepsilon\frac{\partial \hat{e}}{\partial y}(\check{X}_r^\varepsilon, Y_r^\varepsilon)\right) f(\check{X}_r^\varepsilon, Y_r^\varepsilon)dr\right)\\ &+\; \int_s^t \frac{\partial^2 \hat{e}}{\partial x \partial y}(\check{X}_r^\varepsilon, Y_r^\varepsilon)\tau_\varepsilon \left(Z_r^\varepsilon\right)^* dr + \frac{\varepsilon}{2}\int_s^t \frac{\partial^2 \hat{e}}{\partial y^2}(\check{X}_r^\varepsilon, Y_r^\varepsilon)|Z_r^\varepsilon|^2 dr\\ &+\; \int_s^t \frac{\partial \hat{e}}{\partial x}(\check{X}_r^\varepsilon, Y_r^\varepsilon)\gamma\left(\check{X}_r^\varepsilon\right) d\varphi_r^\varepsilon + \int_s^t \left(1 - \varepsilon\frac{\partial \hat{e}}{\partial y}(\check{X}_r^\varepsilon, Y_r^\varepsilon)\right) h(\check{X}_r^\varepsilon, Y_r^\varepsilon)d\varphi_r^\varepsilon\\ &+\; \varepsilon\int_s^t \frac{\partial \hat{e}}{\partial y}(\check{X}_r^\varepsilon, Y_r^\varepsilon)Z_r^\varepsilon d\widehat{B}_r + \int_s^t \widetilde{Z}_r^\varepsilon\left(d\widehat{B}_r - \tau_\varepsilon^*\frac{\partial^2 \hat{e}}{\partial x \partial y}(\check{X}_r^\varepsilon, Y_r^\varepsilon)dr\right).\end{aligned}$$

Since $\dfrac{\partial^2 \hat{e}}{\partial x \partial y}$ is bounded, the Radon-Nikodym derivative $\dfrac{d\widehat{\mathbb{P}}}{d\mathbb{P}^\delta}$ belongs to all spaces $L^q\left(\mathbb{P}^\delta\right)$, then for any $p > 0$, $\sup_\varepsilon \widetilde{\mathbb{E}}\left\|X_t^\varepsilon\right\|^p < \infty$, hence for any $k > 0$,

$$\sup_\varepsilon \widetilde{\mathbb{E}}\left\|g(X_t^\varepsilon)\right\|^k < \infty$$

.

$*$ *Step 2: A priori estimate for* (Y^ϵ, Z^ϵ)

We need to bound appropriate moments of Y^ϵ and Z^ϵ under $\widetilde{\mathbb{P}}$. We first go back to our unperturbed BSDE under the mew Brownian $\widetilde{B}_t$.

$$\begin{aligned}Y_s^\varepsilon =& g\left(X_t^\varepsilon\right) + \int_s^t f(\check{X}_r^\varepsilon, Y_r^\varepsilon)dr + \frac{1}{\varepsilon}\int_s^t e(\check{X}_r^\varepsilon, Y_r^\varepsilon)dr + \int_s^t Z_r^\varepsilon \frac{\partial^2 \hat{e}}{\partial x \partial y}\tau_\varepsilon(\check{X}_r^\varepsilon, Y_r^\varepsilon)dr\\ &+ \int_s^t h(\check{X}_r^\varepsilon, Y_r^\varepsilon)d\varphi_r^\varepsilon - \int_s^t Z_r^\varepsilon d\widetilde{B}_r.\end{aligned}$$

Let $\nu > 0$ be large enough, and apply Itô's formula to $e^{\nu s}\,|Y_s^\epsilon|^3$, we have

$$\begin{aligned}
e^{\nu s}\,|Y_s^\varepsilon|^3 &+ \int_s^t e^{\nu r}\Big(3\,|Y_r^\varepsilon|\,|Z_r^\varepsilon|^2 + \nu|Y_r^\varepsilon|^3\Big)dr\\
&= e^{\nu t}\,|g(X_t^\varepsilon)|^3 + 3\int_s^t e^{\nu r}\,|Y_r^\epsilon|\,Y_r^\epsilon f(\check{X}_r^\varepsilon, Y_r^\varepsilon)dr + \frac{3}{\varepsilon}\int_s^t e^{\nu r}\,|Y_r^\epsilon|\,Y_r^\epsilon e(\check{X}_r^\varepsilon, Y_r^\varepsilon)dr\\
&\quad - 3\int_s^t e^{\nu r}\,|Y_r^\varepsilon|\,Y_r^\varepsilon Z_r^\epsilon d\widetilde{B}_r + 3\int_s^t e^{\nu r}\,|Y_r^\varepsilon|\,Y_r^\varepsilon h(\check{X}_r^\varepsilon, Y_r^\varepsilon)d\varphi_r^\epsilon\\
&\quad + 3\int_s^t e^{\nu r}\,|Y_r^\epsilon|\,Y_r^\varepsilon Z_r^\varepsilon \frac{\partial^2 \hat{e}}{\partial x \partial y}\tau^*(\check{X}_r^\epsilon, Y_r^\epsilon)dr.
\end{aligned}$$

It follows from an argument in Pardoux and Peng (1990) that the expectation of the above stochastic integral is zero. Moreover, from **(H.6)** to **(H.9)** and using similar arguments as those leading to (4.13) in Pardoux (1998), we deduce:

$$\varepsilon\widetilde{\mathbb{E}}\int_0^t e^{\nu r}\Big(\,|Y_r^\varepsilon|\,|Z_r^\varepsilon|^2\Big)dr \leqslant c\Big(\varepsilon + \widetilde{\mathbb{E}}\int_s^t |Y_r^\varepsilon|^2\,dr + \varepsilon\,(\varphi_t^\varepsilon - \varphi_s^\varepsilon)\Big). \tag{2.20}$$

According to Formula (CC) (page 354), we have by applying Itô's formula

$$\begin{aligned}
\left|\widetilde{Y}_t^\epsilon\right|^2 &+ \int_s^t \left|\widetilde{Z}_r^\varepsilon - \varepsilon Z_r^\varepsilon \frac{\partial \hat{e}}{\partial y}(\check{X}_r^\epsilon, Y_r^\epsilon)\right|^2 dr = \left|g(X_t^\epsilon) + \varepsilon\left[\hat{e}(\check{X}_t^\varepsilon, Y_t^\varepsilon) - \hat{e}(\check{X}_s^\varepsilon, Y_s^\varepsilon)\right]\right|^2\\
&+2\int_s^t \widetilde{Y}_r^\varepsilon\left[\frac{\partial \hat{e}}{\partial x}c_\varepsilon - \frac{\partial \hat{e}}{\partial y}.e + (1-\varepsilon\frac{\partial \hat{e}}{\partial y})f + \frac{\partial^2 \hat{e}}{\partial x\partial y}\alpha_\varepsilon\left(\frac{\partial \hat{e}}{\partial x}\right)^*\right](\check{X}_r^\varepsilon, Y_r^\epsilon)dr\\
&+2\int_s^t \widetilde{Y}_r^\varepsilon\left[(1-\varepsilon\frac{\partial \hat{e}}{\partial y})h + \frac{\partial \hat{e}}{\partial x}\gamma_\varepsilon\left(\frac{\partial \hat{e}}{\partial x}\right)^*\right](\check{X}_r^\varepsilon, Y_r^\epsilon)d\varphi_r^\varepsilon\\
&-2\int_s^t \widetilde{Y}_r^\varepsilon \widetilde{Z}_r^\epsilon d\widetilde{B}_r + \varepsilon\int_s^t \frac{\partial^2 \hat{e}}{\partial y^2}(\check{X}_r^\varepsilon, Y_r^\varepsilon)\widetilde{Y}_r^\varepsilon\,|Z_r^\epsilon|^2\,dr\\
&+2\varepsilon\int_s^t \widetilde{Y}_r^\varepsilon Z_r^\varepsilon\frac{\partial \hat{e}}{\partial y}(\check{X}_r^\varepsilon, Y_r^\varepsilon)\Big[d\widetilde{B}_r + \frac{\partial^2 \hat{e}}{\partial x\partial y}(\check{X}_r^\varepsilon, Y_r^\varepsilon)dr\Big].
\end{aligned}$$

To sum up: exploiting in 32 and the inequality (2.20), together with the fact that $1 - \epsilon\dfrac{\partial \hat{e}}{\partial y} \geqslant \dfrac{1}{2}$ for ε small enough, and standard inequalities, one can see :

$$\widetilde{\mathbb{E}}\left(|Y_s^\varepsilon|^2 + \frac{1}{2}\int_s^t \left|\widetilde{Z}_r^\varepsilon\right|^2 dr + \frac{|\beta|}{4}\int_s^t |Y_r^\varepsilon|^2\,d\varphi_r^\varepsilon\right) < C\left(1 + \widetilde{\mathbb{E}}\int_s^t |Y_r^\varepsilon|^2\,dr\right).$$

Hence from Gronwall's theorem

$$\sup_{0\leqslant s\leqslant t} \widetilde{\mathbb{E}}\,|Y_s^\varepsilon|^2 + \frac{1}{2}\int_s^t \left|\widetilde{Z}_r^\varepsilon\right|^2 dr + \frac{|\beta|}{4}\int_s^t \left|\tilde{Y}_r^\epsilon\right|^2 d\varphi_r^\epsilon \leq C.$$

And, in conclusion, from the Davis-Burkholder-Gundy inequality

$$\sup_{0<\varepsilon<\varepsilon_0} \widetilde{\mathbb{E}}\left(\sup_{0\leqslant s\leqslant t} |Y_s^\epsilon|^2 + \int_0^t \left|\widetilde{Z}_r^\epsilon\right|^2 dr + \int_0^t \left|\widetilde{Y}_r^\varepsilon\right|^2 d\varphi_r^\epsilon \right) \leqslant \infty.$$

$*$ *Step 3: Tightness* . Let

$$\mathcal{F}_t^\varepsilon = \mathcal{F}_t^{X^\varepsilon}.$$

We write our reflected BSDE in the form :

$$Y_s^\varepsilon = g(X_t^\varepsilon) + A_t^\varepsilon - A_s^\varepsilon + \widetilde{M}_t^\varepsilon - \widetilde{M}_s^\varepsilon + K_t^\varepsilon - K_s^\varepsilon,$$

where

$$\begin{aligned}
A_s^\varepsilon :=& \int_0^s \left[\frac{\partial \hat{e}}{\partial x}c_\varepsilon - \frac{\partial \hat{e}}{\partial y}.e + \frac{\partial^2 \hat{e}}{\partial x \partial y}\alpha_\varepsilon \left(\frac{\partial \hat{e}}{\partial x}\right)^*\right](\check{X}_r^\varepsilon, Y_r^\varepsilon)dr \\
&+ \int_0^s \left(h + \frac{\partial \hat{e}}{\partial x}\gamma_\varepsilon\right)(\check{X}_r^\varepsilon, Y_r^\epsilon)d\varphi_r^\varepsilon + \int_0^s f(\check{X}_r^\varepsilon, Y_r^\varepsilon)dr \\
\widetilde{M}_s^\varepsilon :=& -\int_0^s \widetilde{Z}_r^\varepsilon d\widetilde{B}_r \\
K_s^\varepsilon :=& \varepsilon\Big(\hat{e}(\check{X}_s^\varepsilon, Y_s^\varepsilon) - \hat{e}(\check{X}_t^\varepsilon, Y_t^\varepsilon)\Big) + \frac{\varepsilon}{2}\int_0^s \frac{\partial^2 \hat{e}}{\partial y^2}(\check{X}_r^\varepsilon, Y_r^\varepsilon)dr \\
&+ \varepsilon \int_0^s \frac{\partial \hat{e}}{\partial y}(\check{X}_r^\varepsilon, Y_r^\varepsilon)Z_r^\varepsilon\left[d\widetilde{B}_r + \left(\frac{\partial^2 \hat{e}}{\partial x \partial y}\tau_\varepsilon\right)(\check{X}_r^\epsilon, Y_r^\epsilon)dr\right] \\
&- \varepsilon \int_0^s \frac{\partial \hat{e}}{\partial y}(\check{X}_r^\varepsilon, Y_r^\varepsilon)f(\check{X}_r^\varepsilon, Y_r^\varepsilon)dr + \varepsilon \int_0^s \frac{\partial \hat{e}}{\partial y}(\check{X}_r^\varepsilon, Y_r^\varepsilon)h(\check{X}_r^\varepsilon, Y_r^\varepsilon)d\varphi_r^\epsilon.
\end{aligned}$$

It is not difficult to check that

$$\widetilde{\mathbb{E}} \sup_{0\leqslant s\leqslant t} |K_s^\varepsilon| \longrightarrow 0,$$

hence, $\sup_{0\leqslant s\leqslant t} |K_s^\varepsilon|$ tends to zero in $\widetilde{\mathbb{P}}$ probability, or equivalently in law.

In order to treat the other terms, we adopt the point of view of the S-topology of Jakubowski (1997). We define the conditional variation of the process A^ϵ on the interval [0, t] as the quantity

$$C\mathcal{V}_t(A^\epsilon) = \sup \widetilde{\mathbb{E}}\left(\sum_i \left| \widetilde{\mathbb{E}}\left(A^\epsilon_{t_{i+1}} - A^\epsilon_{t_i} / \mathcal{F}^\epsilon_{t_i} \right) \right| \right),$$

where the supremum is taken over all the partitions of the interval $[0, t]$. Clearly,

$$C\mathcal{V}_t(A^\epsilon) \leqslant \widetilde{\mathbb{E}}\left(\int_0^t f(\check{X}^\varepsilon_s, Y^\varepsilon_s) ds + \int_0^t h(\check{X}^\varepsilon_s, Y^\varepsilon_s) d\varphi^\epsilon_s \right)$$

and it follows from *Step 2* and the conditions (**H.6**) to (**H.9**) that

$$\sup_\varepsilon \left(C\mathcal{V}_t(A^\varepsilon) + \sup_{0\leqslant s\leqslant t} \widetilde{\mathbb{E}} \left| Y^\varepsilon_s \right| + \sup_{0\leqslant s\leqslant t} \widetilde{\mathbb{E}} \left| \int_0^s \widetilde{Z}^\varepsilon_r d\widetilde{B}_r \right| \right) < \infty$$

hence, the sequences $\left\{ \left(Y^\varepsilon_s, \int_0^t \widetilde{Z}^\varepsilon_s d\widetilde{B}_r \right), 0 \leqslant s \leqslant t \right\}$ satisfy Meyer-Zheng's tightness criterion for quasimartingales under $\widetilde{\mathbb{P}}$ (see for example, Jakubowski (1997), Kurtz (1991), Meyer and Zheng (1984)).

∗ *Step 4: Passage to the limit.*

After extraction of a suitable subsequence, which we omit as an abuse of notations, we have that under $\widetilde{\mathbb{P}}$

$$\left(X^\varepsilon, \widetilde{M}^\varepsilon, \varphi^\varepsilon, Y^\varepsilon, \int_0^\cdot \widetilde{Z}^\varepsilon_r d\widetilde{B}_r \right) \Longrightarrow \left(X, \widetilde{M}^X, \varphi, Y, \widetilde{M} \right)$$

weakly on $C\left([0,t];\mathbb{R}^{d+1}\right) \times D\left([0,t];\mathbb{R}^2\right)$ equipped with the product of the topology of uniform convergence on the first factor, and the S-topology on the second factor, where

$$\widetilde{M}^{X^\varepsilon}_s = \int_0^s \left(I + \nabla\widehat{\beta} \right) \tau \left(\check{X}^\varepsilon_r \right) d\widetilde{B}_r.$$

It remains to recall the two next results:

LEMMA 33. *$\varphi : \mathbb{R}^d \times \mathbb{R} \longrightarrow \mathbb{R}$ be measurable and locally bounded, periodic of period one in each direction, continuous with respect to its second argument, uniformly with respect to the first. Then the sequence of processes $\left\{\int_0^s \varphi\left(\check{X}_r^\varepsilon, Y_r^\varepsilon\right) dr\right\}_{0\le r\le s}$ converges in law under $\widetilde{\mathbb{P}}$ to $\left\{\int_0^s \varphi_0\left(Y_r\right) dr\right\}_{0\le r\le s}$, where*

$$\varphi_0(y) = \int_{\mathbb{T}^d} \varphi(x,y)m(dx).$$

PROOF. see Pardoux (1998), Ouknine and Pardoux (2002). □

LEMMA 34. *$\phi : \mathbb{R}^d \times \mathbb{R} \longrightarrow \mathbb{R}$ be measurable and locally bounded, periodic of period one in each direction with respect to its first argument, continuous with respect to its second argument, uniformly with respect to the first. Then*

$$\sup_{0\le s\le T}\left|\int_0^s \phi\left(\check{X}_r^\varepsilon, Y_r^\varepsilon\right) d\phi_r^\varepsilon - \int_0^s \phi_0\left(Y_r\right) d\phi_r\right| \to 0$$

in $\widetilde{\mathbb{P}}$ probability as $\varepsilon \to 0$,where

$$\phi_0(y) = \int_{\mathbb{T}^{d-1}} \phi(x,y)\widetilde{m}(dx).$$

PROOF. see Ouknine and Pardoux (2002). □

As a result, we pass to the limit in the SDE and the BSDE, and we obtain that for all $0 \leqslant s \leqslant t$,

$$X_t = x + c_0 t + \int_0^t \beta_0\left(Y_s\right) ds + \sqrt{\alpha_0}\widetilde{B}_t + \gamma_0\varphi_t$$

$$Y_t = g\left(X_T\right) + \int_t^T f_0\left(Y_s\right) ds + \int_t^T h_0\left(Y_s\right) d\varphi_s + \widetilde{M}_t - \widetilde{M}_T$$

where

$$\beta_0(y) = \int_{\mathbb{T}^d}\left(I + \nabla\hat{\beta}\right)\alpha(x)\frac{\partial^2\hat{e}}{\partial x\partial y}(x,y)m(dx), \qquad c_0 = \int_{\mathbb{T}^d}\left(I + \nabla\hat{\beta}\right)c(x)m(dx),$$

$$f_0(y) = \int_{\mathbb{T}^d}\left(f + \left[\left\langle\frac{\partial\hat{e}}{\partial x}, c(x)\right\rangle - \left(\frac{\partial\hat{e}}{\partial y}\times e\right) + \frac{\partial^2\hat{e}}{\partial x\partial y}\alpha(x)\left(\frac{\partial\hat{e}}{\partial x}\right)^*\right]\right)(x,y)m(dx),$$

$$h_0 = \int_{\mathbb{T}^{d-1}}\left(h(.,y) + \left\langle\frac{\partial\hat{e}}{\partial x}(.,y),\gamma\right\rangle\right)(x)\widetilde{m}(dx),$$

$$\alpha_0 = \int_{\mathbb{T}^d}\left(I + \nabla\hat{\beta}\right)(x)\alpha(x)\left(I + \nabla\hat{\beta}\right)^*(x)m(dx)\ .$$

Using a similar argument as in Pardoux (1998), one can prove that M and M^X are $\mathcal{F}^{X,Y}$-martingales.

$*$ *Step 4: Identification of the limit*

Let $\left(\overline{Y},\overline{U}\right)$ denote the unique solution of the BSDE

$$\overline{Y}_s = g\left(X_t\right) + \int_s^t f_0\left(\overline{Y}_r\right)dr + \int_s^t h_0\left(\overline{Y}_r\right)d\varphi_r + \int_s^t \overline{U}dM_r^X$$

satisfying

$$\widehat{\mathbb{E}}\left\{\text{Tr}\left(\int_0^t \overline{U}_r d\left\langle M^X\right\rangle_r \overline{U}_r^*\right)\right\} < \infty,$$

and let $\widetilde{M}_s = \displaystyle\int_0^s \overline{U}d\widetilde{M}_r^X$. Since $\overline{Y}$ and $\overline{U}$ are $\mathcal{F}_t^X$ adapted and $\widetilde{M}_r^X$ is a $\mathcal{F}_t^{X,Y}$-martingale , so is also $\widetilde{M}$. It follows from Itô's formula for possibly discontinuous semimartingales that

$$\begin{aligned}\widehat{\mathbb{E}}\left|Y_s-\overline{Y}_s\right|^2 &+ \widehat{\mathbb{E}}\left(\left[M-\widetilde{M}\right]_t - \left[M-\widetilde{M}\right]_s\right)\\ &= 2\widehat{\mathbb{E}}\int_s^t < \left(Y_r-\overline{Y}_r\right), \left(f(Y_r)-f(\overline{Y}_r)\right) > dr\\ &\quad + 2\widehat{\mathbb{E}}\int_s^t < \left(Y_r-\overline{Y}_r\right), \left(h(Y_r)-h(\overline{Y}_r)\right) > d\varphi_r\\ &\leq 2\mu\int_s^t \widehat{\mathbb{E}}\left|Y_r-\overline{Y}_r\right|^2 dr,\end{aligned}$$

(Mind you, we use the fact $\beta \leq 0$).
Hence from Gronwall's lemma $Y_s = \overline{Y}_s$ and $M_s = \widetilde{M}_s,\ 0 \leqslant s \leqslant t,\ Y$ and $\widetilde{M}$ are continuous and have the required properties.

Finally, for $s = 0$,

$$Y_0^\varepsilon = g(X_t^\varepsilon) + A_t^\varepsilon + M_t^\varepsilon + K_t^\varepsilon.$$

Since our subsequence could have been chosen in such a way that for some $a > 0$

$\{M^{\varepsilon}_{s\wedge t}; 0 \leq s \leq t+a\}$ converges weakly to $\{M_{s\wedge t}\ ;\ 0 \leq s \leq t+a\}$ in $D\Big([0,t+a],\mathbb{R}\Big)$ we can assume that

$$M^{\varepsilon}_t \Longrightarrow \widetilde{M}_t.$$

Consequently

$$Y^{\varepsilon}_0 \to Y = g(X_t) + \int_0^t f_0(\overline{Y}_r)dr + \int_0^t h_0(\overline{Y}_r)d\varphi_r - \widetilde{M}_t$$

in probability, since the limit is deterministic. ■

3. Main result

The result of the section 2 permits to us to deduce weak convergence of a sequence Y^{ε} of solutions of BSDEs from weak convergence of the sequence X^{ε}. In a sense, we deduct by the *transformation of drift* and from a probabilistic proof of convergence of semilinear PDEs with nonlinear Neumann boundary condition due to Ouknine and Pardoux (2002), a probabilistic proof of convergence of equations (1.3).

For each $x \in \overline{\boldsymbol{D}}$, let $\{(X^{\epsilon,x}_s, \phi^{\epsilon,x}_s),\ s \geq 0\}$ denote the solution of the SDE (2.1); and let e, f, g and h be as in section 2. For each $(t,x) \in \mathbb{R}_+ \times \overline{\boldsymbol{D}}$, let $u^{\epsilon}(t,x) := Y^{\varepsilon}_0$, where Y^{ε} denotes the solution of the BSDE considered in the section 2. One can easy show as in Pardoux and Zhang (1998) that u^{ε} is a viscosity solution of the semilinear parabolic PDEs (1.3). Therefore, let u be the solution of the homogenized system :

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) = L_0 u(t,x) + f_0\big(u(t,x)\big) + \beta_0\big(u(t,x)\big)\nabla u(t,x)\ ,\ x \in \boldsymbol{D}, \\ \Gamma_0 u(t,x) + h_0\big(u(t,x)\big) = 0\ ,\ x \in \partial\boldsymbol{D},\ t \geq 0, \\ u(0,x) = g(x)\ ,\ x \in \boldsymbol{D}, \end{cases} \tag{3.1}$$

where L_0, Γ_0, β_0, f_0 and h_0 as in section 2.

We now state our main result.

THEOREM 56. *Under* **(H.1)–(H.9)**, $\forall (t,x) \in \mathbb{R}_+ \times \overline{\boldsymbol{D}}$ *we have :*

$$u^{\varepsilon}(t,x) \longrightarrow u(t,x) \ \ \textit{as}\ \ \epsilon \to 0.$$

Bibliography

Anderson R. (1976a) Diffusions with second order boundary conditions Part I. Indiana Univ.Math.J. (**25**) : 367-395.

Anderson R.(1976b) Diffusions with second order boundary conditions, Part II. Indiana Univ.Math.J. (**25**) : 403-441.

Jakubowski, A. (1997). A non-Skorohod topology on the Skorohod space. Electron. J. Probab. 2(4):21[online].

Kurtz, T. G. (1991). Random time changes and convergence in distribution under the Meyer-Zheng conditions. Ann. Probab. 19(3):1010-1034.

Meyer, P. A., Zheng, W. A. (1984). Tightness criteria for laws of semimartingales. Ann. Inst. H. Poincare Probab. Statist. 20(4):353-372.

Ouknine Y., and Pardoux E.(2002). *Generalized BSDEs, weak convergence, and homogenization of semilinear PDEs with the Neumann boundary condition.* In Seminar on Stochastic Analysis, Random Fields and Applications, III (Ascona, 1999). Progress in Probability, (**52**) Basel: Birkhauser, 229-242.

Pardoux E., and Veretennikov Yu A.(2001). *On the Poisson equation and diffusion approximation I*, Annals of Probability, **29**, 1061-1085.

Pardoux, E. and Peng, S. G.(1990). Adapted solution of a backward stochastic differential equation. Systems Control Lett. 14(1):55-61.

Pardoux E., and Zhang S.(1998). Generalized BSDEs and nonlinear boundary value problems. *Prob. Theory and Rel. Fields.* **110**, 535-558.

Pardoux, E., Veretennikov, A. Yu. (1997). Averaging of backward stochastic differential equations, with application to semi-linear PDEs. Stochastics Stochastics Rep. 60(3-4):255-270.

Pardoux, E. (1999). BSDEs, weak convergence and homogenization of semilinear PDEs. In : *Nonlinear Analysis, Differential Equations and Control (Montreal, QC, 1998).* NATO Science Series C, 528. Dordrecht: Kluwer, 503-549.

Stroock D.W., and Varadhan S.R.S.(1971). Diffusion processes with boundary conditions. *Comm. Pure Appl. Math.* (**24**) : 147-225.

Tanaka H.(1984). Homogenization of Diffusion Processes with boundary Conditions. *Stoc. Anal. and ., Adv. Probab. Related Topics*. (**7**) : 411-437.