The Square-root Game

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Abstract

In this work I give an elementary proof of the following: "The absolute 1/2 moment of the beta (1/4, 1/4) distribution about t is independent of t for 0 < t < 1.

Keywords: beta distribution, two person game, Richardson extrapolation.

Theorem 1 The absolute 1/2 moment of the beta (1/4, 1/4) distribution about t is independent of t for 0 < t < 1:

\[ \int_0^1 p(x)(|x - t|^{(1/2)}) \, dx, \]

where

\[ p(x) = [x(1 - x)]^{(-3/4)} , \]

is independent of t for 0 < t < 1.

More generally, for any a with 0 < a < 1, the absolute a-th moment of the beta ((1 - a)/2, (1 - a)/2) distribution about t is independent of t for 0 < t < 1.

This note, which I am pleased to write in honor of my old friend Tom Ferguson, is about the process that led to the Theorem.

Quite a few years ago, shortly after Tom Ferguson got his first PC, I asked him about the Square-root Game:
Square-root Game

Players I and II simultaneously choose numbers $x$ and $y$ in the unit interval. Then II pays I the amount $|x - y|^{(1/2)}$.

Later that day, Tom told me that the value of the game is .59907, to 5 places. I asked him how he got such accuracy, since he could solve games only up to $30 \times 30$ on his machine. He said that he’d used Richardson extrapolation (which I’d never heard of). He told me a bit about Richardson extrapolation, and we turned to other rhings.

Then, in my Fall 1994 game theory class I assigned, as a homework problem, to solve the Square-root Game to 3 places. Several students succeeded, but one group of four students, working together, claimed to have solved the game to 15 places. According to them, if either player used the beta $(1/4,1/4)$ strategy, then Player I’s expected income, as calculated by Mathematica, was constant to 15 places, no matter what the other Player did. They could not prove that a beta $(1/4,1/4)$ strategy gave a constant income, and neither could I.

Later I asked Jim Pitman about the more general case as stated in the Theorem, and he gave a not-quite-elementary proof. Later he found a generalization to higher dimensions in Landkof [1972]. Finally I found an elementary proof, given below.

The method the four students used to get their solution is simple and instructive.

1. They solved a discrete version, restricting each Player to the 21 choices $0, .05, \ldots, .95, 1$. The good strategy for each player was a U-shaped distribution, symmetric about 1/2.

2. They calculated the variance of this distribution, and found the beta distribution symmetric about 1/2 with the same variance. It was beta $(.2613,.2613)$.

3. They guessed that .2613 was trying to be .25, so tried beta$(1/4,1/4)$ as a strategy.

Here is the proof of the Theorem. Fix $a, 0 < a < 1$, and put

$$f(t) = \int_0^1 p(x)(|x - t|^a) \, dx$$

where $p(x) = [x(1 - x)]^{-(a+1)/2}$.
We must show that \( f \) is constant on \( 0 < t < 1 \). Its derivative is

\[
f'(t) = a \int_0^t [(t - x)^{(a-1)}]p(x) \, dx - a \int_t^1 [(x - t)^{(a-1)}]p(x) \, dx
\]

With the change of variable \( u = 1 - x \) in the second integral, we get

\[
f'(t) = a \left[ \int_0^t [(t - x)^{(a-1)}]p(x) \, dx - \int_0^{1-t} [(1 - t - u)^{(a-1)}]p(u) \, du \right]
\]

\[
= a(F(t) - F(1 - t)), \text{ where}
\]

\[
F(t) = \int_0^t [(t - x)^{(a-1)}]p(x) \, dx.
\]

So we must show that \( F(1 - t) = F(t) \). To evaluate \( F \), make the linear fractional change of variable \( z = (t - x)/(t - tx) : x = t(1 - z)/(1 - tz) \) (see Carr, [1970], Formula 2342). We get

\[
F(t) = \left[t(1-t)\right]^\left[(a-1)/2\right] \int_0^1 [(1 - z)^{(-a+1)/2}] [z^{(a-1)}] \, dz
\]

So \( F(1 - t) = F(t) \), proving the Theorem.

So the value of the Square-root game is I's expected income when he chooses \( x \) according to beta \((1/4, 1/4)\) and II chooses \( y = 0 \), namely

\[
\Gamma(1/2)\Gamma(1/4)\Gamma(1/4) \int_0^1 (x^{(1/2)})([x(1 - x)]^{(-3/4)}) \, dx = \Gamma(1/2)\Gamma(3/4)\Gamma(1/4)
\]

\[
= .599070117367796....
\]

So Tom's first five places were correct.

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**References**

