UNBIASED SEQUENTIAL BINOMIAL ESTIMATION

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Abstract

We review the literature on unbiased estimation of some functions of the Bernoulli parameter p in the sequential case. Connections between the so-called efficient and inefficient sampling plans through the well known concept of sufficiency which have been explored recently are also presented.

Introduction

Under the set up of independent identical Bernoulli trials with parameter p, various aspects of unbiased estimation of a parametric function g(p) have been studied in the literature. Early works of Girshick, Mosteller and Savage (1946), Wolfowitz (1946, 1947), Lehmann and Stein (1950), De Groot (1959) and Wasan (1964) are devoted to some general results on sequential binomial estimation. Later works by Gupta (1967), Sinha and Sinha (1975), Sinha and Bhattacharya (1982) and Sinha and Bose (1985) deal with problems related to unbiased estimation of $1/p$. Recently Bose and Sinha (1984) studied the connections between the so-called efficient and inefficient Bernoulli sampling plans through the well known concept of sufficiency of statistical experiments.

Our object in this paper is to present a comprehensive review of most of the available results in this area. We omit proofs of all the results. However, detailed and exact references to various results are provided.

The next section is devoted to setting up the notations, nomenclature, and definition of efficient sampling plans. In the third section, we provide results on efficient sampling plans. The problem of unbiased estimation of $1/p$, which has received considerable amount of attention in the literature, is discussed in fourth section. In fifth section, we discuss the connection between efficient and inefficient sampling plans via the concept of sufficiency. Some concluding remarks are made in the last section.

Notations and Nomenclature

Let $(Z_i,\ i = 1, 2\ldots)$ be an i.i.d. sequence of Bernoulli variates with $P(Z_i = 1) = p$ and $P(Z_i = 0) = 1 - p = q$ (say). We assume $p \in \Omega \subseteq (0, 1)$. Any realization of this process can be exhibited as a lattice path in the (X, Y)-plane, where a particle moves from the origin one step to the right (along X-axis) if the incoming observation is 0 and one step above (along Y-axis) if it is 1. A *stopping rule* can be viewed as a sequence of functions ϕ_k, where ϕ_k is a function of $(Z_1,\ldots,Z_k)$. Each ϕ_k takes the value 0 or 1; given $z_1,\ldots,z_k$, $\phi_k(z_1,\ldots,z_k) = 1$ indicates that we take one more observation and $\phi_k(z_1,\ldots,z_k) = 0$ indicates that we stop at this stage. A point $\alpha = (x, y)$ is a *continuation point* if there exists one sequence of realization $(z_1, z_2,\ldots,z_{x+y})$ leading to α such that $\phi_j(z_1,\ldots,z_j) = 1$ $\forall j \leq x + y$. A point $\alpha = (x, y)$ is a *boundary point* if there exists one sequence of realization $(z_1, z_2,\ldots,z_{x+y})$ leading to α such that $\phi_j(z_1,\ldots,z_j) = 1$ $\forall j < x + y$ and $\phi_j(z_1,\ldots,z_{x+y}) = 0$. A point may be a boundary point or a continuation point depending on the path. A point is an *accessible point* if it is either a boundary point or a continuation point. Points which are not accessible are inaccessible points. For any boundary point $\alpha = (x, y)$, $P(\alpha)$ denotes the probability of stopping at α and is given by

$$P(\alpha) = p^y q^x \sum_{\substack{(z_1,\ldots,z_{x+y}) \\ \text{leading to } (x,y)}} \{1 - \phi_{x+y}(z_1,\ldots,z_{x+y})\}$$

$$= K(\alpha) p^y q^x \text{ (say)} \tag{1}$$

where $K(\alpha)$ is the number of accessible paths from the origin to the point α.

A stopping rule yielding the boundary points together with their probabilities $P(\alpha)$ shall be called a sampling plan P. We say that P is closed iff $\sum_{\alpha \in B} P(\alpha) = 1$ identically in $p \in \Omega$, B denoting the set of all boundary points of P. This refers to eventual termination with probability one. Only closed sampling plans are of interest to the practical experimenter and we shall assume so unless otherwise mentioned.

Given a closed plan P, we say that a parametric function $g(p)$ is unbiasedly estimable if there exists a function $f(\alpha)$ such that

$$g(p) = E_p(f(\alpha)) = \sum_{\alpha \in B} f(\alpha) P(\alpha), \ \forall p \in \Omega. \tag{2}$$

When (2) holds, $f(\alpha)$ is said to define an unbiased estimate of $g(p)$ and it is a proper estimate of $g(p)$ if $f(\alpha) \in$ range of $\{g(p)\colon p \in \Omega\}$ for every $\alpha \in B$. Otherwise, it is said to be improper. We straightaway insist on non-negative estimability of $g(p)$ (i.e., we demand $f(\alpha) \geq 0$) whenever $g(p) \geq 0$, $\forall p \in \Omega$. The reasons for this shall be clear as we proceed. In the same vein, for unbiased estimation of $1/p$, we insist that the estimate $f(\alpha)$ be proper viz., $f(\alpha) \geq 1$, $\forall \alpha \in B$.

Remark 1

Given an arbitrary sampling plan, examining its closure is not always an easy task. Consider plans having boundaries determined through two infinite sequences of points $(0, a_0)$, $(1, a_1)$, $(2, a_2)$,... and $(b_0, 0)$, $(b_1, 1)$, $(b_2, 2)$,.... Here $1 \leq a_0 \leq a_1 \leq a_2 \ldots$ and $1 \leq b_0 \leq b_1 \leq b_2 \leq \ldots$ are two infinite sequences of positive integers. Such plans have been termed *doubly simple* (see Wolfowitz, 1946). For such plans, closure holds whenever $\liminf_{n\to\infty} A(n)/\sqrt{n} < \infty$ where $A(n)$ refers to the number of accessible points of index n. However, an arbitrary unbounded sampling plan need not be doubly simple and, hence, the condition $\liminf_{n\to\infty} A(n)/\sqrt{n} < \infty$ can be substantially improved for other types of unbounded plans. As a matter of fact, plans with $A(n) = 0(n)$ can also be closed. The point to be noted is that the actual value of $A(n)$ is *not always* an important factor to decide on closure or otherwise of a plan. Once an accessible point is reached by a path, only the nature of the remaining part of the sampling plan ahead of this point is relevant for the path to hit a boundary point, and hence, to lead eventually to closure of the plan. The reader is referred to Sinha and Bhattacharya (1982) for examples of various types of unbounded closed plans and other details. The notion of a transformed plan due to Sinha and Sinha (1975) which is also relevant in this context is explained in the fourth section.

Efficient Sampling Plans

DeGroot (1959), under certain regularity conditions, established the validity of the Rao-Cramer lower bound for the variance of an unbiased estimate of any estimable parametric function $g(p)$ based on a sequential sampling design. The concept of efficient sampling plans for unbiased estimation of $g(p)$, as introduced by him, refers to a closed sampling plan P together with an unbiased estimate $f(\cdot)$ such that the sampling variance of $f(\cdot)$ attains its relevant lower bound (which of course depends on $g(p)$ and the particular plan P). He observed that the only efficient sampling plans are the family of Inverse Binomials when $g(p)$ is linear in $1/p$. Of course, trivially the family of Binomials is also efficient when $g(p)$ is linear in p. All other plans may be termed as inefficient. An efficient plan may be seen to maximize the efficiency per unit observation for all $p \in (0, 1)$.

The sampling plans often studied in the literature implicitly (or explicitly) envisage that the decision to stop at a point (or continue) depends only on the point reached (rather than the path traversed in reaching that point). This leaves out a variety of plans obtained by quite interesting and practically suggested stopping rules. A quick example of such a plan is one in which we stop as soon as we obtain two consecutive successes (let us call this plan Plan P1). In this case there would be some boundary points which are exclusively so, namely, the points on the line $Y = X + 2$. There would be other points which would be continuation or boundary points depending on the path or route followed in reaching them. To differentiate the classical sampling plans from such plans, we shall call the former *boundary point* plans and the plans of the type P1 as *route* plans. These two types together form the class of all conceivable plans.

By easy modification of arguments in De Groot (1959) it can be shown that the Rao-Cramer bound remains valid for route plans. Moreover, in case the parameter space Ω is an open subset of (0, 1), the regularity conditions may be replaced by their *local* versions. These indicate that the only parametric functions efficiently estimable are of the form $(a + bq)/(p - \beta q)$ (a, b being arbitrary real numbers and β being an integer ≥ -1). These include p and $1/p$ in particular. The corresponding efficient plans are given by $P(\beta, c) = \{\alpha = (x, y): y = x\beta + c\}$ with β, c integer, $c \geq 0$ and $\beta \geq 1$. Such a plan is closed if $q \leq 1/(\beta + 1)$ when $\beta > 0$, and $\forall p \in (0, 1)$ otherwise. These results have been derived recently by Dutta (1980), who designates such plans as *Generalized Inverse Binomial Plans.*

As regards the inefficient plans, we demonstrate in the fifth section that a large number of them are indeed *sufficient* for the efficient plans.

Sequential Unbiased Estimation of $1/p$

The special problem of sequential unbiased estimation of $1/p$ has been initiated in Gupta (1967) and since then treated extensively in the literature. The central problem has been to characterize all sequential sampling plans which provide unbiased estimation of $1/p$. It may be noted that the analogous problems of unbiased estimation of $1/q$, $1/pq$, etc. can be handled in a similar way.

Gupta (1967) stated a very simple sufficient condition for a sequential sampling plan P to provide an unbiased estimate of $1/p$:

(i) Sufficient condition: if the closed plan P with boundary $B = \{r_i = (x_i, y_i), i = 1, 2, \ldots\}$ be such that by changing its boundary points from r_i to $r'_i = (x_i, y_i + 1)$, we get a closed plan P' with boundary $B' = \{r'_i = (x_i, y_i + 1), i = 1, 2, \ldots\}$, then $1/p$ is estimable for the plan P. An unbiased estimate is given by $f(r) = K'(r')/K(r)$, $r \in B$, where $K'(r')$ is the number of paths from the origin to $r' \in B'$.

Sinha and Sinha (1975) studied the problem in a greater detail and, among other things, put forward the notion of a transformed plan which can be described as follows. For a given plan P with the set B of boundary points α, let (x', y') be an arbitrary but fixed point in the XY-plane. Then the transformed plan $P^T(x', y')$, corresponding to (x', y'), with the set $B^T(x', y')$ of boundary points $\alpha^T(x', y')$ is defined by the following three conditions:

I. Every α^T belonging to B^T also belongs to B necessarily.

II. The points $\{(x, y) : x \geq x', y \geq y'\}$ constitute the *totality* of all points (accessible, boundary and inaccessible) of P^T.

III. Every boundary point $\alpha \in B$ is either a boundary point $\alpha^T \in B^T$ or an inaccessible point in P^T.

Given the plan P with boundary points B, the rules for obtaining the boundary points $\alpha^T \in B^T$ are as follows:

(a) if $(x', y') \in B$, i.e., if $\alpha = (x', y')$, then $\alpha^T = \alpha$ is the only boundary point of B^T;

(b) if $(x', y') \notin B$, then $inf\{\alpha : \alpha = (x, y'), x > x'\}$, for $(x, y') \in B$, is the only point on '$Y = y'$' that belongs to B^T;

(c) if $(x', y') \notin B$, then $inf\{\alpha : \alpha = (x', y), y > y'\}$, for $(x', y) \in B$, is the only point on '$X = x'$' that belongs to B^T;

(d) if $(x', y') \notin B$, any boundary point $\alpha \in B$ also belongs to B^T if and only if it can be reached by a path from (x', y'). Otherwise, it is an inaccessible point of P^T.

It may be noted that whenever the point $t = (x', y')$ is an accessible point of P, we have

$$p^{y'} q^{x'} = \sum_{\alpha \in B^T(x',y')} t(\alpha) p^y q^x$$

$$\text{i.e., } 1 = \sum_{\alpha \in B^T(x',y')} t(\alpha) p^{y-y'} q^{x-x'} \tag{3}$$

where $t(\alpha)$ = total number of ways of passing from t to α only through the accessible points of $P^T(x', y')$. Even when $t = (x', y')$ is an inaccessible point of P, we may use the above definition of $t(\alpha)$ for all $\alpha \in B^T(x', y')$.

The transformed plan $P^T(x', y')$ is defined to be closed *only when* the identity (3) above holds, no matter whether (x', y') is accessible or not. With reference to the problem of unbiased estimation of $1/p$, Sinha and Sinha (1975) came up with the following separate necessary and sufficient conditions.

(ii) Necessary condition: the sampling plan must be unbounded along the X(failure)-direction.

(iii) Sufficient conditions: (a) if no point on the line $Y = 1$ is inaccessible, then $1/p$ is estimable. (b) let $(x_0, 1)$ be the first inaccessible point on the line $Y = 1$. If the transformed plan $P^T(x_0, 1)$ is closed, then $1/p$ is estimable.

It has been demonstrated in Sinha and Sinha (1975) that the sufficient conditions (i) and (iii)(b) are equivalent, and conjectured that the sufficient condition (i) is necessary as well. In Sinha and Bhattacharya (1982), useful notions of finite-step and infinite-step generalizations of the Inverse Binomials have been introduced, and the following results have been deduced. See also Sinha and Bose (1985) in this context.

(iv) All finite-step generalizations of the Inverse Binomials provide unbiased estimation of $1/p$.

(v) Every infinite-step generalized Inverse Binomial, whenever closed, provides unbiased estimation of $1/p$.

Incidentally, an infinite-step generalized Inverse Binomial plan is closed if and only if $\liminf_{n\to\infty} d(n)/n = 0$ where $(n - d(n),\ d(n))$ is the coordinate position of the boundary point on the line $X + Y = n$ $(n = 1, 2, \ldots)$. For a proof, see Bhattacharya and Sinha (1982), Bose and Sinha (1984).

The conjecture relating to a characterization of all sampling plans providing unbiased estimation of $1/p$ has been settled in the affirmative in Sinha and Bose (1985). The result is stated below.

Theorem 1

A plan P provides unbiased estimation of $1/p$ if and only if the plan P' defined in the sufficient condition (i) is closed.

Connections between Efficient and Inefficient Plans

In this section, we demonstrate that a large number of inefficient sampling plans are indeed sufficient for the efficient plans. These results have been established in Bose and Sinha (1984).

The concept of sufficiency in comparing statistical experiments is well known. Roughly speaking, an experiment E resulting in a random variable X having law of distribution $F_\theta(\cdot)$ is said to be sufficient for another experiment E' resulting in a r.v. Y having law of distribution $G_\theta(\cdot)$ if, given an observation x on X, it is possible to generate an observation y on Y using a known randomization procedure, i.e., a known law of distribution $Z(\cdot|x)$, which is independent of θ. If the above holds, we say that X is *sufficient* for Y and write $X > Y$.

Clearly, when $X > Y$, it is enough to observe X to generate Y, if needed. Moreover, it is known (Blackwell and Girshick, 1954) that when $X > Y$, for any estimable parametric function $g(\theta)$, given any unbiased estimate based on Y, one can construct an unbiased estimate based on X which is as good (in the sense of having equal or smaller variance). Applied to the present set up, this would mean that any plan, whenever sufficient for a given inefficient plan, would provide smaller variance (but certainly larger ASN) than the latter.

The following general results on comparison of sampling plans for sufficiency consideration are interesting and illuminating. We consider two arbitrary closed plans P^* and P, and state conditions under which $P^* > P$. In which follows $B^*(B)$ denotes the set of boundary points of $P^*(P)$. We also assume that each of $\Omega(P)$ and $\Omega(P^*)$, the parameter space for closure of P and P^*, is the entire interval (0, 1).

Before we state the results we mention the notion of completeness in this context. Writing $P^*(\alpha^*) = K^*(\alpha^*)p^{y^*}q^{x^*}$ for $\alpha^* = (x^*, y^*) \in B$, a plan P^* is said to be complete if $\sum_{\alpha^* \in B^*} f(\alpha^*)P^*(\alpha^*) = 0, \forall p \in \Omega$ implies $f(\alpha^*) = 0, \forall \alpha^* \in B^*$. The following result (necessity due to Girshick, Mosteller and Savage (1946), sufficiency due to Lehmann and Stein (1950)) gives a characterization of such plans which are useful in the sequel.

Theorem 2

A plan P^* is complete if and only if the following hold:

(a) The plan is simple (i.e., the continuation points of P^* on the line $X + Y = n$ form an interval, $\forall\, n \geq 1$).

(b) The removal of any boundary point destroys closure of the plan.

Following Bose and Sinha (1984), a series of results can be stated.

Theorem 3

(i) A necessary condition for $P^* > P$ is that for every $\alpha = (x, y) \in B$, $p^y q^x$ is estimable under P^*.

(ii) If P^* is complete, then (i) ensures that $P^* > P$.

Bose and Sinha (1984) observed that if P^* is not complete, then the estimability of $p^y q^x$ under P^* for every $\alpha \in B$ may not necessarily yield $P^* > P$. They also noted that the completeness of P^* is not necessary for it to be sufficient for P.

It is clear from the above result that the estimability of $p^y q^x$ for an arbitrary point $\alpha = (x, y) \in B$ of P with reference to P^* arises naturally. Wolfowitz (1946) established its estimability in case α is an accessible point of P^*, though $p^y q^x$ may be estimable even otherwise. The following theorem provides a necessary condition.

In what follows, a point α is defined to line below α^* if α lies in the rectangle formed by the two axes and the point α^*. A point α^* lies above α if it lies in the positive quadrant formed by α as the origin.

Theorem 4

A necessary condition for estimability of $p^y q^x$ under a plan P^* is the existence of at least one boundary point of P^* above (x, y).

As a consequence, we have the following corollary on necessary conditions for P^* to be sufficient for P.

Corollary 1

Two necessary conditions for P^* to be sufficient for P are:

(i) For every $\alpha \in B$, $\exists\ \alpha^* \in B^*$ above α.

(ii) For every $\alpha^* \in B^*$, $\exists\ \alpha \in B$ above α^*.

However, as noted in Bose and Sinha (1984), (i) and (ii) together with even estimability of $p^y q^x$, $\forall\ \alpha \in B$, are not enough to assert $P^* > P$.

We now state a sufficient condition for the estimability of $p^y q^x$ under a plan P^* based on the notion of transformed plans as explained in the last section. Treating (x, y) as the origin, we can derive a transformed form of P^* to be denoted as $P^{**}(x, y)$. In this new plan, the paths emerge from the new origin, and get merged into accessible points or escape them. Note that if (x, y) is itself a boundary point of P^*, the transformed plan does not get started at all. Clearly the set of boundary points of P^* above (x, y) is regarded as the set of boundary points of $P^{**}(x, y)$.

Theorem 5

Whenever $P^{**}(x, y)$ is closed, $p^y q^x$ is estimable.

We conclude this section with another simple sufficient condition for $P^* > P$. Let $K^{**}(\alpha)$ be the number of accessible paths of P^* from origin to α without hitting any other $\alpha' \in B$, leading to α as a continuation point of P^*.

Theorem 6

$K(\alpha) = K^{**}(\alpha)$, $\forall\ \alpha \in B$ implies $P^* > P$.

Specialized to the problem of obtaining plans sufficient for the Binomials, we have the following results.

Theorem 7

(a) A closed plan P^* is sufficient for the Inverse Binomial plan $P(0, c)$ if and only if there exists no boundary point of P^* below the line $Y = c$.

(b) A closed plan P^* is sufficient for the fixed Binomial plan of size n if and only if there is no boundary point of P^* below the line $X + Y = n$.

As a consequence of (a), we have the following result for the First Waiting Time plan.

(c) A plan P^* with no boundary points on the X-axis is sufficient for the plan $P(0, 1)$. Only such plans are sufficient for $P(0, 1)$.

Concluding Remarks

(i) In a recent paper, Bhandari and Bose (1989) have derived conditions on the nature of unbiasedly estimable functions $g(p)$. They have demonstrated that g has to be continuous if it is unbiasedly estimable. Further, if g is nondifferentiable, then it is *not* unbiasedly estimable by a bounded estimator with *finite* expected stopping time for all p. This shows that $g(p) = min(p, 1 - p)$ is *not* estimable by any finite (or bounded) sampling plan though there are plenty of unbounded sampling plans useful for this purpose. An open problem in this context is the following:

Does there exist any proper unbiased estimate of $min(p, 1 - p)$?

(ii) The following problem is also of considerable interest. Fix an integer n and consider the class of all Bernoulli sampling plans P such that for boundary points of the type $\alpha = (x, y)$, $\int_\omega E_p(x + y)d\psi(p) \leq n$ for a prior distribution $\psi(\cdot)$ on p. Does there exist a sampling plan in this class which is the *best* for estimation of p? Here *bestness* refers to minimum prior expectation of posterior variance. In particular, one would be curious to know if the Binomial plan is the best for all or some priors $\psi(\cdot)$.

By a slight modification of the above problem, we may as well search for the best plan among those for which $E_P(x + y) \leq n$, $\forall$ p. Bhandari et al. (1989) have obtained some partial results in this direction.

(iii) Rustagi (1975) has studied some aspects of estimation of p in the simple Markovian set up. Following this, Sinha and Bhattacharya (1982) initiated a study in the dependent set up in the context of sequential estimation. Further research is needed in this area.

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