

A CB (Corporate Bond) Pricing Probabilities and Recovery Rates Model for Deriving Default Probabilities and Recovery Rates

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Abstract: In this paper we formulate a corporate bond (CB) pricing model for deriving the term structure of default probabilities (TSDP) and the recovery rate (RR) for each pair of industry factor and credit rating grade, and these derived TSDP and RR are regarded as what investors imply in forming CB prices in the market at each time. A unique feature of this formulation is that the model allows each firm to run several business lines corresponding to some industry categories, which is typical in reality. In fact, treating all the cross-sectional CB prices simultaneously under a credit correlation structure at each time makes it possible to sort out the overlapping business lines of the firms which issued CBs and to extract the TSDPs for each pair of individual industry factor and rating grade together with the RRs. The result is applied to a valuation of CDS (credit default swap) and a loan portfolio management in banking business.

1. Introduction

In the financial industry credit risk is a central theme of interest in banking, investment and derivatives. In the past financial economic theory and no-arbitrage theory in mathematical finance has provided many important results and thoughts for looking into many credit-risk-related problems. These normative theories are very important as a benchmark for those in financial industry. However, a blind application of the theories to the real world may cause a big problem, which happened in the 2008 Financial Crisis. In fact, rating agencies wrongly applied the theories to valuation of mortgage-backed securities and expanded the serious Sub-prime Problem, while insurance companies issued credit risk swaps beyond their capacity of possibly taking risks and made a serious impact on the US economy, which in turn has been badly affecting world economies. In this paper, the author often emphasizes the importance of testing the empirical validity of “theories” and the limitation of the framework of no-arbitrage theory from a practical viewpoint below. But he never intends to deny the importance and value of theory itself, and thinks it important to now discuss openly about what caused the crisis among academics and practitioners in order not to let it happen again.

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In credit risk analysis in mathematical finance, a time-continuous model is usually formulated together with no-arbitrage concept and it is often the case that a spot rate process is used for instantaneous interest rates, the default-event generation process is associated with such a model as hazard model or Merton-type model and the instantaneous recovery rate at each moment is constant. The resulting model is usually Markovian and univariately stochastic, it individually evaluates the default probability of each product or each firm and the data used in applications is mostly past default rates, rating scores, and firms financial data together with stock prices. It should be pointed out that the Markovian properties, which are the results of the assumption of time-continuous diffusion model, do not hold for credit risk processes in reality in general. It is because business cycles that are closely associated with credit variations of firms are not Markovian, which is a limitation of the model. On the other hand, [Kariya and Liu \(2003\)](#) treated a time-discrete theory to include non-Markovian models.

In addition, since credit risk in reality is related to many factors such as industry business cycles which may depend on import-export relations, exchange rates, resource prices, etc, it does not admit a complete model except for the case in its theoretical assumption, meaning that the no-arbitrage concept in mathematical finance is not necessarily effective in reality. This in turn implies that in practice there is no unique valuation via the no-arbitrage.

In this paper we formulate a corporate bond (CB hereafter) pricing model that enables us to derive the term structures of default probabilities (TSDPs hereafter) and the recovery rates (RRs hereafter) consistently with rating factors and industry factors, where the TSDP is the set of default probabilities that a firm gets defaulted by a future time s where s belongs to a positive interval from 0. Our model is a cross-sectional empirical model valuing all the given CB prices simultaneously at each time.

In this formulation we take into account the fact that each enterprise has a portfolio of multiple business lines corresponding to some different industries, and we explicitly incorporate the business portfolio structures of individual firms together with rating factors into the model in order to derive the TSDPs and RRs implied in CB prices. In our case the portfolio ratios of business lines are measured by the sales ratios of industry-wise business lines. Hence in estimation we need the sales and rating data together with the CB price data and bond attributes (coupon and maturity period).

Our modeling is based on three fundamental assumptions or viewpoints. The first one is about the information content contained in CB prices. The CB prices are assumed to be efficiently formed in the market at each time and so reflect or contain the investors' views on the term structure of future default probabilities for each CB over its maturity period. In fact, their investment decision makings are usually based on sufficient information and analysis on firms and hence almost all the CB prices should be supposedly consistent with the investors' views. Consequently the TSDPs we aim to derive for each firm from all the current CB prices are regarded as the investor's forward-looking TSDPs.

Second, it is viewed in our that the credit condition or credit quality of each firm in general depends on its future cash (profit) flows, which in turn depend on the business portfolio structure. Therefore, the credit condition or credit quality of each firm in general depends on economic trends or business cycles that are often different industry-wise.

The third point is related to the fact that the specification of default correlation is very crucial in credit risk analysis. In our CB price modeling, the default correla-

tions are naturally introduced from the model structure through considering those of stochastic discount functions and the cash flow structure of CBs with defaults. On the other hand, in time-continuous setting, the correlations are often assumed to be constant though they in fact change constantly and increase in the phase of downturn economy where industry-wise business cycles are relevant. Concerning modeling the stochastic correlations, in discrete time series analysis Engle (2009) extensively treats multivariate return processes with conditional stochastic volatilities and conditional stochastic correlations. As terminology, when a credit analysis contains the credit risks of multiple firms, it is often referred to as multi-name case where the correlation structure needs to be specified. Otherwise it is referred to as single-name case.

In applications, the implied TSDP and the implied RR that we derive from the formulation of our CB pricing model can be applied to price such credit derivatives as CDS (credit default swap). In fact, in this paper we also give a formula for valuing a CDS of CB in our discrete-time approach. Since our TSDP depends on the business portfolio structure of the issuer, so does the pricing formula of a CDS, though the form of the formula itself is rather well known.

There is a vast literature in the area of credit risk. Most recent researches take a time-continuous setting. The books by Duffie and Singleton (2003), Lando (2004), and McNeil, Frey and Embrechts (2005) are well known. Most of the articles treat a single-name (univariate) case though some recent works consider a multi-name case, e.g. Filipović, Overbeck and Schmidt (2011). But most of the papers do not take into account the feedback structure of economies and simply assume exogenous processes for credit movements without considering business cycles. Duffie and Kan (1996) viewed interest rate processes as dependent and used a state space model though they assumed a single Markovian process for the univariate state variable. Some papers consider a specific factor for each firm, but the specific factors are simply treated as some exogenous processes independently of a common factor.

In our modeling the stochastic behaviors of the CB prices correspond to those of the attribute-dependent discount functions, which are formally expressed with attribute-specific forward rates. In association with this viewpoint, Collin-Dufresne and Solnik (2001) considers the term structure of default premia in the swap and LIBOR markets. Feldhutter and Lando (2007) decomposes swap rate into a common swap rate and swap spreads that are specific to credit or counter party risk, where the spreads are referred to as convenience yields.

The organization of this paper is as follows. In Section 2, some important problems are discussed in the formulation of credit risk model to be used in practice. In particular, some differences between time-continuous models and our model are discussed from a practical view point. In a line with the arguments in Section 2, Section 3 presents an extension of the government bond (GB) model that Kariya and Tsuda (1994) and Kariya and Tsuda (1996) proposed, where the model is shown to have a one-month ahead predictive power for Japanese Government bond prices. And we discuss about the relation between the stochastic discount functions and spot and forward interest rates which are both attribute-dependent. In Section 4 the non-defaultable bond model is specified in detail for empirical work and an estimation procedure is proposed. The important parts are the specifications of the mean discount function and covariance structure of the bond prices, which separates our model from the other models. In this paper GB and non-defaultable bond are assumed to be synonymous.

In Section 5, we formulate the CB pricing model that enables us to derive the implied TSDPs and RRs. This formulation is quite different from the models in

the literature in that in our modeling the concept of the investors' forward-looking views on the TSDPs is introduced and implemented into the model. In Section 6, we apply our results to credit risk management in banking and pricing credit default swap (CDS) among others.

2. Problems in credit risk analysis

In Section 1, concerning credit risk models developed in mathematical finance, we discussed about some features of the models from a practical viewpoint. Those features come from the continuous-time setting; Markovian, univariate, instantaneous recovery rate, etc. In addition, the following points should be well considered in the formulation:

- (1) Data source to be used for credit-risk modeling—past data or current data;
- (2) Default correlation problem—univariate (single-name) model or multivariate (multi-name) model;
- (3) Conditional model or unconditional model; and
- (4) Information on industry factors and rating factors.

The point (1) is important for practical effectiveness of a model so long as the model is used for a forward-looking decision-making or investment. In fact, economic, financial and technological environments surrounding firms are evolving constantly and rather quickly and hence models using past data on defaults and non-defaults and past financial data of firms do not necessarily give us a forward-looking information to get a future credit perspective for each firm. The model based on past data may tend to deliver a rather backward-looking information, which may be the case of Markovian transition model or hazard rate model, where data on defaults and non-defaults over some past period is often used. On this point, it is noted that each default event in the past is very firm-specific in its nature, and defaulted firms do not exist any longer. Credit risk models that use current market price data on existing firms are more forward-looking as the investors try to be rational and analytical to make gains. Hence the CB prices that are formed in the market are supposed to reflect the views of investors. In this sense our TSDPs whose information is based on current cross-sectional data of CB prices can be regarded as the TSDP of investors' perspective views and so it will be more practically effective for decision makings.

Concerning (2) and (3), note that credit risk factors in credit-related instruments and derivative products are generally significantly correlated through industry factors and business cycles and so credit correlations should be well modeled consistently in valuation. In addition, business cycles in each industry are often different, differently affecting firms that have different portfolios of business lines with some overlapping for each other. Furthermore default processes in practice affected by such business cycles is in general non-Markovian and mutually dependent. In the literature a Merton-type model, which is basically univariate Markovian (conditional) geometric Brown model for stock (firm value) prices, is often extended to a multi-name case where the correlations may be treated as constants with copula functions. It is noted that it is not easy to derive the TSDP from stock prices as stock does not have a finite time horizon.

Considering (4), note that the future cash flow (profit) structure of an enterprise depends on the portfolio structure of business lines associated with industry factors. The cash flow structure matters mostly for the credit quality because most

of defaults occur due to the lack of liquidity or cash, not through the imbalances in the financial balance sheets of firms. Since the industry factors make a correlation structure among different credit risks of firms to a great extent, the portfolio structure of business lines associated with industries should be taken into account for valuing CBs, CDSs and some other credit-related products though the concept of industry is relative and needs to be defined in advance.

Furthermore it is remarked that in reality the recovery rate (RR) after default is determined after a long procedure of asset evaluation and negotiation process among those of interests, which is costly and very time-consuming. In our formulation it is derived as the RR that investors expect at current time and recovery is assumed to takes place at the next coupon-paying time, as will be discussed in Section 5.

3. Pricing non-defaultable bonds

Suppose that there are G government bonds (GBs) or equivalently non-defaultable bonds whose prices are denoted by P_g ($g = 1, \dots, G$). Let $t = 0$ denote the present time and let

$$(1) \quad s_{g1} < s_{g2} < \dots < s_{gM(g)} \quad g = 1, \dots, G$$

denote the future time points at which the g th bond generates the cash flows (coupons or principal) in view of $t = 0$. These s_{gj} values are measured in years, where $s_{gM(g)}$ is the maturity period. In this section and the next section future time points are measured continuously in years. Assume that the face value is 100 (yen or dollar) and let c_g be its coupon rate (yen or dollar). If coupons are paid biannually, the cash flow function $C_g(s)$ of the g th bond is expressed as

$$(2) \quad C_g(s) = \begin{cases} 0.5c_g & s = s_{gm}, m \neq M(g) \\ 100 + 0.5c_g & s = s_{gM(g)} \\ 0 & s \neq s_{s_{gm}} \end{cases}.$$

However, in our argument the cash flow function can be arbitrary, so long as the future cash flows and their time points are given in advance.

Let $D_g(s)$ be the attribute-dependent stochastic discount function of the g th bond defined on $0 < s \leq s_{aM(a)}$ with $s_{aM(a)} = \max_g s_{gM(g)}$, where the whole values of $D_g(s)$ are realized all at $t = 0$ and $D_g(s)$ discounts cash flow $C_g(s)$ by $C_g(s)D_g(s)$. Under these notations, our basic formulation for modeling G bond prices simultaneously at $t = 0$ is based on the following expression:

$$(3) \quad P_g = \sum_{j=1}^{M(g)} C_g(s_{gj})D_g(s_{gj}) \quad g = 1, \dots, G.$$

This is an unconditional cross-sectional expression. In (3), we regard the realization of price P_g as equivalent to the realization of the whole function $\{D_g(s) : 0 \leq s \leq s_{gM(g)}\}$ with $g = 1, \dots, G$. Hence the realizations of G bond prices correspond to those of $D_g s_{am}$, ($m = 1, \dots, s_{aM(a)}$; $g = 1, \dots, G$) and the correlation structure of these stochastic discount functions implies those of prices.

In the spot rate approach in mathematical finance a non-defaultable bond price at $t = 0$ is specified as the conditional expectation with attribute-free discount

function given the past and present information, which is expressed as follows; for each individual price

$$(4) \quad P_g(1) = \sum_{j=1}^{M(g)} C_g(s_{gj}) \bar{D}(s_{gj})$$

with

$$\bar{D}(s_{gj}) = E_0 \left[\exp \left(- \int_0^{s_{gj}} r_u du \right) \right] \equiv H(r_0, s_{gj}, \theta),$$

where $\{r_u : 0 \leq u \leq s_{aM(a)}\}$ is a process of instantaneous spot interest rates $\{r_u\}$ that is common to all the bonds and $E_0[\cdot]$ denotes the conditional expectation given r_0 at 0 with respect to a risk neutral measure. But the measure is not uniquely identified. Here θ denotes a set of possible parameters when a specific model such as CIR (Cox-Ingersoll-Ross) model or Vasicek model is used for the spot rate process $\{r_u\}$. In mathematical finance the conditional expectation can be regarded as being taken under a risk neutral measure and (4) is claimed to hold a.s. for all the bonds in its no-arbitrary theory. In fact, conditioning variable r_0 is the only random variable making all the G bond prices realized. Hence in reality it does not follow that (4) holds a.s. as the bond prices are not functionally related and in fact maturities and coupon rates are different. In such a spot rate approach, modeling the spot rate process yields the conditional discount function through which the zero yield curve $\{R_u : 0 \leq u \leq s_{aM(a)}\}$ defined by $H(r_0, s, \theta) = \exp(-R_s s)$ or equivalently $R_s = (-1/s) \log H(r_0, s, \theta)$ is obtained. In the sequel we use the real measure that generates real data.

Another remark on this approach is that this specification of spot rate process for bond-pricing ignores such bond attributes as coupon rate or maturity. Empirically speaking, it is often observed that bond prices formed in the market depend on such attributes, in which case it is required to take into account such attribute-dependency in the specification of the spot rate process. In modeling a swap rate process [Collin-Dufresne and Solnik \(2001\)](#) and [Feldhutter and Lando \(2007\)](#) take the dependence of swap rates on credit attributes into account and specify a swap rate process as the sum of an abstract risk-free rate process $\{x_{1s}\}$ and a convenience yield process $\{x_{2gs}\}$ where they are assumed to be independent. Here the convenience yield represents such attributes as liquidity premium, credit premium (collateral condition), etc.;

$$r_{gs} = x_{1s} + x_{2gs} .$$

The attribute-dependent convenience process $\{x_{2gs}\}$ can play an adjusting factor for fitting the model as it can be arbitrarily specified. Using this process the discount function in (4) becomes attribute-dependent;

$$\bar{D}_g(s_{gj}) = E_0 \left[\exp \left(- \int_0^{s_{gj}} r_{gs} ds \right) \right] .$$

In this paper we do not use this spot rate approach in (4) with conditional expectation but take a forward rate approach with unconditional expression, and make it the following attribute-dependent model as in (3);

$$(5) \quad P_g = \sum_{j=1}^{M(g)} C_g(s_{gj}) D_g(s_{gj}) \quad \text{with} \quad D_g(s) = \exp \left(- \int_0^s f_{gu} du \right) ,$$

where $\{f_{gs} : 0 \leq s \leq s_{aM(a)}\}$ is an instantaneous forward rate term structure process whose values are realized all at $t = 0$. In other words, for each g a realization of P_g corresponds to that of the whole path $\{f_{gs} : 0 \leq s \leq s_{aM(a)}\}$. Note that (5) is equivalent to (3).

On the other hand, as an attribute-free forward rate process one may use the time-continuous HJM (Heath, Jarrow and Morton (1992)) model, which describes for each individual g a process of term structures $\{f_{ts} : 0 \leq s \leq s_{taM(a)}, t \geq 0\}$ with the discount function $D_t^*(s_{gj}) = \exp(-\int_0^{s_{gj}} f_{ts} ds)$ attribute-free. The HJM model is specified conditionally and the Markovian expression is usually explored with the no-arbitrage argument. Even in this case, one single path $\{f_s : 0 \leq s \leq s_{aM(a)}\}$ does not make (5) hold a.s. for all g either and so we use (5).

Finally it is remarked that a specific cash flow pattern guaranteed by holding some GBs will be a big value to such institutional investors as pension funds or life insurance companies because they need to match cash inflow with cash outflow over a long time horizon. In other words, coupon and maturity are important attributes which affect investment decisions with a future perspective. For example, depending on cash inflow-outflow structures of investors and on future perspectives on movements of interest rates, it may happen that a GB of 2 year maturity and 5% coupon is less preferred to a GB of 6 year maturity and 3% coupon. It is noted that those institutional investors do not necessarily prefer de-coupon or stripped bonds engineered by investment banking as making a portfolio from these stripped bonds to match the cash inflows and outflows is costly and involves additional credit risk of investment bankers.

4. GB pricing model

In this section, we implement an attribute-dependent GB pricing model with a stochastic discount function $D_g(s)$ in (3) or (5). First, let $D_g(s)$ be decomposed into the mean function and the stochastic deviation function as

$$(6) \quad D_g(s) = \bar{D}_g(s) + \Delta_g(s).$$

Substituting this into (5), it follows that

$$(7) \quad P_g = \sum_{m=1}^{M(g)} C_g(s_{gm}) \bar{D}_g(s_{gm}) + \eta_g \quad \eta_g = C'_g \Delta_g = \sum_{m=1}^{M(g)} C_g(s_{gm}) \Delta_g(s_{gm}),$$

where

$$(8) \quad C_g = (C_g(s_{g1}), \dots, C_g(s_{gM(g)}))' \quad \text{and} \quad \Delta_g = (\Delta_g(s_{g1}), \dots, \Delta_g(s_{gM(g)}))'.$$

The expression (7) corresponds to the case in (5), but without specifying the attribute-dependent forward rate process $\{f_{gs} : 0 \leq s \leq s_{aM(a)}\}$, the corresponding mean discount function $\bar{D}_g(s)$ on $[0, s_{aM(a)}]$ is assumed to be continuous in s and then it is uniformly approximated by a p th order polynomial;

$$(9) \quad \bar{D}_g(s) = 1 + (\delta_{11}z_{1g} + \delta_{12}z_{2g} + \delta_{13}z_{3g})s + \dots + (\delta_{p1}z_{1g} + \delta_{p2}z_{2g} + \delta_{p3}z_{3g})s^p,$$

where $z_1 = 1$, $z_2 = c_g$, and $z_3 = s_{gM(g)}$ are the attribute variables of the g th bond. In this specification the parameters are common to all the mean discount functions for $g = 1, \dots, G$ and hence they are estimable with G bond prices, so long as G

is greater than the number of the parameters contained. Substituting (9) into (7) yields

$$\sum_{m=1}^{M(g)} C_g(s_{gm}) \bar{D}_g(s_{gm}) = a_g + \delta_{11}d_{g11} + \delta_{12}d_{g12} + \delta_{13}d_{g13} + \cdots \\ + \delta_{p1}d_{gp1} + \delta_{p2}d_{gp2} + \delta_{p3}d_{gp3},$$

where

$$a_g = \sum_{m=1}^{M(g)} C_g(s_{gm}) \quad \text{and} \quad d_{gij} = \sum_{m=1}^{M(g)} C_g(s_{gm}) z_{gj} s_{gm}^j.$$

Here i in d_{gij} denotes the attribute suffix and j the polynomial order. Thus letting

$$x_g = (d_{g11}, d_{g21}, d_{g31}; d_{g12}, d_{g22}, d_{g32}; \cdots; d_{g1p}, d_{g2p}, d_{g3p})'$$

and

$$X = (x_1, x_2, \dots, x_G)'$$

we have a regression model

$$(10) \quad y = X\beta + \eta,$$

where $y = (y_1, y_2, \dots, y_G)'$ with $y_g = P_g - a_g$, $\eta = (\eta_1, \dots, \eta_G)'$ and

$$\beta = (\delta_{11}, \delta_{12}, \delta_{13}; \delta_{21}, \delta_{22}, \delta_{23}; \dots; \delta_{p1}, \delta_{p2}, \delta_{p3})'.$$

In (10) the specification of the covariance matrix η is crucial since specifying the covariance structure of $P = (P_1, \dots, P_G)'$ or equivalently the covariance structure of η stochastically describes a structure of the joint realizations of G bond prices. In view of (7) the specification is directly related to that of the covariances of the stochastic discount factors $D_g(s_{gj})$ and $D_h(s_{hm})$ at each cash flow point s_{gj} and s_{hm} of the g th and h th bonds. We specify it as

$$(11) \quad Cov(D_g(s_{gj}), D_h(s_{hm})) = \sigma^2 \lambda_{gh} f_{gh \cdot jm},$$

where σ^2 is a common covariance and covariance factor, λ_{gh} is a covariance part related to the differences of maturities and $f_{gh \cdot jm}$ is another covariance part related to the difference of the cash flow points s_{gj} and s_{hm} . These two parts are further specified as

$$(12) \quad \lambda_{gh} = \begin{cases} e_{gg} & g = h \\ \rho e_{gh} & g \neq h \end{cases}$$

with

$$e_{gh} = \exp(-\xi |s_{gM(g)} - s_{hM(h)}|)$$

and

$$(13) \quad f_{gh \cdot jm} = \exp(-\theta |s_{gj} - s_{hm}|),$$

where we assume that $0 \leq \theta, \rho, \xi \leq 1$. These specifications imply;

- (1) as is expressed in e_{gg} of λ_{gg} , the longer the maturity of each bond is, the larger the variance of each price is,

- (2) as is expressed in e_{gh} of λ_{gh} , the larger the difference of the maturities of two bonds, the smaller the covariance is, and
- (3) as is expressed in $f_{gh,jm}$, the closer the two cash flow points are, the larger the covariance of the discount factors $D_g(s_{gj})$ and $D_h(s_{hm})$ is.

Under this specification, the covariance matrix of η is given by

$$(14) \quad Cov(\eta) = (Cov(\eta_g, \eta_h)) = (Cov(P_g, P_h)) = \sigma^2(\lambda_{gh}\varphi_{gh}) \equiv \sigma^2\Phi(\theta, \rho, \xi)$$

with

$$\varphi_{gh} = \sum_{j=1}^{M(g)} \sum_{m=1}^{M(m)} C_g(s_{gj})C_h(s_{hm})f_{gh,jm}.$$

As in [Kariya and Kurata \(2004\)](#), the unknown parameters are efficiently estimated by the GLS (generalized least squares) method, in which we minimize

$$(15) \quad \psi(\beta, \theta, \rho) = [y - X\beta]' [\Phi(\theta, \rho, \xi)]^{-1} [y - X\beta]$$

with respect to the unknown parameters. First, for given (θ, ρ, ξ) , the minimizer of this function with respect to β is known to be the GLSE;

$$\hat{\beta}(\theta, \rho, \xi) = [X'\Phi(\theta, \rho, \xi)^{-1}X]^{-1}X'\Phi(\theta, \rho, \xi)^{-1}y$$

and then the marginally minimized function $\psi(\hat{\beta}, \theta, \rho, \xi)$ with substitution $\hat{\beta}(\theta, \rho, \xi)$ is minimized with respect to (θ, ρ, ξ) , yielding the GLSE $(\hat{\beta}, \hat{\theta}, \hat{\rho}, \hat{\xi})$ where a grid point method for split points of (θ, ρ, ξ) may be used.

In [Kariya and Kurata \(2004, pp 55–63\)](#), an empirical performance due to [Kariya and Tsuda \(1994\)](#) is demonstrated as an example of GLS estimation where the order of the polynomial in (9) is set to 2 with $z_1 = 0$, $z_2 = \text{coupon}$, and $z_3 = \text{maturity}$. Even in this case, the residual standard deviations are 0.338 yen for December 27, 1989 with $G = 70$ and 0.312 yen for January 31, 1990 with $G = 70$ where the face value is 100. Recently, the effectiveness of this model is empirically and comprehensively tested with Japanese bond data in [Kariya et al. \(2011\)](#).

It is noted that in the specification of the attribute-dependent mean discount function in (9), setting $z_1 = 1$, $z_2 = 0$, and $z_3 = 0$ yields the attribute-free specification. In this case the parameters involved are estimated in the same manner to get the attribute-independent mean discount function $\bar{D}(s)$. This $\bar{D}(s)$ is converted to a yield curve by $R_s = -s^{-1} \log \bar{D}(s)$, which is often referred to as a risk-free yield curve. This attribute-free discount function may be used to price a CDS in Section 6 because the cash flows in CDS does not reflect the same cash flow pattern as the corresponding CB.

5. CB pricing model for deriving TSDBs

In this section we propose a formulation of the CB pricing model that enables us to derive the TSDPs and RRs for each pair of industry index and rating index where (1) CB price data, (2) data on industry-wise sales ratios of firms that issued the CBs and (3) credit rating data are assumed to be given at $t = 0$. As has been stated in Sections 1 and 2, in our modeling the differences of the business line portfolios of firms are taken into account in terms of sales ratios of the firms. As a matter of fact, each firm has different exposures to industry-wise business factors and so it

has a different profit structure in association with industry business cycles, which is greatly relevant to analyzing credit qualities of CBs.

To formulate our model, suppose that at $t = 0$ there are K CBs to analyze and let $\{s_{kl}; l = 1, \dots, M(k), k = 1, 2, \dots, K$ with $s_{k1} < s_{k2} < \dots < s_{kM(k)}$ denote the future cash flow time points of those CBs as in the case of GBs. Also let $s_{aM(a)} = \max_k s_{kM(k)}$ and let the cash flow function $C_k(s)$ of the k th CB be defined on $0 < s \leq s_{aM(a)}$ for all k though it is zero except for the above finite points.

On the other hand, if the firm that issued a CB gets defaulted before its maturity, the coupons to be paid thereafter are not paid and some portion of the face value 100 may be paid after a long procedure of legal and practical settlements, where the portion relative to 100 is called recovery rate (RR), which is in general of a stochastic nature. The expected or empirically averaged value of the RR is known to depend on its credit grade via a rating agency. Hence actual cash flows from a CB depend on how likely a firm that issued a CB is to get defaulted and what the expected RR is when it gets defaulted. In the market the TSDPs are evaluated simultaneously and consistently together with the expected RRs and are implicitly reflected in their market prices of CBs.

5.1. Basic formulation of CB pricing model

On this viewpoint, let τ_k be the first random time of the k th CB to default and let

$$(16) \quad L_{ks} = \begin{cases} 0 & \tau_k > s \\ 1 & \tau_k \leq s. \end{cases}$$

Then L_{ks} defined at $t = 0$ is the indicator function of default event $\{\tau_k \leq s\}$ and identifies if the k th CB gets defaulted before or on a future time point s . Then the actual cash flow function $\tilde{C}_k(s_{kj})$ at a future coupon generating time s_{kj} is expressed as

$$(17) \quad \tilde{C}_k(s_{kj}) = C_k(s_{kj})(1 - L_{ks_{kj}}) + 100\gamma(i(k))L_{ks_{kj}}(1 - L_{ks_{kj-1}}).$$

This means that $\tilde{C}_k(s_{kj}) = C_k(s_{kj})$ if $L_{ks_{kj}} = 0$, or equivalently if the firm has not defaulted until s_{kj} , and $\tilde{C}_k(s_{kj}) = 100\gamma(i(k))$ if $L_{ks_{kj}}(1 - L_{ks_{kj-1}}) = 1$, or equivalently if it had not defaulted at s_{kj-1} and has defaulted at s_{kj} . Here $\gamma(i(k))$ is the mean RR of the k th bond with rating grade $i(k)$, where the credit rating is indexed by natural number $i = 1, 2, \dots, I$ and the smaller the number is, the higher the credit grade is. Here this specification implicitly assumes that the recovery payment is made at s_{kj} if a default event occurs in interval $(s_{kj-1}, s_{kj}]$. Note that $s_{kj} - s_{kj-1}$ is typically about a half year.

However, the expression (17) itself exhibits the future relation and has never been realized at $t = 0$. Therefore investors expected cash flow at $t = 0$ is formulated as for interval $(s_{kj-1}, s_{kj}]$

$$(18) \quad \bar{C}_k(s_{kj}) = C_k(s_{kj})[1 - p_k(s_{jk} : i(k))] + 100\gamma(i(k))[p_k(s_{jk} : i(k)) - p_k(s_{kj-1} : i(k))]\chi_k(s_{kj}),$$

where $p_k(s_{jk} : i(k)) = E[L_{ks_{kj}}]$ is the default probability corresponding to the event $\{\tau_k \leq s_{kj}\}$, which is the probability that the k th CB gets defaulted before or on s_{kj} and $\chi_k(s_{kj})$ is the indicator function of the cash flow time points $\{s_{kl}; l =$

$1, \dots, M(k)$ of the k th CB. With investors expected cash flows in (18) we formulate our CB model as

$$(19) \quad V_k = \sum_{j=1}^{M(k)} \bar{C}_k(s_{kj}) D_k(s_{kj}).$$

Before we proceed further, it is remarked that a typical model in the mathematical finance literature is the conditional univariate model that introduces the credit element into the discount part;

$$V_k = \sum_{j=1}^{M(k)} C_k(s_{kj}) E_0 \left[\exp \left(- \int_0^{s_{kj}} (r_u + \lambda_u^k) du \right) \right],$$

where $\{r_u\}$ is a Markovian spot rate process and $\{\lambda_u^k\}$ is an instantaneous Markovian default intensity process that discounts cash flows together with the spot rate process. In this expression it is assumed that $\{r_u\}$ is independent of $\{L_{ks}\}$, which is unlikely but yields

$$E_0 \left[(1 - L_{ks}) \exp \left(- \int_0^s r_u du \right) \right] = E_0 \left[\exp \left(- \int_0^s (r_u + \lambda_u^k) du \right) \right]$$

by the Doob-Meyer Theorem and hence the default intensity $\{\lambda_u\}$ are assumed to be exogenously independent of $\{r_u\}$. It is often the case that such processes as CIR model are assumed for $\{\lambda_u\}$, the discount function $E_0[\cdot]$ is evaluated analytically and then the unknown parameters therein are calibrated or estimated with the present or past data of the k th CB only. This model is sometimes extended to a multivariate case where the correlation of λ_u^k and λ_u^j is often assumed to be constant. Clearly this modeling approach is quite different from ours.

Another remark is that the expression (18) may be regarded as the conditional expectations of investors and in that way the discount function in (19) and hence the model (19) itself may be regarded as a conditional expression to get a dynamic model. Now coming back to our case, the expression in (19) corresponds to the one in (5) and hence the rest of the argument is similar to the non-defaultable case. That is, the stochastic discount function is decomposed as

$$(20) \quad D_k(s) = \bar{D}_k(s) + \Delta_k(s)$$

and as $\bar{D}_k(s)$ we use the mean discount function estimated with G government bond prices where the attributes of the k th bond are inserted into the discount function in evaluation. Consequently

$$(21) \quad \{\bar{D}_k(s_{kj}) : j = 1, \dots, s_{kM(k)}; k = 1, \dots, K\}$$

is a set of known values and it follows from (19) and (20) that the CB pricing model is

$$(22) \quad V_k = \sum_{j=1}^{M(k)} \bar{C}_k(s_{kj}) \bar{D}_k(s_{kj}) + \varepsilon_k \quad \text{with} \quad \varepsilon_k = \sum_{j=1}^{M(k)} \bar{C}_k(s_{kj}) \Delta_k(s_{kj}).$$

It is remarked that a joint model of GBs and CBs can be formulated in which the common mean discount functions are estimated simultaneously, though we do not follow this because of its complexity. Note that there are about 3,000 CBs in the Japanese market and about 50,000 CBs in the US market.

5.2. Specification of TSDPs with business portfolio structures and credit discounts

Next we specify the default probability function of the k th bond in (18) as

$$(23) \quad P_k(s : i(k)) \equiv \sum_{j=1}^J w_k(j) p(s : i(k), j),$$

where $w_k(j) \geq 0$, $\sum_{j=1}^J w_k(j) = 1$. Here

$$(24) \quad p(s : i, j) \quad i = 1, \dots, I, \quad j = 1, \dots, J$$

is the generic or common TSDP with credit grade i and industry j , which is independent of specific CBs. In this paper it is assumed to be approximated by a polynomial of the q th order;

$$(25) \quad p(s : i, j) = \alpha_1^{ij} s + \alpha_2^{ij} s^2 + \dots + \alpha_q^{ij} s^q.$$

On the other hand, $\{w_k(1), \dots, w_k(J)\}$ in (23) is the set of the sales ratios of the k th CB issuer corresponding to the industry indices $j = 1, \dots, J$ and it is regarded as a business portfolio in terms of industry-wise sales, where the industry categorization is determined in advance.

Now from (23) and (25) the TSDP of the k th bond is expressed as

$$(26) \quad \begin{aligned} p_k(s : i(k)) &= s w'_k \alpha_1^{i \cdot} + s^2 w'_k \alpha_2^{i \cdot} + \dots + s^q w'_k \alpha_q^{i \cdot} \\ &= (s w'_k, s^2 w'_k, \dots, s^q w'_k) \beta(i) \\ &\equiv w_k(s)' \beta(i) \end{aligned}$$

where

$$\begin{aligned} w_k &= (w_k(1), \dots, w_k(J))', \\ \alpha_h^{i \cdot} &= (\alpha_h^{i1}, \dots, \alpha_h^{iJ})' \quad h = 1, \dots, q, \\ \beta(i) &\equiv (\alpha_1^{i \cdot}, \dots, \alpha_q^{i \cdot})' \\ w_k(s)' &= (s w'_k, \dots, s^q w'_k). \end{aligned}$$

Also the expected cash flow function in (18) is expressed as

$$(27) \quad \bar{C}_k(s_{kj}) = C_k(s_{kj}) + z_k(s_{kj}, s_{kj-1} : \gamma(i(k)))' \beta(i),$$

where

$$\begin{aligned} z_k(s_{kj}, s_{kj-1} : \gamma(i(k)))' &= -C_k(s_{kj}) w(s_{kj})' \\ &\quad + 100 \gamma(i(k)) \chi_k(s_{kj}) [w(s_{kj})' - w(s_{kj-1})']. \end{aligned}$$

Therefore the pricing model in (22) is given by

$$\begin{aligned} V_k &= \sum_{j=1}^{M(k)} C_k(s_{kj}) \bar{D}_k(s_{kj}) + \sum_{j=1}^{M(k)} \bar{D}_k(s_{kj}) z'_k(s_{kj}, s_{kj-1} : \gamma(i(k))) \beta(i(k)) + \varepsilon_k \\ &= \hat{P}_k + [u_k + \gamma(i(k)) v_k]' \beta(i) + \varepsilon_k, \end{aligned}$$

which yields a regression model

$$(28) \quad y_k^{i(k)} = [u_k + \gamma(i(k))v_k]' \beta(i) + \varepsilon_k$$

with

$$\begin{aligned} y_k^{i(k)} &= V_k - \hat{P}_k, \\ \hat{P}_k &= \sum_{j=1}^{M(k)} C_k(s_{kj}) \bar{D}_k(s_{kj}), \\ u_k &= - \sum_{j=1}^{M(k)} C_k(s_{kj}) \bar{D}_k(s_{kj}) w_k(s_{kj}) \quad \text{and} \\ v_k &= 100 \sum_{j=1}^{M(k)} \bar{D}_k(s_{kj}) \chi_k(s_{kj}) [w_k(s_{kj}) - w_k(s_{kj-1})]. \end{aligned}$$

In this expression, \hat{P}_k is regarded as an expected (or a theoretical) GB price with non-defaultable cash flow $\{C_k(s_{kj}) : j = 1, \dots, M(k)\}$, and hence $y_k^{i(k)}$ is the difference between the k th CB price with credit grade $i(k)$ and the corresponding non-defaultable bond price. The difference $y_k^{i(k)}$ tends to be non-positive, because $\bar{C}_k(s_{kj}) \leq C_k(s_{kj})$ and a CB is of the less creditability and less liquidity than the corresponding GB. We simply call $y_k^{i(k)}$ the credit risk discount of the k th CB price.

Consequently (28) forms a regression model for the credit risk discounts. In fact, supposing that there are K_i CBs of credit grade i and letting $y(i) = (y_1^i, \dots, y_{K_i}^i)'$, $X_1(i) = (u_1^i, \dots, u_{K_i}^i)'$ and $X_2(i) = (v_1^i, \dots, v_{K_i}^i)'$ the pricing model for CBs of credit grade i is reduced to a regression model;

$$(29) \quad y(i) = X(i, \gamma(i)) \beta(i) + \varepsilon(i)$$

with $X(i, \gamma(i)) = X_1(i) + \gamma(i)X_2(i)$, $i = 1, \dots, I$. Here $K_i \geq 2Jq$ is assumed for the identifiability and estimability of $\beta(i)$ and $\gamma(i)$. In fact, $K_i \geq 2Jq$ is a necessary condition for the uniqueness of $(\beta(i), \gamma(i))$ in the sense that $X(i, \gamma(i))\beta(i) = X(i, \gamma(i)^*)\beta(i)^*$ implies $\beta(i) = \beta(i)^*$ and $\gamma(i) = \gamma(i)^*$. It follows from (29) that the credit risk discount vector $y(i)$ of the i th credit grade is explained by the regression matrix $X(\gamma(i))$ which depends on the unknown RR $\gamma(i)$. Combining all the regression models over $i = 1, \dots, I$ yields

$$(30) \quad y = X\beta + \varepsilon$$

with $X \equiv [X(i, \gamma(i))]$ and $[X(i, \gamma(i))]$ is the $K \times IJq$ block-diagonal matrix with the i th block matrix $X(i, \gamma(i))$ and $\beta = (\beta(1)', \dots, \beta(I)')$ with $K = K_1 + \dots + K_I$.

5.3. Specification of covariance structures

To specify the covariance structure, we write the dependency of the mean cash flow function on unknown parameters as $\bar{C}_k(s_{kj}) \equiv \bar{C}_k(s_{kj} : \beta(i), \gamma(i(k)))$. Then the error term in (22) is given by

$$\varepsilon_k = \sum_{j=1}^{M(k)} \bar{C}_k(s_{kj} : \beta(i), \gamma(i(k))) \Delta_k(s_{kj}).$$

Hence the covariance of two error terms of the k th and l th CB prices with rating $i(k)$ and $i(l)$ respectively is assumed to be

$$(31) \quad \begin{aligned} Cov(\varepsilon_k, \varepsilon_l) &= \sum_{j=1}^{M(k)} \sum_{m=1}^{M(m)} \bar{C}_k(s_{kj} : \beta(i), \gamma(i(k))) \bar{C}_l(s_{lm} : \beta(i), \gamma(i(l))) Cov(\Delta_{kj}, \Delta_{lm}) \\ &= \sigma^2 \lambda_{kl} \varphi_{kl}, \end{aligned}$$

where

$$\lambda_{kl} = \begin{cases} e_{kk} & k = l \\ \rho_{ii} e_{kl} & k \neq l, i(k) = i(l) = i \\ \rho_{ij} e_{kl} & k \neq l, i(k) = i, i(l) = j, i \neq j \end{cases}$$

with

$$e_{kl} = \exp(-\xi_{ij} |s_{kM(k)} - s_{lM(l)}|) \quad i(k) = i, i(l) = j,$$

and

$$\varphi_{kl} = \sum_{j=1}^{M(k)} \sum_{m=1}^{M(m)} \bar{C}_k(s_{kj} : \beta(i), \gamma(i(k))) \bar{C}_l(s_{lm} : \beta(i), \gamma(i(l))) b_{kl \cdot jm}$$

with

$$b_{kl \cdot jm} = \exp(-\theta |s_{kj} - s_{lm}|) .$$

In this specification, when the k th CB and the l th CB are of the same credit rate, i.e., $i(k) = i(l) = i$, then the covariance is of the same form as the one in the non-defaultable case. If the two CBs are not in the same credit category, ρ_{ij} will account for the cross-correlation between the two categories. Using the above specifications, the covariance matrices for regression models are obtained;

$$(32) \quad Cov(\varepsilon(i)) = (Cov(\varepsilon_k, \varepsilon_l)) = \sigma^2 (\lambda_{kl} \varphi_{kl}) = \sigma^2 \Phi(\beta(i), \gamma(i), \rho_{ii}, \xi_{ii}) \equiv \sigma^2 \Phi_{ii}$$

$$Cov(\varepsilon(i), \varepsilon(j)) = \sigma^2 \Phi(\beta(i), \beta(j), \gamma(i), \gamma(j), \rho_{ij}, \xi_{ij}) \equiv \sigma^2 \Phi_{ij} \quad i \neq j \quad \text{and}$$

$$(33) \quad Cov(\varepsilon) = (Cov(\varepsilon(i), \varepsilon(j))) = \sigma^2 \{\Phi_{ij}\} .$$

In the CB case, the covariance matrices depend on the regression coefficient $\beta(i)$'s.

5.4. Estimation procedure

Under these formulations we propose the following grid procedure to simplify the model estimation.

- (1) For each credit category, the GLS estimation is pursued: fix i .
 - (a) For each given $(\gamma(i), \rho_{ii}, \xi_{ii})$, each of which moves over $0, 0.1, \dots, 0.9$, $\beta(i)$ is estimated by a repeated procedure. Setting $\beta(i) = 0$ in Φ_{ii} and minimizing

$$\psi \equiv [y(i) - X(i, \gamma(i))\beta(i)]' [\Phi_{ii}(\beta(i), \gamma(i), \rho_{ii}, \xi_{ii})]^{-1} [y(i) - X(i, \gamma(i))\beta(i)]$$

yields the first step GLSE with $\Phi_{ii}^{(0)} = \Phi_{ii}(0, \gamma(i), \rho_{ii}, \xi_{ii})$;

$$\hat{\beta}(i)^{(1)} = [X(i, \gamma(i))' \Phi_{ii}^{(0)-1} X(i, \gamma(i))]^{-1} X(i, \gamma(i))' \Phi_{ii}^{(0)-1} y(i).$$

Substituting this $\hat{\beta}(i)^{(1)}$ into Φ_{ii} to get $\Phi_{ii}^{(1)} = \Phi_{ii}(\beta(i)^{(1)}, \gamma(i), \rho_{ii}, \xi_{ii})$, applying the same procedure yields the second step MLE $\hat{\beta}(i)^{(2)}$. Repeating this procedure a couple of times with a performance evaluation rule, we obtain an approximate minimizer $(\hat{\beta}(i)^{(n)}, \Phi_{ii}^{(n-1)})$ for given $(\gamma(i), \rho_{ii}, \xi_{ii})$.

- (b) Repeating the procedure in (a) over possible values of $(\gamma(i), \rho_{ii}, \xi_{ii})$, we obtain $(\hat{\beta}(i)^*, \Phi_{ii}^*, \gamma(i)^*, \rho_{ii}^*, \xi_{ii}^*)$ that minimizes the objective function approximately. Once we obtain this, we may repeat (a) in the neighborhood of the approximate optimal value by splitting the neighborhood with a finer division.
- (2) A simultaneous estimation procedure for the combined model is obtained as follows. Fix $(\hat{\beta}(i)^*, \hat{\gamma}(i)^*, \hat{\rho}_{ii}^*, \hat{\xi}_{ii}^*)$ $i = 1, \dots, I$ in the matrix

$$\Phi_{ij} = \Phi(\beta(i), \beta(j), \gamma(i), \gamma(j), \rho_{ij}, \xi_{ij}) \quad i \neq j,$$

meaning that the full covariance matrix in (32) carries the $I(I-1)$ unknown parameters $\{\rho_{ij}, \xi_{ij} : i \neq j\}$. Hence in the same way as in (1) we minimize

$$\Psi(\beta, \rho_{ij}, \xi_i) = [y - X\beta]' [\{\Phi_{ij}(\rho_{ij}, \xi_{ij})\}]^{-1} [y - X\beta]$$

to obtain the first approximate simultaneous GLSE $\hat{\beta}_{full}^{[1]}$. One may replace $\hat{\beta}(i)^*$'s in the fixed $(\hat{\beta}(i)^*, \hat{\gamma}(i)^*, \hat{\rho}_{ii}^*, \hat{\xi}_{ii}^*)$ $i = 1, \dots, I$ by $\hat{\beta}_{full}^{[1]}$ and repeat the minimization procedures again to get the second approximate simultaneous GLSE $\hat{\beta}_{full}^{[2]}$.

- (3) This minimization procedures are repeated over the degrees $q = 2, \dots, 6$ of the polynomials in (25) to get an optimal simultaneous GLSE $\hat{\beta}_{full}$.

With this model, an extensive empirical research is under a way with Japanese CB data. A simple empirical work is demonstrated below in a limited model. In this model we do not take into account the fact that each firm has a specific business portfolio over industries. In other words, we set $J = 1$ and pick the AA class of 208 CBs with the assumption of recovery rate $\gamma = 0$. A plot of the residuals $\{\hat{\varepsilon}_k^{AA} = V_k - \hat{V}_k\}$ shows that even in this simplified model the performance are rather good and hence our full model will be more promising as a model to derive the TSDPs. In fact, the standard deviation of $\hat{\varepsilon}_k^{AA}$'s is 0.419 yen.

6. Applications

Once a full estimator $\hat{\beta}_{full}$ is obtained, the following applications are considered.

[1] The whole implied TSDPs in (25) for each pair of credit grade index and industry index are given by

$$(34) \quad \hat{p}(s : i, j) = \hat{\alpha}_1^{ij} s + \hat{\alpha}_2^{ij} s^2 + \dots + \hat{\alpha}_q^{ij} s^q \quad i = 1, \dots, I \quad j = 1, \dots, J$$

together with the implied RRs $\{\hat{\gamma}(i)^* : i = 1, \dots, I\}$. These TSDPs and RRs are basic information sources for credit-related products.

[2] The TSDP of the k th CB or its issuer with credit grade $i(k)$ and business portfolio $\{w_k(1), \dots, w_k(J)\}$ over J industries is obtained from [1] as

$$(35) \quad \hat{p}_k(s : i(k)) = \sum_{j=1}^J w_k(j) \hat{p}(s : i(k), j).$$

This is the TSDP $\{P(\tau_k \leq s) : s > 0\}$ of the issuer of the k th CB where τ_k is the default time as in (16). The expected RR is $\hat{\gamma}(i(k))^* \equiv \hat{\gamma}(i)$ with $i(k) = i$.

[3] Inserting $\hat{\beta}_{full}$ into (28) and using (27), the credit risk discount level of the k th CB or its issuer with credit grade $i(k)$ and business portfolio $\{w_k(1), \dots, w_k(J)\}$ over J industries is given by

$$(36) \quad \begin{aligned} \hat{y}_k^{i(k)} &= [u_k + \hat{\gamma}(i(k))^* v_k]' \hat{\beta}_{full} \\ &= \sum_{j=1}^{M(k)} \bar{D}_k(s_{kj}) W(\hat{p}_k(s_{kj} : i(k)), \hat{p}_k(s_{kj-1} : i(k)); \hat{\gamma}(i(k))) \end{aligned}$$

with

$$(37) \quad \begin{aligned} \hat{W}(s_{kj}) &\equiv W(\hat{p}_k(s_{kj} : i(k)), \hat{p}_k(s_{kj-1} : i(k)); \hat{\gamma}(i(k))) \\ &= [100\hat{\gamma}(i(k))\chi_k(s_{kj}) - C_k(s_{kj})] \hat{p}_k(s_{kj} : i(k)) - 100\hat{\gamma}(i(k)) \\ &\quad \times \hat{p}_k(s_{kj-1} : i(k)) \chi_k(s_{kj}) \end{aligned}$$

where the discount is relative to the fair value $\hat{P}_k = \hat{V}_k + (-\hat{y}_k^{i(k)})$ of the corresponding non-defaultable bond. Here \hat{P}_k is the model value of non-defaultable bond with the same coupon and same maturity as those of the k th CB. Consequently the fair spread rate of the k th CB relative to the corresponding (attribute-dependent) GB is given by

$$(38) \quad s\hat{p}_k \equiv (\hat{P}_k - \hat{V}_k) / \hat{P}_k = -\hat{y}_k^{i(k)} / \hat{P}_k.$$

Hence if the actual spread $(\hat{P}_k - \hat{V}_k) / \hat{P}_k = -\hat{y}_k^{i(k)} / \hat{P}_k$ is too low or too high relative to this fair spread, one may use the information for investment decision making with the expectation that the actual spread eventually converges to the fair level.

[4] CB Portfolio Management. The fair value of a portfolio $\{\varpi_k\}$ of CBs is given by

$$(39) \quad \begin{aligned} \sum_{k=1}^K \varpi_k \hat{V}_k &= \sum_{k=1}^K \varpi_k \sum_{m=1}^{M(a)} \hat{C}_k(s_{am}) \bar{D}_k(s_{am}) = \sum_{k=1}^K \varpi_k [\hat{P}_k + \hat{y}_k] \\ &= \sum_{m=1}^{M(a)} [A_{am} + B_{am}] \equiv \sum_{m=1}^{M(a)} C_{am} \end{aligned}$$

where $\{\varpi_k\}$ is a portfolio of CBs in terms of physical units,

$$(40) \quad A_{am} = \sum_{k=1}^K \varpi_k C_k(s_{am}) \bar{D}_k(s_{am})$$

and

$$(41) \quad B_{am} = \sum_{k=1}^K \varpi_k \bar{D}_k(s_{am}) W(\hat{p}_k(s_{am} : i(k)), \hat{p}_k(s_{am-1} : i(k)); \hat{\gamma}(i(k))).$$

Here A_{am} is the present value of the cash flows at future time s_{am} if none of the bonds are defaulted before or on s_{am} where $\{s_{am}\}$ is the combined set of cash flow points, while B_{am} is the present value of the expected loss at s_{am} , which is generally negative.

What is important in this expression is that these are regarded as fair present values of those of the corresponding loan portfolio in a bank since a fixed-rate loan is regarded as a CB and a variable-rate loan may be converted to an equivalent fixed loan via interest swap rate at the time of evaluation. Though we do not elaborate it, based on this expression, the portfolio can be adjusted to a desirable loss structure relative to industries and rating grades via

$$\hat{p}_k(s; i) \equiv \sum_{j=1}^J w_k(j) \hat{p}(s : i, j),$$

where (37), (39) and (41) are combined to sort out the overlapping structure of firms portfolios and extract the industry positions for each $\hat{p}(s : i, j)$.

Another example is about the durations of expected losses. Putting $b_m = B_{am}/B$ where

$$B = \sum_{m=1}^{M(a)} B_{am},$$

$\{b_m\}$ represents the relative sizes of losses distributed over $\{s_{am}\}$ and hence

$$\sum_{m=1}^M b_m s_{am}$$

is the average duration of the losses. On the other hand, $\{a_m\}$ with $a_m = A_{am}/A$ and

$$A = \sum_{m=1}^M A_{am},$$

represents the relative sizes of the cash inflows distributed over $\{s_{am}\}$ if none of the CBs or loans get defaulted, and so the duration is

$$\sum_{m=1}^M a_m s_{am}.$$

The two durations are in general different and the expected actual duration becomes

$$\sum_{m=1}^M c_m s_{am}$$

with $c_m = C_{am}/C$ and

$$C = \sum_{m=1}^M C_{am},$$

The above argument can be extended to the case where the portfolio contains GBs as well as CBs.

[5] CDS Pricing. Now we apply the result to the valuation of credit default swap (CDS). The CDS is notorious as a credit derivative for enlarging and broadening

the financial crisis because the protection sellers like AIG sold CDSs more than their capacities. However CDS itself is an important instrument because it is an insurance-type product (contract) guaranteeing the principal of a specific CB (or loan) for the protection buyers by covering the loss when the issuer C of the CB gets defaulted. A protection (insurance) buyer B in this contract is the one who pays the protection premiums quarterly or biannually to a protection seller A over a certain period till its contract period or default of C in the period in order to protect the principal. The CDSs of this type are quoted in the market for trades of credit risk on GBs and CBs. The quotation includes the ask-side as well as the offer-side of protection against each bond, from which a market value of the credit risk of a CB is found. Here we first give a price (premium) formula for CDS in a discrete time setting, which is different from those in a continuous setting. As is stated in Section 1, a discrete time setting is more suitable in credit risk analysis than a continuous time setting. For example, a credit variation process is not Markovian and a settlement process after a default takes often 3 months through 6 months. In the discrete-time setting, the uncertainty of the time delay for settlement is directly incorporated into the model if necessary.

Now to price a CDS, let $m = 0, 1, \dots, M$ be daily counted time points at 0 where the time unit h of day is measured in year as $h = 1/365$. Time point m corresponds to mh years from 0. Let

$$(42) \quad \Upsilon = \{0 \leq m_1 \leq m_2 < \dots < m_k\}$$

be the biannual premium payment time points counted in days viewed at 0, and at these time points the protection buyer pays premium x for principal value 100. Further let N^C be the default time of firm C as in (16) and let the default process be denoted by

$$\{L_m\} \quad \text{with} \quad L_m = \chi_{\{N^C \leq m\}}.$$

To derive a theoretical premium x , we distinguish the two cases; $m \notin \Upsilon$ or $m \in \Upsilon$.

When $m \notin \Upsilon$ the cash flow at m of the protection buyer B occurs from the protection seller A only if firm C gets defaulted at m . Hence the payoff of A is expressed as

$$(43) \quad U_m^A = -100(1 - L_{m-1})L_m + 100\gamma_m(1 - L_{m-1})L_m.$$

In (43), if C had defaulted at $m - 1$, $L_{m-1} = 1$ and so $U_m^A = 0$, i.e., no cash flow at m . Here this expression assumes that on the day of default, protection seller A pays 100 and receives recovery $100\gamma_m$, which is not realistic, but we can change it in a way as we wish. In fact, if the protection seller A pays the principal to B in 14 days after default and A gets the recovery money in 90 days after default, the expression can be modified as for the payoff of A at m

$$(44) \quad U_m^A = -100(1 - L_{m-1-14})L_{m-14} \quad \text{and} \quad U_{m+90}^A = 100\gamma_{m+90}(1 - L_{m-1-14})L_{m-14}.$$

However, for simplicity we here assume (43) in which the payments are made at m when the default occurs in $(m - 1, m]$.

When $m \in \Upsilon$, B pays $100x$ to A if the default has not occurred until m and hence combining this with (43) the payoff of A becomes

$$(45) \quad U_m^A = -100(1 - \gamma_m)(1 - L_{m-1})L_m + 100x(1 - L_m).$$

Note that by the definition of L_m , for $m < n$, $L_m = 1$ implies $L_n = 1$ and hence $U_m^A = 0$. Now to value a theoretical premium x in a way as in no-arbitrage theory,

each payoff U_m at m is regarded as a derivative and is valued relative to cash managed up to m with daily spot rate process $\{r_j\}$;

$$(46) \quad B_m = \exp \left(\sum_{j=0}^{m-1} r_j h \right) .$$

Then the martingale condition gives the value at 0 of U_n by

$$(47) \quad \nu^A(m) = E_0^*[U_m/B_m] = E_0^*[d(m)U_m],$$

where in the theory the measure is risk neutral though it is not uniquely identifiable. If the interest rate process is independent of U_n , which would not hold in reality, the value is expressed as

$$(48) \quad \nu^A(m) = 100\bar{D}(m)[-(1 - \hat{\gamma}(i(C)))Q(N^C = m) + \delta(m)xQ(N^C > m)],$$

where $\delta(m) = 1$ if $m \in \Upsilon$, and it is 0 otherwise. Here we use an attribute-free discount function;

$$\bar{D}(m) = E_0[1/B_m] = E_0[d(m)] .$$

This discount function can be chosen as the one stated at the end of Section 4. Therefore the expected payoff at 0 of A is

$$V^A = \sum_{m=1}^M \nu^A(m) .$$

On the other hand, the payoff of B should be fair to that of A, meaning that it is equal to V^A and $V^B = -V^A$. Consequently the premium x must satisfy $V^A = 0$.

The CB-CDS premium described above is valued as a fair value;

$$(49) \quad x = \frac{(1 - \gamma(i(C))) \sum_{m=1}^M \bar{D}(m)Q(N^C = m)}{\sum_{k=1}^K Q(N^C > m_k)\bar{D}(m_k)} ,$$

where $\gamma(i(C))$ is the recovery rate of the CB issuer C and $\{m_k\}$ is the set of premium payment times. In our case, $\gamma(i(C))$ is the implied recovery rate and the default probability is the implied TSDP,

$$Q(N^C \leq m) = p_C(mh : i(C)) = \sum_{j=1}^J w_k(C)p(mh : i(C), j) ,$$

where the industry-wise business structure of the issuer C is taken into account. In (36), one may replace

$$\sum_{m=1}^M Q(N^C = m)\bar{D}_C(m)$$

by the continuous time expression

$$\int_0^{S_{M(C)}} \bar{D}(s)p'_C(s : i(C)) ds ,$$

where $p'_C(s : i(C))$ is the derivative of the TSDP p_C .

7. Conclusion

In this paper, we formulated a CB pricing model from which the TSDPs are derived for each industry index and each rating index and the recovery rates are derived for each rating index. A notable feature of this model is to take into account the fact that each firm has a different portfolio structure of business lines over industries. The TSDPs and recovery rates are basic inputs in credit derivatives and credit risk management. Though we wait for a thorough empirical work, the model itself and derived outputs will be very useful for practical decision makings. Further these results can be also applied to pricing some multi-name credit derivatives.

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