IMS Collections
Nonparametrics and Robustness in Modern Statistical Inference and Time Series Analysis: A Festschrift in honor of Professor Jana Jurečková
Vol. 7 (2010) 105–112

DOI: 10.1214/10-IMSCOLL711

A class of multivariate distributions related to distributions with a Gaussian component

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Abstract: A class of random vectors $(\mathbf{X},\, \mathbf{Y})$, $\mathbf{X} \in \mathbb{R}^j$, $\mathbf{Y} \in \mathbb{R}^k$ with characteristic functions of the form

$$h(\mathbf{s},\, \mathbf{t}) = f(\mathbf{s})g(\mathbf{t})\exp\{\mathbf{s}'\mathbf{C}\mathbf{t}\}$$

where C is a $(j \times k)$-matrix and prime stands for transposition is introduced and studied. The class contains all Gaussian vectors and possesses some of their properties. A relation of the class to random vectors with Gaussian components is of a particular interest. The problem of describing all pairs of characteristic functions $f(\mathbf{s})$, $g(\mathbf{t})$ such that $h(\mathbf{s},\, \mathbf{t})$ is a characteristic function is open.

1. Introduction

In the paper we study properties of random vectors $(\mathbf{X},\, \mathbf{Y})$ taking values in $\mathbb{R}^m$, $m = j + k$ with characteristic functions $h(\mathbf{s},\, \mathbf{t}) = E\exp i\{\mathbf{s}'\mathbf{X} + \mathbf{t}'\mathbf{Y}\}$ of the form

$$h(\mathbf{s},\, \mathbf{t}) = f(\mathbf{s})\mathbf{g}(\mathbf{t})\exp\{\mathbf{s}'\mathbf{C}\mathbf{t}\}. \tag{1}$$

Here $\mathbf{s} \in \mathbb{R}^j$, $\mathbf{t} \in \mathbb{R}^k$, C is a $(j \times k)$-matrix, prime stands for transposition, and $f(\mathbf{s})$, $\mathbf{g}(\mathbf{t})$ are the (marginal) characteristic functions of $\mathbf{X}$ and $\mathbf{Y}$.
The class of m-variate distributions with characteristic functions (1) includes all Gaussian distributions and, trivially, all distributions of independent $\mathbf{X}$ and $\mathbf{Y}$ (for the latter $C = 0$). The dependence between $\mathbf{X}$ and $\mathbf{Y}$ is, in a sense, concentrated in the matrix C and it seems natural to call this form of dependence *Gaussian-like*. Note that if $E(|\mathbf{X}|^{\mathbf{2}}) < \infty$, $\mathbf{E}(|\mathbf{Y}^{\mathbf{2}}) < \infty$, $-\mathbf{C}$ is the covariance matrix of $\mathbf{X}$ and $\mathbf{Y}$, $-C = \mathrm{cov}(\mathbf{X},\, \mathbf{Y})$. We call the distributions with characteristic functions (1) *GL-distributions*.

When $f(\mathbf{s}), \mathbf{g}(\mathbf{t})$ are characteristic functions, (1) is, in general, not a characteristic function. For example, in case of $j = k = 1$ if $f(s) = \sin s/s$ is the characteristic function of a uniform distribution on $(-1,\, 1)$, then for any characteristic function $g(\mathbf{t})$ (1) is not a characteristic function unless $C = 0$ (if $C \neq 0$, $h(s, t)$ is unbounded).

In the next section it is shown that if $f(\mathbf{s})$, $\mathbf{g}(\mathbf{t})$ have Gaussian components, (1) is a characteristic function for all C with sufficiently small elements. We know of no other examples of $f(\mathbf{s})$, $\mathbf{g}(\mathbf{t})$ when $h(\mathbf{s},\, \mathbf{t})$ is a characteristic function. Note

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AMS 2000 subject classifications: Primary 60E05, 60E10
Keywords and phrases: Gaussian-like dependence, uncorrelatedness and independence, Fréchet classes.

in passing that the absence of Gaussian components plays an important role in problem of the arithmetic of characteristic functions (see, e. g., [3]).

The vectors $(\mathbf{X},\ \mathbf{Y})$ with characteristic functions (1) have some nice properties.

2. Properties of the GL-distributions

Proposition 1. *If* $(\mathbf{X_1},\ \mathbf{Y_1})$, $(\mathbf{X_2},\ \mathbf{Y_2})$ *are independent random vectors having GL-distributions and* a, b *constants,* $(\mathbf{X},\ \mathbf{Y}) = a(\mathbf{X_1},\ \mathbf{Y_1}) + b(\mathbf{X_2},\ \mathbf{Y_2})$ *also has a GL-distribution.*

Proposition 2. *If* $(\mathbf{X},\ \mathbf{Y})$ *has a GL-distribution and* $\mathbf{X}_1$ *(resp.* $\mathbf{Y}_1$*) is a subvector of* $\mathbf{X}$ *(resp.* $\mathbf{Y}$*), then* $(\mathbf{X_1},\ \mathbf{Y_1})$ *also has a GL-distribution.*

Proof. Assuming $\mathbf{X}_1$ (resp. $\mathbf{Y}_1$) consisting of the first j_1 (resp. k_1) components of $\mathbf{X}$ (resp. $\mathbf{Y}$) and denoting C_1 the submatrix of the first j_1 rows and k_1 columns of the matrix C from the characteristic function (1) of $(\mathbf{X},\ \mathbf{Y})$, $\mathbf{s}_1$ (resp. $\mathbf{t}_1$) the vector of the first j_1 (resp. k_1) components of $\mathbf{s}$ (resp. $\mathbf{t}$), the characteristic function of $\mathbf{X_1},\ \mathbf{Y_1})$ is

$$h_1(\mathbf{s}_1\ \mathbf{t}_1) = f_1(\mathbf{s}_1)g_1(\mathbf{t}_1)\exp\{\mathbf{s}_1' C_1 \mathbf{t}_1\}$$

with $f_1(\mathbf{s}_1) = f(\mathbf{s_1}, \mathbf{0})$, $g_1(\mathbf{t_1}) = g(\mathbf{t_1}, \mathbf{0})$. □

Proposition 3. *Let* $(\mathbf{X},\ \mathbf{Y})$ *have a GL-distribution and* $E(|\mathbf{X}|^2) < \infty$, $E(|\mathbf{Y}|^2) < \infty$. *If linear forms* $L_1 = \mathbf{a}'\mathbf{X}$, $L_2 = \mathbf{b}'\mathbf{Y}$ *where* $\mathbf{a} \in \mathbb{R}^j$, $\mathbf{b} \in \mathbb{R}^k$ *are constant vectors, are uncorrelated, they are independent.*

Proof. In the characteristic function (1), $-C = \mathrm{cov}(\mathbf{X},\ \mathbf{Y})$ whence $\mathrm{cov}(L_1,\ L_2) = -\mathbf{a}'C\mathbf{b}$. Thus, uncorrelatedness of L_1 and L_2 means $\mathbf{a}'C\mathbf{b} = 0$. But then for $u, v \in \mathbb{R}$

$$E\exp\{i(uL_1 + vL_2)\} = f(u\mathbf{a})g(v\mathbf{b})\exp\{uv\mathbf{a}'C\mathbf{b}\} = f(u\mathbf{a})g(v\mathbf{b}).$$

□

Proposition 3 is related to Vershik's (see, [5]) characterization of of Gaussian vectors. Let $\mathbf{Z}$ be an m-variate random vector with covariance matrix V of rank ≥ 2. If any two uncorrelated linear forms $\mathbf{a'Z}$, $\mathbf{b'Z}$ are independent, $\mathbf{Z}$ is a Gaussian vector [5]. The reverse is a well known property of Gaussian vectors.

The property stated in Proposition 3 is not characteristic of the random vectors with GL-distributions. However, if to assume additionally that $(\mathbf{X},\ \mathbf{Y})$ are the vectors of the first and second components, respectively, of independent (not necessarily identically distributed) bivariate random vectors $(X_1,\ Y_1), \ldots, (X_n,\ Y_n)$, the GL-distributions are characterized by "uncorrelatedness of $\mathbf{a'X}$ and $\mathbf{b'Y}$ implies their independence" property. The following result holds.

Theorem 2.1. *If* $E(X_j^2 + Y_j^2) < \infty$, $j = 1, \ldots, n$ *and any two uncorrelated linear forms*

$$L_1 = a_1X_1 + \ldots + a_nX_n,\ L_2 = b_1Y_1 + \ldots + b_nY_n$$

are independent, then (i) $\mathrm{cov}(X_j,\ Y_j) = 0$ *implies independence of* X_j *and* Y_j *(a trivial part), (ii) if, additionally,* $\#\{i : \mathrm{cov}(X_i,\ Y_i) \neq 0\} \geq 3$, *the characteristic function* $h_j(s,\ t)$ *of any uncorrelated* $(X_j,\ Y_j)$ *in a vicinity of* $s = t = 0$ *has the form of*

$$h_j(s,\ t) = f_j(s)g_j(t)\exp\{C_j st\} \tag{2}$$

for some constant C_j, (iii) if neither of those $h_j(s,\, t)$ vanishes, (2) holds for all $s,\, t \in \mathbb{R}$.

Proof. See [1] $\square$

Theorem 2.1 and the next result also proved in [1] show that some characteristic properties of the Gaussian distributions, after being modified for the setup of partitioned random vectors, become characteristic properties of the GL-distributions.

Theorem 2.2. *If $(X_1,\, Y_1), \ldots, (X_n,\, Y_n)$ is a sample of size $n \geq 3$ from a bivariate population and the sample mean $\bar{X}$ of the first components is independent of the vector of the residuals $(Y_1 - \bar{Y}, \ldots, Y_n - \bar{Y})$ of the second components and (not or) $\bar{Y}$ is independent of $(X_1 - \bar{X}, \ldots, X_n - \bar{X})$, then the population characteristic function $h(s,\, t)$ in a vicinity of $s = t = 0$ has the form*

$$h(s,\, t) = f(s)g(t)\exp\{Cst\} \tag{3}$$

for some C. If $h(s,\, t)$ does not vanish, (3) holds for all $s,\, t \in \mathbb{R}$.

The next two properties demonstrate the role of Gaussian components in GL-distributions.
Recall that a random vector ξ with values in $\mathbb{R}^s$ has a Gaussian component if

$$\xi = \eta + \zeta \tag{4}$$

where η and ζ are independent random vectors and ζ has an s-variate Gaussian distribution. In terms of characteristic functions, if $f(\mathbf{u}),\, \mathbf{u} \in \mathbb{R}^{\mathbf{s}}$ is the characteristic function of ξ, (4) is equivalent to

$$f(\mathbf{u}) = f_1(\mathbf{u})\exp\{-\mathbf{u}'V\mathbf{u}/2\} \tag{5}$$

where V is a Hermitian $(s \times s)$-matrix and $f_1(\mathbf{u})$ is a characteristic function. In view of (5), they say also that $f(\mathbf{u})$ has a Gaussian component.

Theorem 2.3. *If $f(\mathbf{s}),\, \mathbf{s} \in \mathbb{R}^j,\, g(\mathbf{t}),\, \mathbf{t} \in \mathbb{R}^k$ are characteristic functions having Gaussian components and $C = [c_{rq}]$ is a $(j \times k)$-matrix, then for sufficiently small $|c_{rq}|,\, r = 1, \ldots, j;\, q = 1, \ldots, k$ the function*

$$h(\mathbf{s},\, \mathbf{t}) = f(\mathbf{s})g(\mathbf{t})\exp\{\mathbf{s}'C\mathbf{t}\}.$$

is the characteristic function of a random vector $(\mathbf{X},\, \mathbf{Y})$ with values in $\mathbb{R}^m$, $m = j + k$. Plainly,

$$h(\mathbf{s},\, \mathbf{0}) = f(\mathbf{s}),\ h(\mathbf{0},\, \mathbf{t}) = g(\mathbf{t})$$

are the (marginal) characteristic functions of $\mathbf{X}$ and $\mathbf{Y}$.

Note that if $\mathcal{F}(F,\, G)$ is the Fréchet class of m-variate distribution functions $H(\mathbf{x},\, \mathbf{y})$ with $H(\mathbf{x},\, \infty) = F(\mathbf{x})$, $H(\infty,\, \mathbf{y}) = G(\mathbf{y})$, Theorem 2.3 means that if $\mathbf{X} \sim F(\mathbf{x})$ and $\mathbf{Y} \sim G(\mathbf{y})$ have Gaussian components, the class $\mathcal{F}(F,\, G)$ contains $H(\mathbf{x},\, \mathbf{y})$ with the characteristic function

$$h(\mathbf{s},\, \mathbf{t}) = \int_{\mathbb{R}^m} \exp\{i(\mathbf{s}'\mathbf{x} + \mathbf{t}'\mathbf{y})\}\, \mathrm{d}H(\mathbf{x},\, \mathbf{y})$$

of the form (3) for all C with sufficiently small elements.

Proof. By assumption,

$$f(\mathbf{s}) = f_1(\mathbf{s})\exp\{-\mathbf{s}'V_1\mathbf{s}/2\},\ g(\mathbf{t}) = g_1(\mathbf{t})\exp\{-\mathbf{t}'V_2\mathbf{t}/2\}$$

where V_1, V_2 are $(j\times j)$ and $(k\times k)$ Hermitian matrices respectively, and $f_1(\mathbf{s})$, $g_1(\mathbf{t})$ are characteristic functions. Let now $\zeta' = (\zeta_1,\ \zeta_2)'$ be an m-dimensional Gaussian vector with mean vector zero and covariance matrix

$$V = \begin{bmatrix} V_1 & C \\ C' & V_2 \end{bmatrix}$$

where V_i is the covariance matrix of ζ_i, $i = 1, 2$ and $C = [c_{rq}] = \mathrm{cov}(\zeta_1, \zeta_2)$ is a $(j \times k)$-matrix.

The matrix

$$V = \begin{bmatrix} V_1 & 0 \\ 0 & V_2 \end{bmatrix}$$

is positive definite. Hence, for all sufficiently small $|c_{rq}|$ (their smallness is determined by $V_1,\ V_2$) the matrix

$$V = \begin{bmatrix} V_1 & C \\ C' & V_2 \end{bmatrix} + \begin{bmatrix} 0 & C \\ C' & 0 \end{bmatrix} \tag{6}$$

is also positive definite so that (6) is Hermitian and may be chosen as a covariance matrix. Indeed, the property of a matrix to be positive definite is determined by positivity of a (finite) number of submatrices and plainly is preserved under small additive perturbations as in (6).

Now one sees that the function (3) rewritten as

$$h(\mathbf{s},\ \mathbf{t}) = f_1(\mathbf{s})g_1(\mathbf{t})\exp\left\{-\frac{1}{2}(\mathbf{s}'V_1\mathbf{s} - 2\mathbf{s}'C\mathbf{t} + \mathbf{t}'V_2\mathbf{t})\right\}$$

is a product of three characteristic functions, $f_1(\mathbf{s})$, $g_1(\mathbf{t})$ and

$$\varphi(\mathbf{s},\ \mathbf{t}) = \exp\left\{-\frac{1}{2}(\mathbf{s}'V_1\mathbf{s} - 2\mathbf{s}'C\mathbf{t} + \mathbf{t}'V_2\mathbf{t})\right\},$$

the latter being the characteristic function of an m-variate Gaussian distribution $N(\mathbf{0},\ V)$, and thus is a characteristic function itself. □

Remark. In case of $j = k = 1$, the smallness of $|C|$ required in Theorem 2.3 can be quantified. Namely, if the variances of the Gaussian components ζ_1 and ζ_2 are σ_1^2 and σ_2^2, suffice to assume $|C| < \sigma_1\sigma_2$. In this case, $C = \rho\sigma_1\sigma_2$ for some ρ, $|\rho| < 1$ and

$$h(s,\ t) = f_1(s)g_1(t)\exp\left\{-\frac{1}{2}(\sigma_1^2 s^2 - 2\rho\sigma_1\sigma_2 st + \sigma_2^2 t^2)\right\}$$

with the third factor on the right being the characteristic function of a bivariate Gaussian distribution.

Theorem 2.4. *If* $(\mathbf{X},\ \mathbf{Y})$ *has a GL-distribution with* $C \neq 0$ *and* $\mathbf{X}$ *is a Gaussian vector, then any linear form* $\mathbf{b}'\mathbf{Y}$ *either is independent of* $\mathbf{X}$ *or has a Gaussian component.*

Proof. Fix $\mathbf{b} \in \mathbb{R}^k$. If for any $\mathbf{a} \in \mathbb{R}^j$, $\mathbf{a}'C\mathbf{b} = 0$, then for any $u \in \mathbb{R}$

$$E\exp\{iu(\mathbf{a}'\mathbf{X} + \mathbf{b}'\mathbf{Y})\} = h(u\mathbf{a}) = f(u\mathbf{a})g(u\mathbf{b})\exp\{u^2\mathbf{a}'C\mathbf{b}\} = f(u\mathbf{a})g(u\mathbf{b}).$$

Thus, in this case $\mathbf{b}'\mathbf{Y}$ is independent of any $\mathbf{a}'\mathbf{X}$ implying independence of $\mathbf{b}'\mathbf{Y}$ and $\mathbf{X}$. Indeed, for any $u \in \mathbb{R}$, $u \neq 0$ and $\mathbf{v} \in \mathbb{R}^j$,

$$E\exp\{i(\mathbf{v}'\mathbf{X} + u\mathbf{b}'\mathbf{Y})\} = E\exp\{iu(\mathbf{a}'\mathbf{X} + \mathbf{b}'\mathbf{Y})\} = f(u\mathbf{a})g(u\mathbf{b}) = f(\mathbf{v})g(u\mathbf{b}).$$

Suppose now that there exists an $\mathbf{a} \in \mathbb{R}^j$ such that $\mathbf{a}'C\mathbf{b} \neq 0$. Then, denoting V the covariance matrix of $\mathbf{X}$,

$$E\exp\{iu(\mathbf{a}'\mathbf{X} + \mathbf{b}'\mathbf{Y})\} = g(u\mathbf{b})\exp\{-\frac{u^2}{2}(\mathbf{a}'V\mathbf{a} - 2\mathbf{a}'C\mathbf{b})\}.$$

One can always choose $|\mathbf{b}|$ large enough (replacing, if necessary, $\mathbf{b}$ with $\lambda\mathbf{b}$) so that

$$\mathbf{a}'V\mathbf{a} - 2\mathbf{a}'C\mathbf{b} = -\sigma^2 < 0.$$

Now

$$g(u\mathbf{b}) = h(u\mathbf{a},\, u\mathbf{b})\exp\{-\sigma^2u^2/2\}$$

and since $h(u\mathbf{a},\, u\mathbf{b})$ is a characteristic function, the random variable $\mathbf{b}'\mathbf{Y}$ with the characteristic function $g(u\mathbf{b})$ has a Gaussian component. □

As a direct corollary of Theorem 2.4 note that in case $j = k = 1$, if $(X,\, Y)$ has a GL-distribution and X is Gaussian, either Y is independent of X (in which case its distribution may be arbitrary) or it has a Gaussian component.

Cramér classical theorem (see, e. g., Linnik and Ostrovskii (1977) [4]) claims that the components of a Gaussian random vector are necessarily Gaussian (the components of a Poisson random variable are necessarily Poisson and the components of the sums of independent Poisson and Gaussian random variables are necessarily of the same form so that the above is not a characteristic property of the Gaussian distribution). A corollary of Theorems 2.3 and 2.4 shows that the class of GL-distributions is not closed with respect to deconvolution.

Corollary 1. *There exist independent bivariate vectors* $(X_1,\, Y_1)$, $(X_2,\, Y_2)$ *whose distributions are not GL while their sum* $(X_1 + Y_1,\, X_2 + Y_2)$ *has a GL-distribution.*

Proof. There are examples of independent random variables Y_1, Y_2 without Gaussian components whose sum $Y_1 + Y_2$ has a Gaussian component. In [4] was shown that independent identically distributed random variables Y_1, Y_2 with the characteristic function $f(t) = (1 - t^2)e^{-t^2/2}$ have no Gaussian component while their sum $Y_1 + Y_2$ whose characteristic function is $(1 - t^2)^2e^{-t^2}$ has a Gaussian component with the characteristic function $e^{-t^2/4}$.

Il'inskii [2] showed that any non-trivial (i. e., with $ab \neq 0$) linear combination $aY_1 + bY_2$ of the above Y_1, Y_2 has a Gaussian component. It leads to that any vector $(aX_1 + bY_1, aX_2 + bY_2)$ with $ab \neq 0$, Y_1, Y_2 from Il'inski'si example and Gaussian X_1, X_2 has a GL-distribution.

Let now $(X_1,\, Y_1)$, $(X_2,\, Y_2)$ be independent random vectors with Gaussian first components and such that X_i and Y_i, $i = 1, 2$ are not independent (their dependence may be arbitrary). Due to Theorem 2.4, in case of $j = k = 1$ the distributions of the vectors $(X_i,\, Y_i)$, $i = 1,\, 2$ are not GL. At the same time, both components of their sum $(X,\, Y) = (X_1 + X_2,\, Y_1 + Y_2)$ have Gaussian components so that due to Theorem 2.3 the vector $(X,\, Y)$ has a GL-distribution. □

Combining Theorems 2.1 and 2.4 leads to a characterization of distributions with a Gaussian component by a property of linear forms.

Corollary 2. *Let $(X_1,\ Y_1),\dots,(X_n,\ Y_n)$, $n\ge 3$ be independent random vectors with Gaussian first components. Assume that for $i=1,\dots,n$*

$$E|Y_i|^2<\infty,\ \ \mathrm{cov}(X_i,\ Y_i)\ne 0$$

and the characteristic functions $h_i(s,\ t)$ of $(X_i,\ Y_i)$ do not vanish. Then uncorrelatedness of pairs $L_1=a_1X_1+\ldots+a_nX_n$, $L_2=b_1Y_1+\ldots+b_nY_n$ in the first and second components is equivalent to their independence if and only if $Y_1,\dots,Y_n$ have Gaussian components.

Proof. From Theorem 2.1 (assuming $E(X_i)=0$),

$$h_i(s,\ t)=e^{-\sigma_i^2s^2/2}g_i(t)\exp\{C_ist\}$$

where $\sigma_i^2=E(X_i^2)$, $g_i(t)$ is the characteristic function of Y_i and $C_i=-\mathrm{cov}(X_i,Y_i)\ne 0$. Then by Theorem 2.4 Y_i has a Gaussian component. For the sufficiency part see Proposition 3. □

To the best of the authors' knowledge, it is the first example of characterization of distributions with Gaussian components.

For simplicity, let us consider the case of two-dimensional vector (X,Y) with a GL-distribution.

Hypothesis *Vector (X,Y) has GL-distribution if and only if both X and Y have Gaussian components.*

To support this Hypothesis note that it is true for infinitely divisible characteristic function $h(s,t)$.

This fact is rather simple, and its proof follows from Lévy Chinchine representation for infinitely divisible characteristic functions.

Let us give another example of characterization of distributions with a Gaussian component, supporting the **Hypothesis**. To this aim consider a set $\xi_1,\dots,\xi_n$ of independent random variables, and two sets $a_1,\dots,a_n$, $b_1,\dots,b_n$ of real constants. Denote

$$J=\{j:\ a_jb_j\ne 0\},\ \ J_1=\{1,\dots,n\}\setminus J. \tag{7}$$

Theorem 2.5. *Let*

$$X=\sum_{j=1}^n a_j\xi_j,\ \ Y=\sum_{j=1}^n b_j\xi_j.$$

Denote by $h(s,t)$ the characteristic function of the pair (X,Y) and suppose that the set $J\ne\emptyset$. The pair (X,Y) has a GL-distribution if and only if all ξ_j with $j\in J$ have Gaussian distribution. In this case

$$h(s,t)=f(s)g(t)\exp\{cst\}, \tag{8}$$

where both f and g have Gaussian components or are Gaussian.

Proof. Let us calculate $h(s,t)$. We have

$$h(s,t)=E\exp\{isX+itY\}=E\exp\left\{\sum_{j=1}^n i(sa_j+tb_j)\xi_j\right\}=\prod_{j=1}^n h_j(sa_j+tb_j), \tag{9}$$

where h_j is the characteristics function of ξ_j $(j = 1, \dots, n)$. From (8) and (9) it follows that

$$\prod_{j=1}^{n} h_j(sa_j + tb_j) = f(s)g(t)\exp\{cst\}. \tag{10}$$

The equation (10) is very similar to that appearing in known Skitovich–Darmois Theorem. The same method shows us that the functions h_j with $j \in J$ are characteristic functions of Gaussian distributions. Therefore, the functions $f(s)$ and $g(t)$ are represented as the products of Gaussian characteristic functions (h_j with $j \in J$) and some other functions (h_j with $j \in J_1$).

Reverse statement is trivial. □

GL-distributions may be of some interest for the theory of statistical models. Let $F(\mathbf{x})$ and $G(\mathbf{y})$ be a j- and k-variate distribution functions with $\int |\mathbf{x}|^2 dF(\mathbf{x}) < \infty$, $\int |\mathbf{y}|^2 \,\mathrm{d}G(\mathbf{y}) < \infty$. Does there exist an m-variate, $m = j + k$, distribution function $H(\mathbf{x}, \mathbf{y})$ with marginals F and G such that if $(\mathbf{X}, \mathbf{Y}) \sim H$, the covariance matrix $\mathrm{cov}(\mathbf{X}, \mathbf{Y})$ is a given $(j \times k)$-matrix C? In other words, is it possible to assume as a statistical model the triple $(F, G; C)$?

Since for any variables ξ, η with $\sigma_\xi^2 = \mathrm{var}(\xi) < \infty$, $\sigma_\eta^2 = \mathrm{var}(\eta) < \infty$,

$$|\mathrm{cov}(\xi, \eta)| \le \sigma_\xi \sigma_\eta, \tag{11}$$

the elements of C must satisfy conditions à la (11). Even in case of $j = k = 1$, (11) is not always (i. e., not for all F, G) sufficient.

Proposition 4. *If $\mathbf{X} \sim F$, $\mathbf{Y} \sim G$ have Gaussian components with covariance matrices V_1, V_2, there exist models $(F, G; C)$ for all C with sufficiently small elements, their smallness is determined by V_1 and V_2.*

Proof. As shown in Theorem 2.3, the function

$$h(\mathbf{s}, \mathbf{t}) = f(\mathbf{s})g(\mathbf{t})\exp\{\pm \mathbf{s}' C \mathbf{t}\}$$

where $f(\mathbf{s})$, $g(\mathbf{t})$ are the characteristic functions of $\mathbf{X}$ and $\mathbf{Y}$, is for all C with sufficiently small elements the characteristic function of a distribution $H(\mathbf{x}, \mathbf{y})$ with marginals F and G. Simple calculation shows that $\mathrm{cov}(\mathbf{X}, \mathbf{Y}) = \mp C$. □

Certainly, the presence of Gaussian components in $\mathbf{X}$ and $\mathbf{Y}$ is an artificial condition for the existence of a model $(F, G; C)$. And besides, the statistician would prefer to work with the distribution function or the density and not with the characteristic function.

H. Furstenberg, Y. Katznelson and B.Weiss (private communication) showed that if j- and k-dimensional random vectors $\mathbf{X} \sim F$, $\mathbf{Y} \sim G$ with finite second moments are such that for any unit vectors $\mathbf{a} \in \mathbb{R}^j$, $\mathbf{b} \in \mathbb{R}^k$

$$E(|\mathbf{a}'\mathbf{X}|) > A, \; E(|\mathbf{b}'\mathbf{Y}|) > A,$$

then for all sufficiently small (depending on A) absolute values of the elements of an $(j \times k)$-matrix C there exists an $m = (j + k)$-variate distribution $H(\mathbf{x}, \mathbf{y})$ with marginals $F(\mathbf{x})$ and $G(\mathbf{y})$ and $\mathrm{cov}(\mathbf{X}, \mathbf{Y}) = C$. Their proof is based on the convexity of the Fréchet class $\mathcal{F}(F, G)$ that allows constructing the required H as a convex combination of $H_{rq} \in \mathcal{F}(F, G)$ where for a given pair(r, q),

$$\int_{\mathbb{R}}^{m} x_r y_q \,\mathrm{d}H_{rq}(\mathbf{x}\,, \mathbf{y}) = \pm\epsilon$$

for some $\epsilon > 0$ while for all other pairs $(r', q') \neq (r, q)$,

$$\int_{\mathbb{R}}^{m} x_{r'} y_{q'} \, \mathrm{d}H_{rq}(\mathbf{x}, \mathbf{y}) = 0.$$

The resulting H, though given in an explicit form, is not handy for using in applications and would be interesting to construct (in case of absolutely continuous F and G) an absolutely continuous H with the required property.

Acknowledgement

The first author worked on the paper when he was visiting Department of Statistics, the Hebrew University in Jerusalem as a Forchheimer professor. The work of the second author was supported by the Grant MSM 002160839 from the Ministry of Education of Czech Republic and by Grant IAA 101120801 from the Academy of Sciences of Czech Republic.

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