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Conditional expectations and martingales in the fractional Brownian field

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Abstract: Conditional expectations of a fractional Brownian motion with Hurst index H respect to the filtration of a fractional Brownian motion with Hurst index $H^{'}$, both contained in the fractional Brownian field, are studied. A stochastic integral representation of those processes is constructed from the covariance structure of the underlying fractional Brownian field. As processes, the conditional expectations contain martingale components and for dual pairs of Hurst indices the processes become pure martingales which, up to a multiplicative constant, coincide with the fundamental martingales of fractional Brownian motions.

1. Introduction

In this paper all fractional Brownian motions $(Z_H(t))_{t\geq 0}$, $H \in (0,1)$ are imbedded in the fractional Gaussian field

$$Z = (Z_H(t))_{(t,H)\in[0,\infty)\times(0,1)}$$

whose covariance is given by

$$E\left(Z_H\left(t\right)Z_{H'}\left(s\right)\right) = a_{H,H'}\left\{\frac{|t|^{H+H^{'}} + |s|^{H+H^{'}} - |t-s|^{H+H^{'}}}{2}\right\}, \tag{1}$$

where for $H + H^{'} \neq 1$

$$\begin{aligned} a_{H,H'} = &-2\frac{\sqrt{\Gamma\left(2H+1\right)\sin\left(\pi H\right)}\sqrt{\Gamma\left(2H'+1\right)\sin\left(\pi H'\right)}}{\pi} \\ &\times \Gamma\left(-\left(H+H^{'}\right)\right)\cos\left(\left(H^{'}-H\right)\frac{\pi}{2}\right)\cos\left(\left(H+H^{'}\right)\frac{\pi}{2}\right), \end{aligned}$$

and for $H + H^{'} = 1$

$$a_{H,H'} = \sqrt{\Gamma\left(2H+1\right)\Gamma\left(3-2H\right)}\sin^2\left(\pi H\right) =: a_H = a_{H'}. \tag{2}$$

In the case when $H + H^{'} = 1$, a pair $(H, H^{'})$ will be called dual pair. Existence of that field was established by Dobric and Ojeda [4].

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This paper focuses on processes defined by

$$(3) \qquad X_{H,H'} = \left(X_{H,H'}(t)\right)_{t\geq 0} = \left(E\left(Z_H(t) \mid \mathcal{F}_t^{H'}\right)\right)_{t\geq 0},$$

where $\mathcal{F}_t^{H'} = \sigma(Z_{H'}(r) : 0 \leq r \leq t)$. Essentially the same proof as in [4] for even and odd fractional Gaussian fields can be used to show that $X_{H,H'}$ is a centered Gaussian H-selfsimilar process, and when (H, H') is a dual pair then $X_{H,H'}$ is a $(\mathcal{F}_t^{H'})_{t\geq 0}$ martingale (alternative proofs for these assertions are provided at the end of the present article).

In this paper we have obtained a stochastic integral representation of the processes $(X_{H,H'}(t))_{t\geq 0} = (E(Z_H(t) \mid \mathcal{F}_t^{H'}))_{t\geq 0}$ for an arbitrary pair of $(H, H') \in (0,1)^2$. In the case of dual pairs, integrands in the stochastic representations are constructed directly from the definition of the process. This study was triggered by our discovery that in the case when (H, H') is a dual pair, the conditional expectation process $(E(Z_H(t) \mid \mathcal{F}_t^{H'}))_{t\geq 0}$ is a martingale. The proof of that fact follows directly from (3) using (1). It turned out that the dual pair martingale $X_{H,H'}$ is in fact, up to a multiplicative constant, the fundamental martingale of Molchan ([8], [9]). Norros, Valkeila and Virtamo [10] have rediscovered that martingale while studying Girsanov formula for fractional Brownian motions. Results in [8], [9] and [10] were obtained by considering a single fractional Brownian motion process. Since any Gaussian martingale has independent increments, reformulated, our results state that when the process $(Z_H(t))_{t\geq 0}$ is orthogonally projected to the increasing sequence of Hilbert spaces $(\mathcal{H}_t^{H'})_{t\geq 0}$, where $\mathcal{H}_t^{H'} = \overline{L\{Z_{H'}(r) \mid 0 \leq r \leq t\}}$, its increments decorrelate. Therefore the pair $(Z_H(t), Z_{H'}(t))_{t\geq 0}$ contains information sufficient to transform each of them to a new process with independent increments. In particular, this implies that the covariance structure of Z contains additional information about $(Z_H(t))_{t\geq 0}$ which comes into light by studying $(Z_H(t))_{t\geq 0}$ as a subfamily of Z. It is natural to ask what happens to $X_{H,H'}$ if (H, H') is not a dual pair. In that case, as we expected, $X_{H,H'}$ decomposes into a martingale and an additional process, which is getting closer to a martingale when $H + H'$ approaches to 1.

The paper is organized as follows: In the second section we have concentrated our attention to the dual pair (martingale case) and built the kernel of a stochastic integral representation for this martingale directly from the definition. Hypergeometric functions turn out to be the main tool for the construction. The proof does not depend on whether H is greater or less than 1/2. We have taken a step further and decomposed each fractional Brownian motion into the sum of two processes one being Gaussian-Markov process. For $H > 0.4$ the Gaussian Markov process approximates fractional Brownian motion surprisingly well. A discussion in this direction is contained in [10].

The last section of the paper is devoted to deriving a stochastic integral representation for the general case of $X_{H,H'}$. Proof is based on the theory of fractional integrals and derivatives and hypergeometric functions.

Samko et al. [12] is the main source for all properties of fractional integrals and derivatives used in this paper, while Abramowitz and Stegun [1] and Andrews et al. [2] are the source for properties of hypergeometric functions.

2. Martingale case

Fundamental martingale of fractional Brownian motion was discovered by Molchan ([8], [9]) at the end of 1960s as a stochastic integral respect to time dependent kernel. Norros, Valkeila and Virtamo [10] have obtained fundamental martingales while studying Girsanov formula for fractional Brownian motions. In this section, from the definition of dual pair martingales, we have built the kernel, which when integrated with respect to a fractional Brownian motion recovers, up to a constant, the fundamental martingale of Molchan associated to that fractional Brownian motion.

Let $M_H = (M_H(t))_{t\geq 0}$ be the process defined by

$$M_H(t) = E(Z_{H'}(t) \mid \mathcal{F}_t^H) \tag{4}$$

where $H + H^{'} = 1$. As already mentioned that process is a martingale. In this section our attention is focused on projections of a fractional Brownian motion. In order to simplify notation we have exchanged the roles of H and $H^{'}$. In the section that follows we will return to the original notation. Since $M_H(t)$ is a conditional expectation of $Z_{H'}(t)$ respect to $\mathcal{F}_t^H$, it is an element of the Gaussian space generated by $\{Z_H(s) \mid 0 \leq s \leq t\}$. Any element of the Gaussian space determined by $\{Z_H(s) \mid 0 \leq s < \infty\}$ is a stochastic integral

$$\int_0^\infty f(u)\, dZ_H(u)$$

of some f in an appropriate Hilbert space Γ_H (see Huang and Cambanis [6]). Therefore, for every $t >$ there is $f_t \in \Gamma_H$ so that

$$M_H(t) = \int_0^\infty f_t(u)\, dZ_H(u).$$

Our goal is to identify f_t. Since $(M_H(t))_{t\in[0,\infty)}$ is a martingale respect to the filtrations $(\mathcal{F}_t^H)_{t\in[0,\infty)}$, $f_t(u) = 0$ when $u > t$.

An important property of f_t is easily derived by considering finite-dimensional approximations of $E(Z_{H'}(t) \mid \mathcal{F}_t^H)$. For a fixed t let n be an integer, $\Delta u = t/n$, $u_i = i\Delta u$, $i = 0, 1, \ldots, n$. Since

$$\begin{aligned} M_{H,n}(t) &= E\left(Z_{H'}(t) \mid Z_H(u_i),\ i = 0, \ldots, n\right) \\ &= E\left(Z_{H'}(t) \mid Z_H(u_i) - Z_H(u_{i-1}),\ i = 1, \ldots, n\right), \end{aligned}$$

we have

$$M_{H,n}(t) = \sum_{i=1}^n \alpha_i^*(Z_H(u_i) - Z_H(u_{i-1})),$$

where $\alpha^* = (\alpha_i^*)$ is an element of $\mathbb{R}^n$ for which the minimum, defined by

$$\min\left\{ E\left(Z_{H'}(t) - \sum_{i=1}^n \alpha_i(Z_H(u_i) - Z_H(u_{i-1}))\right)^2 \middle| (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n \right\},$$

is achieved. That minimum is a solution of the system of linear equations

$$\begin{aligned} &\sum_{j=1}^n \alpha_j^* E\left((Z_H(u_i) - Z_H(u_{i-1}))(Z_H(u_j) - Z_H(u_{j-1})))\right) \\ &= E(Z_{H'}(t)(Z_H(u_i) - Z_H(u_{i-1}))),\ i = 1, \ldots, n. \end{aligned}$$

That system can be rewritten as

$$\sum_{j=1}^n \alpha_j^* \left(\frac{|i-j-1|^{2H} + |i-j+1|^{2H}}{2} - |i-j|^{2H} \right) = a_H \Delta u^{1-2H},\ i = 1, \ldots, n. \tag{5}$$

The equations i and $n-i$ of that system do not change if (α_i^*) and replaced by (α_{n-i}^*). Since the matrix

$$A = \left[\frac{|i-j-1|^{2H} + |i-j+1|^{2H}}{2} - |i-j|^{2H} \right]_{n\times n}$$

is regular, the solution of the system (5) is unique and consequently $(\alpha_i^*) = (\alpha_{n-i}^*)$. So, if $0 < u < u + \Delta u < t$, the contribution of $Z_H(u+\Delta u) - Z_H(u)$ to $M_H(t)$ is the same as of $Z_H(t-u) - Z_H(t-u-\Delta u)$, which implies that

$$f_t(u) = f_t(t-u), \; u \in (0,t). \tag{6}$$

Since

$$\left(E\left(Z_H(s) \int_0^t f_t(u) dZ_H(u) \right) \right)^2 \le s^{2H} E\left(\int_0^t f_t(u) dZ_H(u) \right)^2$$

and

$$\begin{aligned} &E\left(Z_H(s)(Z_H(t_i) - Z_H(t_{i-1}))\right) \\ &= \frac{1}{2}(t_i^{2H} - |t_i - s|^{2H} - \left(t_{i-1}^{2H} - |t_{i-1} - s|^{2H}\right) \\ &= H\left(u_i^{2H-1} + \operatorname{sgn}(s-u_i)\,|s-u_i|^{2H-1}\right)(t_i - t_{i-1}) \end{aligned}$$

for some $u_i \in [t_{i-1}, t_i]$, and all intervals not containing s, it follows, with some care, that

$$E\left(Z_H(s) \int_0^t f_t(u) dZ_H(u) \right) = H \int_0^t f_t(u) \left(u^{2H-1} + \operatorname{sgn}(s-u)\,|s-u|^{2H-1} \right) du. \tag{7}$$

When $s \le t$ then

$$E(M_H(t)Z_H(s)) = E((Z_{H'}(t) \mid \mathcal{F}_t^H)Z_H(s)) = E(Z_{H'}(t)Z_H(s)) = a_H s,$$

and the equation (7) becomes

$$a_H s = H \int_0^t f_t(u) \left(u^{2H-1} + \operatorname{sgn}(s-u)\,|s-u|^{2H-1} \right) du. \tag{8}$$

Setting $s = t$ in (8) and applying (6) yields

$$a_H t = H \int_0^t f_t(u) \left(u^{2H-1} + |t-u|^{2H-1} \right) du = 2H \int_0^t f_t(u) u^{2H-1} du.$$

Therefore determining f_t boils down to solving the following integral equation

$$a_H \left(s - \frac{t}{2} \right) = H \int_0^t f_t(u) \operatorname{sgn}(s-u)\,|s-u|^{2H-1} du. \tag{9}$$

Equation (9) can be further simplified. Since $(M_H(t))_{t\ge 0}$ is a $(1-H)$-selfsimilar process, that is,

$$M_H(ht) \stackrel{d}{=} h^{1-H} M_H(t),$$

or

$$M_H(ht) = \int_0^{ht} f_{ht}(u) dZ_H(u) = \int_0^{ht} f_{ht}(hv) dZ_H(hv) \overset{d}{=} h^{1-H} \int_0^t f_t(v) dZ_H(v),$$

and since $(Z_H(t))_{t \geq 0}$ is H-selfsimilar we have

$$h^H \int_0^t f_{ht}(hv) dZ_H(v) \overset{d}{=} h^{1-H} \int_0^t f_t(v) dZ_H(v).$$

The last equation holds true if

$$f_{ht}(hv) = h^{1-2H} f_t(v). \tag{10}$$

Based on (6), one of possible candidates for f_t is of the form $c_H(t-u)^p u^p$, and if so $p = \frac{1}{2} - H$ (by (10)). We are left to prove that for any $v \in [0,1]$, and some constant $c = a_H / H c_H$, the following is true

$$\int_0^1 (1-u)^{\frac{1}{2}-H} u^{\frac{1}{2}-H} \operatorname{sgn}(v-u) \left|v-u\right|^{2H-1} du = c(v - \frac{1}{2}). \tag{11}$$

The last equation can be transformed, which is done below, into a hypergeometric function identity. The subject of hypergeometric function identities is more than 250 years old. There are thousand of hypergeometric identities, hundreds of them having simple right-hand side expression. We have consulted the most common handbooks that include hypergeometric identities (Abramowitz and Stegun [1], Gradshteyn and Ryznik [5]), searched the web, but have found none matching our equation (11). The following result might be a new hypergeometric identity.

Theorem 1 (Hypergeometric identity). *For any v, $0 \leq v \leq 1$, the following identity holds*

$$\begin{aligned} &\int_0^1 (1-u)^{\frac{1}{2}-H} u^{\frac{1}{2}-H} sgn(v-u) \left|v-u\right|^{2H-1} du \\ &= 2\Gamma(3/2 - H) \Gamma(H + 1/2) \left(v - \frac{1}{2}\right). \end{aligned}$$

Proof. The main tool that connects different expected values associated to fractional Brownian motions and hypergeometric functions ${}_2F_1$ is the famous Euler's Integral Representation. Proved in 1769, it states that for Re (c) >Re(b) and $z \in \mathbb{C} \setminus [1, \infty)$ (see [1] formula 15.3.1) the following holds

$${}_2F_1(a,b,c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 y^{b-1} (1-y)^{c-b-1} (1-zy)^{-a} dy. \tag{12}$$

All hypergeometric functions in this paper are ${}_2F_1$ functions, and therefore we will suppress indices in F. For the proof we will need the following hypergeometric identities:

a) ([1], formula 15.3.7)

$$\begin{aligned} F(a,b,c;z) = &\frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)} (-z)^{-a} F\left(a, 1-c+a, 1-b+a; \frac{1}{z}\right) \\ &+ \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)} (-z)^{-b} F\left(b, 1-c+b, 1-a+b; \frac{1}{z}\right), \end{aligned} \tag{13}$$

which holds for $|\arg(-z)| < \pi$, $z \notin (0,1)$,

b) the well known Pfaff identity from 1797 ([1], formula 15.3.4)

$$F\left(a,b,c;z\right)=\left(1-z\right)^{-a}F\left(a,c-b,c;\frac{z}{z-1}\right), \tag{14}$$

c) ([1], formula 15.4.1)

$$F(-m,b,c;z)=\sum_{n=0}^{m}\frac{(-m)_n(b)_n}{(c)_n}\frac{z^n}{n!}, \tag{15}$$

and

d) ([1], formula 15.3.3)

$$F(a,b,c;z)=(1-z)^{c-a-b}F\left(c-a,c-b,c;z\right). \tag{16}$$

Set

$$\begin{aligned}I&=\int_0^v(1-u)^{\frac12-H}u^{\frac12-H}(v-u)^{2H-1}du\\&\quad-\int_v^1(1-u)^{\frac12-H}u^{\frac12-H}(u-v)^{2H-1}du\\&=I_1-I_2.\end{aligned}$$

Substituting $u=tv$ in I_1, and then applying Euler's representation (12), we obtain

$$\begin{aligned}I_1&=v^{H+1/2}\int_0^1t^{1/2-H}\left(1-vt\right)^{1/2-H}\left(1-t\right)^{2H-1}dt\\&=\frac{\Gamma\left(3/2-H\right)\Gamma\left(2H\right)}{\Gamma\left(H+3/2\right)}v^{H+1/2}F(H-1/2,3/2-H,H+3/2;v).\end{aligned}$$

Transformation of I_2 into a hypergeometric function is done by substitutions, $u=x+v$ first, then $x=(1-v)y$, followed by the Euler's Representation formula, yielding

$$\begin{aligned}I_2&=(1-v)^{H+1/2}\int_0^1\left((1-v)\,t+v\right)^{1/2-H}\left(1-t\right)^{1/2-H}t^{2H-1}dt\\&=(1-v)^{H+1/2}v^{1/2-H}\int_0^1t^{2H-1}\left(1-t\right)^{1/2-H}\left(1-t\frac{v-1}{v}\right)^{1/2-H}dt\\&=\frac{\Gamma\left(2H\right)\Gamma\left(3/2-H\right)}{\Gamma\left(H+3/2\right)}\left(1-v\right)^{H+1/2}v^{1/2-H}\times \qquad (17)\\&\quad F\left(H-1/2,2H,H+3/2;\frac{v-1}{v}\right).\end{aligned}$$

The hypergeometric function in the above formula, when $H\in(0,1)\setminus\{1/2\}$, by identity (13) can be rewritten as

$$\begin{aligned}&F\left(H-1/2,2H,H+3/2;\frac{v-1}{v}\right) \qquad (18)\\&=\frac{\Gamma\left(H+\frac32\right)\Gamma\left(H+\frac12\right)}{\Gamma\left(2H\right)}\left(\frac{1-v}{v}\right)^{\frac12-H}F\left(H-\frac12,-1,\frac12-H;\frac{v}{v-1}\right)\\&\quad+\frac{\Gamma\left(H+\frac32\right)\Gamma\left(-\frac12-H\right)}{\Gamma\left(H-\frac12\right)\Gamma\left(\frac32-H\right)}\left(\frac{1-v}{v}\right)^{-2H}F\left(2H,H-\frac12,\frac32+H;\frac{v}{v-1}\right).\end{aligned}$$

An application of the Euler's reflection formula

$$\Gamma(z)\Gamma(1-z)=\frac{\pi}{\sin(\pi z)},$$

yields

$$\frac{\Gamma(H+3/2)\Gamma(-1/2-H)}{\Gamma(H-1/2)\Gamma(3/2-H)}=\frac{\frac{\pi}{\sin(\pi(H+3/2))}}{\frac{\pi}{\sin(\pi(H-1/2))}}=1.$$

One of elementary properties of hypergeometric functions $F(a,b,c;z)=F(b,a,c;z)$ together with (15) simplifies the second hypergeometric function in (18) to

$$F\left(H-1/2,-1,1/2-H;\frac{v}{v-1}\right)=\sum_{n=0}^{1}\frac{(-1)_n(H-1/2)_n}{(1/2-H)_n}\frac{\left(\frac{v}{v-1}\right)^n}{n!}=1+\frac{v}{v-1}.$$

By Pfaff's formula (14)

$$F\left(2H,H-1/2,3/2+H;\frac{v}{v-1}\right)=(1-v)^{2H}F(2H,2,3/2+H;v),$$

and an application of (16) assures that

$$\begin{aligned}F(2H,2,3/2+H;v)&=v^{H+1/2}F(3/2-H,H-1/2,3/2+H;v)\\&=v^{H+1/2}F(H-1/2,3/2-H,3/2+H;v).\end{aligned}$$

After all transformations performed on I_2, for $H\in(0,1)\setminus\{\frac{1}{2}\}$, we have

$$\begin{aligned}I_2&=-\Gamma(3/2-H)\Gamma(H+1/2)(2v-1)\\&\quad+\frac{\Gamma(2H)\Gamma(3/2-H)}{\Gamma(H+3/2)}v^{H+1/2}F(H-1/2,3/2-H,3/2+H;v)\\&=-\Gamma(3/2-H)\Gamma(H+1/2)(2v-1)+I_1,\end{aligned}$$

and in the light of $I=I_1-I_2$ the proof follows. □

Therefore we have proved the following Theorem:

Theorem 2. *Let $H+H'=1$. Then*

$$M_H(t)=E(Z_{H'}(t)\mid\mathcal{F}_t^H)=c_H\int_0^t(1-u)^{\frac{1}{2}-H}u^{\frac{1}{2}-H}dZ_H(u)$$

where

$$c_H=\frac{a_H}{2H\Gamma(3/2-H)\Gamma(H+1/2)}=\frac{\sqrt{\Gamma(3-2H)\Gamma(2H+1)}\sin^2(\pi H)}{2H\Gamma(3/2-H)\Gamma(H+1/2)}.$$

The kernel in the theorem above turns out to be the same as in [10] except for the constant. Our constant is $(2H)^2a_H$ times larger.

2.1. Gaussian Markov processes inside fractional Brownian motion

In the Gaussian field $(Z_H(t))_{(t,H)\in[0,\infty)\times(0,1)}$ dual pairs (H,H'), $H+H'=1$ play an important role. Orthogonal projection of $(Z_{H'}(t))_{t\geq0}$ to $(\mathcal{F}_t^H)_{t\geq0}$ decorrelates increments of $(Z_H(t))_{t\geq0}$. But, how "much of the Markovian" property of $(Z_H(t))_{t\geq0}$

does $(Z_{H'}(t))_{t\geq 0}$ encode? The answer is surprising, a lot. Let us make this statement more specific.

The function $\alpha_H : [0,\infty) \to \mathbb{R}$ defined by

$$\alpha_H(t) = \frac{E(Z_H(t)M_H(t))}{EM_H^2(t)}.$$

minimizes

$$E\left(Z_H(t) - \alpha(t)M_H(t)\right)^2.$$

Since

$$E(Z_H(t)M_H(t)) = E(Z_H(t)Z_{H'}(t)) = a_H t, \tag{19}$$

and

$$EM_H^2(t) = E(M_H(t)Z_{H'}(t)),$$

by Theorem 2 and substitution $u = tv$ it follows

(20)

$$EM_H^2(t) = c_H a_H t^{2-2H} \int_0^1 u^{\frac{1}{2}-H}(t-u)^{\frac{1}{2}-H} du = c_H a_H B\left(\frac{3}{2} - H, \frac{3}{2} - H\right) t^{2-2H}.$$

Therefore by (19) and (20)

$$\alpha_H(t) = \frac{\Gamma(3-2H)}{c_H \Gamma^2(\frac{3}{2} - H)} t^{2H-1}.$$

Set

$$Y_H(t) = Z_H(t) - \alpha_H(t)M_H(t),$$

and note that by equations (19) and (20)

$$\begin{aligned} EY_H^2(t) &= t^{2H} - \frac{\left(E(Z_H(t)M_H(t))\right)^2}{EM_H^2(t)} \\ &= t^{2H}\left(1 - \frac{a_H \Gamma(3-2H)}{c_H \Gamma^2(\frac{3}{2} - H)}\right) = t^{2H} d_H^2. \end{aligned}$$

Recall

$$a_H = 2Hc_H B\left(\frac{1}{2} + H, \frac{3}{2} - H\right),$$

which establishes

$$d_H^2 = 1 - 2H \frac{\Gamma\left(\frac{1}{2} + H\right)\Gamma(3-2H)}{\Gamma\left(\frac{3}{2} - H\right)}.$$

The graph of d_H is shown on Figure 1.

For $H \in (0.4, 1)$ the Gaussian-Markov process

$$G_H = (G_H(t))_{t\geq 0} = (\alpha_H(t)M_H(t))_{t\geq 0}$$

approximates Z_H with a relative L^2 error of at most 12%. As H decreases below 0.4 the approximation worsens. The connection between $G_H(t)$ and $Z_H(t)$ has been observed in [10].

Let us find the covariance of Y_H. If $t > s$, by the martingale property of M_H,

$$E(Y_H(s)Y_H(t)) = E(Y_H(s)Z_H(t)) = E(Z_H(s)Z_H(t)) - \alpha_H(s)E(M_H(s)Z_H(t)).$$

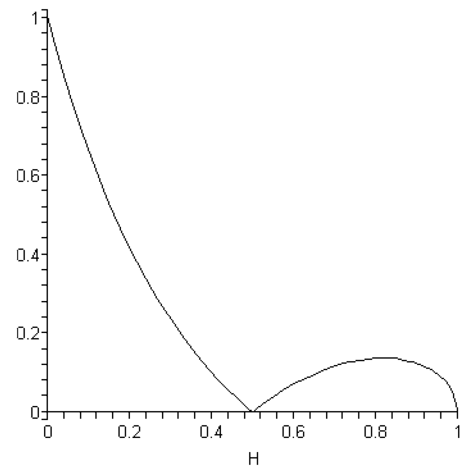

FIG 1. *Graph of d_H.*

Using Theorem 2 we obtain

$$E(M_H(s)Z_H(t)) = c_H H \int_0^s u^{\frac{1}{2}-H}(s-u)^{\frac{1}{2}-H}\left(u^{2H-1}+(t-u)^{2H-1}\right)du.$$

By substitution $u = sv$, and the Euler's representation formula (12) it readily follows

$$\begin{aligned}E(Z_H(t)M_H(s)) &= c_H H s \int_0^1 v^{\frac{1}{2}-H}(1-v)^{\frac{1}{2}-H}v^{2H-1}dv \\ &\quad + c_H H s^{2-2H}\int_0^1 v^{\frac{1}{2}-H}(1-v)^{\frac{1}{2}-H}(t-sv)^{2H-1}dv \\ &= c_H H s \left(B\left(H+\frac{1}{2},\frac{3}{2}-H\right)\right. \\ &\quad \left. + \left(\frac{s}{t}\right)^{1-2H}\frac{\Gamma^2\left(\frac{3}{2}-H\right)}{\Gamma\left(3-2H\right)}F\left(1-2H,\frac{3}{2}-H,3-2H;\frac{s}{t}\right)\right),\end{aligned}$$

and consequently the second term in the correlation expression becomes

$$\begin{aligned}\alpha_H(s)E(Z_H(t)M_H(s)) &= s^{2H}H\left(\frac{\Gamma(3-2H)\Gamma(H+\frac{1}{2})}{\Gamma(\frac{3}{2}-H)}\right. \\ &\quad \left. + \left(\frac{s}{t}\right)^{1-2H}F\left(1-2H,\frac{3}{2}-H,3-2H;\frac{s}{t}\right)\right) \\ &= s^{2H}H\frac{\Gamma(3-2H)\Gamma(H+\frac{1}{2})}{\Gamma\left(\frac{3}{2}-H\right)} + s^{2H}f_H\left(\frac{s}{t}\right).\end{aligned}$$

So the covariance of Y_H is given by

$$\begin{aligned}E(Y_H(s)Y_H(t)) &= E(Y_H(s)Y_H(t)) - (s\wedge t)^{2H}H\frac{\Gamma(3-2H)\Gamma\left(H+\frac{1}{2}\right)}{\Gamma\left(\frac{3}{2}-H\right)} \\ &\quad - s^{2H}f_H\left(\frac{s\wedge t}{s\vee t}\right).\end{aligned}$$

Notice that middle term of the covariance of Y_H is a martingale part.

It is trivial to prove, but worthwhile to observe, that the decomposition

$$Z_H(t) = G_H(t) + Y_H(t)$$

of a fractional Brownian motion into the Gaussian-Markov process G_H and the Y_H process has one interesting feature, namely for each t the processes $(Y_H(s))_{s\in[0,t]}$ and $(G_H(s))_{s\in[t,\infty)}$ are independent.

3. General pairs

This section is devoted to finding, for an arbitrary pair $(H, H^{'}) \in (0,1)\times(0,1)$, an explicit representation of the process $X_{H,H'}(t) = E(Z_H(t) \mid \mathcal{F}_t^{H'})$.

It is possible to find a Brownian motion $W^{H^{'}}$ such that the filtration $(\mathcal{F}^{H^{'}})_{t\geq 0}$, up to sets of measure zero, coincides with the filtration $(\sigma(W_s^{H^{'}} \mid 0 \leq s \leq t))_{t\geq 0}$, and such that $Z_{H'}(t)$ can be expressed as

$$\begin{aligned}Z_{H'}(t) = \int_0^t C_{H'} \Bigg(\left(\frac{t}{s}\right)^{H^{'}-\frac{1}{2}} (t-s)^{H^{'}-\frac{1}{2}} \\ -\left(H^{'}-\frac{1}{2}\right) s^{\frac{1}{2}-H^{'}} \int_s^t u^{H^{'}-\frac{3}{2}}(u-s)^{H^{'}-\frac{1}{2}} du\Bigg)\, dW_s^{H^{'}},\end{aligned} \tag{21}$$

where $C_{H'}$ is a suitable constant (see for example [10]). Moreover, the kernel is equal to

$$C_{H'}(t-s)^{H^{'}-\frac{1}{2}} F\left(\frac{1}{2}-H^{'}, H^{'}-\frac{1}{2}, H^{'}+\frac{1}{2}, 1-\frac{t}{s}\right), \tag{22}$$

where $F = {}_2F_1$ is the Gauss Hypergeometric function (see Decreusefond [3]). The kernel can be rewritten in terms of fractional integrals too, see Pipiras and Taqqu [11], and then $Z_{H'}(r)$, for $0 \leq r \leq t$, can be expressed as

$$\begin{aligned}Z_{H'}(r) &= \sigma_{H'} \int_0^t s^{\frac{1}{2}-H^{'}} I_{t-}^{H^{'}-\frac{1}{2}} \left((\cdot)^{H^{'}-\frac{1}{2}} 1_{[0,r)}\right)(s)\, dW_s^{H^{'}} \\ &= \sigma_{H'} \int_0^r s^{\frac{1}{2}-v} I_{r-}^{H^{'}-\frac{1}{2}} \left((\cdot)^{H^{'}-\frac{1}{2}}\right)(s)\, dW_s^{H^{'}},\end{aligned} \tag{23}$$

where

$$\sigma_{H'}^2 = \frac{\pi\left(H^{'}-\frac{1}{2}\right)\left(2\left(H^{'}-\frac{1}{2}\right)+1\right)}{\Gamma\left(1-2\left(H^{'}-\frac{1}{2}\right)\right)\sin\left(\pi\left(H^{'}-\frac{1}{2}\right)\right)}.$$

For each t there is exist a unique function $g_{H,H'}(\cdot,t) \in L^2[0,t]$, such that

$$X_{H,H'}(t) = \int_0^t g_{H,H'}(s,t)\, dW_s^{H^{'}}. \tag{24}$$

From definition (3), equations (23), (24), and the covariance of the the fractional Brownian field (1) it follows that, for $0 \leq r \leq t$,

$$\begin{aligned}E\left(X_{H,H'}(t)\, Z_{H'}(r)\right) &= \sigma_{H'} \int_0^r g_{H,H'}(s,t)\, s^{\frac{1}{2}-H^{'}} I_{r-}^{H^{'}-\frac{1}{2}} \left((\cdot)^{H^{'}-\frac{1}{2}}\right)(s)\, ds \\ &= a_{H,H'} \frac{t^{H+H^{'}} + r^{H+H^{'}} - (t-r)^{H+H^{'}}}{2}.\end{aligned} \tag{25}$$

Uniqueness of conditional expectations assures that a function $g_{H,H'}(\cdot,t)$ solving (25) is unique.

Theorem 3. *The process $X_{H,H'}$ can be represented as*

$$X_{H,H'}(t) = \int_0^t g_{H,H'}(s,t)dW_s^{H'},$$

where for $0 \le r \le t$,

$$g_{H,H'}(r,t) = \alpha_{H,H'} r^{H'-\frac{1}{2}} I_{0+}^{\frac{1}{2}-H'} \left(u^{H-\frac{1}{2}} + u^{\frac{1}{2}-H'} (t-u)^{H+H'-1}\right)(r),$$

and

$$\alpha_{H,H'} = \frac{\left(H+H'\right) a_{H,H'}}{2\sigma_{H'}}. \tag{26}$$

Proof. When $H' = \frac{1}{2}$, it is straightforward to verify that $g_{H,H'}$ given by the previous expression satisfies equation (25).

Proof for $H' > 1/2$

First step toward solving the integral equation (25) is to rewrite the integral in (25) by applying the fractional integration by parts formula (see equation (2.20) in [12]). Conditions for applying the formula are met since $s \to g_{H,H'}(s)s^{\frac{1}{2}-H'}$ is in $L^1[0,r]$ (by Cauchy-Schwarz), and $s \to s^{H'-\frac{1}{2}}$ is in $L^q[0,r]$ for all $q \ge 1$. In particular $s \to s^{H'-\frac{1}{2}}$ is in $L^q[0,r]$ for q satisfying $\frac{1}{1}+\frac{1}{q} < 1+(H'-\frac{1}{2})$. Consequently

$$E\left(X_{H,H'} Z_{H'}(r)\right) = \sigma_{H'} \int_0^r s^{H'-\frac{1}{2}} I_{0+}^{H'-\frac{1}{2}} \left((\cdot)^{\frac{1}{2}-H'} g_{H,H'}(\cdot)\right)(s)\, ds,\ 0 \le r \le t,$$

and after differentiation respect to r, the equation (25) turns into the following a.e. identity

$$I_{0+}^{H'-\frac{1}{2}} \left((\cdot)^{\frac{1}{2}-H'} g_{H,H'}(\cdot,t)\right)(r) = \alpha_{H,H'} \frac{r^{H+H'-1} + (t-r)^{H+H'-1}}{r^{H'-\frac{1}{2}}},\ r \in [0,t], \tag{27}$$

where $\alpha_{H,H'}$ is given by equation (26). It is left to invert the integral operator $I_{0+}^{H'-\frac{1}{2}}$. Since $(\cdot)^{\frac{1}{2}-H'} g_{H,H'}(\cdot,t) \in L^1[0,t]$, by Theorem 2.4 [12] $I_{0+}^{\frac{1}{2}-H'}$ inverts $I_{0+}^{H'-\frac{1}{2}}$, consequently for $0 \le r \le t$

$$g_{H,H'}(r,t) = \alpha_{H,H'} r^{H'-\frac{1}{2}} I_{0+}^{\frac{1}{2}-H'} \left(u^{H-\frac{1}{2}} + u^{\frac{1}{2}-H'} (t-u)^{H+H'-1}\right)(r). \tag{28}$$

Proof for $H' < 1/2$

This case will be proved by verifying that $g_{H,H'}$ defined by (28) satisfies (25). Set $\widetilde{g}$ to be the right-hand-side of (28). First, let us show that $\widetilde{g} \in L^2[0,t]$. By formula (2.44) in [12]

$$I_{0+}^{\frac{1}{2}-H'} \left(u^{H-\frac{1}{2}}\right)(r) = \frac{\Gamma\left(H+\frac{1}{2}\right)}{\Gamma\left(1+H-H'\right)} r^{H-H'}, \tag{29}$$

and therefore

$$r^{H'-\frac{1}{2}} I_{0+}^{\frac{1}{2}-H'} \left(u^{H-\frac{1}{2}}\right)(r) = \frac{\Gamma\left(H+\frac{1}{2}\right)}{\Gamma\left(1+H-H'\right)} r^{H-\frac{1}{2}} \in L^2\left[0,t\right].$$

We are left to establish that the second summand of $\widetilde{g}$,

$$r \to r^{H'-\frac{1}{2}} I_{0+}^{\frac{1}{2}-H'} \left(u^{\frac{1}{2}-H'} \left(t-u\right)^{H+H'-1}\right)(r)$$

belongs to $L^2[0,t]$. For that step hypergeometric functions are the right tool. By formula (2.46) in [12] for $0<r<t$,

$$\begin{aligned} I_{0+}^{\frac{1}{2}-H'} \left(u^{\frac{1}{2}-H'} \left(t-u\right)^{H+H'-1}\right)(r) &= t^{H+H'-1} \frac{\Gamma\left(\frac{3}{2}-H'\right)}{\Gamma\left(2-2H'\right)} r^{1-2H'} \\ &\times F\left(1-H-H', \frac{3}{2}-H', 2-2H'; \frac{r}{t}\right) \end{aligned} \tag{30}$$

and hence the problem boils down to the study of

$$h_H(r) = r^{\frac{1}{2}-H'} F\left(1-H-H', \frac{3}{2}-H', 2-2H'; \frac{r}{t}\right), \ 0<r<t. \tag{31}$$

A hypergeometric series

$$F\left(a,b,c;z\right) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} z^n$$

converges absolutely for $|z|<1$ and by Theorem 2.1.2 in [2] if $c-a-b>0$ for $|z|\le 1$. For hypergeometric function factor in (31) $c-a-b = H-1/2$. Therefore, if $H>1/2$, $h_H \in L^2[0,t]$, since its hypergeometric factor is continuous on $[0,t]$ and $r^{\frac{1}{2}-H'} \in L^2[0,t]$. The case $c-a-b = H-1/2<0$ is a bit more complex. Theorem 2.1.3 in [2] states that if $c-a-b<0$ then

$$\lim_{x\to 1^-} \frac{F\left(a,b,c;x\right)}{(1-x)^{c-a-b}} = \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)},$$

and if $c-a-b=0$ then

$$\lim_{x\to 1^-} \frac{F\left(a,b,c;x\right)}{\log(1/(1-x))} = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}.$$

Consequently when $H<1/2$ the hypergeometric function in (31), when r is close to t, $r<t$, behaves like $(t-r)^{H-1/2}$, hence $h_H \in L^2[0,t]$. For $H=1/2$, the function $F(1-H-H', \frac{3}{2}-H', 2-2H'; \frac{r}{t})$ behaves as $\log(1-\frac{r}{t})$ for r is close to $t, r<t$, implying $h_H \in L^2[0,t]$. So it is verified that $\widetilde{g} \in L^2[0,t]$.

It is left to check that $\widetilde{g}$ satisfies (25). Note that

$$\begin{aligned} & E\left(\int_0^t \widetilde{g}\left(s\right) dW_s dZ_{H'}(r)\right) \\ &= \sigma_{H'} \alpha_{H,H'} \int_0^r I_{0+}^{\frac{1}{2}-H'} \left(u^{H-\frac{1}{2}} + u^{\frac{1}{2}-H'} \left(t-u\right)^{H+H'-1}\right)(s) \\ &\times I_{r-}^{H'-\frac{1}{2}} \left(u^{H'-\frac{1}{2}}\right)(s)\, ds. \end{aligned} \tag{32}$$

Our goal is to apply the formula of fractional integration by parts to move the integral $I_{0+}^{\frac{1}{2}-H^{'}}$. This formula can be applied in (32) if

$$u^{H-\frac{1}{2}} + u^{\frac{1}{2}-H^{'}}\,(t-u)^{H+H^{'}-1} \in L^{p}\,[0,r]\,,$$

$$I_{r-}^{H^{'}-\frac{1}{2}}\left(u^{H^{'}-\frac{1}{2}}\right)(s) \in L^{q}\,[0,r]\,,$$

and

$$\frac{1}{p} + \frac{1}{q} \leq 1 + \frac{1}{2} - H^{'}. \tag{33}$$

By formula (3.4) and Lemma 3.4 in Jost [7]

$$I_{r-}^{H^{'}-\frac{1}{2}}\left(u^{H^{'}-\frac{1}{2}}\right)(s) = \frac{1}{\Gamma\left(H^{'}+\frac{1}{2}\right)}\,(r-s)^{H^{'}-\frac{1}{2}}\,r^{H^{'}-\frac{1}{2}} \times F\left(\frac{1}{2}-H^{'},1,H^{'}+\frac{1}{2};\frac{r-s}{r}\right). \tag{34}$$

For the hypergeometric factor in (34) $c-a-b = 2H^{'}-1 < 0$, so by Theorem 2.1.3 in [2]

$$\lim_{s\to 0^{+}} \frac{F\left(\frac{1}{2}-H^{'},1,H^{'}+\frac{1}{2};\frac{r-s}{r}\right)}{(1-\frac{r-s}{r})^{2H^{'}-1}} = \frac{\Gamma(H^{'}+\frac{1}{2})\Gamma(1-2H^{'})}{\Gamma(\frac{1}{2}-H^{'})},$$

$s \to I_{r-}^{H^{'}-\frac{1}{2}}(u^{H^{'}-\frac{1}{2}})(s)$ behaves as $s^{2H^{'}-1}$ when s is close to 0. Since the hypergeometric function in (34) it is bounded when s is away from 0,

$$s \to I_{r-}^{H^{'}-\frac{1}{2}}\left(u^{H^{'}-\frac{1}{2}}\right)(s) \in L^{q}[0,r]$$

for any q, $0<q<\min\{1/(1-2H^{'}),2\}$. Observe that, for $0<r<t$,

$$u \to u^{H-\frac{1}{2}} + u^{\frac{1}{2}-H^{'}}\,(t-u)^{H+H^{'}-1} \in L^{2}\,[0,r]\,.$$

Choosing q so that

$$\frac{1}{1-H^{'}} < q < \min\left\{\frac{1}{1-2H^{'}},2\right\}$$

assures that both condition (33) and $0<q<1/(1-2H^{'})$ are satisfied. Applying the fractional integration by parts formula to the right-hand side of (32) that expression becomes

$$\sigma_{H^{'}}\alpha_{H,H^{'}}\int_{0}^{r}\left(s^{H-\frac{1}{2}} + s^{\frac{1}{2}-H^{'}}\,(t-s)^{H+H^{'}-1}\right) I_{r-}^{\frac{1}{2}-H^{'}}\, I_{r-}^{H^{'}-\frac{1}{2}}\left(u^{H^{'}-\frac{1}{2}}\right)(s)\,ds,$$

and it is further simplified by formula (3.9) in [7] to

$$\begin{aligned}&\sigma_{H^{'}}\alpha_{H,H^{'}}\int_{0}^{r}\left(s^{H-\frac{1}{2}} + s^{\frac{1}{2}-H^{'}}\,(t-s)^{H+H^{'}-1}\right) s^{H^{'}-\frac{1}{2}}ds\\ &= \frac{\sigma_{H^{'}}\alpha_{H,H^{'}}}{H+H^{'}}\left(r^{H+H^{'}} + t^{H+H^{'}} - (t-r)^{H+H^{'}}\right).\end{aligned}$$

Recalling that $\alpha_{H,H^{'}} = \frac{(H+H^{'})a_{H,H^{'}}}{2\sigma_{H^{'}}}$ completes the proof of the theorem. □

3.1. An alternative expression for the kernel

Using identities (29) and (30), valid for $0 < H^{'}, H < 1$, the kernel $g_{H,H'}(r,t)$, $0 \leq r \leq t$, can be rewritten as

$$
\begin{aligned}
(35) \qquad g_{H,H'}(r,t) = {} & \alpha_{H,H'} \frac{\Gamma\left(H+\frac{1}{2}\right)}{\Gamma\left(1+H-H'\right)} r^{H-\frac{1}{2}} \\
& + \alpha_{H,H'} \frac{\Gamma\left(\frac{3}{2}-H^{'}\right)}{\Gamma\left(2-2H'\right)} \\
& \times t^{H+H^{'}-1} r^{\frac{1}{2}-H^{'}} F\left(1-H-H', \frac{3}{2}-H^{'}, 2-2H^{'}; \frac{r}{t}\right).
\end{aligned}
$$

Remark 4. Since $X_{H,H'}(t) = E(Z_H(t) \mid \mathcal{F}_t^{H'}) = \int_0^t g_{H,H'}(r,t)\, dW_r^{H^{'}}$ holds for any pair $(H, H^{'}) \in (0,1)^2$, setting $H = H^{'}$, gives $X_{H,H'}(t) = Z_{H'}(t)$. Therefore, by uniqueness, $g_{H',H'}$ coincides with the kernel that represents fBm as a stochastic integral on a finite interval (see equations (21), (22) and (23)). This statement could also be verified using special functions identities.

Remark 5. As we have seen in section 2, in the case $H = 1 - H^{'}$, $X_{H,H'}(t)$ is, up to a constant multiple, the fundamental martingale of $Z_{H'}$. This also follows from (35). Recall that the fundamental martingale of a fBm $Z_{H'}$ given by (21) is the process $M_{H'}(t) = \gamma_{H'} \int_0^t s^{\frac{1}{2}-H^{'}}\, dW_s^{H^{'}}$, where $\gamma_{H'}$ is a suitable constant (see [10]). To see that $X_{1-H',H'}$ is the fundamental martingale, up to a constant multiple, it is enough to substitute $H = 1-H^{'}$ in (35) and observe that $F(0, \frac{3}{2}-H^{'}, 2-2H^{'}; \frac{r}{t})$ is a constant.

Remark 6. It is straightforward to see using (35) that the centered Gaussian process $X_{H,H'}$ is H-selfsimilar.

Remark 7. From equation (35) it follows that $X_{H,H'}$ decomposes into a sum of two processes, one of them being a martingale.

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