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CLASSICALLY INTEGRABLE TWO-DIMENSIONAL NON-LINEAR SIGMA MODELS

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Abstract. We give a master equation expressing the zero curvature representation of the equations of motion of a two-dimensional non-linear sigma models. The geometrical properties of this equation are highlighted.

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1. Introduction

This note is a summary and a selection of a longer paper which deals with the issue of integrability in two-dimensional non-linear sigma models [14]. The interest in this subject stems from the fact that, in the past, only few of such theories were known to be integrable. These are the principal chiral model [19], the Wess-Zumino-Witten model and their various modifications. However, recently, more studies have been devoted to this problem and more integrable non-linear sigma models have been discovered [1–4, 7, 10–13, 17, 18]. These theories were found by pure guesses or by brute force. The aim of this short contibution is to provide a systematic method for searching for integrable two-dimensional non-linear sigma models. The result of this work is a 'master equation' whose solutions yields all the, so far, known integrable non-linear sigma models.

2. Zero Curvature Representation of Non-Linear Sigma Models

A two-dimensional non-linear sigma model is an interacting theory for some scalar fields $\varphi^i\,(z\,,\bar{z})$ as described by the action[1]

$$S = \int \mathrm{d}z\mathrm{d}\bar{z}\, Q_{ij}\,(\varphi)\,\partial\varphi^i\bar{\partial}\varphi^j. \tag{1}$$

The metric and the anti-symmetric tensor fields of this theory are defined as

$$g_{ij} = \frac{1}{2}\,(Q_{ij} + Q_{ji})\,, \qquad b_{ij} = \frac{1}{2}\,(Q_{ij} - Q_{ji})\,. \tag{2}$$

We will assume that the metric g_{ij} is invertible and its inverse is denoted g^{ij}. Indices are raised and lowered using this metric. We will also define, respectively, the Christoffel symbols, the torsion and the generalised connection as follows

$$\begin{aligned} \Gamma^k_{ij} &= \frac{1}{2}g^{kl}\,(\partial_i g_{lj} + \partial_j g_{li} - \partial_l g_{ij}) \\ H^k_{ij} &= \frac{1}{2}g^{kl}\,(\partial_l b_{ij} + \partial_j b_{li} + \partial_i b_{jl}) \\ \Omega^k_{ij} &= \Gamma^k_{ij} - H^k_{ij}. \end{aligned} \tag{3}$$

The equations of motion of this theory can be written as

$$\mathcal{E}^l \;\equiv\; \bar{\partial}\partial\varphi^l + \Omega^l_{ij}\partial\varphi^i\bar{\partial}\varphi^j = 0. \tag{4}$$

The derivative $\partial_i = \frac{\partial}{\partial\varphi^i}$ and summation is implied over repeated indices.

Let us now construct a linear system whose consistency conditions are equivalent to these equations of motion (a zero curvature representation). By looking at the form of the equations of motion, this linear system must have the following form

$$\left[\partial + \frac{1}{1+\lambda}\,(K_i - L_i)\,\partial\varphi^i\right]\Psi = 0, \qquad \left[\bar{\partial} + \frac{1}{1-\lambda}\,(K_j + L_j)\,\bar{\partial}\varphi^j\right]\Psi = 0 \tag{5}$$

where the matrices $K_i(\varphi)$ and $L_i(\varphi)$ are functions of the field φ^i but are *independent* of the spectral parameter λ.

The compatibility condition of the linear system (the zero curvature condition) is found by acting with $\bar{\partial}$ and ∂, respectively, on the first equation and second

[1]Here, the two-dimensional coordinates are (τ, σ) with $\partial_0 = \frac{\partial}{\partial\tau}$ and $\partial_1 = \frac{\partial}{\partial\sigma}\cdot$ In the rest of the paper, however, we will use the complex coordinates ($z = \tau + \mathrm{i}\sigma$, $\bar{z} = \tau - \mathrm{i}\sigma$) together with $\partial = \frac{\partial}{\partial z}$ and $\bar{\partial} = \frac{\partial}{\partial\bar{z}}\cdot$

equation of the set (5) and demanding that $\bar{\partial}\partial\Psi = \partial\bar{\partial}\Psi$. This leads to the condition

$$\begin{aligned}\mathcal{F} \equiv\ & \frac{1}{1-\lambda^2}\left\{2L_i\,\partial\bar{\partial}\varphi^i + (\partial_i K_j - \partial_j K_i + \partial_i L_j + \partial_j L_i\right. \\ & \left. +\ [K_i - L_i\,,\, K_j + L_j])\,\partial\varphi^i\bar{\partial}\varphi^j\right\} \\ & + \frac{\lambda}{1-\lambda^2}\left\{2K_i\,\partial\bar{\partial}\varphi^i + (\partial_i K_j + \partial_j K_i + \partial_i L_j - \partial_j L_i)\,\partial\varphi^i\bar{\partial}\varphi^j\right\} = 0.\end{aligned} \tag{6}$$

The non-linear sigma model is integrable if the linear system (5) is compatible only when the equations of motion of the non-linear sigma model are obeyed. This means that

$$\mathcal{F}\ = \mathcal{E}^i\mu_i = 0 \tag{7}$$

for some linearly independent matrices $\mu_i\,(\varphi)$. By comparing the terms involving $\partial\bar{\partial}\varphi^i$ and $\partial\varphi^i\bar{\partial}\varphi^j$ on both sides of (7) we deduce that

$$\mu_i = \frac{2}{1-\lambda^2}\,(L_i + \lambda\,K_i) \tag{8}$$

and we must have

$$\begin{aligned}&\frac{1}{1-\lambda^2}\,(\partial_i K_j - \partial_j K_i + \partial_i L_j + \partial_j L_i + [K_i - L_i\,,\, K_j + L_j]) \\ &\qquad + \frac{\lambda}{1-\lambda^2}\,(\partial_i K_j + \partial_j K_i + \partial_i L_j - \partial_j L_i) = \frac{2}{1-\lambda^2}\Omega^l_{ij}\,(L_l + \lambda\,K_l)\,.\end{aligned} \tag{9}$$

This relation must hold *for all values* of the spectral parameter λ. Therefore one gets the equations

$$\begin{aligned}\partial_i K_j - \partial_j K_i + \partial_i L_j + \partial_j L_i + [K_i - L_i\,,\, K_j + L_j] &= 2\Omega^l_{ij}L_l \\ \partial_i K_j + \partial_j K_i + \partial_i L_j - \partial_j L_i &= 2\Omega^l_{ij}K_l.\end{aligned} \tag{10}$$

We name this set of equations the 'master equation' behind the integrability of two-dimensional non-linear sigma models. In this set of equations the unkowns are the matrices K_i and L_i as well as the Christoffel symbols Γ^i_{jk} and the torsion H_{ijk}. We will provide below some known solutions to these equations.

2.1. Symmetries

We would like now to spell out the geometry and the special features behind the non-linear sigma models admitting the Lax representation given in (5). In terms of the tensor $Q_{ij} = g_{ij} + b_{ij}$ of the non-linear sigma model, the two sets of equations in (10) can be cast in the form

$$\begin{aligned}\mathcal{L}_L\, Q_{ij} \;=& -\left[\partial_i\left(K_j + b_{jl}L^l\right) - \partial_j\left(K_i + b_{il}L^l\right)\right] \\ & - \left[K_i - L_i\,,\, K_j + L_j\right]\end{aligned} \tag{11}$$

$$\mathcal{L}_K\, Q_{ij} \;= -\left[\partial_i\left(L_j + b_{jl}K^l\right) - \partial_j\left(L_i + b_{il}K^l\right)\right] \tag{12}$$

where $K^i = g^{ij}K_j$ and $L^i = g^{ij}L_j$ and the Lie derivative with respect to K^i is

$$\mathcal{L}_K\, Q_{ij} = K^l\partial_l Q_{ij} + Q_{lj}\partial_i K^l + Q_{il}\partial_j K^l. \tag{13}$$

A similar expression holds for $\mathcal{L}_L\, Q_{ij}$ with L^l replacing K^l.

The relation in (12) says that the non-linear sigma model (1) possesses the isometry symmetry [8, 9]

$$\delta\varphi^i = \alpha^{AB}K^i_{AB} \tag{14}$$

where K^i_{AB} are the entries of the matrix K^i and α^{AB} are constant infinitesimal parameters. This is a major requirement for a non-linear sigma model to accept a Lax pair representation of the form (5).

2.2. Solutions

Let us now define the two currents

$$J = (K_i - L_i)\,\partial\varphi^i, \qquad \bar{J} = (K_j + L_j)\,\bar{\partial}\varphi^j. \tag{15}$$

Using the set of equations in (10), these two currents satisfy

$$\partial\bar{J} + \bar{\partial}J = 2K_i\,\mathcal{E}^i, \qquad \partial\bar{J} - \bar{\partial}J + \left[J\,,\,\bar{J}\right] = 2L_i\,\mathcal{E}^i. \tag{16}$$

where $\mathcal{E}^l \;\equiv\; \bar{\partial}\partial\varphi^l + \Omega^l_{ij}\partial\varphi^i\bar{\partial}\varphi^j = 0$ are the equations of motion of the non-linear sigma model.

This last set of equations suggests the study of three different cases:

1) $K_i = 0$ and $L_i \neq 0$

   In this case the two currents J and $\bar{J}$ satisfy the equation $\partial\bar{J} + \bar{\partial}J = 0$ independently of the equations of motion of the non-linear sigma model. Therefore, the two currents J and $\bar{J}$ are topological currents. Furthermore, the equations in (10) reduce to

$$\partial_i L_j - \Gamma^l_{ij}\, L_l = 0, \qquad 2H^l_{ij}\, L_l = \left[L_i\,,\, L_j\right]. \tag{17}$$

   This set has a unique solution given by $L_i = 2\kappa T_i$, where T_i satisfy the Lie algebra $[T_i\,,\,T_j] = f^k_{ij}T_k$ and the integrable non-linear sigma model is

$$S = \int \mathrm{d}z\mathrm{d}\bar{z}\,\left(\eta_{ij} + \kappa\frac{2}{3}\eta_{kl}\, f^l_{ij}\varphi^k\right)\partial\varphi^i\bar{\partial}\varphi^j \tag{18}$$

where κ is a constant and η_{ij} satifies $\eta_{ij} f^j_{kl} + \eta_{kj} f^j_{il} = 0$. The properties of this theory were investigated by Nappi [15].

2) $L_i = 0$ and $K_i \neq 0$
Here the two currents J and $\bar{J}$ are the conserved currents corresponding to the isometry symmetry $\delta\varphi^i = \alpha^{AB} K^i_{AB}$ of the non-linear sigma model and satisfy, independently of the equations of motion, the Bianchi identity $\partial\bar{J} - \bar{\partial}J + \left[J\,,\,\bar{J}\right] = 0$. In this case the set (10) gives

$$\begin{aligned} \partial_i K_j + \partial_j K_i - 2\Gamma^l_{ij}\, K_l &= 0 \\ 2H^l_{ij}\, K_l &= 0 \\ \partial_i K_j - \partial_j K_i + [K_i\,,\, K_j] &= 0. \end{aligned} \tag{19}$$

We immediately see that the last equation admits the solution $K_i = g^{-1}\partial_i g$, for some Lie group element $g(\varphi)$, and the integrable theory is the principal chiral non-linear sigma model

$$S\,(g) = \int \mathrm{d}z \mathrm{d}\bar{z}\, \mathrm{Tr}\left[\left(g^{-1}\partial g\right)\left(g^{-1}\bar{\partial} g\right)\right]. \tag{20}$$

Here $g^{-1}\partial g = K_i \partial\varphi^i$ and $g^{-1}\bar{\partial} g = K_j \bar{\partial}\varphi^j$.

3) $K_i \neq 0$ and $L_i \neq 0$
This case means that the integrable non-linear sigma model must possess the isometry symmetry (14) whose conserved currents are J and $\bar{J}$ (namely, $\partial\bar{J} + \bar{\partial}J = 0$ on shell). Moreover, the field strength $\partial\bar{J} - \bar{\partial}J + \left[J\,,\,\bar{J}\right]$ vanishes only when the equations of motion are obeyed and is no longer a Bianchi identity as in the previous case.

As seen earlier, the first two cases are unique and lead to known integrable non-linear sigma models. Therefore, any new integrable non-linear sigma model (having (5) as a Lax pair) must fit in this third class. The integrable non-linear sigma models found in [1–4, 7, 10–13, 17, 18] enter all in this third category. Further solutions are also found in the longer version of this note [14].

Finally, we should mention that integrable non-linear sigma models are of relevance to string theory as advocated in [5, 6, 16].

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