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On high-dimensional sign tests

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Sign tests are among the most successful procedures in multivariate nonparametric statistics. In this paper, we consider several testing problems in multivariate analysis, directional statistics and multivariate time series analysis, and we show that, under appropriate symmetry assumptions, the fixed-p multivariate sign tests remain valid in the high-dimensional case. Remarkably, our asymptotic results are *universal*, in the sense that, unlike in most previous works in high-dimensional statistics, p may go to infinity in an arbitrary way as n does. We conduct simulations that (i) confirm our asymptotic results, (ii) reveal that, even for relatively large p, chi-square critical values are to be favoured over the (asymptotically equivalent) Gaussian ones and (iii) show that, for testing i.i.d.-ness against serial dependence in the high-dimensional case, Portmanteau sign tests outperform their competitors in terms of validity-robustness.

Keywords: high-dimensional tests; Portmanteau tests; sign tests; universal asymptotics

1. Introduction

Sign procedures that discard the magnitude of the observations to rather focus on their direction from a given location are among the most popular nonparametric techniques. Multivariate sign tests, in particular, have been extensively studied in the last decades. Multivariate location was considered in Randles [33,34], Möttönen and Oja [26] and Hallin and Paindaveine [15], whereas problems on (normalized) covariance or scatter matrices were considered in Tyler [41], Dümbgen [10], Hallin and Paindaveine [16] and Hallin, Paindaveine and Verdebout [17]. Multivariate sign tests were also developed, for example, for the problem of testing i.i.d.-ness against serial dependence (see Paindaveine [30]), or for testing for multivariate independence (see Taskinen, Kankainen and Oja [39] and Taskinen, Oja and Randles [40]). Most references above actually focus on so-called *spatial sign tests*, that is, on tests that are based on (a possibly standardized version of) the signs $\mathbf{U}_i = \mathbf{X}_i/\|\mathbf{X}_i\|$, $i = 1, \ldots, n$, obtained by projecting the p-variate observations $\mathbf{X}_i$, $i = 1, \ldots, n$ on the unit sphere of $\mathbb{R}^p$. In the sequel, sign tests will refer to spatial sign tests.

Multivariate sign tests enjoy many desirable properties. First, they are robust, since they do not require stringent parametric assumptions, nor any moment conditions. The above projection on the unit sphere also guarantees that robustness holds with respect to possible outliers (observations with large magnitudes). Second, for fixed p, sign tests further enjoy uniformly high asymptotic efficiency unless p is small: for p-dimensional location and serial problems, the *lower bound* of the asymptotic relative efficiencies of sign tests with respect to their classical

Gaussian competitors is $((p-1)/p)^2$, while for p-dimensional problems involving normalized scatter matrices (testing for sphericity, PCA, etc.), this lower bound is $(p/(p+2))^2$. This shows that, for moderate-to-large p, using sign tests instead of classical Gaussian tests may barely have any cost in terms of asymptotic efficiency. Even better, in all problems above, there are no finite upper bounds, so that the asymptotic efficiency gain of using sign tests may be arbitrarily large. These remarkable asymptotic efficiency properties of sign tests that may be puzzling at first sight are actually in line with the fact that, as $p \to \infty$, the signs $\mathbf{U}_i$, $i = 1, \dots, n$ asymptotically contain all relevant information since data points concentrate more and more on a common sphere (see, e.g., Hall, Marron and Neeman [14]).

The asymptotic results in the previous paragraph, however, relate to the fixed-p large-n setup, hence are not directly interpretable in an (n, p)-asymptotics framework. In this paper, we therefore study the asymptotic null behaviour of several multivariate sign tests in the high-dimensional setup. Actually we show that the classical sign tests, based on their usual fixed-p asymptotic chi-square critical values, remain valid in the high-dimensional case. In this sense, sign tests are robust to high-dimensionality (see Section 2). Beyond validating for the first time the use of several sign tests in the high-dimensional setup, our results put the emphasis on an interesting robustness property of sign tests, namely the fact that they allow for *universal* (n, p)-asymptotics, in the sense that the respective null asymptotic distributions hold whenever $\min(n, p) \to \infty$. In contrast, (n, p)-asymptotic results in the literature usually restrict in a stringent way how p may go to infinity as a function of n – typically, it is imposed that $p/n \to c$ for some c belonging to a given convex set $C \subset [0, \infty)$ (most often, $C = [0, 1)$ or $C = (1, +\infty)$). Some asymptotic investigations cover all (n, p)-"regimes", but different regimes provide different asymptotic distributions, or lead to different test statistics, which jeopardizes practical implementation. To the best of our knowledge, thus, sign tests are the only tests that can be applied without making (i) strong restrictions on (n, p) and (ii) severe distributional or moment assumptions.

A huge amount of research has been dedicated in the last decade to high-dimensional hypothesis testing. Location problems have been investigated in, for example, Srivastava and Fujikoshi [37], and Srivastava and Kubokawa [38] (see also the references therein), while numerous papers have considered problems related to covariance or scatter matrices; see, among many others, Ledoit and Wolf [20], Onatski, Moreira and Hallin [29] and Jiang and Yang [18]. In this paper, we study the (n, p)-asymptotic null distribution of sign tests for various problems.

First, we tackle problems related with high-dimensional directional data, which are more and more common, for example, in magnetic resonance (Dryden [8]) or gene-expression (Banerjee *et al.* [1]). In Section 2.1, we provide the (n, p)-asymptotic null distribution of the Rayleigh [35] test statistic that addresses the problem of testing uniformity on the unit sphere. High-dimensional tests for this problem were recently proposed in Cai and Jiang [5] and Cai, Fan and Jiang [4]. We show that the Rayleigh test, unlike the latter competitors, is robust to high-dimensionality in a universal way (in the sense explained above) and can therefore be used for any (n, p)-regime. In the same section, we treat another important problem that is standard in directional statistics, namely the spherical location problem in the context of rotationally symmetric distributions (see Mardia and Jupp [25] and Ley *et al.* [21]), and show that the Paindaveine and Verdebout [31] sign test is also robust to high-dimensionality.

Then in Section 2.2, we consider the problem of testing for i.i.d.-ness against serial dependence in the general multivariate case, which is arguably the most fundamental goodness-of-fit testing problem in multivariate time series analysis. To the best of our knowledge, we provide here the first high-dimensional result for this problem by deriving, under appropriate symmetry assumptions, the universal (n, p)-asymptotic null distribution of the Paindaveine [30] Portmanteau-type sign test. Finally, in Section 2.3, we tackle the problem of testing for multivariate independence and the problem of testing for sphericity about a specified center.

In Section 3, we conduct different simulations. In Section 3.1, a Monte-Carlo study confirms that, when properly standardized, sign test statistics are (n, p)-asymptotically normal under the null. Yet, as we also show through simulations in Section 3.2, the traditional chi-square critical values better approximate the exact ones than their (actually, liberal) Gaussian counterparts, even for relatively large p, hence should be favoured for practical purposes. Finally, in Section 3.3, we show that, unlike its classical competitors, the Portmanteau sign test from Paindaveine [30] has null rejection frequencies that are robust to high-dimensionality and heavy tails. We end the paper with the Appendix that contains proofs of technical results. For the sake of completeness, the proofs for the independence and sphericity problems are provided in the supplemental article Paindaveine and Verdebout [32] that also reports some additional simulation results.

2. High-dimensional sign tests

Consider some generic testing problem involving the null hypothesis $\mathcal{H}_0$ and the alternative hypothesis $\mathcal{H}_1$, to be addressed on the basis of p-variate observations $\mathbf{X}_i$, $i = 1, \ldots, n$. Let $Q_p^{(n)}$ be a test statistic for this problem, that, for fixed p, is asymptotically $\chi^2_{d_p}$ under the null, where $d_p \to \infty$ as $p \to \infty$ (all sign test statistics considered in this paper meet this property). Then the corresponding fixed-p test, $\phi_p^{(n)}$ say, rejects the null at asymptotic level α whenever

$$Q_p^{(n)} > \chi^2_{d_p, 1-\alpha},$$

where $\chi^2_{d,1-\alpha}$ stands for the upper α-quantile of the chi-square distribution with d degrees of freedom. Now, it is well known that, if Z_d is chi-square with d degrees of freedom, then $(Z_d - d)/\sqrt{2d}$ weakly converges to the standard normal distribution as $d \to \infty$. Since we assumed that $d_p \to \infty$ as $p \to \infty$, it may then be *expected* that the test $\phi_{\mathcal{N},p}^{(n)}$ that rejects the null whenever

$$Q_{\mathcal{N},p}^{(n)} = \frac{Q_p^{(n)} - d_p}{\sqrt{2d_p}} > \Phi^{-1}(1 - \alpha)$$

(throughout, Φ denotes the c.d.f. of the standard normal distribution) has asymptotic level α under the null when both n and p converge to infinity. Clearly, the larger p, the closer the tests $\phi_p^{(n)}$ and $\phi_{\mathcal{N},p}^{(n)}$, which may then be considered equivalent in the (n, p)-asymptotic framework.

Of course, establishing that the sequence of tests $\phi_{\mathcal{N},p}^{(n)}$ – hence also the sequence of tests $\phi_p^{(n)}$ – is valid in the high-dimensional setup, that is, has asymptotic level α under the null when both n and p go to infinity, requires a formal proof. All the more so that the result does not always hold true. Some test statistics indeed need to be appropriately corrected to obtain valid (n, p)-asymptotic results, while some others do not (see, e.g., Ledoit and Wolf [20]). The test statistics that do not need be corrected can be called *high-dimensional (HD-)robust*. In the setup above, the test statistic $Q_p^{(n)}$ is thus HD-robust if, under the null,

$$\frac{Q_p^{(n)} - d_p}{\sqrt{2d_p}}$$

converges weakly to a standard Gaussian random variable as n and p go to infinity.

The main goal of this paper is to show that for the problems enumerated in the Introduction, the traditional sign test statistics are, under appropriate symmetry conditions, *universally* HD-robust, in the sense that HD-robustness is achieved without imposing any constraint on the way p goes to infinity as n does.

2.1. Testing uniformity on the unit sphere

Let the random p-vectors $\mathbf{U}_1, \ldots, \mathbf{U}_n$ be mutually independent and identically distributed, with a common distribution that is supported on the unit sphere $\mathcal{S}^{p-1} = \{\mathbf{x} \in \mathbb{R}^p\colon\ \|\mathbf{x}\| = \sqrt{\mathbf{x}'\mathbf{x}} = 1\}$ of $\mathbb{R}^p$. An important problem in directional statistics consists in testing the null hypothesis $\mathcal{H}_0$ that this common distribution is the uniform on $\mathcal{S}^{p-1}$. As already mentioned in the Introduction, this problem has been recently considered in the high-dimensional case in Cai and Jiang [5] and Cai, Fan and Jiang [4]. Arguably, the most classical test for this problem is the Rayleigh [35] test that rejects the null for large values of

$$R_p^{(n)} = \frac{p}{n} \sum_{i,j=1}^{n} \mathbf{U}_i'\mathbf{U}_j; \tag{2.1}$$

see, for example, Mardia and Jupp [25], Section 6.3. Under $\mathcal{H}_0$, $\mathbf{U}_i$ has mean zero and covariance matrix $\frac{1}{p}\mathbf{I}_p$ (see Lemma A.2), where $\mathbf{I}_\ell$ denotes the $\ell \times \ell$ identity matrix, so that the multivariate CLT readily implies that, for any fixed p, the asymptotic null distribution of $R_p^{(n)}$ is χ^2_p. Therefore, Rayleigh's test rejects the null, at asymptotic level α, whenever $R_p^{(n)} > \chi^2_{p,1-\alpha}$.

Obviously, applying Rayleigh's test of uniformity is only possible when the sample size n is large enough, compared to p, to make the CLT approximation reasonable. We now derive the asymptotic distribution of Rayleigh's test statistic $R_p^{(n)}$ in the high-dimensional case when both $p = p_n$ and n go to infinity. As announced in the Introduction, our approach is *universal*, in the sense that, unlike in most works on high-dimensional statistics, p_n may go to infinity in a totally arbitrary way (the only restriction being that both p_n and n go to infinity). Note in particular that the asymptotic null distribution of the tests of uniformity proposed in Cai, Fan and Jiang [4], which are based on statistics of the form $\min_{i,j} \arccos(\mathbf{U}_i'\mathbf{U}_j)$, depends on the (n, p)-regime considered.

Basically, we will show that the (n, p)-asymptotic distribution of (the standardized version of) $R_p^{(n)}$ is universally standard normal. To do so, rewrite Rayleigh's statistic as

$$R_p^{(n)} = \frac{p}{n}\left(n + \sum_{1\le i\ne j\le n}^{n} \mathbf{U}_i'\mathbf{U}_j\right) = p + \frac{2p}{n} \sum_{1\le i<j\le n} \mathbf{U}_i'\mathbf{U}_j, \tag{2.2}$$

and consider the standardized statistic

$$R^{(n)}_{\mathcal{N},p} = \frac{R^{(n)}_p - p}{\sqrt{2p}} = \frac{\sqrt{2p}}{n} \sum_{1\le i<j\le n} \mathbf{U}_i'\mathbf{U}_j. \tag{2.3}$$

As we recalled above, the fixed-p asymptotic null distribution of $R^{(n)}_p$ is χ^2_p, hence has mean p and variance $2p$, which makes the standardization in (2.3) most natural. The main result of this section is the following.

Theorem 2.1. *Let p_n be an arbitrary sequence of positive integers converging to $+\infty$ as $n \to \infty$. Assume that $\mathbf{U}_{ni}$, $i = 1, \dots, n$, $n = 1, 2, \dots,$ is a triangular array such that for any n, the random p_n-vectors $\mathbf{U}_{ni}$, $i = 1, \dots, n$ are i.i.d. uniform on $\mathcal{S}^{p_n-1}$. Then*

$$R^{(n)}_{\mathcal{N},p} = \frac{R^{(n)}_p - p_n}{\sqrt{2p_n}} = \frac{\sqrt{2p_n}}{n} \sum_{1\le i<j\le n} \mathbf{U}_{ni}'\mathbf{U}_{nj} \tag{2.4}$$

converges in distribution to the standard normal as $n \to \infty$.

See the Appendix for the proof. As we now explain, we are also able to deal with another famous testing problem in directional statistics, namely the spherical location problem. The relevant distributional setup for this problem is the class of so-called *rotationally symmetric distributions* on $\mathcal{S}^{p-1}$; see Mardia and Jupp [25] or Ley *et al.* [21] for details. In the absolutely continuous case, this corresponds to the semiparametric class of densities (with respect to the surface area measure on $\mathcal{S}^{p-1}$) of the form

$$\mathbf{u} \mapsto c_{p,f}\, f\big(\mathbf{u}'\boldsymbol{\theta}\big), \qquad \mathbf{u} \in \mathcal{S}^{p-1}, \tag{2.5}$$

where $\boldsymbol{\theta} \in \mathcal{S}^{p-1}$ is a location parameter, $c_{p,f}$ (> 0) is a normalization constant and $f : [-1, 1] \to \mathbb{R}^+$ is some monotone increasing function. The spherical location problem consists in testing that $\boldsymbol{\theta}$ is equal to some given vector $\boldsymbol{\theta}_0 \in \mathcal{S}^{p-1}$, on the basis of a random sample $\mathbf{U}_1, \dots, \mathbf{U}_n$ from (2.5). Signed-rank tests for this problem were recently proposed in Paindaveine and Verdebout [31] (while Ley *et al.* [21] developed the corresponding estimators of $\boldsymbol{\theta}$). In particular, the sign-based version of these tests rejects the null hypothesis $\mathcal{H}_0$: $\boldsymbol{\theta} = \boldsymbol{\theta}_0$ whenever

$$R^{(n)}_p(\boldsymbol{\theta}_0) = \frac{p-1}{n} \sum_{i,j=1}^{n} \mathbf{U}_i'(\boldsymbol{\theta}_0)\mathbf{U}_j(\boldsymbol{\theta}_0) > \chi^2_{p-1,1-\alpha},$$

where

$$\mathbf{U}_i(\boldsymbol{\theta}_0) = \frac{(\mathbf{I}_p - \boldsymbol{\theta}_0\boldsymbol{\theta}_0')\mathbf{U}_i}{\|(\mathbf{I}_p - \boldsymbol{\theta}_0\boldsymbol{\theta}_0')\mathbf{U}_i\|}, \qquad i = 1, \dots, n$$

is the multivariate sign of the projection of $\mathbf{U}_i$ onto the tangent space to $\mathcal{S}^{p-1}$ at $\boldsymbol{\theta}_0$ (note that since the $\mathbf{U}_i$'s have an absolutely continuous distribution on the sphere, the corresponding $\mathbf{U}_i(\boldsymbol{\theta}_0)$'s are well-defined almost surely).

Irrespective of the underlying infinite-dimensional nuisance f, the $\mathbf{U}_i(\boldsymbol{\theta}_0)$'s, under the null, are i.i.d. with a common distribution that is uniform on the hypersphere $\mathcal{S}^{p-2}_{\boldsymbol{\theta}_0} = \{\mathbf{u} \in \mathcal{S}^{p-1}\colon \mathbf{u}'\boldsymbol{\theta}_0 = 0\}$. The following result then follows from Theorem 2.1.

Theorem 2.2. *Let p_n be an arbitrary sequence of positive integers converging to $+\infty$ as $n \to \infty$. Assume that $\mathbf{U}_{ni}$, $i = 1, \dots, n$, $n = 1, 2, \dots,$ is a triangular array such that for any n, the random p_n-vectors $\mathbf{U}_{ni}$, $i = 1, \dots, n$, are i.i.d. with a common rotationally symmetric density over $\mathcal{S}^{p_n-1}$. Then, letting $\mathbf{U}_{ni}(\boldsymbol{\theta}_0) = (\mathbf{I}_p - \boldsymbol{\theta}_0\boldsymbol{\theta}_0')\mathbf{U}_{ni}/\|(\mathbf{I}_p - \boldsymbol{\theta}_0\boldsymbol{\theta}_0')\mathbf{U}_{ni}\|$, we have that*

$$R^{(n)}_{\mathcal{N},p}(\boldsymbol{\theta}_0) = \frac{R^{(n)}_p(\boldsymbol{\theta}_0) - (p_n - 1)}{\sqrt{2(p_n-1)}} = \frac{\sqrt{2(p_n-1)}}{n} \sum_{1 \le i < j \le n} \mathbf{U}'_{ni}(\boldsymbol{\theta}_0)\mathbf{U}_{nj}(\boldsymbol{\theta}_0)$$

converges in distribution to the standard normal as $n \to \infty$.

Of course, the resulting (universal) test of spherical location rejects the null $\mathcal{H}_0\colon \boldsymbol{\theta} = \boldsymbol{\theta}_0$ at asymptotic level α whenever $R^{(n)}_{\mathcal{N},p}(\boldsymbol{\theta}_0)$ exceeds the upper α-quantile $\Phi^{-1}(1-\alpha)$ of the standard Gaussian distribution. At the same level, the universal test of uniformity on $\mathcal{S}^{p-1}$ rejects the null when $R^{(n)}_{\mathcal{N},p}$ exceeds $\Phi^{-1}(1-\alpha)$. As discussed in the beginning of Section 2, one may alternatively perform the original fixed-p chi-square tests, since chi-square and Gaussian critical values are asymptotically equivalent as $p \to \infty$. The objective of Section 3.2 is to compare both types of critical values through simulations.

We end this section by stressing that Theorem 2.1 is also relevant in the context of high-dimensional location testing, in a framework where the p-variate observations $\mathbf{X}_i$, $i = 1, \dots, n$ have *independent spherical directions about* $\boldsymbol{\theta}$ ($\in \mathbb{R}^p$). Throughout the paper, this will mean that $(\mathbf{X}_1', \dots, \mathbf{X}_n')'$ is equal in distribution to $(\boldsymbol{\theta}' + R_1\mathbf{U}_1', \dots, \boldsymbol{\theta}' + R_n\mathbf{U}_n')'$, where (i) the $\mathbf{U}_i$'s are i.i.d. uniform over $\mathcal{S}^{p-1}$ and (ii) the R_i's are arbitrary random variables such that $\mathrm{P}[R_i = 0] = 0$ for all i. In particular, the R_i's may fail to be mutually independent and/or may be dependent of the $\mathbf{U}_i$'s. Also, parallel to the *generalized spherical distributions* from Frahm [11] and Frahm and Jaekel [12,13], the R_i's do not need to be nonnegative. In the sequel, instead of "*the p-variate random vectors $\mathbf{X}_i$, $i = 1, \dots, n$ have independent spherical directions about the origin of* $\mathbb{R}^p$", we will simply write "*the p-variate random vectors $\mathbf{X}_i$, $i = 1, \dots, n$ have independent spherical directions*".

Assume then that $\mathbf{X}_{ni}$, $i = 1, \dots, n$, $n = 1, 2, \dots$ is a triangular array such that for any n, the random p_n-vectors $\mathbf{X}_{ni}$, $i = 1, \dots, n$ have independent spherical directions about $\boldsymbol{\theta} \in \mathbb{R}^{p_n}$, and consider the (location) testing problem associated with the null $\mathcal{H}_0\colon \boldsymbol{\theta} = \mathbf{0}$ and the alternative $\mathcal{H}_1\colon \boldsymbol{\theta} \neq \mathbf{0}$. It readily follows from Theorem 2.1 that the test rejecting the null whenever

$$\frac{R^{(n)}_p(\mathbf{X}_1/\|\mathbf{X}_1\|, \dots, \mathbf{X}_n/\|\mathbf{X}_n\|) - p_n}{\sqrt{2p_n}} > \Phi^{-1}(1-\alpha)$$

has (n, p)-asymptotic level α, irrespective of the way p_n goes to infinity with n. This settles the high-dimensional null distribution of the so-called *spatial sign location test*; see, for example, Oja [28], Chapter 6.

One might consider that the assumption that the observations have independent spherical directions is restrictive. We point out, however, that it is less stringent than the assumptions that observations are mutually independent with a common spherically symmetric distribution, and that testing for sphericity is one of the most treated problems in high-dimensional hypothesis testing (which would not be the case if sphericity never holds). If the null of sphericity is not rejected, then practitioners may resort to the location tests above (and to the tests we propose in the next sections). Now, if observations fail to have a spherically symmetric distribution (more generally, if they do not have independent spherical directions), performing marginal standardization may bring us closer to spherical symmetry or independent spherical directions. A thorough investigation of the impact of such a "whitening" step on the asymptotic null behaviour of the tests we consider is however beyond the scope of this paper, and we therefore leave this for future research.

2.2. Testing for i.i.d.-ness against serial dependence

In univariate time series analysis, the daily practice for location models such as ARMA or ARIMA is deeply rooted in the so-called *Box and Jenkins methodology*; see, for example, Brockwell and Davis [3] for details. An important role in this methodology is played by diagnostic checking procedures, such as *Portmanteau tests*, that address the null that the residuals of the model at hand are not serially correlated (one often speaks of the null hypothesis of *randomness*). These tests typically reject the null for large values of $\sum_{h=1}^{H}(n-h)(r(h))^2$, where $r(h)$ denotes the lag-h sample autocorrelation in the residual series. If autocorrelations are computed in the series of residual signs rather than in the series of residuals themselves, one obtains the "generalized runs tests" of Dufour, Hallin and Mizera [9], that are robust to heteroscedasticity (for $H=1$, these tests reduce to the celebrated *runs* test of randomness, which justifies the terminology).

Diagnostic checking also belongs to daily practice of multivariate time series analysis, where Portmanteau tests are based on sums of squared norms of lag-h autocorrelation matrices; see, for example, Lütkepohl [24]. The corresponding sign tests, that can be seen as multivariate (generalized) runs tests, were developed in Paindaveine [30]. To the best of our knowledge, Portmanteau-type tests have not yet been studied in the high-dimensional case.

Let $\mathbf{X}_1,\dots,\mathbf{X}_n$ be random p-vectors and consider the problem of testing the null hypothesis of randomness (white noise) versus the alternative of serial dependence. This problem can be addressed by considering the sign-based autocorrelation matrices

$$\mathbf{r}(h)=\frac{p}{n-h}\sum_{t=h+1}^{n}\mathbf{U}_t\mathbf{U}_{t-h}',\qquad h=1,\dots,H,$$

with $\mathbf{U}_t=\mathbf{X}_t/\|\mathbf{X}_t\|$, $t=1,\dots,n$. More precisely, the resulting (fixed-p) test is the Paindaveine [30] test, that rejects the null of randomness at asymptotic level α whenever

$$T_p^{(n)}=\sum_{h=1}^{H}(n-h)\|\mathbf{r}(h)\|_{\mathrm{Fr}}^2>\chi^2_{Hp^2,1-\alpha},\tag{2.6}$$

where $\|\mathbf{A}\|_{\mathrm{Fr}} = (\mathrm{Trace}(\mathbf{AA}'))^{1/2}$ is the Frobenius norm of $\mathbf{A}$. This test is a natural sign-based multivariate extension of the univariate Portmanteau tests described above.

Following the same methodology as in Section 2.1, we will study the universal (n, p)-asymptotic null behaviour of a standardized version of $T_p^{(n)}$, under the assumption that the observations have independent spherical directions. Since

$$\begin{aligned}(n-h)\|\mathbf{r}(h)\|_{\mathrm{Fr}}^2 &= \frac{p^2}{n-h}\sum_{s,t=h+1}^{n}\big(\mathbf{U}_{s-h}'\mathbf{U}_{t-h}\big)\big(\mathbf{U}_s'\mathbf{U}_t\big)\\ &= p^2 + \frac{2p^2}{n-h}\sum_{h+1\le s<t\le n}\big(\mathbf{U}_{s-h}'\mathbf{U}_{t-h}\big)\big(\mathbf{U}_s'\mathbf{U}_t\big),\end{aligned}$$

we will consider the standardization of $T_p^{(n)}$ given by

$$T_{\mathcal{N},p}^{(n)} = \frac{T_p^{(n)} - Hp^2}{\sqrt{2Hp^2}} = \frac{\sqrt{2p^2}}{\sqrt{H}}\sum_{h=1}^{H}\frac{1}{n-h}\sum_{h+1\le s<t\le n}\big(\mathbf{U}_{s-h}'\mathbf{U}_{t-h}\big)\big(\mathbf{U}_s'\mathbf{U}_t\big).$$

Again, this standardization is in line with the fixed-p/large-n (chi-square) null asymptotic distribution of $T_p^{(n)}$ in (2.6).

Theorem 2.3. *Let p_n be an arbitrary sequence of positive integers converging to $+\infty$ as $n \to \infty$. Assume that $\mathbf{X}_{nt}$, $t = 1, \dots, n$, $n = 1, 2, \dots$, is a triangular array such that for any n, the random p_n-vectors $\mathbf{X}_{nt}$, $t = 1, \dots, n$ have independent spherical directions. Then, letting $\mathbf{U}_{nt} = \mathbf{X}_{nt}/\|\mathbf{X}_{nt}\|$ for any n, t, we have that*

$$T_{\mathcal{N},p}^{(n)} = \frac{T_p^{(n)} - Hp_n^2}{\sqrt{2Hp_n^2}} = \frac{\sqrt{2p_n^2}}{\sqrt{H}}\sum_{h=1}^{H}\frac{1}{n-h}\sum_{h+1\le s<t\le n}\big(\mathbf{U}_{n,s-h}'\mathbf{U}_{n,t-h}\big)\big(\mathbf{U}_{ns}'\mathbf{U}_{nt}\big), \tag{2.7}$$

converges in distribution to the standard normal as $n \to \infty$.

As a direct consequence of Theorem 2.3 above, the Paindaveine [30] test statistic is universally HD-robust in the sense described in the beginning of Section 2. As we will show through simulations in Section 3.3, this is not the case for its classical competitors.

2.3. Testing for multivariate independence, testing for sphericity

Consider now the problem of testing that the p-variate marginal $\mathbf{X}$ and q-variate marginal $\mathbf{Y}$ of the random vector $(\mathbf{X}', \mathbf{Y}')'$ are independent, on the basis of a random sample $(\mathbf{X}_1', \mathbf{Y}_1')', \dots, (\mathbf{X}_n', \mathbf{Y}_n')'$. A sign test for this problem was introduced in Taskinen, Kankainen and Oja [39]. The spherical version of this test is based on sign covariance matrices of the form

$$\mathbf{C}_n = \frac{1}{n}\sum_{i=1}^{n}\mathbf{U}_i\mathbf{V}_i',$$

where $\mathbf{U}_i := \mathbf{X}_i/\|\mathbf{X}_i\|$ and $\mathbf{V}_i := \mathbf{Y}_i/\|\mathbf{Y}_i\|$ $i = 1, \ldots, n$, are the multivariate signs associated with $\mathbf{X}_i$ and $\mathbf{Y}_i$, respectively. More precisely, for fixed p and q, the resulting test rejects the null of multivariate independence at asymptotic level α whenever

$$I_{p,q}^{(n)} = npq\|\mathbf{C}_n\|_{\mathrm{Fr}}^2 = \frac{pq}{n} \sum_{i,j=1}^{n} \big(\mathbf{U}_i'\mathbf{U}_j\big)\big(\mathbf{V}_i'\mathbf{V}_j\big) > \chi^2_{pq,1-\alpha}. \tag{2.8}$$

Below, we obtain the universal asymptotic distribution of a standardized version of $I_{p,q}^{(n)}$, when the $\mathbf{X}_i$'s and the $\mathbf{Y}_i$'s have independent spherical directions.

Adopting the same approach as in the previous sections, decompose the test statistic $I_{p,q}^{(n)}$ into

$$I_{p,q}^{(n)} = pq + \frac{2pq}{n} \sum_{1\le i<j\le n} \big(\mathbf{U}_i'\mathbf{U}_j\big)\big(\mathbf{V}_i'\mathbf{V}_j\big),$$

and consider the standardized statistic

$$I_{\mathcal{N},p,q}^{(n)} = \frac{I_{p,q}^{(n)} - pq}{\sqrt{2pq}} = \frac{\sqrt{2pq}}{n} \sum_{1\le i<j\le n} \big(\mathbf{U}_i'\mathbf{U}_j\big)\big(\mathbf{V}_i'\mathbf{V}_j\big). \tag{2.9}$$

We then have the following universal (n, p)-asymptotic normality result.

Theorem 2.4. *Let p_n and q_n be arbitrary sequences of positive integers such that* $\max(p_n, q_n)$ *converges to $+\infty$ as $n \to \infty$. Assume that $(\mathbf{X}_{ni}', \mathbf{Y}_{ni}')'$, $i = 1, \ldots, n$, $n = 1, 2, \ldots$, is a triangular array such that* (i) *for any n, the p_n-variate random vectors $\mathbf{X}_{ni}$, $i = 1, \ldots, n$ and q_n-variate marginals $\mathbf{Y}_{ni}$, $i = 1, \ldots, n$ have independent spherical directions, and such that* (ii) *$\mathbf{U}_{ni} = \mathbf{X}_{ni}/\|\mathbf{X}_{ni}\|$ and $\mathbf{V}_{ni} = \mathbf{Y}_{ni}/\|\mathbf{Y}_{ni}\|$ are independent for all i, n. Then*

$$I_{\mathcal{N},p,q}^{(n)} = \frac{I_{p,q}^{(n)} - p_nq_n}{\sqrt{2p_nq_n}} = \frac{\sqrt{2p_nq_n}}{n} \sum_{1\le i<j\le n} \mathbf{U}_{ni}'\mathbf{U}_{nj}\mathbf{V}_{ni}'\mathbf{V}_{nj} \tag{2.10}$$

converges in distribution to the standard normal as $n \to \infty$.

For the sake of completeness, the result is proved in the supplemental article Paindaveine and Verdebout [32]. Note that asymptotic normality is obtained even when only one of the dimensions p_n, q_n goes to infinity. Testing for multivariate independence is intimately related to testing block-diagonality of covariance matrices (the sign test in (2.8) rejects the null when the off-diagonal blocks of an empirical sign-based covariance matrix is too large, in Frobenius norm).

Finally, another classical problem in multivariate analysis that is linked to covariance matrices (in this case, the null of interest if that the covariance matrix is proportional to the identity matrix) is the problem of testing for sphericity. Since the seminal paper Ledoit and Wolf [20], this problem – that consists in testing that the common distribution of i.i.d. random p-vectors $\mathbf{X}_1, \ldots, \mathbf{X}_n$ is spherically symmetric – has been treated in many papers on high-dimensional inference; see, for example, Chen, Zhang and Zhong [6], Jiang and Yang [18] and Zou *et al.* [42]. When testing for sphericity about a specified center (without loss of generality, about the origin of $\mathbb{R}^p$), the natural fixed-p sign test of sphericity rejects the null at asymptotic level α whenever

$$S_p^{(n)} = \frac{p(p+2)}{2n} \sum_{i,j=1}^{n} \left(\left(\mathbf{U}_i' \mathbf{U}_j \right)^2 - \frac{1}{p} \right) > \chi^2_{d_p, 1-\alpha}, \tag{2.11}$$

with $\mathbf{U}_i = \mathbf{X}_i / \|\mathbf{X}_i\|$, $i = 1, \dots, n$, and $d_p = (p-1)(p+2)/2$; see Hallin and Paindaveine [16] and Sirkiä *et al.* [36]. Using the methodology proposed above, it can be shown that, under the null, the universal (n, p)-asymptotic distribution of a standardized version of $S_p^{(n)}$ is standard normal under extremely mild assumptions. More precisely, we have the following result (see the supplemental article Paindaveine and Verdebout [32] for a proof).

Theorem 2.5. *Let p_n be an arbitrary sequence of positive integers converging to $+\infty$ as $n \to \infty$. Assume that $\mathbf{X}_{ni}$, $i = 1, \dots, n$, $n = 1, 2, \dots,$ is a triangular array such that for any n, the random p_n-vectors $\mathbf{X}_{ni}$, $i = 1, \dots, n$ have independent spherical directions. Then, letting $\mathbf{U}_{ni} = \mathbf{X}_{ni} / \|\mathbf{X}_{ni}\|$ for any n, i, we have that*

$$S_{\mathcal{N},p}^{(n)} = \frac{S_p^{(n)} - \ell(p_n)}{\sqrt{2\ell(p_n)}} = \frac{p_n \sqrt{p_n + 2}}{n \sqrt{p_n - 1}} \sum_{1 \le i < j \le n} \left(\left(\mathbf{U}_{ni}' \mathbf{U}_{nj} \right)^2 - \frac{1}{p_n} \right) \tag{2.12}$$

converges in distribution to the standard normal as $n \to \infty$.

Performing the test in (2.11) on centered observations $\mathbf{X}_i - \hat{\boldsymbol{\theta}}$, $i = 1, \dots, n$ of course provides a test of sphericity about an unspecified center $\boldsymbol{\theta}$. Recently, Zou *et al.* [42] showed that this test is not robust to high dimensionality, and proposed a robustified test that allows p_n to increase to infinity at most as fast as n^2 (hence, this test does not allow for universal (n, p)-asymptotics).

3. Simulations

In this section, we first conduct a Monte-Carlo study to check the validity of our universal asymptotic results in Theorems 2.1 and 2.3. Then we investigate whether or not, for practical purposes, the resulting Gaussian critical values should be favoured over the (asymptotically equivalent) chi-square ones. Finally, we show that the classical competitors of the Paindaveine [30] sign test are based on statistics that severely fail to be HD-robust.

3.1. Checking universal asymptotics

For every $(n, p) \in \{4, 30, 200, 1000\}^2$, we generated $M = 10\,000$ independent random samples of the form

$$\mathbf{X}_i^{(p)}, \qquad i = 1, \dots, n,$$

from the p-dimensional standard normal distribution. Then we evaluated the standardized statistics $T^{(n)}_{\mathcal{N},p}$ (see (2.7)) on each of these M samples, and we also computed the standardized statistic $R^{(n)}_{\mathcal{N},p}$ (see (2.4)) on the corresponding samples of unit vectors

$$\mathbf{U}_i^{(p)} = \frac{\mathbf{X}_i^{(p)}}{\|\mathbf{X}_i^{(p)}\|}, \qquad i = 1, \ldots, n.$$

Clearly, in each case, samples are generated from the respective null model, so that, according to our asymptotic results, the resulting empirical distributions of both standardized test statistics considered should be close to the standard normal for *virtually any* combination of "large" n and p values ("virtually any" here translates the universal asymptotics). To assess this, Figures 2 and 3 in supplemental article Paindaveine and Verdebout [32] provide, for each (n, p), histograms of the $M = 10\,000$ corresponding values of $R^{(n)}_{\mathcal{N},p}$ and $T^{(n)}_{\mathcal{N},p}$, respectively. Inspection of these figures reveals the following:

(i) The empirical distributions of $R^{(n)}_{\mathcal{N},p}$ and $T^{(n)}_{\mathcal{N},p}$ are clearly compatible with Theorems 2.1 and 2.3. For both statistics, the Gaussian approximation is valid for moderate to large values of n and p, irrespective of the ratio p/n. This confirms our universal asymptotic results.

(ii) For small n ($n = 4$), $R^{(n)}_{\mathcal{N},p}$ seems to be asymptotically Gaussian when $p \to \infty$, which illustrates the fixed-n asymptotic results from Chikuse [7]. On the contrary, $T^{(n)}_{\mathcal{N},p}$ cannot be well approximated by a Gaussian distribution for small n.

(iii) The empirical distributions of $R^{(n)}_{\mathcal{N},p}$ and $T^{(n)}_{\mathcal{N},p}$ are approximately (standardized) chi-square distributions for small p and moderate-to-large n (i.e., $p = 4$ and $n \geq 30$), which is consistent with the classical fixed-p n-asymptotic results.

For the sake of completeness, the results of a similar study for the statistic $I^{(n)}_{\mathcal{N},p,q}$ (see (2.10)) are reported in the supplemental article Paindaveine and Verdebout [32].

3.2. Comparing critical values

For the sake of clarity, we focus here on the problem of testing uniformity on the unit sphere, that was considered in Section 2.1. Our main result there (Theorem 2.1) justifies the use, in high dimensions, of two tests:

- The first test, $\phi^{(n)}_{\mathcal{N},p}$ say, is based on *Gaussian critical values*, and rejects the null at asymptotic level α whenever $R^{(n)}_{\mathcal{N},p} > \Phi^{-1}(1-\alpha)$.
- The second test, that we will denote as $\phi^{(n)}_p$, is the standard fixed-p sign test, based on *chi-square critical values*. This test, that rejects the null at asymptotic level α if

$$R^{(n)}_p > \chi^2_{p,1-\alpha} \qquad \text{equivalently, if } R^{(n)}_{\mathcal{N},p} > \frac{\chi^2_{p,1-\alpha} - p}{\sqrt{2p}},$$

is of course (n, p)-asymptotically equivalent to $\phi^{(n)}_{\mathcal{N},p}$.

To investigate what type of critical values should be favoured depending on the (n, p) configuration at hand, we performed the following numerical exercise. In the exact same way as in the simulations of Section 3.1, we generated, for each $(n, p) \in \{30, 200, 1000\} \times \{50, 100, 150, \ldots, 950, 1000\}$, $M = 100\,000$ independent random samples

$$\mathbf{U}_i^{(p)}, \qquad i = 1, \ldots, n$$

from the uniform distribution over $\mathcal{S}^{p-1}$. For each (n, p) considered, the statistic $R^{(n)}_{\mathcal{N},p}$ was evaluated on the corresponding M independent samples; for such a large M, the sample upper α-quantile, $\hat{q}_{n,p,1-\alpha}$ say, of these M independent replications of $R^{(n)}_{\mathcal{N},p}$ of course provides a very accurate estimate of the exact upper α-quantile of this test statistic. The appropriateness of Gaussian and chi-square critical values may thus be evaluated by looking at how much these differ from $\hat{q}_{n,p,1-\alpha}$. This evaluation is made possible in the six upper panels of Figure 1, which plot, for the various n considered, the sample quantiles $\hat{q}_{n,p,0.95}$ and $\hat{q}_{n,p,0.99}$ as a function of p, along with the corresponding Gaussian and chi-square critical values. Clearly, unless n is small ($n = 30$), chi-square quantiles are to be favoured over Gaussian ones that tend to underestimate the exact critical values.

The lower panels of Figure 1 provide empirical rejection frequencies of the Gaussian tests $\phi^{(n)}_{\mathcal{N},p}$ and of the chi-square tests $\phi^{(n)}_p$. In line with the results above, we conclude that, when n and p are moderate to large (irrespective of the ratio p/n), chi-square tests tend to achieve empirical type 1 risks that are much closer to the nominal level than their Gaussian counterparts that tend to be too liberal.

As a conclusion, while our universal asymptotic results broadly validates the use of Gaussian – hence, also of chi-square – sign tests, our recommendation is to avoid basing practical implementation on Gaussian tests; instead, chi-square tests should be used in practice. Finally, we point out that this conclusion is not only valid for the problem of testing uniformity on the sphere, but does extend to the other problems considered in this paper, as we checked by performing similar numerical exercises – with smaller M values, though, as the computational burden for the other tests is more severe than for the Rayleigh test (this clearly makes (n, p)-asymptotic critical values crucial for practical implementation).

3.3. Comparing Portmanteau tests

In this last simulation exercise, we compare the high-dimensional behaviour of the Portmanteau sign test described in Section 2.2 with those of some classical (fixed-p) competitors, namely the Ljung and Box [23] test (LB) and the Li and McLeod [22] test (LM). To the best of our knowledge, no tests are currently available in the high-dimensional case for this problem.

In order to do so, we generated, for every $p \in \{3, 30, 100\}$, $M = 1000$ independent random samples of the form

$$\mathbf{X}_{jt}^{(p)}, \qquad j = 1, 2, 3, t = 1, \ldots, n = 150.$$

The $\mathbf{X}_{1t}^{(p)}$'s are standard Gaussian, whereas the $\mathbf{X}_{2t}^{(p)}$'s (resp., the $\mathbf{X}_{3t}^{(p)}$'s) are (standard) student with 8 degrees of freedom (resp., with one degree of freedom). For all of these (null) samples, we

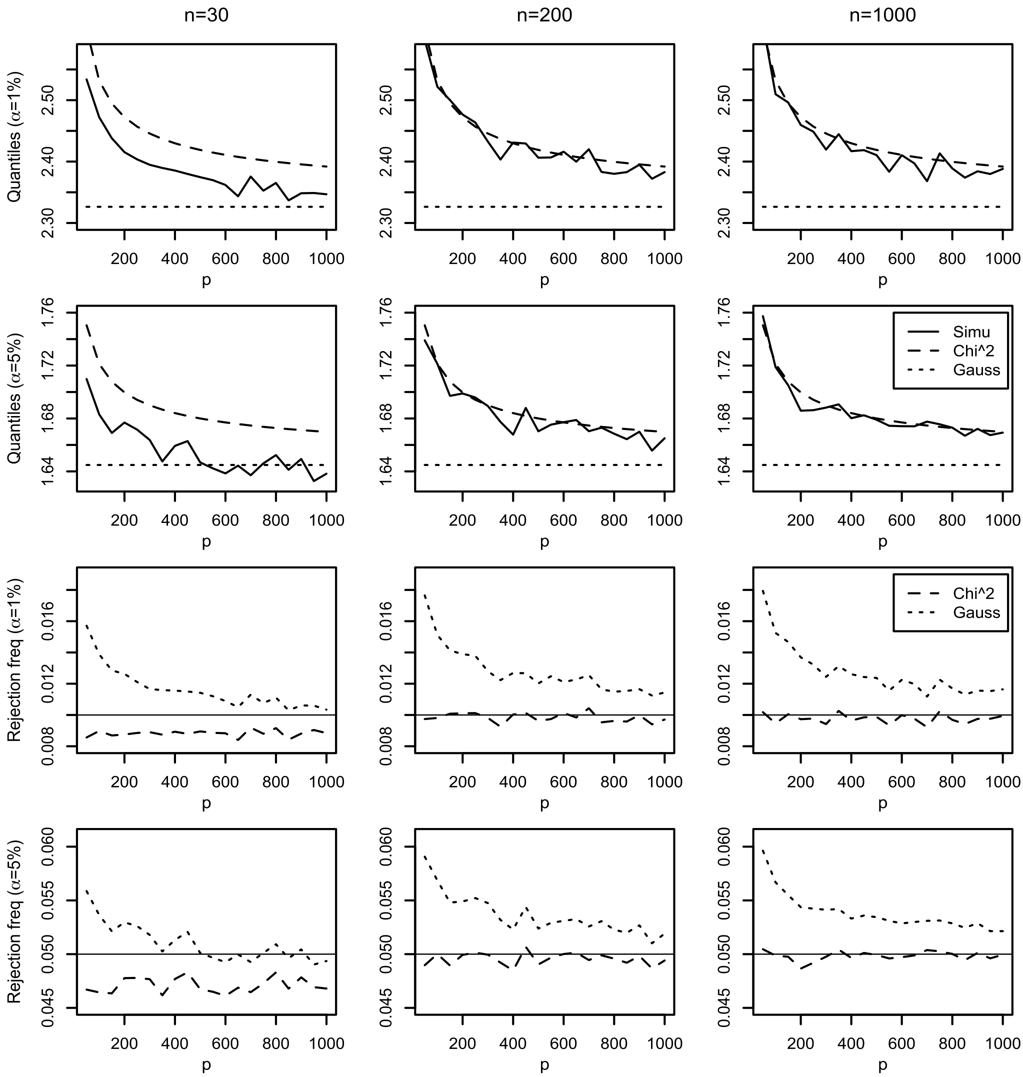

Figure 1. For $\alpha = 1\%$ and $\alpha = 5\%$, the six upper panels correspond to $n = 30, 200$ and 1000 and report the estimates $\hat{q}_{n,p,1-\alpha}$ of the exact upper α-quantile (solid line) of the statistic $R^{(n)}_{\mathcal{N},p}$, for $p = 50, 100, 150, \ldots, 1000$; the Gaussian (dotted line) and chi-square (dashed line) approximations of these exact quantiles are also provided. The six lower panels plot the corresponding empirical rejection frequencies of the Gaussian sign test $\phi^{(n)}_{\mathcal{N},p}$ (dotted line) and chi-square sign test $\phi^{(n)}_{p}$ (dashed line). Results are based on $M = 100\,000$ independent replications; see Section 3.2 for details.

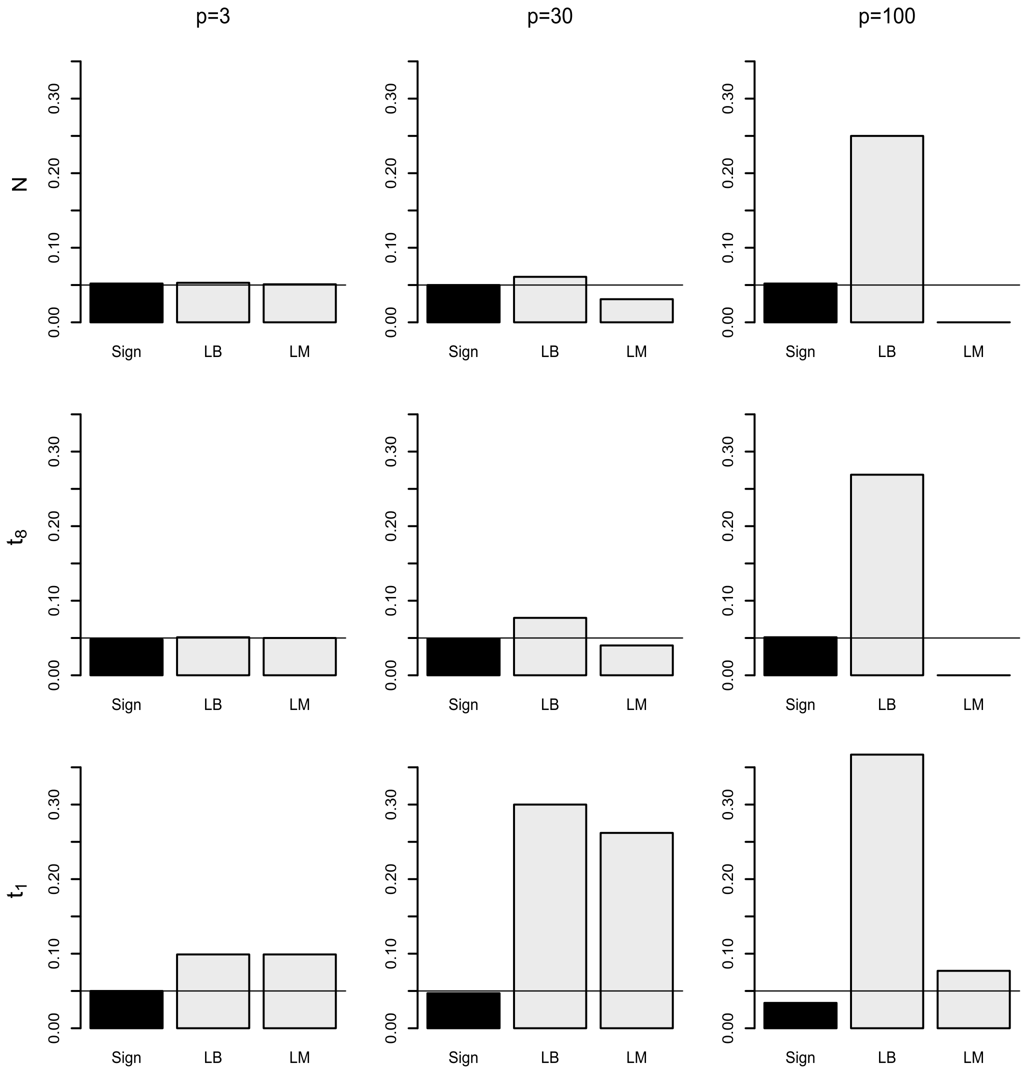

Figure 2. Empirical rejection frequencies of the chi-square Paindaveine [30] sign test (in black) and of the LM and LB tests (in light grey). All tests were performed at the nominal level 5%. Results are based on $M = 1000$ replications and the sample size is throughout $n = 150$.

performed the three Portmanteau tests mentioned above at nominal level $\alpha = 5\%$. The resulting rejection frequencies are represented in Figure 2.

Inspection of this figure reveals that the Paindaveine [30] sign test is the only test that is robust to high-dimensionality (and to heavy tails). In the high-dimensional case, the LB test is extremely liberal, while the LM test is liberal under the Cauchy but basically never rejects the null under the

Gaussian and the t_8. It should be noted that the reason why we restricted here to cases for which $n > p$ is that the LB and LM tests collapse when $n < p$ (in the sense that the corresponding test statistics then cannot even be computed). In contrast, the simulations of Section 3.1 showed that the sign test perfectly can deal with such cases.

Appendix A: Some preliminary lemmas

Lemma A.1. *Let* $\mathbf{U}_1, \dots, \mathbf{U}_n$ *be i.i.d. uniform on* $\mathcal{S}^{p-1}$, *and write* $\rho_{ij} = \mathbf{U}_i'\mathbf{U}_j$. *Then, for any* i, j, (i) $\rho_{ij}^2 \sim \mathrm{Beta}(1/2, (p-1)/2)$, *that is,* ρ_{ij}^2 *follows the Beta distribution with parameters* $1/2$ *and* $(p-1)/2$; (ii) *for any odd positive integer* m, $\mathrm{E}[\rho_{ij}^m] = 0$; (iii) *for any even positive integer* m,

$$\mathrm{E}\big[\rho_{ij}^m\big] = \prod_{r=0}^{m/2}\left(\frac{1+2r}{p+2r}\right),$$

so that

$$\mathrm{E}\big[\rho_{ij}^2\big] = \frac{1}{p}, \qquad \mathrm{E}\big[\rho_{ij}^4\big] = \frac{3}{p(p+2)}, \qquad \mathrm{E}\big[\rho_{ij}^6\big] = \frac{15}{p(p+2)(p+4)}$$

and

$$\mathrm{E}\big[\rho_{ij}^8\big] = \frac{105}{p(p+2)(p+4)(p+6)};$$

(iv) *the* ρ_{ij}*'s,* $i < j$, *are pairwise independent* (*hence uncorrelated*); (v) *fix* $h \in \{1, 2, \dots, n-2\}$. *Then, for any* $i, j, s, t \in \{h+1, \dots, n\}$ *with* $i < j$ *and* $s < t$, $\rho_{i-h,j-h}\rho_{i,j}$ *and* $\rho_{s-h,t-h}\rho_{s,t}$ *are uncorrelated, unless* $(i, j) = (s, t)$, *in which case* $\mathrm{Cov}[\rho_{i-h,j-h}\rho_{i,j}, \rho_{s-h,t-h}\rho_{s,t}] = 1/p^2$.

Proof. (i) Rotational invariance of the uniform distribution on $\mathcal{S}^{p-1}$ readily implies that ρ_{ij} is equal in distribution to $\mathbf{e}_{p,1}'\mathbf{U}_j$, where $\mathbf{e}_{p,\ell}$ denotes the ℓth vector in the canonical basis of $\mathbb{R}^p$. The result then follows from the fact that $(\mathbf{e}_1'\mathbf{U}_j)^2 \sim \mathrm{Beta}(1/2, (p-1)/2)$; see, for example, Muirhead [27], Theorem 1.5.7(ii). (ii) This is a trivial corollary of the fact that $\mathbf{U}_i$ and $-\mathbf{U}_i$, hence also ρ_{ij} and $-\rho_{ij}$, are equal in distribution. (iii) This directly follows from (i) and the fact that, if $Y \sim \mathrm{Beta}(\alpha, \beta)$, then

$$\mathrm{E}\big[Y^s\big] = \prod_{r=0}^{s-1}\left(\frac{\alpha + r}{\alpha+\beta+r}\right),$$

for any positive integer s; see, for example, Johnson, Kotz and Balakrishnan [19], equation (25.14). (iv) If $i, j, r, s \in \{1, 2, \dots, n\}$, are pairwise different, ρ_{ij} and ρ_{rs} are trivially independent. Let then i, j, s be three different integers in $\{1, 2, \dots, n\}$. Then the rotational invariance of the uniform distribution on $\mathcal{S}^{p-1}$ entails that the joint distribution of (ρ_{ij}, ρ_{ir}) coincides with that of $(\mathbf{e}_{p,1}'\mathbf{U}_j, \mathbf{e}_{p,1}'\mathbf{U}_r)$, which has independent marginals. The result follows. (v) From parts (iv) and (ii) of the lemma, we obtain

$$\mathrm{Cov}[\rho_{i-h,j-h}\rho_{i,j}, \rho_{s-h,t-h}\rho_{s,t}] = \mathrm{E}[\rho_{i-h,j-h}\rho_{s-h,t-h}\rho_{i,j}\rho_{s,t}]. \tag{A.1}$$

If $j \neq t$, this expectation is equal to zero, since

$$(\mathbf{U}_1, \dots, \mathbf{U}_{\max(j,t)-1}, \pm\mathbf{U}_{\max(j,t)}, \mathbf{U}_{\max(j,t)+1}, \dots, \mathbf{U}_n)$$

are equal in distribution. Similarly, using the fact that

$$(\mathbf{U}_1, \dots, \mathbf{U}_{\min(i-h,s-h)-1}, \pm\mathbf{U}_{\min(i-h,s-h)}, \mathbf{U}_{\min(i-h,s-h)+1}, \dots, \mathbf{U}_n)$$

are equal in distribution, we obtain that the expectation in (A.1) is equal to zero. Thus, to obtain a nonzero covariance, we need to have $(i, j) = (s, t)$, which leads to

$$\mathrm{Cov}[\rho_{i-h,j-h}\rho_{i,j}, \rho_{s-h,t-h}\rho_{s,t}] = \mathrm{Var}[\rho_{s-h,t-h}\rho_{s,t}] = \mathrm{E}\big[\rho^2_{s-h,t-h}\big]\mathrm{E}\big[\rho^2_{s,t}\big] = 1/p^2. \qquad \square$$

Let $\mathbf{J}_p = \sum_{i,j=1}^{p} (\mathbf{e}_{p,i}\mathbf{e}'_{p,j}) \otimes (\mathbf{e}_{p,i}\mathbf{e}'_{p,j}) = (\mathrm{vec}\,\mathbf{I}_p)(\mathrm{vec}\,\mathbf{I}_p)'$ and consider the *commutation matrix* $\mathbf{K}_p = \sum_{i,j=1}^{p} (\mathbf{e}_{p,i}\mathbf{e}'_{p,j}) \otimes (\mathbf{e}_{p,j}\mathbf{e}'_{p,i})$. We then have the following result.

Lemma A.2. *Let* $\mathbf{U}, \mathbf{V}, \mathbf{W}$ *be mutually independent and uniformly distributed on* $\mathcal{S}^{p-1}$, $\mathcal{S}^{q-1}$, *and* $\mathcal{S}^{m-1}$, *respectively. Then* (i) $\mathrm{E}[\mathbf{U}] = \mathbf{0}$, (ii) $\mathrm{Var}[\mathbf{U}] = \frac{1}{p}\mathbf{I}_p$, (iii) $\mathrm{E}[\mathrm{vec}(\mathbf{U}\mathbf{U}')(\mathrm{vec}(\mathbf{U}\mathbf{U}'))'] = \frac{1}{p(p+2)}(\mathbf{I}_{p^2} + \mathbf{K}_p + \mathbf{J}_p)$, (iv) $\mathrm{E}[\mathrm{vec}(\mathbf{U}\mathbf{V}')(\mathrm{vec}(\mathbf{U}\mathbf{V}'))'] = \frac{1}{pq}\mathbf{I}_{pq}$, *and* (v) $\mathrm{E}[\mathrm{vec}(\mathbf{U}\mathbf{V}') \times (\mathrm{vec}(\mathbf{U}\mathbf{W}'))'] = \mathbf{0}_{pq\times pm}$, *where* $\mathbf{0}_{k\times\ell}$ *denotes the* $k \times \ell$ *zero matrix.*

Proof. (i)–(ii) These identities follow directly from the orthogonal invariance of $\mathbf{U}$ and the fact that $\|\mathbf{U}\| = 1$ almost surely. (iii) See Tyler [41], page 244. (iv) The independence of $\mathbf{U} = (U_1, \dots, U_p)'$ and $\mathbf{V} = (V_1, \dots, V_q)'$ readily gives

$$\mathrm{E}\big[\mathrm{vec}\big(\mathbf{U}\mathbf{V}'\big)\big(\mathrm{vec}\big(\mathbf{U}\mathbf{V}'\big)\big)'\big] = \sum_{i,j=1}^{p}\sum_{r,s=1}^{q} \mathrm{E}[U_iU_j]\mathrm{E}[V_rV_s]\,\mathrm{vec}\big(\mathbf{e}_{p,i}\mathbf{e}'_{q,r}\big)\big(\mathrm{vec}\big(\mathbf{e}_{q,j}\mathbf{e}'_{q,s}\big)\big)',$$

which yields the result since $\mathrm{E}[U_iU_j] = \frac{1}{p}\delta_{ij}$ and $\mathrm{E}[V_rV_s] = \frac{1}{q}\delta_{rs}$ (see (ii)). (v) This directly follows from the fact that $\pm\mathbf{W}$ are equal in distribution. $\square$

Lemma A.3. *For any* $a > 0$, $\sum_{\ell=1}^{n} \ell^a = \mathrm{O}(n^{a+1})$ *as* $n \to \infty$.

Proof. The result follows from the fact that the sequence (c_n), defined by $c_n = \frac{1}{n}\sum_{\ell=1}^{n}(\frac{\ell}{n})^a$, is a sequence of Riemann sums for $\int_0^1 x^a\,\mathrm{d}x$, hence converges in $\mathbb{R}$. $\square$

Appendix B: Proofs of Theorems 2.1 and 2.3

PROOF OF THEOREM 2.1. To prove this result (and the corresponding results in the next two sections), we adopt an approach that exploits the martingale difference structure of some process. In that framework, the key result we will use is Theorem 35.12 from Billingsley [2]. Since this result plays a crucial role in the paper, we state it here, in a form that is suitable for our purposes.

Theorem B.1 (Billingsley [2], Theorem 35.12). *Let $D_{n\ell}$, $\ell = 1, \dots, n$, $n = 1, 2, \dots$, be a triangular array of random variables such that, for any n, $D_{n1}, D_{n2}, \dots, D_{nn}$ is a martingale difference sequence with respect to some filtration $\mathcal{F}_{n1}, \mathcal{F}_{n2}, \dots, \mathcal{F}_{nn}$. Assume that, for any n, ℓ, $D_{n\ell}$ has a finite variance. Letting $\sigma_{n\ell}^2 = \mathrm{E}[D_{n\ell}^2 | \mathcal{F}_{n,\ell-1}]$ (with $\mathcal{F}_{n0}$ being the trivial σ-algebra $\{\varnothing, \Omega\}$ for all n), further assume that, as $n \to \infty$,*

$$\sum_{\ell=1}^{n} \sigma_{n\ell}^2 \xrightarrow{\mathrm{P}} 1 \tag{B.1}$$

(*where $\xrightarrow{\mathrm{P}}$ denotes convergence in probability*), *and*

$$\sum_{\ell=1}^{n} \mathrm{E}\big[D_{n\ell}^2 \mathbb{I}\big[|D_{n\ell}| > \varepsilon\big]\big] \to 0. \tag{B.2}$$

Then $\sum_{\ell=1}^{n} D_{n\ell}$ is asymptotically standard normal.

In order to apply this result, we define $\mathcal{F}_{n\ell}$ as the σ-algebra generated by $\mathbf{U}_{n1}, \dots, \mathbf{U}_{n\ell}$, and, writing $\mathrm{E}_{n\ell}$ for the conditional expectation with respect to $\mathcal{F}_{n\ell}$, we let

$$D_{n\ell}^R = \mathrm{E}_{n\ell}\big[R_{\mathcal{N},p}^{(n)}\big] - \mathrm{E}_{n,\ell-1}\big[R_{\mathcal{N},p}^{(n)}\big] = \frac{\sqrt{2p_n}}{n} \sum_{i=1}^{\ell-1} \mathbf{U}_{ni}' \mathbf{U}_{n\ell},$$

for any $\ell = 1, 2, \dots$ (throughout, sums over empty set of indices are defined as being equal to zero). Clearly, we have that

$$R_{\mathcal{N},p}^{(n)} = \frac{\sqrt{2p_n}}{n} \sum_{1 \le i < j \le n} \mathbf{U}_{ni}' \mathbf{U}_{nj} = \sum_{\ell=1}^{n} D_{n\ell}^R.$$

Since $|D_{n\ell}^R| \le \sqrt{2p_n}(\ell - 1)/n$ almost surely, every $D_{n\ell}^R$ has a finite-variance. Therefore, to establish Theorem 2.1, it is sufficient to prove the following two lemmas, that show that (B.1)–(B.2) are fulfilled in the present context.

Lemma B.1. *Letting $\sigma_{n\ell}^2 = \mathrm{E}_{n,\ell-1}[(D_{n\ell}^R)^2]$, $\sum_{\ell=1}^{n} \sigma_{n\ell}^2$ converges to one in quadratic mean as $n \to \infty$.*

Lemma B.2. *For any $\varepsilon > 0$, $\sum_{\ell=1}^{n} \mathrm{E}[(D_{n\ell}^R)^2 \mathbb{I}[|D_{n\ell}^R| > \varepsilon]] \to 0$ as $n \to \infty$.*

The proofs of these lemmas make intensive use of the properties of the inner products $\rho_{n,ij} = \mathbf{U}_{ni}' \mathbf{U}_{nj}$; see Lemma A.1 in Appendix A. For notational simplicity, we will systematically drop the dependence on n in $\rho_{n,ij}$, $\mathbf{U}_{ni}$, $\mathrm{E}_{n\ell}$, E_n and Var_n; here, E_n and Var_n stand for the unconditional expectation and unconditional variance computed with respect to the joint distribution of the $\mathbf{U}_{ni}$'s, $i = 1, \dots, n$.

Proof of Lemma B.1. Using Lemma A.2(i)–(ii), we obtain

$$\sigma_{n\ell}^2 = \mathrm{E}_{\ell-1}\big[\big(D_{n\ell}^R\big)^2\big] = \frac{2p_n}{n^2}\sum_{i,j=1}^{\ell-1}\mathbf{U}_i'\mathrm{E}\big[\mathbf{U}_\ell\mathbf{U}_\ell'\big]\mathbf{U}_j = \frac{2}{n^2}\sum_{i,j=1}^{\ell-1}\rho_{ij},$$

which yields (Lemma A.1(ii))

$$\mathrm{E}\big[\sigma_{n\ell}^2\big] = \frac{2(\ell-1)}{n^2} \qquad \big(=\mathrm{E}\big[\big(D_{n\ell}^R\big)^2\big]\big). \tag{B.3}$$

Therefore, as $n \to \infty$,

$$\mathrm{E}\left[\sum_{\ell=1}^n \sigma_{n\ell}^2\right] = \frac{2}{n^2}\sum_{\ell=1}^n(\ell-1) = \frac{n-1}{n} \to 1.$$

Using Lemma A.1(iv), then the fact that $\mathrm{Var}[\rho_{ij}] = 1/p_n$ (which follows from Lemma A.1(ii)–(iii)), we obtain that

$$\begin{aligned}
\mathrm{Var}\left[\sum_{\ell=1}^n \sigma_{n\ell}^2\right] &= \frac{16}{n^4}\mathrm{Var}\left[\sum_{\ell=3}^n \sum_{1\le i<j\le \ell-1}\rho_{ij}\right] = \frac{16}{n^4}\mathrm{Var}\left[\sum_{1\le i<j\le n}(n-j)\rho_{ij}\right] \\
&= \frac{16}{n^4p_n}\sum_{1\le i<j\le n}(n-j)^2 = \frac{16}{n^4p_n}\sum_{j=2}^n(j-1)(n-j)^2 \\
&= \frac{16}{n^4p_n}\sum_{j=1}^{n-1}j(n-j-1)^2 \le \frac{16}{n^2p_n}\sum_{j=1}^{n-1}j = \frac{8(n-1)}{np_n},
\end{aligned}$$

which is o(1) as $n \to \infty$. The result follows. □

Proof of Lemma B.2. Applying first the Cauchy–Schwarz inequality, then the Chebyshev inequality (note that $D_{n\ell}^R$ has zero mean), yields

$$\begin{aligned}
\sum_{\ell=1}^n \mathrm{E}\big[\big(D_{n\ell}^R\big)^2\mathbb{I}\big[\big|D_{n\ell}^R\big| > \varepsilon\big]\big] &\le \sum_{\ell=1}^n\sqrt{\mathrm{E}\big[\big(D_{n\ell}^R\big)^4\big]}\sqrt{\mathrm{P}\big[\big|D_{n\ell}^R\big| > \varepsilon\big]} \\
&\le \frac{1}{\varepsilon}\sum_{\ell=1}^n\sqrt{\mathrm{E}\big[\big(D_{n\ell}^R\big)^4\big]}\sqrt{\mathrm{Var}\big[D_{n\ell}^R\big]}.
\end{aligned}$$

From (B.3), we readily obtain that $\mathrm{Var}[D_{n\ell}^R] \le 2(\ell-1)/n^2$, which provides

$$\sum_{\ell=1}^n \mathrm{E}\big[\big(D_{n\ell}^R\big)^2\mathbb{I}\big[\big|D_{n\ell}^R\big| > \varepsilon\big]\big] \le \frac{\sqrt{2}}{\varepsilon n}\sum_{\ell=1}^n\sqrt{(\ell-1)\mathrm{E}\big[\big(D_{n\ell}^R\big)^4\big]}. \tag{B.4}$$

Now, Lemma A.1(iv) yields

$$\mathrm{E}\big[(D^R_{n\ell})^4\big] = \frac{4p_n^2}{n^4}\mathrm{E}\Bigg[\Bigg(\sum_{i=1}^{\ell-1}\rho_{i\ell}\Bigg)^4\Bigg] = \frac{4p_n^2}{n^4}\sum_{i,j,r,s=1}^{\ell-1}\mathrm{E}[\rho_{i\ell}\rho_{j\ell}\rho_{r\ell}\rho_{s\ell}]$$
$$= \frac{4p_n^2}{n^4}\big\{(\ell-1)\mathrm{E}\big[\rho_{i\ell}^4\big] + 3(\ell-1)(\ell-2)\mathrm{E}\big[\rho_{1\ell}^2\big]\mathrm{E}\big[\rho_{2\ell}^2\big]\big\},$$

so that, using Lemma A.1(iii), we obtain

$$\mathrm{E}\big[(D^R_{n\ell})^4\big] = \frac{4p_n^2}{n^4}\left\{\frac{3(\ell-1)}{p_n(p_n+2)} + \frac{3(\ell-1)(\ell-2)}{p_n^2}\right\} \le \frac{24(\ell-1)^2}{n^4}. \tag{B.5}$$

Plugging (B.5) into (B.4), we conclude that

$$\sum_{\ell=1}^n \mathrm{E}\big[(D^R_{n\ell})^2\mathbb{I}\big[|D^R_{n\ell}|>\varepsilon\big]\big] \le \frac{\sqrt{2}}{\varepsilon n}\sum_{\ell=1}^n\sqrt{\frac{24(\ell-1)^3}{n^4}}$$
$$\le \frac{\sqrt{48}}{\varepsilon n^3}\sum_{\ell=1}^n(\ell-1)^{3/2},$$

which is $\mathrm{O}(n^{-1/2})$ (see Lemma A.3). The result follows. □

Proof of Theorem 2.3. We define $\mathcal{F}_{n\ell}$ as the σ-algebra generated by $\mathbf{X}_{n1},\dots,\mathbf{X}_{n\ell}$, and we let

$$D^T_{n\ell} = \mathrm{E}_{n\ell}\big[T^{(n)}_{\mathcal{N},p}\big] - \mathrm{E}_{n,\ell-1}\big[T^{(n)}_{\mathcal{N},p}\big] = \frac{\sqrt{2p_n^2}}{\sqrt{H}}\sum_{h=1}^H\frac{1}{n-h}\sum_{s=h+1}^{\ell-1}\rho_{n,s-h,\ell-h}\rho_{n,s\ell}$$

(recall that sums over empty sets of indices are defined as zero), where we wrote $\rho_{n,s\ell} = \mathbf{U}'_{ns}\mathbf{U}_{nt}$ and where $\mathrm{E}_{n\ell}$ still denotes conditional expectation with respect to $\mathcal{F}_{n\ell}$. This provides $T^{(n)}_{\mathcal{N},p} = \sum_{\ell=1}^n D^T_{n\ell}$, where $D^T_{n\ell}$ is almost surely bounded, hence has a finite-variance. As in the previous section, asymptotic normality is then proved by applying Theorem B.1, which is based on both following lemmas. □

Lemma B.3. *Letting* $\sigma^2_{n\ell} = \mathrm{E}_{\ell-1}[(D^T_{n\ell})^2]$, $\sum_{\ell=1}^n\sigma^2_{n\ell}$ *converges to one in quadratic mean as* $n\to\infty$.

Lemma B.4. *For any* $\varepsilon>0$, $\sum_{\ell=1}^n \mathrm{E}[(D^T_{n\ell})^2\mathbb{I}[|D^T_{n\ell}|>\varepsilon]]\to 0$ *as* $n\to\infty$.

In the proofs, we use the same notational shortcuts as in the proofs of Lemmas B.1–B.2, that is, we write ρ_{st}, $\mathbf{U}_t$, E_ℓ, E, and Var, instead of $\rho_{n,st}$, $\mathbf{U}_{nt}$, $\mathrm{E}_{n\ell}$, E_n and Var_n, respectively. For any $r\times s$ matrix $\mathbf{A}$, we denote as usual by $\operatorname{vec}\mathbf{A}$ the (rs)-vector obtained by stacking the columns of $\mathbf{A}$ on top of each other. Recall that we then have $(\operatorname{vec}\mathbf{A})'(\operatorname{vec}\mathbf{B}) = \mathrm{Trace}[\mathbf{A}'\mathbf{B}]$.

Proof of Lemma B.3. First note that, for any $s, t \in \{1, \dots, \ell-1\}$ and any $h, g \in \{1, \dots, H\}$, the $\mathrm{E}_{\ell-1}$ expectation of

$$\begin{aligned}&\rho_{s-h,\ell-h}\rho_{s,\ell}\rho_{t-g,\ell-g}\rho_{t,\ell}\\&\quad=\big(\mathrm{vec}\big(\mathbf{U}_s\mathbf{U}_{s-h}'\big)\big)'\,\mathrm{vec}\big(\mathbf{U}_\ell\mathbf{U}_{\ell-h}'\big)\big(\mathrm{vec}\big(\mathbf{U}_\ell\mathbf{U}_{\ell-g}'\big)\big)'\,\mathrm{vec}\big(\mathbf{U}_t\mathbf{U}_{t-g}'\big)\end{aligned}$$

is given by (see Lemma A.2(iv)–(v))

$$\begin{aligned}&\big(\mathrm{vec}\big(\mathbf{U}_s\mathbf{U}_{s-h}'\big)\big)'\mathrm{E}\big[\mathrm{vec}\big(\mathbf{U}_\ell\mathbf{U}_{\ell-h}'\big)\big(\mathrm{vec}\big(\mathbf{U}_\ell\mathbf{U}_{\ell-g}'\big)\big)'\big]\,\mathrm{vec}\big(\mathbf{U}_t\mathbf{U}_{t-g}'\big)\\&\quad=\big(\mathrm{vec}\big(\mathbf{U}_s\mathbf{U}_{s-h}'\big)\big)'\mathrm{E}\big[\mathrm{vec}\big(\mathbf{U}_\ell\mathbf{U}_{\ell-h}'\big)\big(\mathrm{vec}\big(\mathbf{U}_\ell\mathbf{U}_{\ell-g}'\big)\big)'\big]\,\mathrm{vec}\big(\mathbf{U}_t\mathbf{U}_{t-g}'\big)\\&\quad=\frac{1}{p_n^2}\big(\mathrm{vec}\big(\mathbf{U}_s\mathbf{U}_{s-h}'\big)\big)'\,\mathrm{vec}\big(\mathbf{U}_t\mathbf{U}_{t-g}'\big)\delta_{h,g},\end{aligned}$$

where $\delta_{h,g}$ is equal to one if $h=g$ and to zero otherwise. Therefore, we have that

$$\begin{aligned}\sigma_{n\ell}^2&=\frac{2p_n^2}{H}\sum_{h,g=1}^{H}\frac{1}{(n-h)(n-g)}\sum_{s=h+1}^{\ell-1}\sum_{t=g+1}^{\ell-1}\mathrm{E}_{\ell-1}[\rho_{s-h,\ell-h}\rho_{s,\ell}\rho_{t-g,\ell-g}\rho_{t,\ell}]\\&=\frac{2}{H}\sum_{h=1}^{H}\frac{1}{(n-h)^2}\sum_{s,t=h+1}^{\ell-1}\big(\mathrm{vec}\big(\mathbf{U}_s\mathbf{U}_{s-h}'\big)\big)'\,\mathrm{vec}\big(\mathbf{U}_t\mathbf{U}_{t-h}'\big)\qquad\text{(B.6)}\\&=\frac{2}{H}\sum_{h=1}^{H}\frac{1}{(n-h)^2}\sum_{s,t=h+1}^{\ell-1}\rho_{s-h,t-h}\rho_{s,t}.\end{aligned}$$

From Lemma A.1(iv), we then obtain

$$\mathrm{E}\big[\sigma_{n\ell}^2\big]=\frac{2}{H}\sum_{h=1}^{H}\frac{(\ell-h-1)_+}{(n-h)^2}\qquad\big(=\mathrm{E}\big[\big(D_{n\ell}^T\big)^2\big]\big),\tag{B.7}$$

where we let $m_+=\max(m,0)$. This implies that

$$\mathrm{E}\Bigg[\sum_{\ell=1}^{n}\sigma_{n\ell}^2\Bigg]=\frac{2}{H}\sum_{h=1}^{H}\frac{1}{(n-h)^2}\sum_{\ell=h+2}^{n}(\ell-h-1)=\frac{1}{H}\sum_{h=1}^{H}\frac{n-h-1}{n-h}\to 1,$$

as $n\to\infty$. Using the identity $\mathrm{Var}[\sum_{h=1}^H Z_h]\le H\sum_{h=1}^H\mathrm{Var}[Z_h]$, (B.6) directly yields

$$\begin{aligned}\mathrm{Var}\Bigg[\sum_{\ell=1}^{n}\sigma_{n\ell}^2\Bigg]&\le\frac{4}{H^2}\times H\times\sum_{h=1}^{H}\frac{1}{(n-h)^4}\,\mathrm{Var}\Bigg[\sum_{\ell=1}^{n}\sum_{s,t=h+1}^{\ell-1}\rho_{s-h,t-h}\rho_{s,t}\Bigg]\\&\le\frac{16}{H(n-H)^4}\sum_{h=1}^{H}\mathrm{Var}\Bigg[\sum_{\ell=1}^{n}\sum_{h+1\le s<t\le\ell-1}\rho_{s-h,t-h}\rho_{s,t}\Bigg].\end{aligned}\tag{B.8}$$

From Lemma A.1(v), we obtain

$$
\begin{aligned}
&\mathrm{Var}\Bigg[\sum_{\ell=1}^{n}\sum_{h+1\le s<t\le \ell-1}\rho_{s-h,t-h}\rho_{s,t}\Bigg]\\
&\le \mathrm{Var}\Bigg[\sum_{h+1\le s<t\le n}(n-t)\rho_{s-h,t-h}\rho_{st}\Bigg]\\
&= \frac{1}{p_n^2}\sum_{h+1\le s<t\le n}(n-t)^2 = \frac{1}{p_n^2}\sum_{t=h+2}^{n}(t-h-1)(n-t)^2\\
&\le \frac{1}{p_n^2}\sum_{t=1}^{n-h-1}t(n-h-1)^2\\
&= \frac{1}{p_n^2}\sum_{t=1}^{n-h-1}t(n-t-h-1)^2 \le \frac{n^2}{p_n^2}\sum_{t=1}^{n-h-1}t = \frac{n^2(n-h-1)(n-h)}{2p_n^2}.
\end{aligned}
$$

By plugging this into (B.8), we obtain

$$
\mathrm{Var}\Bigg[\sum_{\ell=1}^{n}\sigma_{n\ell}^2\Bigg] \le \frac{16}{H(n-H)^4}\sum_{h=1}^{H}\frac{n^2(n-h-1)(n-h)}{2p_n^2} \le \frac{8n^4}{(n-H)^4p_n^2},
$$

which is $\mathrm{o}(1)$ as $n\to\infty$. The result follows. □

Proof of Lemma B.4. Proceeding as in the proof of Lemma B.2, we obtain

$$
\begin{aligned}
\sum_{\ell=1}^{n}\mathrm{E}\big[(D_{n\ell}^T)^2\mathbb{I}\big[|D_{n\ell}^T|>\varepsilon\big]\big] &\le \frac{1}{\varepsilon}\sum_{\ell=1}^{n}\sqrt{\mathrm{E}\big[(D_{n\ell}^T)^4\big]}\sqrt{\mathrm{Var}\big[D_{n\ell}^T\big]}\\
&\le \frac{\sqrt{2}}{\varepsilon\sqrt{H}}\sum_{h=1}^{H}\frac{1}{n-h}\sum_{\ell=h+2}^{n}\sqrt{(\ell-h-1)\mathrm{E}\big[(D_{n\ell}^T)^4\big]} \qquad \text{(B.9)}\\
&\le \frac{\sqrt{2}}{\varepsilon(n-H)\sqrt{H}}\sum_{h=1}^{H}\sum_{\ell=h+2}^{n}\sqrt{(\ell-h-1)\mathrm{E}\big[(D_{n\ell}^T)^4\big]},
\end{aligned}
$$

where we have used the fact that (see (B.7))

$$
\mathrm{Var}\big[D_{n\ell}^T\big] \le \mathrm{E}\big[(D_{n\ell}^T)^2\big] = \frac{2}{H}\sum_{h=1}^{H}\frac{(\ell-h-1)_+}{(n-h)^2}.
$$

Note that

$$
\begin{aligned}
&\mathrm{E}\Bigg[\Bigg(\sum_{s=h+1}^{\ell-1}\rho_{s-h,\ell-h}\rho_{s\ell}\Bigg)^4\Bigg]\\
&\quad=\sum_{s,t,i,j=h+1}^{\ell-1}\mathrm{E}[\rho_{s-h,\ell-h}\rho_{t-h,\ell-h}\rho_{i-h,\ell-h}\rho_{j-h,\ell-h}\rho_{s\ell}\rho_{t\ell}\rho_{i\ell}\rho_{j\ell}]\\
&\quad=(\ell-h-1)_+\mathrm{E}\big[\rho^4_{1,\ell-h}\big]\mathrm{E}\big[\rho^4_{h+1,\ell}\big]\\
&\qquad+3(\ell-h-1)_+(\ell-h-2)_+\mathrm{E}\big[\rho^2_{1,\ell-h}\big]\mathrm{E}\big[\rho^2_{2,\ell-h}\big]\mathrm{E}\big[\rho^2_{h+1,\ell}\big]\mathrm{E}\big[\rho^2_{h+2,\ell}\big]\\
&\quad=\frac{3(\ell-h-1)_+}{p_n^2(p_n+2)^2}+\frac{3(\ell-h-1)_+(\ell-h-2)_+}{p_n^4}\le\frac{6(\ell-h-1)_+^2}{p_n^4},
\end{aligned}
$$

which yields

$$
\begin{aligned}
\mathrm{E}\big[\big(D^T_{n\ell}\big)^4\big]&\le\frac{4p_n^4}{H^2}\times H^3\times\sum_{h=1}^{H}\frac{1}{(n-h)^4}\mathrm{E}\Bigg[\Bigg(\sum_{s=h+1}^{\ell-1}\rho_{s-h,\ell-h}\rho_{s\ell}\Bigg)^4\Bigg]\\
&\le\frac{24H}{(n-H)^4}\sum_{h=1}^{H}(\ell-h-1)_+^2.
\end{aligned}
$$

Plugging into (B.9), we conclude that

$$
\begin{aligned}
\sum_{\ell=1}^{n}\mathrm{E}\big[\big(D^T_{n\ell}\big)^2\mathbb{I}\big[\big|D^T_{n\ell}\big|>\varepsilon\big]\big]&\le\frac{\sqrt{48}}{\varepsilon(n-H)^3}\sum_{h,g=1}^{H}\sum_{\ell=h+2}^{n}\sqrt{(\ell-h-1)(\ell-g-1)_+^2}\\
&\le\frac{\sqrt{48}H^2}{\varepsilon(n-H)^3}\sum_{\ell=3}^{n}(\ell-2)^{3/2}.
\end{aligned}
$$

In view of Lemma A.3, this is $\mathrm{O}(n^{-1/2})$, which establishes the result. □

As an alternative to the tests in (2.6), one may consider (see Paindaveine [30]) the lower-rank multivariate runs tests rejecting the null at asymptotic level α whenever

$$
\tilde{T}_p^{(n)}=\sum_{h=1}^{H}(n-h)\big(\tilde{\mathbf{r}}(h)\big)^2=\sum_{h=1}^{H}\frac{p}{n-h}\sum_{s,t=h+1}^{n}\big(\mathbf{U}'_{s-h}\mathbf{U}_s\big)\big(\mathbf{U}'_{t-h}\mathbf{U}_t\big)>\chi^2_{H,1-\alpha},
$$

where $\tilde{\mathbf{r}}(h)=\frac{\sqrt{p}}{n-h}\sum_{t=h+1}^{n}\mathbf{U}'_t\mathbf{U}_{t-h}$. Using similar arguments as above, one may then show that, under the same assumptions as in Theorem 2.3, the universal (n,p)-asymptotic distribution of $\tilde{T}^{(n)}_{\mathcal{N},p}=(\tilde{T}_n-H)/\sqrt{2H}$ is standard normal.

Supplementary Material

Supplement to "High-dimensional sign tests" (DOI: 10.3150/15-BEJ710SUPP; .pdf). The supplement article contains the proofs of Theorems 2.4 and 2.5 together with simulation results related to the sign test for independence. It also provides histograms from the simulations of Section 3.1.

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