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Syntactic Preservation Theorems for Intuitionistic Predicate Logic

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Abstract We define notions of homomorphism, submodel, and sandwich of Kripke models, and we define two syntactic operators analogous to universal and existential closure. Then we prove an intuitionistic analogue of the generalized (dual of the) Lyndon-Łoś-Tarski Theorem, which characterizes the sentences preserved under inverse images of homomorphisms of Kripke models, an intuitionistic analogue of the generalized Łoś-Tarski Theorem, which characterizes the sentences preserved under submodels of Kripke models, and an intuitionistic analogue of the generalized Keisler Sandwich Theorem, which characterizes the sentences preserved under sandwiches of Kripke models. We also define several intuitionistic formula hierarchies analogous to the classical formula hierarchies $\forall_n (= \Pi^0_n)$ and $\exists_n (= \Sigma^0_n)$, and we show how our generalized syntactic preservation theorems specialize to these hierarchies. Each of these theorems implies the corresponding classical theorem in the case where the Kripke models force classical logic.

1 Introduction

1.1 Statement of the Problem In classical model theory, much attention has been devoted to characterizing the connection between classes of models and their first-order syntactic descriptions. The most well-known characterization of this sort is Gödel's completeness theorem. Other well-known characterizations are the *syntactic preservation theorems* of classical model theory. The Łoś-Tarski Theorem states that a classical theory is axiomatizable by universal sentences if and only if it is preserved under submodels. The Lyndon-Łoś-Tarski Theorem (sometimes called the homomorphism preservation theorem) states that a classical theory is axiomatizable by existential-positive sentences if and only if it is preserved under homomorphisms

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of models. The Chang-Łoś-Suszko Theorem and Keisler Sandwich Theorem state that a classical theory is axiomatizable by universal-existential sentences if and only if it is preserved under unions of chains of models if and only if it is preserved under sandwiches of models. See [4] and [9].

Formally, intuitionistic predicate logic IQC is a proper subsystem of classical predicate logic CQC. The completeness theorem for intuitionistic predicate logic states that an intuitionistic theory Δ is axiomatizable by a set of sentences Γ if and only if Δ is satisfied by all Kripke models of Γ. See [5] and [12]. In this paper, we define notions of homomorphism, submodel, and sandwich of Kripke models for IQC, and we define two syntactic operators $\mathcal{U}(\cdot,\cdot)$ and $\mathcal{E}(\cdot)$ analogous to universal and existential closure. Then we prove some corresponding generalized syntactic preservation theorems. The first is an intuitionistic analogue of the generalized (dual of the) Lyndon-Łoś-Tarski Theorem, which characterizes the sentences preserved under inverse images of homomorphisms of Kripke models. The second is an intuitionistic analogue of the generalized Łoś-Tarski Theorem, which characterizes the sentences preserved under submodels of Kripke models. The third is an intuitionistic analogue of the generalized Keisler Sandwich Theorem. We define intuitionistic formula hierarchies $\mathcal{U}_n$ and $\mathcal{E}_n$, which are analogous to the classical formula hierarchies $\forall_n (= \Pi^0_n)$ and $\exists_n (= \Sigma^0_n)$, and we obtain an intuitionistic analogue of the Keisler Sandwich Theorem for sentences in these hierarchies. The $\mathcal{U}_n$ and $\mathcal{E}_n$ formula hierarchies each provide a normal-form theorem for IQC, and they are equivalent over CQC to the $\forall_n$ and $\exists_n$ hierarchies for $n \geq 0$ and $n \geq 1$, respectively. We also define formula hierarchies $\mathcal{U}^*_n$ and $\mathcal{U}^{\mathcal{S}}_n$, the first of which is provably equivalent over Heyting Arithmetic to a variant of the hierarchy introduced by Burr in [3], and we obtain intuitionistic analogues of the Keisler Sandwich Theorem for sentences in these hierarchies as well.

1.2 Status of the Problem While preservation theorems have been proven for intuitionistic propositional logic [14], often in the context of general modal propositional logic [1], there are few examples of published preservation theorems for Kripke models and intuitionistic predicate logic. One such theorem is due to Visser [13]. This theorem states that an intuitionistic theory is axiomatizable by semi-positive sentences if and only if it is preserved under submodels of Kripke models obtained by restricting the frame of the original Kripke model. Visser mentions several notions of a submodel of a Kripke model, including the notion adopted in this paper (which is the same as in [6]). A different notion of submodel is used by Bagheri and Moniri in [2], but they do not prove any preservation theorems.

It is not the case that every formula is equivalent over IQC to a formula in prenex-normal-form. Thus, it is not obvious how intuitionistic formula hierarchies analogous to $\forall_n$ and $\exists_n$ should even be defined. Other authors have already given some attention to defining intuitionistic formula hierarchies with suitable normal-form theorems. For the propositional logic case, the authors of [14] define a formula hierarchy based on implication depth. For the predicate logic case, Burr [3] defines a formula hierarchy Φ_n over Heyting Arithmetic HA that is analogous to the $\forall_n$ hierarchy. The Φ_n formula hierarchy provides a normal-form theorem for IQC over HA, and it is equivalent over CQC to the $\forall_n$ hierarchy for $n = 0$ and $n \geq 2$. A generalization of the Burr Hierarchy can be found in [11].

A Kripke model for IQC can be viewed as a collection of classical models and (classical) homomorphisms, with intuitionistic truth defined by the usual forcing relation. A classical model is essentially equivalent to a one-node Kripke model with only the identity homomorphism. In this case, intuitionistic truth in the Kripke model corresponds exactly with classical truth at the single node. Conversely, if a Kripke model forces CQC, then intuitionistic truth coincides with classical truth at every node of the Kripke model, and all of the homomorphisms are (classical) elementary embeddings. If this Kripke model is rooted, then intuitionistic truth in the Kripke model coincides with classical truth at the root. Using this "translation," each of the preservation theorems in this paper implies the corresponding classical theorem. One intuitionistic preservation theorem with this property has already appeared in [6]. This theorem is an intuitionistic analogue of the Łoś-Tarski Theorem, and it is included in this paper with a different proof (see Corollary 4.12).

1.3 A first-order language We consider a first-order language $\mathcal{L}$ to be the set of all formulas that can be built from a symbol set (relation, function, and constant symbols, and variables) using $\top$, $\bot$, $\wedge$, $\vee$, $\rightarrow$, $\exists$, and $\forall$. In this paper, we consider only languages that include $=$ as a binary relation, interpreted as real equality in every model. Symbols $\top$ and $\bot$ are both atoms and nullary connectives. Negation $\neg\varphi$ is short for $\varphi \rightarrow \bot$, and bi-implication $\varphi \leftrightarrow \psi$ is short for $(\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$. A list of constant symbols or variables $t_1, \ldots, t_n$ is abbreviated as $\mathbf{t}$. If C is an arbitrary set of constant symbols, then $\mathcal{L}(C)$ is the language $\mathcal{L}$ extended by all constant symbols in C. $\mathcal{A}\mathrm{t} \subseteq \mathcal{L}$ is the set of atomic formulas in $\mathcal{L}$. Analogously, $\mathcal{A}\mathrm{t}(C) \subseteq \mathcal{L}(C)$ is the set of atomic formulas in $\mathcal{L}(C)$, and so on. Fraktur letters, $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ ..., represent either classical models or Kripke models. If $\mathfrak{A}$ is a classical model, then the domain of $\mathfrak{A}$ is denoted by the corresponding Latin letter A, and $\mathcal{L}(A)$ is the language $\mathcal{L}$ extended by a new constant symbol for every element in A. The symbol $\vDash$ denotes classical satisfaction in a model and is defined for sentences (closed formulas) only. If $\mathfrak{A}$ is a classical model, then $\mathrm{Th}(\mathfrak{A}) = \{\varphi \in \mathcal{L}(A) : \mathfrak{A} \vDash \varphi\}$ is the elementary diagram of $\mathfrak{A}$. This notation is convenient, since we can write $\mathrm{Th}(\mathfrak{A}) \cap \mathcal{L}$ for the complete theory of $\mathfrak{A}$ over $\mathcal{L}$, we can write $\mathrm{Th}(\mathfrak{A}) \cap \mathcal{A}\mathrm{t}(A)$ for the positive atomic diagram of $\mathfrak{A}$, and so on. The symbol $\vdash$ denotes intuitionistic derivability and is defined for sentences only. If $\Gamma \subseteq \mathcal{L}$ is a set of sentences, then $\mathrm{Th}(\Gamma) = \{\varphi \in \mathcal{L} : \Gamma \vdash \varphi\}$ is the deductive closure of Γ over $\mathcal{L}$. An (intuitionistic) theory is a set of sentences Γ such that $\mathrm{Th}(\Gamma) = \Gamma$.

2 Kripke Models

Let $\mathcal{L}$ be a first-order language, and let $\mathbf{M}(\mathcal{L})$ be the category of all classical models for the language $\mathcal{L}$, with all homomorphisms between them. That is, a morphism in this category is a classical homomorphism in the sense of [9] and [5]. Let $\mathbf{A}$ be an arbitrary small category (in practice, $\mathbf{A}$ is often taken to be a small poset category). A Kripke model $\mathfrak{A}$ is a functor $\mathfrak{A} : \mathbf{A} \rightarrow \mathbf{M}(\mathcal{L})$. So for every object $i \in |\mathbf{A}|$, there is an associated classical model $\mathfrak{A}(i) = \mathfrak{A}_i$ in $\mathbf{M}(\mathcal{L})$, and for every arrow $f : i \rightarrow j$ in $\mathbf{A}$, there is an associated morphism $\mathfrak{A}(f) = \mathfrak{A}f : \mathfrak{A}_i \rightarrow \mathfrak{A}_j$. The fact that we use homomorphisms rather than embeddings allows us to interpret the equality predicate as real equality in each node structure $\mathfrak{A}_i$, contrary to [12].

As an alternative to saying that a sentence is intuitionistically true in a Kripke model, we say that the sentence is forced. Let $\mathfrak{A} : \mathbf{A} \rightarrow \mathbf{M}(\mathcal{L})$ be a Kripke model.

For every node $i \in |\mathbf{A}|$ and for every sentence $\varphi \in \mathcal{L}(A_i)$, we define the forcing relation $\Vdash^{\mathfrak{A}}$ inductively as follows:

1. $i \Vdash^{\mathfrak{A}} \varphi \;\Leftrightarrow\; \mathfrak{A}_i \vDash \varphi$, for all (atomic) sentences $\varphi \in \mathcal{A}t(A_i)$,
2. $i \Vdash^{\mathfrak{A}} \varphi \wedge \psi \;\Leftrightarrow\; i \Vdash^{\mathfrak{A}} \varphi$ and $i \Vdash^{\mathfrak{A}} \psi$,
3. $i \Vdash^{\mathfrak{A}} \varphi \vee \psi \;\Leftrightarrow\; i \Vdash^{\mathfrak{A}} \varphi$ or $i \Vdash^{\mathfrak{A}} \psi$,
4. $i \Vdash^{\mathfrak{A}} \varphi \to \psi \;\Leftrightarrow\;$ for all $f : i \to j$, if $j \Vdash^{\mathfrak{A}} \varphi^f$ then $j \Vdash^{\mathfrak{A}} \psi^f$,
5. $i \Vdash^{\mathfrak{A}} \forall x\, \varphi(x) \;\Leftrightarrow\;$ for all $f : i \to j$ and for all $a \in A_j$, $j \Vdash^{\mathfrak{A}} \varphi^f(a)$, and
6. $i \Vdash^{\mathfrak{A}} \exists x\, \varphi(x) \;\Leftrightarrow\; i \Vdash^{\mathfrak{A}} \varphi(a)$ for some $a \in A_i$,

where $\varphi^f \in \mathcal{L}(A_j)$ is constructed from $\varphi \in \mathcal{L}(A_i)$ by replacing all constant symbols $a \in A_i$ in φ by $(\mathfrak{A}f)(a) \in A_j$.

A sentence $\varphi \in \mathcal{L}(A_i)$ is *true* at node $i \in |\mathbf{A}|$ if $\mathfrak{A}_i \vDash \varphi$. A sentence $\varphi \in \mathcal{L}(A_i)$ is *forced* at node $i \in |\mathbf{A}|$ if $i \Vdash^{\mathfrak{A}} \varphi$. A sentence $\varphi \in \mathcal{L}$ is forced in the Kripke model $\mathfrak{A}$, written $\mathfrak{A} \Vdash \varphi$, if $i \Vdash^{\mathfrak{A}} \varphi$ for all $i \in |\mathbf{A}|$. If $\Gamma \subseteq \mathcal{L}$ is a set of sentences, then $\mathfrak{A} \Vdash \Gamma$ if and only if $\mathfrak{A} \Vdash \varphi$ for all $\varphi \in \Gamma$. It is easy to verify that sentences in Kripke models are *persistent*: that is, for all $f : i \to j$ in $\mathbf{A}$ and for all $\varphi \in \mathcal{L}(A_i)$, if $i \Vdash^{\mathfrak{A}} \varphi$, then $j \Vdash^{\mathfrak{A}} \varphi^f$. For the purposes of this paper, we say that a Kripke model $\mathfrak{A}$ is *rooted* if there exists a unique $r \in |\mathbf{A}|$ such that for all $i \in |\mathbf{A}|$, there exists an $f : r \to i$ in $\mathbf{A}$. If $\mathfrak{A}$ is a rooted Kripke model with root r, then for all sentences $\varphi \in \mathcal{L}$, we have $\mathfrak{A} \Vdash \varphi$ if and only if $r \Vdash^{\mathfrak{A}} \varphi$. If we want to refer to the root node of a rooted Kripke model explicitly, we will write $(\mathfrak{A}, r)$ to denote a rooted Kripke model with root node r.

Definition 2.1 Let $\mathfrak{A}$ be a Kripke model. Then for all $i \in |\mathbf{A}|$, we define $\mathrm{Th}(\mathfrak{A}, i) = \{\varphi \in \mathcal{L}(A_i) : i \Vdash^{\mathfrak{A}} \varphi\}$.

If $(\mathfrak{A}, r)$ is a rooted Kripke model, then $\mathrm{Th}(\mathfrak{A}, r) = \{\varphi \in \mathcal{L}(A_r) : (\mathfrak{A}, r) \Vdash \varphi\}$.

A consistent theory Γ over $\mathcal{L}$ is called *prime* if for all sentences $\varphi, \psi \in \mathcal{L}$, we have $\Gamma \vdash \varphi \vee \psi$ if and only if $\Gamma \vdash \varphi$ or $\Gamma \vdash \psi$. Let C be a set of constant symbols in $\mathcal{L}$. A consistent theory Γ over $\mathcal{L}$ is called *C-Henkin* if for all sentences $\exists x\, \varphi(x) \in \mathcal{L}$, we have $\Gamma \vdash \exists x\, \varphi(x)$ if and only if there is a $c \in C$ such that $\Gamma \vdash \varphi(c)$. A theory is called *C-Henkin prime* if it is both C-Henkin and prime. The following two lemmas lead to a completeness theorem for intuitionistic predicate logic.

Lemma 2.2 *Let $\mathcal{L}$ be a first-order language, let C be a set of constant symbols not in $\mathcal{L}$, with $|C| \geq |\mathcal{L}|$, let $\varphi \in \mathcal{L}$ be a sentence, and let Γ be a theory over $\mathcal{L}$ such that $\Gamma \nvdash \varphi$. Then there is a C-Henkin prime theory Γ' over $\mathcal{L}(C)$ such that $\Gamma \subseteq \Gamma'$ and $\Gamma' \nvdash \varphi$.*

Proof See [5], Section 5.3. □

Definition 2.3 Let $\mathcal{L}$ be a first-order language, and let X be a set of constant symbols not in $\mathcal{L}$ such that $|X| \geq |\mathcal{L}|$. Let $C \subseteq X$ such that $|X \setminus C| = |X|$, and let Γ be a C-Henkin prime theory. Let r be the ordered pair $\langle \Gamma, C \rangle$. The *canonical rooted Kripke model* $(\mathfrak{A}, r)$ with constants from X and root node $r = \langle \Gamma, C \rangle$ is defined as follows:

1. Let $\mathbf{A}$ be the poset category with objects all pairs $\langle \Gamma', C' \rangle$ such that Γ' is a C'-Henkin prime theory, $\Gamma \subseteq \Gamma'$, $C \subseteq C' \subseteq X$, and $|X \setminus C'| = |X|$, and with $\langle \Gamma', C' \rangle \preceq \langle \Gamma'', C'' \rangle$ if and only if $\Gamma' \subseteq \Gamma''$ and $C' \subseteq C''$.

2. For every $i = \langle \Gamma_i, C_i \rangle \in |\mathbf{A}|$, let $\mathfrak{A}_i$ be the classical model over $\mathcal{L}(C_i)$ defined by
   (a) $A_i = (C_i / \equiv)$, where $a \equiv b$ if and only if $\Gamma_i \vdash a = b$;
   (b) $\mathfrak{A}_i \models \varphi$ if and only if $\Gamma_i \vdash \varphi$, for all $\varphi \in \mathcal{A}t(C_i)$.
3. For every $f : i \to j$ in $\mathbf{A}$ with $i = \langle \Gamma_i, C_i \rangle$, let $\mathfrak{A}f : \mathfrak{A}_i \to \mathfrak{A}_j$ be defined by $\mathfrak{A}f(c^{\mathfrak{A}_i}) = c^{\mathfrak{A}_j}$ for all $c \in C_i$.

Given the canonical rooted Kripke model $(\mathfrak{A}, r)$, we define the *canonical tree Kripke model* $(\mathfrak{A}', r)$ with constants from X and root node $r = \langle \Gamma, C \rangle$ as follows:

1. Let $\mathbf{A}'$ be the poset category with objects all finite sequences $\langle i_0, i_1, \ldots, i_n \rangle$ such that $i_0 = r$, $i_k \in |\mathbf{A}|$, and $i_k \precneqq i_{k+1}$ for $0 \leq k < n$, and with the usual order by string extension.
2. For every $i = \langle i_0, i_1, \ldots, i_n \rangle \in |\mathbf{A}'|$, let $\mathfrak{A}'_i = \mathfrak{A}_{i_n}$.
3. For every $f : i \to j$ in $\mathbf{A}'$ with $i = \langle i_0, \ldots, i_m \rangle$ and $j = \langle i_0, \ldots, i_m, \ldots, i_n \rangle$ and $g : i_m \to i_n$ the unique arrow from i_m to i_n in $\mathbf{A}$, let $\mathfrak{A}'f = \mathfrak{A}g$.

The second half of Definition 2.3 is a special case of the general unraveling construction. Note that if $(\mathfrak{A}', r)$ is a canonical tree model, then $\mathbf{A}'$ is a tree poset of countable height. Also note that every path in $\mathbf{A}'$ is infinite. This has some technical advantages for the induction proofs in Section 4.

Lemma 2.4 *Let $\mathcal{L}$ be a first-order language, let C be a set of constant symbols, and let Γ be a C-Henkin prime theory over $\mathcal{L}(C)$. Let $(\mathfrak{A}, r)$ be a canonical tree model with $r = \langle \Gamma, C \rangle$. Then for all $i = \langle i_0, i_1, \ldots, i_n \rangle \in |\mathbf{A}|$, with $i_n = \langle \Gamma_i, C_i \rangle$, we have*

$$i \Vdash^{\mathfrak{A}} \varphi \Leftrightarrow \Gamma_i \vdash \varphi, \text{ for all } \varphi \in \mathcal{L}(C_i).$$

In particular, we have

$$(\mathfrak{A}, r) \Vdash \varphi \Leftrightarrow \Gamma \vdash \varphi, \text{ for all } \varphi \in \mathcal{L}(C).$$

Proof See [5], Section 5.3. □

Given the fact that IQC is sound for the class of all Kripke models, the Kripke completeness theorem follows as an easy corollary of Lemma 2.2 and Lemma 2.4.

Theorem 2.5 *Let $\mathcal{L}$ be a first-order language, and let Γ be a theory over $\mathcal{L}$. Let $\varphi \in \mathcal{L}$ be a sentence. The following are equivalent:*

1. $\Gamma \vdash \varphi$.
2. *For all Kripke models $\mathfrak{A}$, if $\mathfrak{A} \Vdash \Gamma$, then $\mathfrak{A} \Vdash \varphi$.*
3. *For all canonical tree models $\mathfrak{A}$, if $\mathfrak{A} \Vdash \Gamma$, then $\mathfrak{A} \Vdash \varphi$.*

In view of Theorem 2.5, we could essentially restrict our attention to Kripke models defined over tree posets of countable height. In this paper, however, a Kripke model will continue to be defined over an arbitrary category, as in the definition at the beginning of this section. We use canonical tree models to facilitate our main proof technique, which is induction on the height of nodes in the canonical tree models. We will always clearly state when a Kripke model is a canonical tree model.

Next we define our relations between Kripke models. We define a Kripke submodel as in [6].

Definition 2.6 Let $\mathfrak{A} : \mathbf{A} \to \mathbf{M}(\mathcal{L})$ and $\mathfrak{B} : \mathbf{B} \to \mathbf{M}(\mathcal{L})$ be Kripke models. Then $\mathfrak{A}$ is a *submodel* of $\mathfrak{B}$, written $\mathfrak{A} \subseteq \mathfrak{B}$, if and only if

1. $\mathbf{A}$ is a subcategory of $\mathbf{B}$,
2. for all $i \in |\mathbf{A}|$, the structure $\mathfrak{A}_i$ is a classical submodel of $\mathfrak{B}_i$, and
3. for all $f : i \to j$ in $\mathbf{A}$, $\mathfrak{A}f = (\mathfrak{B}f \restriction A_i)$.

We also say that $\mathfrak{B}$ is an *extension* of $\mathfrak{A}$.

We introduce the following notion of a homomorphism of Kripke models as a generalization of the Kripke submodel.

Definition 2.7 Let $\mathfrak{A} : \mathbf{A} \to \mathbf{M}(\mathcal{L})$ and $\mathfrak{B} : \mathbf{B} \to \mathbf{M}(\mathcal{L})$ be Kripke models. A *homomorphism* of Kripke models $\mathcal{F} : \mathfrak{A} \to \mathfrak{B}$ consists of

1. a functor $F : \mathbf{A} \to \mathbf{B}$, and
2. for all $i \in |\mathbf{A}|$, a classical homomorphism $\mathcal{F}_i : \mathfrak{A}_i \to \mathfrak{B}_{Fi}$ such that
3. for all $f : i \to j$ in $\mathbf{A}$, the following diagram commutes:

$$\begin{array}{ccc} \mathfrak{A}_j & \xrightarrow{\mathcal{F}_j} & \mathfrak{B}_{Fj} \\ \uparrow{\scriptstyle \mathfrak{A}f} & & \uparrow{\scriptstyle \mathfrak{B}Ff} \\ \mathfrak{A}_i & \xrightarrow{\mathcal{F}_i} & \mathfrak{B}_{Fi} \end{array}$$

If each $\mathcal{F}_i : \mathfrak{A}_i \to \mathfrak{B}_{Fi}$ is a classical embedding, we say that $\mathcal{F} : \mathfrak{A} \to \mathfrak{B}$ is a *local embedding* of Kripke models. If, in addition, $F : \mathbf{A} \to \mathbf{B}$ is an embedding, we say that $\mathcal{F}$ is an *embedding* of Kripke models.

In category-theoretic terms, a homomorphism of Kripke models $\mathcal{F} : \mathfrak{A} \to \mathfrak{B}$ is a natural transformation $\mathcal{F} : \mathfrak{A} \stackrel{\cdot}{\to} \mathfrak{B}F$. Note that $\mathfrak{A} \subseteq \mathfrak{B}$ if and only if there exists an *identity embedding* of Kripke models $\mathcal{I}_{\mathfrak{A}} : \mathfrak{A} \to \mathfrak{B}$ such that $I_{\mathfrak{A}} = \mathrm{id}_{\mathbf{A}}$ and $(\mathcal{I}_{\mathfrak{A}})_i = \mathrm{id}_{A_i}$ for every $i \in |\mathbf{A}|$.

We also define a notion of an elementary submodel of a Kripke model.

Definition 2.8 Let $\mathfrak{A}$ and $\mathfrak{B}$ be Kripke models. Then $\mathfrak{A}$ is an *elementary submodel* of $\mathfrak{B}$, written $\mathfrak{A} \preceq \mathfrak{B}$, if and only if $\mathfrak{A} \subseteq \mathfrak{B}$ and for all $i \in |\mathbf{A}|$, $\mathrm{Th}(\mathfrak{B}, i) \cap \mathcal{L}(A_i) = \mathrm{Th}(\mathfrak{A}, i)$.

In particular, if $(\mathfrak{A}, r) \preceq (\mathfrak{B}, r)$ are rooted Kripke models with $\mathbf{A}$ and $\mathbf{B}$ sharing their root node r, then for all $\varphi \in \mathcal{L}(A_r)$, $(\mathfrak{A}, r) \Vdash \varphi$ if and only if $(\mathfrak{B}, r) \Vdash \varphi$.

Finally, we define a sandwich of Kripke models.

Definition 2.9 Let $\mathfrak{A}$, $\mathfrak{B}$, and $\mathfrak{C}$ be Kripke models. Then $\mathcal{F} : \langle \mathfrak{A}, \mathfrak{B}, \mathfrak{C} \rangle$ is a *sandwich* if $\mathfrak{A} \subseteq \mathfrak{B}$, $\mathfrak{A} \preceq \mathfrak{C}$, and $\mathcal{F} : (\mathfrak{B} \restriction \mathbf{A}) \to \mathfrak{C}$ is a homomorphism of Kripke models such that $F = \mathrm{id}_{\mathbf{A}}$ and $(\mathcal{F}_i \restriction A_i) = \mathrm{id}_{A_i}$ for all $i \in |\mathbf{A}|$.

If $(\mathfrak{B} \restriction \mathbf{A}) \subseteq \mathfrak{C}$, then we write $\langle \mathfrak{A}, \mathfrak{B}, \mathfrak{C} \rangle$ for $\mathcal{I}_{(\mathfrak{B} \restriction \mathbf{A})} : \langle \mathfrak{A}, \mathfrak{B}, \mathfrak{C} \rangle$. Note that if $\mathcal{F} : \langle \mathfrak{A}, \mathfrak{B}, \mathfrak{C} \rangle$ is a sandwich and $\mathcal{F} : (\mathfrak{B} \restriction \mathbf{A}) \to \mathfrak{C}$ is an embedding of Kripke models, then we may assume up to isomorphism that $\langle \mathfrak{A}, \mathfrak{B}, \mathfrak{C} \rangle$ is a sandwich.

3 Formula Classes

In this section, we define the subsets of $\mathcal{L}$ for which we prove syntactic preservation theorems in Section 4. In particular, we define two syntactic operators $\mathcal{U}(\cdot, \cdot)$ and $\mathcal{E}(\cdot)$ that act like universal and existential closure in an intuitionistic setting, and we use these operators to define formula classes and formula hierarchies for IQC. Each of these formula hierarchies reduces to the corresponding classical formula hierarchy $\forall_n$ or $\exists_n$ in the presence of classical logic (for sufficiently large n), in a way that will be made precise below. To compare sets of formulas, we introduce the following definition.

Definition 3.1 Let $\mathcal{L}$ be a first-order language. Let $\Gamma \subseteq \mathcal{L}$ be a set of sentences, and let $\Phi \subseteq \mathcal{L}$ and $\Psi \subseteq \mathcal{L}$ be sets of formulas. We say that Φ is *equivalent over* Γ to Ψ if for every $\varphi(\mathbf{x}) \in \Phi$, there is a $\psi(\mathbf{x}) \in \Psi$ such that $\Gamma \vdash \forall \mathbf{x}\,(\varphi(\mathbf{x}) \leftrightarrow \psi(\mathbf{x}))$, and for every $\psi(\mathbf{x}) \in \Psi$, there is a $\varphi(\mathbf{x}) \in \Phi$ such that $\Gamma \vdash \forall \mathbf{x}\,(\varphi(\mathbf{x}) \leftrightarrow \psi(\mathbf{x}))$. If Φ is equivalent over $\varnothing$ to Ψ, we simply say that Φ is *equivalent* to Ψ.

Let $\mathcal{L}$ be a first-order language, and let CQC denote the set of all classical tautologies over $\mathcal{L}$. In this paper, we will assume that all sets of formulas $\Phi \subseteq \mathcal{L}$ are closed under proper substitutions of variables for free variables. If $\Phi \subseteq \mathcal{L}$ is a set of formulas and $A \subseteq \{\wedge, \vee, \rightarrow, \forall, \exists\}$, then $A\Phi$ is the closure of the formulas in Φ under the connectives in A. We write $\neg\Phi$ to denote the set of negations of formulas in Φ. That is, $\neg\Phi = \{\varphi \rightarrow \bot : \varphi \in \Phi\}$. Note that $\neg\Phi$ does not denote the closure of Φ under negation, and that Φ and $\neg\Phi$ may be disjoint. Also note that $A\Phi$ and $\neg\Phi$ remain closed under proper substitutions of variables for free variables.

Definition 3.2 Let $\Phi \subseteq \mathcal{L}$. Let the set $\mathcal{E}(\Phi)$ of *existential-Φ formulas* be defined as $\mathcal{E}(\Phi) := \{\wedge, \vee, \exists\}\Phi$.

Observe that $\mathcal{E}(\mathcal{E}(\Phi)) = \mathcal{E}(\Phi)$, and if $\Phi \subseteq \Phi'$ then $\mathcal{E}(\Phi) \subseteq \mathcal{E}(\Phi')$.

Definition 3.3 Let $\Phi, \Psi \subseteq \mathcal{L}$. Let the set $\mathcal{U}(\Phi, \Psi)$ of *universal-(Φ, Ψ) formulas* be the smallest set such that

1. $\Phi \subseteq \mathcal{U}(\Phi, \Psi)$,
2. $\varphi, \varphi' \in \mathcal{U}(\Phi, \Psi) \Rightarrow \varphi \wedge \varphi',\ \varphi \vee \varphi' \in \mathcal{U}(\Phi, \Psi)$,
3. $\psi \in \Psi,\ \varphi \in \mathcal{U}(\Phi, \Psi) \Rightarrow \psi \rightarrow \varphi \in \mathcal{U}(\Phi, \Psi)$, and
4. $\varphi \in \mathcal{U}(\Phi, \Psi) \Rightarrow \forall x\, \varphi \in \mathcal{U}(\Phi, \Psi)$.

It is easy to prove that $\mathcal{U}(\Phi, \Psi)$ is equivalent over CQC to $\{\wedge, \vee, \forall\}(\Phi \cup \neg\Psi)$. Observe that $\mathcal{U}(\mathcal{U}(\Phi, \Psi), \Psi) = \mathcal{U}(\Phi, \Psi)$, and if $\Phi \subseteq \Phi'$ and $\Psi \subseteq \Psi'$ then $\mathcal{U}(\Phi, \Psi) \subseteq \mathcal{U}(\Phi', \Psi')$. Also note that $\mathcal{U}(\Phi, \Psi)$ remains closed under proper substitutions of variables for free variables.

Lemma 3.4 *$\mathcal{U}(\Phi, \Psi)$ is equivalent to $\mathcal{U}(\Phi, \mathcal{E}(\Psi))$.*

Proof This follows from the following intuitionistic tautologies:

1. $\vdash ((\psi \wedge \psi') \rightarrow \varphi) \leftrightarrow (\psi \rightarrow (\psi' \rightarrow \varphi))$,
2. $\vdash ((\psi \vee \psi') \rightarrow \varphi) \leftrightarrow ((\psi \rightarrow \varphi) \wedge (\psi' \rightarrow \varphi))$, and
3. $\vdash (\exists x\, \psi(x) \rightarrow \varphi) \leftrightarrow \forall x\, (\psi(x) \rightarrow \varphi)$, where x is not free in φ. $\square$

The following formula classes are applications of the operators $\mathcal{E}(\cdot)$ and $\mathcal{U}(\cdot, \cdot)$ to some special subsets of $\mathcal{L}$.

Definition 3.5 Let $\mathcal{E}^+ := \mathcal{E}(\mathcal{A}t)$ be the set of *existential-positive formulas*.

Note that $\mathcal{E}^+$ is equivalent over CQC to the set of classical existential-positive formulas $\exists^+$.

Definition 3.6 Let $\mathcal{U}^- := \mathcal{U}(\{\bot\}, \mathcal{A}t)$ be the set of *universal-negative formulas*.

Note that $\mathcal{U}^-$ is equivalent over CQC to $\{\wedge, \vee, \forall\}(\neg\, \mathcal{A}t)$, which is equivalent over CQC to $\neg\exists^+$.

Definition 3.7 Let $\mathcal{U} := \mathcal{U}(\mathcal{A}\mathrm{t}, \mathcal{A}\mathrm{t})$ be the set of *universal formulas*.

Note that $\mathcal{U}$ is equivalent over CQC to $\forall_1$. Moreover, the set $\mathcal{U}$ defined here is identical to the set $\mathcal{U}$ defined in [6].

Definition 3.8 Let $\mathcal{U}\mathcal{E}^+ := \mathcal{U}(\mathcal{E}(\mathcal{A}\mathrm{t}), \mathcal{A}\mathrm{t})$.

Note that $\mathcal{U}\mathcal{E}^+$ is equivalent over CQC to $\{\wedge, \vee, \forall\}(\exists^+ \cup \neg\exists^+)$.

There is no prenex-normal-form theorem for intuitionistic predicate logic. In searching for intuitionistic analogues of the $\forall_n$ and $\exists_n$ hierarchies, we may expect to find more than one useful notion. From a model-theoretic point of view, it seems desirable to obtain intuitionistic formula hierarchies that, in addition to providing a normal-form theorem, have preservation properties similar to those of the classical $\forall_n$ and $\exists_n$ hierarchies. Here is one possibility.

Definition 3.9 ($\mathcal{U}_n$ and $\mathcal{E}_n$ formula hierarchies) Let $\mathcal{E}_0 := \{\wedge, \vee\}\,\mathcal{A}\mathrm{t}$, and let $\mathcal{U}_0$ be the smallest set such that

1. $\mathcal{A}\mathrm{t} \subseteq \mathcal{U}_0$,
2. $\varphi, \varphi' \in \mathcal{U}_0 \;\Rightarrow\; \varphi \wedge \varphi',\ \varphi \vee \varphi' \in \mathcal{U}_0$, and
3. $\psi \in \mathcal{A}\mathrm{t},\ \varphi \in \mathcal{U}_0 \;\Rightarrow\; \psi \to \varphi \in \mathcal{U}_0$.

For all $n \geq 1$, let $\mathcal{E}_n := \mathcal{E}(\mathcal{U}_{n-1})$, and let $\mathcal{U}_n := \mathcal{U}(\mathcal{E}_{n-1}, \mathcal{E}_{n-1})$.

Note that $\mathcal{U}_1$ is equivalent to $\mathcal{U}$. Also note that $\mathcal{E}_n$ is equivalent over CQC to $\exists_n$ for all $n \geq 1$, and that $\mathcal{U}_n$ is equivalent over CQC to $\forall_n$ for all $n \geq 0$. It is easy to prove by induction on n that $\mathcal{U}_{n-1} \subseteq \mathcal{U}_n$ and $\mathcal{E}_{n-1} \subseteq \mathcal{E}_n$ for all $n \geq 1$. Moreover, we have the following normal-form theorem.

Theorem 3.10 $\bigcup_{n<\omega} \mathcal{U}_n = \bigcup_{n<\omega} \mathcal{E}_n = \mathcal{L}$.

Proof Apply the following closure properties, which are immediate from the definitions:

1. $\mathcal{A}\mathrm{t} \subseteq \mathcal{E}_0 \subseteq \mathcal{U}_0$
2. $\mathcal{U}_{n-1} \subseteq \mathcal{E}_n$
3. $\mathcal{E}_{n-1} \subseteq \mathcal{U}_n$
4. $\varphi, \varphi' \in \mathcal{E}_n \;\Rightarrow\; \varphi \wedge \varphi', \varphi \vee \varphi' \in \mathcal{E}_n$
5. $\varphi, \varphi' \in \mathcal{U}_n \;\Rightarrow\; \varphi \wedge \varphi', \varphi \vee \varphi' \in \mathcal{U}_n$
6. $\varphi \in \mathcal{E}_n \;\Rightarrow\; \exists x\, \varphi \in \mathcal{E}_n$
7. $\varphi \in \mathcal{U}_n \;\Rightarrow\; \forall x\, \varphi \in \mathcal{U}_n$
8. $\psi \in \mathcal{E}_{n-1}, \varphi \in \mathcal{U}_n \;\Rightarrow\; \psi \to \varphi \in \mathcal{U}_n$ □

As an alternative to Definition 3.9, we define another formula hierarchy as follows.

Definition 3.11 ($\mathcal{U}_n^*$ and $\mathcal{E}_n^*$ formula hierarchies) Let $\mathcal{E}_0^* := \mathcal{E}_0$ and $\mathcal{U}_0^* := \mathcal{E}_0$. For all $n \geq 1$, let $\mathcal{E}_n^* := \mathcal{E}(\mathcal{U}_{n-1}^*)$, and let $\mathcal{U}_n^* := \mathcal{U}(\mathcal{E}_{n-1}^*, \mathcal{E}_n^*)$.

Note that $\mathcal{E}_1^* = \mathcal{E}^+$ and $\mathcal{U}_1^* = \mathcal{U}_1$. Also note that $\mathcal{E}_n^*$ is equivalent over CQC to $\exists_n$ for all $n \geq 2$, and that $\mathcal{U}_n^*$ is equivalent over CQC to $\forall_n$ for all $n \geq 1$. It is easy to prove by induction on n that $\mathcal{U}_{n-1}^* \subseteq \mathcal{U}_n^*$ and $\mathcal{E}_{n-1}^* \subseteq \mathcal{E}_n^*$ for all $n \geq 1$. We also have $\neg\, \mathcal{U}_{n-1}^* \subseteq \mathcal{U}_n^*$ for all $n \geq 1$, and again $\bigcup_{n<\omega} \mathcal{U}_n^* = \bigcup_{n<\omega} \mathcal{E}_n^* = \mathcal{L}$.

The Φ_n formula hierarchy of Burr [3] can be defined using the $\mathcal{U}(\cdot,\cdot)$ and $\mathcal{E}(\cdot)$ operators as follows: $\Phi_0 :=$ all quantifier-free formulas, $\Phi_1 := \exists_1$, $\Phi_2 := \forall_2$, and $\Phi_n := \mathcal{U}(\Phi_{n-1} \cup \mathcal{E}(\Phi_{n-2}), \Phi_{n-1})$ for all $n \geq 3$. Let us define the following variant of the Burr Hierarchy: $\Phi_0^* :=$ all quantifier-free formulas, $\Phi_1^* := \forall_1$, $\Phi_2^* := \forall_2$, and $\Phi_n^* := \mathcal{U}(\Phi_{n-1}^* \cup \mathcal{E}(\Phi_{n-2}^*), \Phi_{n-1}^*)$ for all $n \geq 3$. Then we have the following theorem.

Theorem 3.12 *Let* $\mathrm{HA}^<$ *be the theory of Heyting Arithmetic over a first-order language* $\mathcal{L}$ *containing* $<$. *Then* $\mathcal{U}_n^*$ *is equivalent over* $\mathrm{HA}^<$ *to* Φ_n^* *for all* $n \geq 0$.

Proof The proof is by induction on n. It can be shown that every quantifier-free formula in $\mathcal{L}$ is equivalent over $\mathrm{HA}^<$ to a formula in $\{\wedge, \vee\}\,\mathcal{A}\mathrm{t}$. Using this fact, it is easy to see that $\mathcal{U}_0^*$, $\mathcal{U}_1^*$, and $\mathcal{U}_2^*$ are equivalent over $\mathrm{HA}^<$ to Φ_0^*, Φ_1^*, and Φ_2^*, respectively. Let $n \geq 3$, and suppose that $\mathcal{U}_{n-2}^*$ and $\mathcal{U}_{n-1}^*$ are equivalent over $\mathrm{HA}^<$ to Φ_{n-2}^* and Φ_{n-1}^*, respectively. Then, since $\mathcal{U}_n^*$ is equivalent to $\mathcal{U}(\mathcal{U}_{n-1}^* \cup \mathcal{E}(\mathcal{U}_{n-2}^*), \mathcal{U}_{n-1}^*)$, we have that $\mathcal{U}_n^*$ is equivalent over $\mathrm{HA}^<$ to $\mathcal{U}(\Phi_{n-1}^* \cup \mathcal{E}(\Phi_{n-2}^*), \Phi_{n-1}^*) := \Phi_n^*$. So $\mathcal{U}_n^*$ is equivalent over $\mathrm{HA}^<$ to Φ_n^* for all $n \geq 0$. □

It is easy to define other formula hierarchies having similar properties using the $\mathcal{U}(\cdot,\cdot)$ and $\mathcal{E}(\cdot)$ operators. For example, we can define the following hierarchy of semi-positive formulas.

Definition 3.13 ($\mathcal{U}_n^{\mathcal{S}}$ and $\mathcal{E}_n^{\mathcal{S}}$ formula hierarchies) Let $\mathcal{E}_0^{\mathcal{S}} := \mathcal{E}_0$ and $\mathcal{U}_0^{\mathcal{S}} := \mathcal{U}_0$. For all $n \geq 1$, let $\mathcal{E}_n^{\mathcal{S}} := \mathcal{E}(\mathcal{U}_{n-1}^{\mathcal{S}})$, and let $\mathcal{U}_n^{\mathcal{S}} := \mathcal{U}(\mathcal{E}_{n-1}^{\mathcal{S}}, \mathcal{A}\mathrm{t})$.

Note that $\mathcal{U}_n^{\mathcal{S}} = \mathcal{U}_n \cap \mathcal{S}$ and $\mathcal{E}_n^{\mathcal{S}} = \mathcal{E}_n \cap \mathcal{S}$ for all $n \geq 0$, where $\mathcal{S} \subseteq \mathcal{L}$ is the set of semi-positive formulas in $\mathcal{L}$, as in [13]. Again, note that $\mathcal{E}_n^{\mathcal{S}}$ is equivalent over CQC to $\exists_n$ for all $n \geq 1$ and that $\mathcal{U}_n^{\mathcal{S}}$ is equivalent over CQC to $\forall_n$ for all $n \geq 0$. In this case, however, we have $\bigcup_{n<\omega} \mathcal{U}_n^{\mathcal{S}} = \bigcup_{n<\omega} \mathcal{E}_n^{\mathcal{S}} = \mathcal{S}$. Note that $\mathcal{S}$ is equivalent over CQC to $\mathcal{L}$.

4 Syntactic Preservation Theorems

In this section, we prove four generalized syntactic preservation theorems. The first is for sentences in $\mathcal{U}(\Phi, \Psi)$ with $\mathcal{A}\mathrm{t} \subseteq \Psi$. The second is for sentences in $\mathcal{U}(\Phi, \Psi)$ with $\mathcal{A}\mathrm{t} \subseteq \Phi$ and $\mathcal{A}\mathrm{t} \subseteq \Psi$. The third is for sentences in $\mathcal{U}(\mathcal{E}(\Phi), \Psi)$ with $\mathcal{A}\mathrm{t} \subseteq \Phi$ and $\mathcal{A}\mathrm{t} \subseteq \Psi$. The fourth is for sentences in $\mathcal{U}(\mathcal{E}(\Phi), \Psi)$ with $(\mathcal{A}\mathrm{t} \cup \neg\,\mathcal{A}\mathrm{t}) \subseteq \Phi$ and $\mathcal{A}\mathrm{t} \subseteq \Psi$. As corollaries of these theorems, we obtain syntactic preservation theorems for the formula classes $\mathcal{U}^-$, $\mathcal{U}$, $\mathcal{U}\mathcal{E}^+$, and the hierarchies $\mathcal{U}_n$, $\mathcal{U}_n^*$, and $\mathcal{U}_n^{\mathcal{S}}$ defined in Section 3. We begin by stating the following well-known result.

Theorem 4.1 *Let* $\mathfrak{A}$ *be a Kripke model. Then for every* $i \in |\mathbf{A}|$ *and for every sentence* $\varphi \in \mathcal{E}^+(A_i)$,

$$i \Vdash^{\mathfrak{A}} \varphi \Leftrightarrow \mathfrak{A}_i \models \varphi.$$

Proof This follows directly from the definition of forcing. □

The following definition and subsequent lemma are from [12].

Definition 4.2 Let Γ be an intuitionistic theory, and let $\Sigma \subseteq \mathcal{L}$ be a set of sentences. (Γ, Σ) is *consistent* if $\Gamma \nvdash \bigvee_{\alpha<n} \sigma_\alpha$ for all $\{\sigma_\alpha\}_{\alpha<n} \subseteq \Sigma$. The empty disjunction (the case $n = 0$) is taken to be $\perp$.

Lemma 4.3 *Let* $\mathcal{L}$ *be a first-order language, let* C *be a set of constant symbols not in* $\mathcal{L}$, *with* $|C| \geq |\mathcal{L}|$, *let* $\Sigma \subseteq \mathcal{L}$ *be a set of sentences, and let* Γ *be a theory over* $\mathcal{L}$

such that (Γ, Σ) *is consistent. Then there is a* C*-Henkin prime theory* Γ' *over* $\mathcal{L}(C)$ *such that* $\Gamma \subseteq \Gamma'$ *and* (Γ', Σ) *is consistent.*

Proof A straightforward modification of the proof of Lemma 2.2. □

The following lemma is an intuitionistic version of Lemma 3.2.1 in [4]. Like its classical counterpart, Lemma 4.4 is useful for proving general syntactic preservation theorems. If $\Gamma \subseteq \Delta$ are intuitionistic theories over $\mathcal{L}$ and $\Psi \subseteq \mathcal{L}$ is a set of sentences, we say that Δ is *axiomatizable by* Ψ*-sentences over* Γ if $\Delta = \mathrm{Th}(\Gamma \cup (\Delta \cap \Psi))$. If Γ is a theory over $\mathcal{L}$, then Γ^c is the set of sentences in $\mathcal{L} \setminus \Gamma$.

Lemma 4.4 *Let* $\Psi \subseteq \mathcal{L}$ *be closed under disjunction, and let* $\Gamma \subseteq \Delta$ *be intuitionistic theories over* $\mathcal{L}$*. Let* Y *be an arbitrary set of constant symbols. The following are equivalent:*

1. Δ *is axiomatizable by* $\Psi(Y)$*-sentences over* Γ.
2. *For all canonical tree models* $(\mathfrak{A}, r) \Vdash \Gamma$ *and* $(\mathfrak{B}, s) \Vdash \Delta$ *with constants from the same set of constant symbols* X *and root nodes* $r = \langle \Gamma_r, C\rangle$ *and* $s = \langle \Delta_s, D\rangle$ *such that* $C \cap D = Y$*, if* $\Delta_s \cap \Psi(Y) \subseteq \Gamma_r$ *then* $(\mathfrak{A}, r) \Vdash \Delta$.

Proof (1 $\Rightarrow$ 2) follows from Theorem 2.5.

(2 $\Rightarrow$ 1) Suppose (2) holds. Let X be a set of constant symbols not in $\mathcal{L}$ such that $|X| \geq |\mathcal{L}|$, and let $(\mathfrak{A}, r)$ be a canonical tree model with constants from X and root node $r = \langle \Gamma_r, C\rangle$ such that $(\mathfrak{A}, r) \Vdash \Gamma \cup (\Delta \cap \Psi(Y))$.

Claim $(\Delta, \Gamma_r^c \cap \Psi(Y))$ is consistent.

Assume the contrary. Then $\Delta \vdash \bigvee_{\alpha<n} \psi_\alpha$ for some $\{\psi_\alpha\}_{\alpha<n} \subseteq \Gamma_r^c \cap \Psi(Y)$. Since Ψ is closed under disjunction, $\bigvee_{\alpha<n} \psi_\alpha \in \Delta \cap \Psi(Y)$. So $\Gamma_r \vdash \bigvee_{\alpha<n} \psi_\alpha$. Thus, since Γ_r is prime, we have $\Gamma_r \vdash \psi_\alpha$ for some α, where $\psi_\alpha \in \Gamma_r^c$, which is a contradiction. So the claim is proven. Let $D \subseteq X$ be a set of constant symbols such that $|X \setminus D| = |X|$, $C \cap D = Y$, and $|D| \geq |\mathcal{L}(C)|$. By Lemma 4.3, there exists a canonical tree model $(\mathfrak{B}, s)$ with constants from X and root node $s = \langle \Delta_s, D\rangle$ such that $\Delta_s \supseteq \Delta$ and $(\Delta_s, \Gamma_r^c \cap \Psi(Y))$ is consistent. So $\Gamma_r^c \cap \Psi(Y) \subseteq \Delta_s^c$. Thus, we have $\Delta_s \cap \Psi(Y) \subseteq \Gamma_r$. So, by (2), $(\mathfrak{A}, r) \Vdash \Delta$. Thus, by Theorem 2.5, $\Gamma \cup (\Delta \cap \Psi(Y)) \vdash \Delta$. □

In particular, Lemma 4.4 applies when $Y = \varnothing$.

Definition 4.5 Let $\mathcal{L}$ be a first-order language, and let $\Phi, \Psi \subseteq \mathcal{L}$. Let $\mathfrak{A}$ and $\mathfrak{B}$ be Kripke models over $\mathcal{L}$, and let $\mathcal{F} : \mathfrak{A} \to \mathfrak{B}$ be a homomorphism of Kripke models. We say that $\mathcal{F}$ is Φ*-reflecting*, written $\mathcal{F} : \mathfrak{A} \Leftarrow_\Phi \mathfrak{B}$, if for all $i \in |\mathbf{A}|$ and $\varphi \in \Phi(A_i)$,

$$Fi \Vdash^{\mathfrak{B}} \varphi^{\mathcal{F}_i} \;\Rightarrow\; i \Vdash^{\mathfrak{A}} \varphi,$$

where $\varphi^{\mathcal{F}_i} \in \mathcal{L}(B_{Fi})$ is constructed from $\varphi \in \mathcal{L}(A_i)$ by replacing all constant symbols $a \in A_i$ in φ by $\mathcal{F}_i(a) \in B_{Fi}$. We say that $\mathcal{F}$ is Ψ*-preserving*, written $\mathcal{F} : \mathfrak{A} \Rightarrow_\Psi \mathfrak{B}$, if for all $i \in |\mathbf{A}|$ and $\psi \in \Psi(A_i)$,

$$i \Vdash^{\mathfrak{A}} \psi \;\Rightarrow\; Fi \Vdash^{\mathfrak{B}} \psi^{\mathcal{F}_i}.$$

If $\mathfrak{A} \subseteq \mathfrak{B}$, then we write $\mathfrak{A} \Leftarrow_\Phi \mathfrak{B}$ for $\mathcal{I}_{\mathfrak{A}} : \mathfrak{A} \Leftarrow_\Phi \mathfrak{B}$, and we write $\mathfrak{A} \Rightarrow_\Psi \mathfrak{B}$ for $\mathcal{I}_{\mathfrak{A}} : \mathfrak{A} \Rightarrow_\Psi \mathfrak{B}$.

Lemma 4.6 *Let $\Phi, \Psi \subseteq \mathcal{L}$, and let $\mathcal{F} : \mathfrak{A} \to \mathfrak{B}$ be a homomorphism of Kripke models such that $\mathcal{F} : \mathfrak{A} \Lleftarrow_{\Phi} \mathfrak{B}$ and $\mathcal{F} : \mathfrak{A} \Rrightarrow_{\Psi} \mathfrak{B}$. Then*

1. $\mathcal{F} : \mathfrak{A} \Rrightarrow_{\mathcal{E}(\Psi)} \mathfrak{B}$, *and*
2. $\mathcal{F} : \mathfrak{A} \Lleftarrow_{\mathcal{U}(\Phi,\Psi)} \mathfrak{B}$.

Proof (1) follows directly from the definition of forcing. We prove (2) by induction on the complexity of φ, for all i simultaneously. Let $i \in |\mathbf{A}|$, and let $\varphi \in \mathcal{U}(\Phi, \Psi)(A_i)$ be a sentence. Suppose $\varphi \in \Phi(A_i)$, and $Fi \Vdash^{\mathfrak{B}} \varphi^{\mathcal{F}_i}$. Since $\mathcal{F} : \mathfrak{A} \Lleftarrow_{\Phi} \mathfrak{B}$ we have that $i \Vdash^{\mathfrak{A}} \varphi$. The induction steps for $\varphi := \theta \wedge \theta'$ and $\varphi := \theta \vee \theta'$ are obvious. Suppose $Fi \Vdash^{\mathfrak{B}} \psi^{\mathcal{F}_i} \to \theta^{\mathcal{F}_i}$, where $\psi \in \Psi(A_i)$ and $\theta \in \mathcal{U}(\Phi, \Psi)(A_i)$. Let $g : i \to j$ be in $\mathbf{A}$. Suppose $j \Vdash^{\mathfrak{A}} \psi^g$. Since $\psi^g \in \Psi(A_j)$ and $\mathcal{F} : \mathfrak{A} \Rrightarrow_{\Psi} \mathfrak{B}$, we have $Fj \Vdash^{\mathfrak{B}} (\psi^g)^{\mathcal{F}_j}$. Since $\mathcal{F}$ is a homomorphism of Kripke models, we also have that Fg is in $\mathbf{B}$, and $(\psi^g)^{\mathcal{F}_j} = (\psi^{\mathcal{F}_i})^{Fg}$. Thus, since $Fi \Vdash^{\mathfrak{B}} \psi^{\mathcal{F}_i} \to \theta^{\mathcal{F}_i}$ and $Fj \Vdash^{\mathfrak{B}} (\psi^{\mathcal{F}_i})^{Fg}$, we have $Fj \Vdash^{\mathfrak{B}} (\theta^g)^{\mathcal{F}_j}$. So, by induction hypothesis, $j \Vdash^{\mathfrak{A}} \theta^g$. So for all $g : i \to j$ in $\mathbf{A}$, if $j \Vdash^{\mathfrak{A}} \psi^g$ then $j \Vdash^{\mathfrak{A}} \theta^g$. Thus, $i \Vdash^{\mathfrak{A}} \psi \to \theta$. Now suppose $Fi \Vdash^{\mathfrak{B}} \forall x\, \theta^{\mathcal{F}_i}(x)$, where $\theta(x) \in \mathcal{U}(\Phi, \Psi)(A_i)$. Let $g : i \to j$ be in $\mathbf{A}$. Let $a \in A_j$. Since $\mathcal{F}$ is a homomorphism of Kripke models, Fg is in $\mathbf{B}$, $\mathcal{F}_j(a) \in B_{Fj}$, and $(\theta^g)^{\mathcal{F}_j} = (\theta^{\mathcal{F}_i})^{Fg}$. So $Fj \Vdash^{\mathfrak{B}} (\theta^g)^{\mathcal{F}_j}(a)$. By induction hypothesis, we have $j \Vdash^{\mathfrak{A}} \theta^g(a)$. So for all $g : i \to j$ in $\mathbf{A}$ and for all $a \in A_j$, we have $j \Vdash^{\mathfrak{A}} \theta^g(a)$. Thus, $i \Vdash^{\mathfrak{A}} \forall x\, \theta(x)$. This completes the induction on the complexity of φ. □

Let $\mathcal{F} : \mathfrak{A} \to \mathfrak{B}$ be a homomorphism of Kripke models. Then $\mathcal{F} : \mathfrak{A} \Rrightarrow_{\mathcal{A}t} \mathfrak{B}$ by Theorem 4.1, and $\mathcal{F} : \mathfrak{A} \Lleftarrow_{\{\perp\}} \mathfrak{B}$ trivially. Thus, by Lemma 4.6, we have $\mathcal{F} : \mathfrak{A} \Rrightarrow_{\mathcal{E}^+} \mathfrak{B}$ and $\mathcal{F} : \mathfrak{A} \Lleftarrow_{\mathcal{U}^-} \mathfrak{B}$.

The following lemma is our first main result. Used together with Lemma 4.4, it provides the groundwork for all of the syntactic preservation theorems in this paper.

Lemma 4.7 *Let $\mathcal{L}$ be a first-order language, and let $\Phi, \Psi \subseteq \mathcal{L}$ such that $\mathcal{A}t \subseteq \Psi$. Let X be a set of constant symbols not in $\mathcal{L}$, with $|X| \geq |\mathcal{L}|$. Let $(\mathfrak{A}, r)$ and $(\mathfrak{B}, s)$ be canonical tree models with constants from X and root nodes $r = \langle \Gamma_r, C_r \rangle$ and $s = \langle \Delta_s, D_s \rangle$ such that $C_r \cap D_s = Y$. If $\Delta_s \cap \mathcal{U}(\Phi, \Psi)(Y) \subseteq \Gamma_r$, then there exists a homomorphism of Kripke models $\mathcal{F} : (\mathfrak{A}, r) \to (\mathfrak{B}, s)$ such that $\mathcal{F} : (\mathfrak{A}, r) \Lleftarrow_{\mathcal{U}(\Phi,\Psi)} (\mathfrak{B}, s)$ and $\mathcal{F} : (\mathfrak{A}, r) \Rrightarrow_{\Psi} (\mathfrak{B}, s)$.*

Proof Let X be a set of constant symbols not in $\mathcal{L}$ such that $|X| \geq |\mathcal{L}|$. Let $(\mathfrak{A}, r)$ and $(\mathfrak{B}, s)$ be canonical tree models with constants from X and root nodes $r = \langle \Gamma_r, C_r \rangle$ and $s = \langle \Delta_s, D_s \rangle$ such that $C_r \cap D_s = Y$. We denote an arbitrary node $i \in |\mathbf{A}|$ as $i = \langle r, i_1, \ldots, i_m \rangle$ with $i_m = \langle \Gamma_i, C_i \rangle$ and an arbitrary node $j \in |\mathbf{B}|$ as $j = \langle s, j_1, \ldots, j_n \rangle$ with $j_n = \langle \Delta_j, D_j \rangle$. Then for every $i \in |\mathbf{A}|$, $\mathfrak{A}_i$ is a classical model defined over the language $\mathcal{L}(C_i)$, and for every $j \in |\mathbf{B}|$, $\mathfrak{B}_j$ is a classical model defined over the language $\mathcal{L}(D_j)$. Suppose that $\Delta_s \cap \mathcal{U}(\Phi, \Psi)(Y) \subseteq \Gamma_r$. We define a homomorphism of Kripke models $\mathcal{F} : (\mathfrak{A}, r) \to (\mathfrak{B}, s)$ with $\mathcal{F} : (\mathfrak{A}, r) \Lleftarrow_{\mathcal{U}(\Phi,\Psi)} (\mathfrak{B}, s)$ and $\mathcal{F} : (\mathfrak{A}, r) \Rrightarrow_{\Psi} (\mathfrak{B}, s)$, by induction on the height of nodes in $\mathbf{A}$.

Basis for the induction

Claim 1 $(\Delta_s \cup (\Gamma_r \cap \Psi(C_r)), \Gamma_r^c \cap \mathcal{U}(\Phi, \Psi)(C_r))$ is consistent.

Assume not. Then, by compactness, there are $\varphi(\mathbf{xy}), \psi(\mathbf{xy}) \in \mathcal{L}$, $\mathbf{a} \in Y$, and $\mathbf{b} \in C_r \setminus Y$ such that $\psi(\mathbf{ab}) := \bigwedge_\alpha \psi_\alpha(\mathbf{ab})$ with $\psi_\alpha(\mathbf{ab}) \in \Gamma_r \cap \Psi(C_r)$ for all α, $\varphi(\mathbf{ab}) := \bigvee_\beta \varphi_\beta(\mathbf{ab})$ with $\varphi_\beta(\mathbf{ab}) \in \Gamma_r^c \cap \mathcal{U}(\Phi, \Psi)(C_r)$ for all β, and $\Delta_s \cup \{\psi(\mathbf{ab})\} \vdash \varphi(\mathbf{ab})$. So $\Delta_s \vdash \psi(\mathbf{ab}) \to \varphi(\mathbf{ab})$. Thus, since $\mathbf{b} \notin \mathcal{L}(D_s)$, $\Delta_s \vdash \forall \mathbf{y}\,(\psi(\mathbf{ay}) \to \varphi(\mathbf{ay}))$. Thus, by the definition of $\mathcal{U}(\Phi, \mathcal{E}(\Psi))$ and Lemma 3.4, we have $\forall \mathbf{y}\,(\psi(\mathbf{ay}) \to \varphi(\mathbf{ay})) \in \Delta_s \cap \mathcal{U}(\Phi, \mathcal{E}(\Psi))(Y) \subseteq \Gamma_r$. So $\Gamma_r \vdash \forall \mathbf{y}\,(\psi(\mathbf{ay}) \to \varphi(\mathbf{ay}))$. Thus, since $\Gamma_r \vdash \psi(\mathbf{ab})$, we have $\Gamma_r \vdash \varphi(\mathbf{ab})$. So, since Γ_r is prime, $\Gamma_r \vdash \varphi_\beta(\mathbf{ab})$ for some β, which is a contradiction. So Claim 1 is proven.

Thus, by Lemma 4.3, there is a node $r' \in |\mathbf{B}|$ such that $D_{r'} \supseteq C_r \cup D_s$, $\Delta_{r'} \supseteq \Delta_s \cup (\Gamma_r \cap \Psi(C_r))$, and $(\Delta_{r'}, \Gamma_r^c \cap \mathcal{U}(\Phi, \Psi)(C_r))$ is consistent. Since $\mathcal{A}\mathrm{t} \subseteq \Psi$, we have $\Gamma_r \cap \mathcal{A}\mathrm{t}(C_r) \subseteq \Delta_{r'}$. Thus, $\mathfrak{B}_{r'} \vDash \mathrm{Th}(\mathfrak{A}_r) \cap \mathcal{A}\mathrm{t}(C_r)$ by Theorem 4.1. So there is a classical homomorphism $\mu : \mathfrak{A}_r \to \mathfrak{B}_{r'}$ defined by $\mu(a) = a^{\mathfrak{B}_{r'}}$ for $a \in C_r$. Let $Fr = r'$ and let $\mathcal{F}_r = \mu$. Then $\mathcal{F}_r : \mathfrak{A}_r \to \mathfrak{B}_{Fr}$ is a classical homomorphism, $\Delta_{Fr} \cap \mathcal{U}(\Phi, \Psi)(C_r) \subseteq \Gamma_r$, and $\Gamma_r \cap \Psi(C_r) \subseteq \Delta_{Fr}$. For the induction step, let $i \in |\mathbf{A}|$, and let $j \in |\mathbf{A}|$ be any immediate successor of i. Without loss of generality, we may assume that $(C_j \setminus C_i) \cap D_{Fi} = \varnothing$. Suppose that $\mathcal{F}_i : \mathfrak{A}_i \to \mathfrak{B}_{Fi}$ is a classical homomorphism, $\Delta_{Fi} \cap \mathcal{U}(\Phi, \Psi)(C_i) \subseteq \Gamma_i$, and $\Gamma_i \cap \Psi(C_i) \subseteq \Delta_{Fi}$.

Induction step

Claim 2 $(\Delta_{Fi} \cup (\Gamma_j \cap \Psi(C_j)), \Gamma_j^c \cap \mathcal{U}(\Phi, \Psi)(C_j))$ is consistent.

Assume not. Then, by compactness, there are $\varphi(\mathbf{xy}), \psi(\mathbf{xy}) \in \mathcal{L}$, $\mathbf{a} \in C_i$, and $\mathbf{b} \in C_j \setminus C_i$ such that $\psi(\mathbf{ab}) := \bigwedge_\alpha \psi_\alpha(\mathbf{ab})$ with $\psi_\alpha(\mathbf{ab}) \in \Gamma_j \cap \Psi(C_j)$ for all α, $\varphi(\mathbf{ab}) := \bigvee_\beta \varphi_\beta(\mathbf{ab})$ with $\varphi_\beta(\mathbf{ab}) \in \Gamma_j^c \cap \mathcal{U}(\Phi, \Psi)(C_j)$ for all β, and $\Delta_{Fi} \cup \{\psi(\mathbf{ab})\} \vdash \varphi(\mathbf{ab})$. So $\Delta_{Fi} \vdash \psi(\mathbf{ab}) \to \varphi(\mathbf{ab})$. Since $\mathbf{b} \notin \mathcal{L}(D_{Fi})$, $\Delta_{Fi} \vdash \forall \mathbf{y}\,(\psi(\mathbf{ay}) \to \varphi(\mathbf{ay}))$. Thus, by the definition of $\mathcal{U}(\Phi, \mathcal{E}(\Psi))$ and Lemma 3.4, we have $\forall \mathbf{y}\,(\psi(\mathbf{ay}) \to \varphi(\mathbf{ay})) \in \Delta_{Fi} \cap \mathcal{U}(\Phi, \mathcal{E}(\Psi))(C_i) \subseteq \Gamma_i$ by induction hypothesis. So $\Gamma_j \vdash \forall \mathbf{y}\,(\psi(\mathbf{ay}) \to \varphi(\mathbf{ay}))$ by persistence. Thus, since $\Gamma_j \vdash \psi(\mathbf{ab})$, we have $\Gamma_j \vdash \varphi(\mathbf{ab})$. So, since Γ_j is prime, $\Gamma_j \vdash \varphi_\beta(\mathbf{ab})$ for some β, which is a contradiction. So Claim 2 is proven.

Thus, by Lemma 4.3, there is a node $j' \in |\mathbf{B}|$ such that $D_{j'} \supseteq C_j \cup D_{Fi}$, $\Delta_{j'} \supseteq \Delta_{Fi} \cup (\Gamma_j \cap \Psi(C_j))$, and $(\Delta_{j'}, \Gamma_j^c \cap \mathcal{U}(\Phi, \Psi)(C_j))$ is consistent. Since $\mathcal{A}\mathrm{t} \subseteq \Psi$, we have $\Gamma_j \cap \mathcal{A}\mathrm{t}(C_j) \subseteq \Delta_{j'}$. Thus, $\mathfrak{B}_{j'} \vDash \mathrm{Th}(\mathfrak{A}_j) \cap \mathcal{A}\mathrm{t}(C_j)$ by Theorem 4.1. So there is a classical homomorphism $\mu : \mathfrak{A}_j \to \mathfrak{B}_{j'}$ defined by $\mu(a) = a^{\mathfrak{B}_{j'}}$ for $a \in C_j$. Let $Fj = j'$ and let $\mathcal{F}_j = \mu$. Then $\mathcal{F}_j : \mathfrak{A}_j \to \mathfrak{B}_{Fj}$ is a classical homomorphism, $\Delta_{Fj} \cap \mathcal{U}(\Phi, \Psi)(C_j) \subseteq \Gamma_j$, and $\Gamma_j \cap \Psi(C_j) \subseteq \Delta_{Fj}$. Thus, by induction on the height of nodes in $\mathbf{A}$, there exists a homomorphism of Kripke models $\mathcal{F} : (\mathfrak{A}, r) \to (\mathfrak{B}, s)$ such that $\mathcal{F} : \mathfrak{A} \Leftarrow_{\mathcal{U}(\Phi,\Psi)} \mathfrak{B}$ and $\mathcal{F} : \mathfrak{A} \Rrightarrow_{\Psi} \mathfrak{B}$. $\square$

We are now in a position to prove our most general syntactic preservation theorem, which is an intuitionistic analogue of Theorem 2 in [10].

Theorem 4.8 *Let $\Gamma \subseteq \Delta$ be intuitionistic theories over a first-order language $\mathcal{L}$, and let $\Phi, \Psi \subseteq \mathcal{L}$ such that $\mathcal{A}\mathrm{t} \subseteq \Psi$. The following are equivalent:*

1. *Δ is axiomatizable by $\mathcal{U}(\Phi, \Psi)$-sentences over Γ.*

2. *For all Kripke models* $\mathfrak{A} \Vdash \Gamma$ *and* $\mathfrak{B} \Vdash \Delta$*, if there exists a homomorphism of Kripke models* $\mathcal{F} : \mathfrak{A} \to \mathfrak{B}$ *such that* $\mathcal{F} : \mathfrak{A} \Leftarrow_\Phi \mathfrak{B}$ *and* $\mathcal{F} : \mathfrak{A} \Rrightarrow_\Psi \mathfrak{B}$*, then* $\mathfrak{A} \Vdash \Delta$.

Proof (1 $\Rightarrow$ 2) Suppose that Δ is axiomatized by $\mathcal{U}(\Phi, \Psi)$-sentences over Γ. Let $\mathfrak{A}$ and $\mathfrak{B}$ be Kripke models such that $\mathfrak{A} \Vdash \Gamma$ and $\mathfrak{B} \Vdash \Delta$, and let $\mathcal{F} : \mathfrak{A} \to \mathfrak{B}$ be a homomorphism of Kripke models such that $\mathcal{F} : \mathfrak{A} \Leftarrow_\Phi \mathfrak{B}$ and $\mathcal{F} : \mathfrak{A} \Rrightarrow_\Psi \mathfrak{B}$. Then, by Lemma 4.6, we have $\mathcal{F} : \mathfrak{A} \Leftarrow_{\mathcal{U}(\Phi,\Psi)} \mathfrak{B}$. Since $\mathfrak{B} \Vdash \Delta$, it follows that $\mathfrak{A} \Vdash \Gamma \cup (\Delta \cap \mathcal{U}(\Phi, \Psi))$. So $\mathfrak{A} \Vdash \Delta$.

(2 $\Rightarrow$ 1) Suppose that (2) holds. Since $\mathcal{U}(\Phi, \Psi)$ is closed under disjunction, we may apply Lemma 4.4. Let X be a set of constant symbols not in $\mathcal{L}$, with $|X| \geq |\mathcal{L}|$. Let $(\mathfrak{A}, r)$ and $(\mathfrak{B}, s)$ be canonical tree models with constants from X and root nodes $r = \langle \Gamma_r, C_r \rangle$ and $s = \langle \Delta_s, D_s \rangle$ such that $C_r \cap D_s = \varnothing$. Suppose that $(\mathfrak{A}, r) \Vdash \Gamma$, $(\mathfrak{B}, s) \Vdash \Delta$, and $\Delta_s \cap \mathcal{U}(\Phi, \Psi) \subseteq \Gamma_r$. Then, by Lemma 4.7, there exists a homomorphism of Kripke models $\mathcal{F} : (\mathfrak{A}, r) \to (\mathfrak{B}, s)$ such that $\mathcal{F} : (\mathfrak{A}, r) \Leftarrow_\Phi (\mathfrak{B}, s)$ and $\mathcal{F} : (\mathfrak{A}, r) \Rrightarrow_\Psi (\mathfrak{B}, s)$. So, by (2), $(\mathfrak{A}, r) \Vdash \Delta$. Thus, by Lemma 4.4, Δ is axiomatizable by $\mathcal{U}(\Phi, \Psi)$-sentences over Γ. $\square$

The classical Lyndon-Łoś-Tarski Theorem (or the homomorphism preservation theorem) states that a theory $\Delta \supseteq \Gamma \supseteq \mathrm{CQC}$ is axiomatizable by existential-positive sentences over Γ if and only if Δ is preserved under homomorphisms of Γ-models. (Note that these are classical homomorphisms in the sense of [9], and need not be onto.) In dual form, a theory $\Delta \supseteq \Gamma \supseteq \mathrm{CQC}$ is axiomatizable by universal-negative sentences over Γ if and only if for all classical models $\mathfrak{A} \vDash \Gamma$ and $\mathfrak{B} \vDash \Delta$, if there exists a classical homomorphism $f : \mathfrak{A} \to \mathfrak{B}$, then $\mathfrak{A} \vDash \Delta$. As a corollary to Theorem 4.8 we obtain the following direct intuitionistic analogue of the dual of the Lyndon-Łoś-Tarski Theorem.

Corollary 4.9 *Let* $\Gamma \subseteq \Delta$ *be intuitionistic theories over a first-order language* $\mathcal{L}$. *The following are equivalent:*

1. Δ *is axiomatizable by universal-negative sentences over* Γ.
2. *For all Kripke models* $\mathfrak{A} \Vdash \Gamma$ *and* $\mathfrak{B} \Vdash \Delta$*, if there exists a homomorphism of Kripke models* $\mathcal{F} : \mathfrak{A} \to \mathfrak{B}$*, then* $\mathfrak{A} \Vdash \Delta$.

Proof Immediate from Theorem 4.8 and the comments following Lemma 4.6. $\square$

Next we obtain an intuitionistic analogue of the generalized Łoś-Tarski Theorem, which is Theorem 3.5.11(i) in [8]. To state the generalized Łoś-Tarski Theorem, we need to introduce some terminology and notation. A set of formulas is called *regular* if it contains all atoms and negated atoms, is closed under $\wedge$ and $\vee$, and is closed under proper substitutions of variables for free variables. If $\Psi \subseteq \mathcal{L}$ is a set of formulas and $\mathfrak{M}$ and $\mathfrak{N}$ are classical models over $\mathcal{L}$ such that $\mathfrak{M} \subseteq \mathfrak{N}$, then we write $\mathfrak{M} \Rrightarrow_\Psi \mathfrak{N}$ if for all $\psi \in \Psi(M)$, if $\mathfrak{M} \vDash \psi$ then $\mathfrak{N} \vDash \psi$.

The generalized Łoś-Tarski Theorem states that if $\Gamma \subseteq \Delta$ are classical theories and $\Psi \subseteq \mathcal{L}$ is regular, then Δ is axiomatizable by $(\forall\neg\Psi)$-sentences over Γ if and only if for all classical models $\mathfrak{M} \vDash \Gamma$ and $\mathfrak{N} \vDash \Delta$, if $\mathfrak{M} \subseteq \mathfrak{N}$ and $\mathfrak{M} \Rrightarrow_\Psi \mathfrak{N}$, then $\mathfrak{M} \vDash \Delta$. Before we prove an intuitionistic analogue of the generalized Łoś-Tarski Theorem, we need a lemma.

Lemma 4.10 *Let X be a set of constant symbols not in $\mathcal{L}$, with $|X| \geq |\mathcal{L}|$. Let $(\mathfrak{A}, r)$ and $(\mathfrak{B}, s)$ be canonical tree models with constants from X, and let $\mathcal{F} : (\mathfrak{A}, r) \to (\mathfrak{B}, s)$ be a local embedding of Kripke models. Then there exists a canonical tree model $(\mathfrak{B}', s)$ such that $(\mathfrak{B}, s) \preceq (\mathfrak{B}', s)$ and an embedding of Kripke models $\mathcal{F}' : (\mathfrak{A}, r) \to (\mathfrak{B}', s)$.*

Proof Note that if $X \subseteq X'$ are sets of constant symbols and $(\mathfrak{B}, s)$ and $(\mathfrak{B}', s)$ are canonical tree models with constants from X and X', respectively, with $\mathbf{B}$ and $\mathbf{B}'$ sharing their root node s, then $(\mathfrak{B}, s) \preceq (\mathfrak{B}', s)$. Let X be a set of constant symbols not in $\mathcal{L}$ such that $|X| \geq |\mathcal{L}|$. Let $(\mathfrak{A}, r)$ and $(\mathfrak{B}, s)$ be canonical tree models with constants from X, and let $\mathcal{F} : (\mathfrak{A}, r) \to (\mathfrak{B}, s)$ be a local embedding of Kripke models. Choose a sufficiently large set of constant symbols $X' \supseteq X$ not in $\mathcal{L}$ such that $|X'| \geq 2^{|X|}$. Let $(\mathfrak{B}', s)$ be the canonical tree model with constants from X' and root node s.

We define an embedding of Kripke models $\mathcal{F}' : (\mathfrak{A}, r) \to (\mathfrak{B}', s)$ by induction on the height of nodes in $\mathbf{A}$. Let $F'r = Fr$ and $\mathcal{F}'_r = \mathcal{F}_r$. Suppose that $F'i$ and $\mathcal{F}'_i$ are defined for some $i \in |\mathbf{A}|$, and let $\Xi \subseteq |\mathbf{A}|$ be the set of all immediate successors of i. We write $F'i = \langle s, s'_1, \ldots, s'_m \rangle \in |\mathbf{B}'|$ with $s'_m = \langle \Delta'_i, D'_i \rangle$, and for every $j \in \Xi$ we write $Fj = \langle s, s^j_1, \ldots, s^j_n \rangle \in |\mathbf{B}|$ with $s^j_n = \langle \Delta_j, D_j \rangle$. Since $|\Xi| \leq 2^{|X|}$, for every $j \in \Xi$ we may choose a set of new constant symbols $E_j \subseteq X' \setminus X$ such that $|E_j| = |D_j \setminus D_i|$, $E_j \cap E_k = \varnothing$ for all $j \neq k$, and $|X' \setminus \bigcup\{E_j : j \in \Xi\}| = |X'|$. Let $D'_j = D'_i \cup E_j$. Then there is an obvious bijection between D_j and D'_j. Let Δ'_j be the theory over D'_j isomorphic to Δ_j with the isomorphism induced by the bijection between D_j and D'_j. Clearly Δ'_j is D'_j-Henkin prime. Let $F'j \in |\mathbf{B}'|$ be the concatenation of $F'i$ with $\langle \Delta'_j, D'_j \rangle$, and let $\mathcal{F}'_j$ be obtained from $\mathcal{F}_j$ by replacing every element in the range of $\mathcal{F}_j$ by its isomorphic copy in $\mathfrak{B}'_{F'j}$. By induction on the height of nodes in $\mathbf{A}$, we obtain an embedding of Kripke models $\mathcal{F}' : (\mathfrak{A}, r) \to (\mathfrak{B}', s)$. $\square$

The following theorem is a direct intuitionistic analogue of the generalized Łoś-Tarski Theorem.

Theorem 4.11 *Let $\Gamma \subseteq \Delta$ be intuitionistic theories over a first-order language $\mathcal{L}$, and let $\Phi, \Psi \subseteq \mathcal{L}$ such that $\mathcal{A}\mathrm{t} \subseteq \Phi$ and $\mathcal{A}\mathrm{t} \subseteq \Psi$. The following are equivalent:*

1. *Δ is axiomatizable by $\mathcal{U}(\Phi, \Psi)$-sentences over Γ.*
2. *For all Kripke models $\mathfrak{A} \Vdash \Gamma$ and $\mathfrak{B} \Vdash \Delta$, if $\mathfrak{A} \subseteq \mathfrak{B}$, $\mathfrak{A} \Lleftarrow_\Phi \mathfrak{B}$, and $\mathfrak{A} \Rrightarrow_\Psi \mathfrak{B}$, then $\mathfrak{A} \Vdash \Delta$.*

Proof (1 $\Rightarrow$ 2) Immediate from Theorem 4.8.

(2 $\Rightarrow$ 1) Suppose that (2) holds. Let $\mathfrak{A} \Vdash \Gamma$ and $\mathfrak{B} \Vdash \Delta$ be Kripke models, and let $\mathcal{F} : \mathfrak{A} \to \mathfrak{B}$ be a homomorphism of Kripke models such that $\mathcal{F} : \mathfrak{A} \Lleftarrow_\Phi \mathfrak{B}$ and $\mathcal{F} : \mathfrak{A} \Rrightarrow_\Psi \mathfrak{B}$. Since $\mathcal{A}\mathrm{t} \subseteq \Phi$, $\mathcal{F}$ is a local embedding by Theorem 4.1. Thus, by Lemma 4.10, there exists a Kripke model $\mathfrak{B}'$ such that $\mathfrak{B} \preceq \mathfrak{B}'$ and an embedding of Kripke models $\mathcal{F}' : \mathfrak{A} \to \mathfrak{B}'$. So we may assume up to isomorphism that $\mathfrak{A} \subseteq \mathfrak{B}'$, $\mathfrak{A} \Lleftarrow_\Phi \mathfrak{B}'$, and $\mathfrak{A} \Rrightarrow_\Psi \mathfrak{B}'$. So, by (2), $\mathfrak{A} \Vdash \Delta$. Thus, by Theorem 4.8, Δ is axiomatizable by $\mathcal{U}(\Phi, \Psi)$-sentences over Γ. $\square$

The classical Łoś-Tarski Theorem states that a theory $\Delta \supseteq \Gamma \supseteq \mathrm{CQC}$ is axiomatizable by universal sentences over Γ if and only if Δ is preserved under Γ-submodels. That is, a theory $\Delta \supseteq \Gamma \supseteq \mathrm{CQC}$ is axiomatizable by universal sentences over Γ if and only if for all classical models $\mathfrak{A} \models \Gamma$ and $\mathfrak{B} \models \Delta$, if $\mathfrak{A} \subseteq \mathfrak{B}$, then $\mathfrak{A} \models \Delta$. As a corollary to Theorem 4.11 we obtain the following direct intuitionistic analogue of the Łoś-Tarski Theorem.

Corollary 4.12 *Let $\Gamma \subseteq \Delta$ be intuitionistic theories over a first-order language $\mathcal{L}$. The following are equivalent:*

1. *Δ is axiomatizable by universal sentences over Γ.*
2. *For all Kripke models $\mathfrak{A} \Vdash \Gamma$ and $\mathfrak{B} \Vdash \Delta$, if $\mathfrak{A} \subseteq \mathfrak{B}$, then $\mathfrak{A} \Vdash \Delta$.*

Proof Since $\mathfrak{A} \subseteq \mathfrak{B}$ implies $\mathfrak{A} \Leftarrow_{\mathcal{A}t} \mathfrak{B}$ and $\mathfrak{A} \Rightarrow_{\mathcal{A}t} \mathfrak{B}$, this is immediate from Theorem 4.11. □

For an alternate proof of Corollary 4.12, see [6]. In [7], Corollary 4.12 is used to determine the universal fragment of the intuitionistic theory of a special class of Kripke models. See [7], Theorem 5.6.

We now turn our attention to proving a generalized Sandwich Theorem. We need the following extension of Definition 4.5.

Definition 4.13 Let $\mathcal{L}$ be a first-order language, and let $\Psi \subseteq \mathcal{L}$. Let $\mathfrak{A}$ and $\mathfrak{B}$ be Kripke models over $\mathcal{L}$, let $\mathbf{X} \subseteq \mathbf{A}$, and let $\mathcal{F} : (\mathfrak{A} \restriction \mathbf{X}) \to \mathfrak{B}$ be a homomorphism of Kripke models. Then we write $\mathcal{F} : \mathfrak{A} \Rrightarrow^{\mathbf{X}}_{\Psi} \mathfrak{B}$ if for all $i \in |\mathbf{X}|$ and $\psi \in \Psi(A_i)$,

$$i \Vdash^{\mathfrak{A}} \psi \;\Rightarrow\; Fi \Vdash^{\mathfrak{B}} \psi^{\mathcal{F}_i}.$$

If $(\mathfrak{A} \restriction \mathbf{X}) \subseteq \mathfrak{B}$, then we write $\mathfrak{A} \Rrightarrow^{\mathbf{X}}_{\Psi} \mathfrak{B}$ for $\mathcal{I}_{(\mathfrak{A} \restriction \mathbf{X})} : \mathfrak{A} \Rrightarrow^{\mathbf{X}}_{\Psi} \mathfrak{B}$.

Note that $\mathcal{F} : \mathfrak{A} \Rrightarrow^{\mathbf{X}}_{\Psi} \mathfrak{B}$ does not mean the same thing as $\mathcal{F} : (\mathfrak{A} \restriction \mathbf{X}) \Rrightarrow_{\Psi} \mathfrak{B}$, since the latter means that for all $i \in |\mathbf{X}|$ and $\psi \in \Psi(A_i)$, if $i \Vdash^{(\mathfrak{A} \restriction \mathbf{X})} \psi$ then $Fi \Vdash^{\mathfrak{B}} \psi^{\mathcal{F}_i}$.

We also need the following extension of part 1 of Lemma 4.6.

Lemma 4.14 *Let $\Psi \subseteq \mathcal{L}$, let $\mathfrak{A}$ and $\mathfrak{B}$ be Kripke models over $\mathcal{L}$, let $\mathbf{X} \subseteq \mathbf{A}$, and let $\mathcal{F} : (\mathfrak{A} \restriction \mathbf{X}) \to \mathfrak{B}$ be a homomorphism of Kripke models such that $\mathcal{F} : \mathfrak{A} \Rrightarrow^{\mathbf{X}}_{\Psi} \mathfrak{B}$. Then $\mathcal{F} : \mathfrak{A} \Rrightarrow^{\mathbf{X}}_{\mathcal{E}(\Psi)} \mathfrak{B}$.*

Proof This follows directly from the definition of forcing. □

Note that the analogous extension of part 2 of Lemma 4.6 is not true.

The following lemma tells us under what conditions we can build a sandwich from two canonical tree models. It is analogous (and structurally similar) to Lemma 4.7.

Lemma 4.15 *Let $\mathcal{L}$ be a first-order language, and let $\Phi \subseteq \mathcal{L}$ such that $\mathcal{A}t \subseteq \Phi$. Let X be a set of constant symbols not in $\mathcal{L}$, with $|X| \geq |\mathcal{L}|$. Let $(\mathfrak{A}, r)$ and $(\mathfrak{B}, s)$ be canonical tree models with constants from X and root nodes $r = \langle \Gamma_r, X_r \rangle$ and $s = \langle \Delta_s, Y_s \rangle$. Suppose that $(\mathfrak{A}, r) \subseteq (\mathfrak{B}, s)$ and $(\mathfrak{A}, r) \Lleftarrow_{\mathcal{E}(\Phi)} (\mathfrak{B}, s)$. Then there exists a Kripke model $(\mathfrak{C}, r)$ and a homomorphism of Kripke models $\mathcal{F}$ such that $\mathcal{F} : \langle (\mathfrak{A}, r), (\mathfrak{B}, s), (\mathfrak{C}, r) \rangle$ is a sandwich and $\mathcal{F} : (\mathfrak{B}, s) \Rrightarrow^{\mathbf{A}}_{\Phi} (\mathfrak{C}, r)$.*

Proof Let X be a set of constant symbols not in $\mathcal{L}$, with $|X| \geq |\mathcal{L}|$. Let $(\mathfrak{A}, r) \subseteq (\mathfrak{B}, s)$ be canonical tree models with constants from X and root nodes $r = \langle \Gamma_r, X_r \rangle$ and $s = \langle \Delta_s, Y_s \rangle$ such that $(\mathfrak{A}, r) \Lleftarrow_{\mathcal{E}(\Phi)} (\mathfrak{B}, s)$. We first construct a Kripke model $(\mathfrak{D}, t)$ and a local embedding $\mathcal{G} : ((\mathfrak{B}, s) \restriction \mathbf{A}) \to (\mathfrak{D}, t)$ such that $\mathcal{G} : (\mathfrak{B}, s) \Rrightarrow^{\mathbf{A}}_{\Phi} (\mathfrak{D}, t)$ and $\mathrm{Th}(\mathfrak{D}, Gi) \cap \mathcal{L}(A_i) = \mathrm{Th}(\mathfrak{A}, i)$ for all $i \in |\mathbf{A}|$. We proceed by induction on the height of nodes in $\mathbf{A}$. We denote an arbitrary node

$i \in |\mathbf{A}|$ as $i = \langle r, i_1, \ldots, i_m \rangle$ with $i_m = \langle \Gamma_i, X_i \rangle$, and an arbitrary node $j \in |\mathbf{B}|$ as $j = \langle s, j_1, \ldots, j_n \rangle$ with $j_n = \langle \Delta_j, Y_j \rangle$.

Basis for the induction

Claim 1 $(\Gamma_r \cup (\Delta_r \cap \Phi(Y_r)), \Gamma_r^c)$ is consistent.

Assume not. Then, by compactness, there are $\varphi(\mathbf{xy}), \psi(\mathbf{x}) \in \mathcal{L}$, $\mathbf{a} \in X_r$, and $\mathbf{b} \in Y_r \setminus X_r$ such that $\varphi(\mathbf{ab}) := \bigwedge_\alpha \varphi_\alpha(\mathbf{ab})$ with $\varphi_\alpha(\mathbf{ab}) \in \Delta_r \cap \Phi(Y_r)$, $\psi(\mathbf{a}) := \bigvee_\beta \psi_\beta(\mathbf{a})$ with $\psi_\beta(\mathbf{a}) \in \Gamma_r^c$, and $\Gamma_r \cup \{\varphi(\mathbf{ab})\} \vdash \psi(\mathbf{a})$. Since $\mathbf{b} \notin \mathcal{L}(X_r)$, we have $\Gamma_r \cup \{\exists \mathbf{x}\, \varphi(\mathbf{ax})\} \vdash \psi(\mathbf{a})$. Since $\exists \mathbf{x}\, \varphi(\mathbf{ax}) \in \Delta_r \cap \mathcal{E}(\Phi)(X_r)$ and $(\mathfrak{A}, r) \Lleftarrow_{\mathcal{E}(\Phi)} (\mathfrak{B}, s)$, we have $\exists \mathbf{x}\, \varphi(\mathbf{ax}) \in \Gamma_r$. So $\Gamma_r \vdash \psi(\mathbf{a})$. Thus, since Γ_r is prime, $\Gamma_r \vdash \psi_\beta(\mathbf{a})$ for some β, which is a contradiction. So Claim 1 is proven.

It follows by Lemma 4.3 that there exists a canonical tree model $(\mathfrak{D}, t)$ with constants from X and root node $t = \langle \Xi_t, Z_t \rangle$ such that $Z_t \supseteq X_r \cup Y_r$, $\Xi_t \supseteq \Gamma_r \cup (\Delta_r \cap \Phi(Y_r))$, and (Ξ_t, Γ_r^c) is consistent. We denote an arbitrary node $i \in |\mathbf{D}|$ as $i = \langle t, i_1, \ldots, i_n \rangle$ with $i_n = \langle \Xi_i, Z_i \rangle$. Since $\Delta_r \cap \Phi(Y_r) \subseteq \Xi_t$ and $\mathcal{A}\mathrm{t}(Y_r) \subseteq \Phi(Y_r)$, we have $\mathfrak{D}_t \vDash \mathrm{Th}(\mathfrak{B}_r) \cap \mathcal{A}\mathrm{t}(Y_r)$ by Theorem 4.1. So there is a classical homomorphism $\mu : \mathfrak{B}_r \to \mathfrak{D}_t$ defined by $\mu(b) = b^{\mathfrak{D}_t}$ for $b \in Y_r$. Let $Gr = t$ and $\mathcal{G}_r = \mu$. Then $\mathcal{G}_r : \mathfrak{B}_r \to \mathfrak{D}_{Gr}$ is a classical homomorphism, $\Delta_r \cap \Phi(Y_r) \subseteq \Xi_{Gr}$, and $\Xi_{Gr} \cap \mathcal{L}(X_r) = \Gamma_r$. For the induction step, let $i \in |\mathbf{A}|$, and let $j \in |\mathbf{A}|$ be any immediate successor of i. Suppose that $\mathcal{G}_i : \mathfrak{B}_i \to \mathfrak{D}_{Gi}$ is a classical homomorphism, $\Delta_i \cap \Phi(Y_i) \subseteq \Xi_{Gi}$, and $\Xi_{Gi} \cap \mathcal{L}(X_i) = \Gamma_i$. We may assume up to isomorphism that $(Y_j \setminus Y_i) \cap Z_{Gi} = \varnothing$ and $(X_j \setminus X_i) \cap Z_{Gi} = \varnothing$.

Induction step

Claim 2 $(\Xi_{Gi} \cup \Gamma_j \cup (\Delta_j \cap \Phi(Y_j)), \Gamma_j^c)$ is consistent.

Assume not. Then, by compactness, there are $\theta(\mathbf{xyz}), \varphi(\mathbf{xy}), \psi(\mathbf{x}) \in \mathcal{L}$, $\mathbf{a} \in X_j$, $\mathbf{b} \in Y_j \setminus X_j$, and $\mathbf{c} \in Z_{Gi} \setminus Y_j$ such that $\theta(\mathbf{abc}) \in \Xi_{Gi}$, $\varphi(\mathbf{ab}) := \bigwedge_\alpha \varphi_\alpha(\mathbf{ab})$ with $\varphi_\alpha(\mathbf{ab}) \in \Delta_j \cap \Phi(Y_j)$, $\psi(\mathbf{a}) := \bigvee_\beta \psi_\beta(\mathbf{a})$ with $\psi_\beta(\mathbf{a}) \in \Gamma_j^c$, and $\Gamma_j \cup \{\theta(\mathbf{abc})\} \cup \{\varphi(\mathbf{ab})\} \vdash \psi(\mathbf{a})$. Since $\mathbf{c} \notin \mathcal{L}(Y_j) \supseteq \mathcal{L}(X_j)$ and $\mathbf{b} \notin \mathcal{L}(X_j)$, we have $\Gamma_j \cup \{\exists \mathbf{yz}\, \theta(\mathbf{ayz})\} \cup \{\exists \mathbf{y}\, \varphi(\mathbf{ay})\} \vdash \psi(\mathbf{a})$. Since $\exists \mathbf{yz}\, \theta(\mathbf{ayz}) \in \Xi_{Gi} \cap \mathcal{L}(X_j) = \Gamma_i$, we have $\exists \mathbf{yz}\, \theta(\mathbf{ayz}) \in \Gamma_j$ by persistence. Also, since $\exists \mathbf{y}\, \varphi(\mathbf{ay}) \in \Delta_j \cap \mathcal{E}(\Phi)(X_j)$ and $(\mathfrak{A}, r) \Lleftarrow_{\mathcal{E}(\Phi)} (\mathfrak{B}, s)$, we have $\exists \mathbf{y}\, \varphi(\mathbf{ay}) \in \Gamma_j$. So $\Gamma_j \vdash \psi(\mathbf{a})$. Thus, since Γ_j is prime, we have $\Gamma_j \vdash \psi_\beta(\mathbf{a})$ for some β, which is a contradiction. So Claim 2 is proven.

Thus, by Lemma 4.3, there is a node $j' \in |\mathbf{D}|$ such that $Z_{j'} \supseteq Y_j \cup Z_{Gi}$, $\Xi_{j'} \supseteq \Xi_{Gi} \cup \Gamma_j \cup (\Delta_j \cap \Phi(Y_j))$, and $(\Xi_{j'}, \Gamma_j^c)$ is consistent. Since $\Delta_j \cap \Phi(Y_j) \subseteq \Xi_{j'}$ and $\mathcal{A}\mathrm{t}(Y_j) \subseteq \Phi(Y_j)$, we have $\mathfrak{D}_{j'} \vDash \mathrm{Th}(\mathfrak{B}_j) \cap \mathcal{A}\mathrm{t}(Y_j)$ by Theorem 4.1. So there is a classical homomorphism $\mu : \mathfrak{B}_j \to \mathfrak{D}_{j'}$ defined by $\mu(b) = b^{\mathfrak{D}_{j'}}$ for $b \in Y_j$. Let $Gj = j'$ and let $\mathcal{G}_j = \mu$. Then $\mathcal{G}_j : \mathfrak{B}_j \to \mathfrak{D}_{Gj}$ is a classical homomorphism, $\Delta_j \cap \Phi(Y_j) \subseteq \Xi_{Gj}$, and $\Xi_{Gj} \cap \mathcal{L}(X_j) = \Gamma_j$. By induction on the height of nodes in $\mathbf{A}$, there exists a homomorphism of Kripke models $\mathcal{G} : ((\mathfrak{B}, s) \restriction \mathbf{A}) \to (\mathfrak{D}, t)$ such that $\mathcal{G} : (\mathfrak{B}, s) \Rrightarrow_\Phi^{\mathbf{A}} (\mathfrak{D}, t)$ and $\mathrm{Th}(\mathfrak{D}, Gi) \cap \mathcal{L}(A_i) = \mathrm{Th}(\mathfrak{A}, i)$ for all $i \in |\mathbf{A}|$. Since $(\mathfrak{A}, r) \subseteq (\mathfrak{B}, s)$, $\mathcal{G} : (\mathfrak{A}, r) \to (\mathfrak{D}, t)$ is a local embedding of Kripke models. By a construction similar to the one used in the proof of Lemma 4.10, there exists a canonical tree model $(\mathfrak{C}, t)$ such that $(\mathfrak{D}, t) \preceq (\mathfrak{C}, t)$ and a homomorphism of Kripke models $\mathcal{F} : ((\mathfrak{B}, s) \restriction \mathbf{A}) \to (\mathfrak{C}, t)$ such that $\mathcal{F} : (\mathfrak{A}, r) \to (\mathfrak{C}, t)$ is an embedding, $Fr = t$, and $\mathrm{Th}(\mathfrak{C}, Fi) \cap \mathcal{L}(A_i) = \mathrm{Th}(\mathfrak{A}, i)$ for all $i \in |\mathbf{A}|$. We may assume up to isomorphism that $t = r$ and $F = \mathrm{id}_{\mathbf{A}}$. So we have $(\mathfrak{A}, r) \preceq (\mathfrak{C}, r)$ and $(\mathcal{F}_i \restriction A_i) = \mathrm{id}_{A_i}$ for all $i \in |\mathbf{A}|$. So $\mathcal{F} : \langle(\mathfrak{A}, r), (\mathfrak{B}, s), (\mathfrak{C}, r)\rangle$ is a sandwich and $\mathcal{F} : (\mathfrak{B}, s) \Rrightarrow^{\mathbf{A}}_{\Phi} (\mathfrak{C}, r)$. □

We use Lemma 4.15 to prove the following generalized Sandwich Theorem, which is an intuitionistic analogue of Lemma 1 in [10].

Theorem 4.16 *Let $\Gamma \subseteq \Delta$ be intuitionistic theories over a first-order language $\mathcal{L}$, and let $\Phi, \Psi \subseteq \mathcal{L}$ such that $\mathcal{A}\mathrm{t} \subseteq \Phi$ and $\mathcal{A}\mathrm{t} \subseteq \Psi$. The following are equivalent:*

1. *Δ is axiomatizable by $\mathcal{U}(\mathcal{E}(\Phi), \Psi)$-sentences over Γ.*
2. *For all Kripke models $\mathfrak{A} \Vdash \Gamma$ and $\mathfrak{B} \Vdash \Delta$, if there exists a sandwich $\mathcal{F} : \langle\mathfrak{A}, \mathfrak{B}, \mathfrak{C}\rangle$ such that $\mathfrak{A} \Rrightarrow_{\Psi} \mathfrak{B}$ and $\mathcal{F} : \mathfrak{B} \Rrightarrow^{\mathbf{A}}_{\Phi} \mathfrak{C}$, then $\mathfrak{A} \Vdash \Delta$.*

Proof (1 ⇒ 2) Suppose that Δ is axiomatized by $\mathcal{U}(\mathcal{E}(\Phi), \Psi)$-sentences over Γ. Let $\mathfrak{A} \Vdash \Gamma$ and $\mathfrak{B} \Vdash \Delta$ be Kripke models, and suppose there exists a sandwich $\mathcal{F} : \langle\mathfrak{A}, \mathfrak{B}, \mathfrak{C}\rangle$ such that $\mathfrak{A} \Rrightarrow_{\Psi} \mathfrak{B}$ and $\mathcal{F} : \mathfrak{B} \Rrightarrow^{\mathbf{A}}_{\Phi} \mathfrak{C}$. By Theorem 4.11, we need only show that $\mathfrak{A} \Lleftarrow_{\mathcal{E}(\Phi)} \mathfrak{B}$. By Lemma 4.14, we have $\mathcal{F} : \mathfrak{B} \Rrightarrow^{\mathbf{A}}_{\mathcal{E}(\Phi)} \mathfrak{C}$. So, since $\mathfrak{A} \preceq \mathfrak{C}$ and $F = \mathrm{id}_{\mathbf{A}}$ and $(\mathcal{F}_i \restriction A_i) = \mathrm{id}_{A_i}$ for all $i \in |\mathbf{A}|$, we have $\mathfrak{A} \Lleftarrow_{\mathcal{E}(\Phi)} \mathfrak{B}$.

(2 ⇒ 1) Suppose that (2) holds. Since $\mathcal{U}(\mathcal{E}(\Phi), \Psi)$ is closed under disjunction, we may apply Lemma 4.4. Let X be a set of new constant symbols, with $|X| \geq |\mathcal{L}|$. Let $(\mathfrak{A}, r) \Vdash \Gamma$ and $(\mathfrak{B}, s) \Vdash \Delta$ be canonical tree models with constants from X and root nodes $r = \langle\Gamma_r, X_r\rangle$ and $s = \langle\Delta_s, Y_s\rangle$ such that $X_r \cap Y_s = \varnothing$. Suppose that $\Delta_s \cap \mathcal{U}(\mathcal{E}(\Phi), \Psi) \subseteq \Gamma_r$. Then, by Lemma 4.7, there exists a homomorphism of Kripke models $\mathcal{G} : (\mathfrak{A}, r) \to (\mathfrak{B}, s)$ such that $\mathcal{G} : (\mathfrak{A}, r) \Lleftarrow_{\mathcal{E}(\Phi)} (\mathfrak{B}, s)$ and $\mathcal{G} : (\mathfrak{A}, r) \Rrightarrow_{\Psi} (\mathfrak{B}, s)$. Since $\mathcal{A}\mathrm{t} \subseteq \Phi \subseteq \mathcal{E}(\Phi)$, $\mathcal{G}$ is a local embedding. By Lemma 4.10, there exists a canonical tree model $(\mathfrak{B}', s)$ such that $(\mathfrak{B}, s) \preceq (\mathfrak{B}', s)$ and an embedding of Kripke models $\mathcal{G}' : (\mathfrak{A}, r) \to (\mathfrak{B}', s)$. We may assume up to isomorphism that $(\mathfrak{A}, r) \subseteq (\mathfrak{B}', s)$, $(\mathfrak{A}, r) \Lleftarrow_{\mathcal{E}(\Phi)} (\mathfrak{B}', s)$, and $(\mathfrak{A}, r) \Rrightarrow_{\Psi} (\mathfrak{B}', s)$. By Lemma 4.15, there exists a sandwich $\mathcal{F} : \langle(\mathfrak{A}, r), (\mathfrak{B}', s), (\mathfrak{C}, r)\rangle$ such that $\mathcal{F} : (\mathfrak{B}', s) \Rrightarrow^{\mathbf{A}}_{\Phi} (\mathfrak{C}, r)$. So, by (2), $(\mathfrak{A}, r) \Vdash \Delta$. Thus, by Lemma 4.4, Δ is axiomatizable by $\mathcal{U}(\mathcal{E}(\Phi), \Psi)$-sentences over Γ. □

If $\Gamma \subseteq \Delta$ are classical theories, then the following are equivalent: (1) Δ is axiomatizable by $\{\wedge, \vee, \forall\}(\exists^{+} \cup \neg\exists^{+})$-sentences over Γ. (2) For all classical models $\mathfrak{A} \vDash \Gamma$ and $\mathfrak{B} \vDash \Delta$, if there exists a sandwich of classical models $\mathcal{F} : \langle\mathfrak{A}, \mathfrak{B}, \mathfrak{C}\rangle$, then $\mathfrak{A} \vDash \Delta$. Note that our notion of a sandwich of classical models is more general than that of [4]. As a corollary to Theorem 4.16, we obtain the following intuitionistic analogue.

Corollary 4.17 *Let $\Gamma \subseteq \Delta$ be intuitionistic theories over a first-order language $\mathcal{L}$. The following are equivalent:*

1. *Δ is axiomatizable by $\mathcal{U}\mathcal{E}^{+}$-sentences over Γ.*
2. *For all Kripke models $\mathfrak{A} \Vdash \Gamma$ and $\mathfrak{B} \Vdash \Delta$, if there exists a sandwich $\mathcal{F} : \langle\mathfrak{A}, \mathfrak{B}, \mathfrak{C}\rangle$, then $\mathfrak{A} \Vdash \Delta$.*

Proof Immediate from Theorem 4.16 and the comments following Lemma 4.6. □

From Theorem 4.16 and Lemma 4.10, we obtain the following generalized Sandwich Theorem.

Theorem 4.18 *Let $\Gamma \subseteq \Delta$ be intuitionistic theories over a first-order language $\mathcal{L}$, and let $\Phi, \Psi \subseteq \mathcal{L}$ such that $(\mathcal{A}\mathrm{t} \cup \neg \mathcal{A}\mathrm{t}) \subseteq \Phi$ and $\mathcal{A}\mathrm{t} \subseteq \Psi$. The following are equivalent:*

1. *Δ is axiomatizable by $\mathcal{U}(\mathcal{E}(\Phi), \Psi)$-sentences over Γ.*
2. *For all Kripke models $\mathfrak{A} \Vdash \Gamma$ and $\mathfrak{B} \Vdash \Delta$, if there exists a sandwich $\langle \mathfrak{A}, \mathfrak{B}, \mathfrak{C} \rangle$ such that $\mathfrak{A} \Rrightarrow_{\Psi} \mathfrak{B}$ and $\mathfrak{B} \Rrightarrow^{\mathbf{A}}_{\Phi} \mathfrak{C}$, then $\mathfrak{A} \Vdash \Delta$.*

Proof (1 $\Rightarrow$ 2) Immediate from Theorem 4.16.

(2 $\Rightarrow$ 1) Suppose that (2) holds. Let $\mathfrak{A} \Vdash \Gamma$ and $\mathfrak{B} \Vdash \Delta$ be Kripke models, and let $\mathcal{F} : \langle \mathfrak{A}, \mathfrak{B}, \mathfrak{C} \rangle$ be a sandwich such that $\mathfrak{A} \Rrightarrow_{\Psi} \mathfrak{B}$ and $\mathcal{F} : \mathfrak{B} \Rrightarrow^{\mathbf{A}}_{\Phi} \mathfrak{C}$. Since $\neg \mathcal{A}\mathrm{t} \subseteq \Phi$, $\mathcal{F}$ is a local embedding. Thus, by Lemma 4.10, there exists a Kripke model $\mathfrak{C}'$ with $\mathfrak{C} \preceq \mathfrak{C}'$ and an embedding of Kripke models $\mathcal{F}' : (\mathfrak{B} \restriction \mathbf{A}) \to \mathfrak{C}'$ such that $\mathcal{F}' : \langle \mathfrak{A}, \mathfrak{B}, \mathfrak{C}' \rangle$ is a sandwich. So we may assume up to isomorphism that $\langle \mathfrak{A}, \mathfrak{B}, \mathfrak{C}' \rangle$ is a sandwich, and that $\mathfrak{B} \Rrightarrow^{\mathbf{A}}_{\Phi} \mathfrak{C}'$. So, by (2), $\mathfrak{A} \Vdash \Delta$. Thus, by Theorem 4.16, Δ is axiomatizable by $\mathcal{U}(\mathcal{E}(\Phi), \Psi)$-sentences over Γ. $\square$

The classical Keisler Sandwich Theorem can be stated as follows. If $\Gamma \subseteq \Delta$ are classical theories, then the following are equivalent for all $n \geq 2$: (1) Δ is axiomatizable by $\forall_n$-sentences over Γ; (2) For all classical models $\mathfrak{A} \vDash \Gamma$ and $\mathfrak{B} \vDash \Delta$, if there exists a sandwich of classical models $\langle \mathfrak{A}, \mathfrak{B}, \mathfrak{C} \rangle$ such that $\mathfrak{A} \Rrightarrow_{\forall_{n-2}} \mathfrak{B}$ and $\mathfrak{B} \Rrightarrow_{\forall_{n-2}} \mathfrak{C}$, then $\mathfrak{A} \vDash \Delta$. As a corollary to Theorem 4.18, we obtain the following intuitionistic analogue of the Keisler Sandwich Theorem for the $\mathcal{U}_n$ hierarchy.

Corollary 4.19 *Let $\Gamma \subseteq \Delta$ be intuitionistic theories over a first-order language $\mathcal{L}$. The following are equivalent for all $n \geq 2$:*

1. *Δ is axiomatizable by $\mathcal{U}_n$-sentences over Γ.*
2. *For all Kripke models $\mathfrak{A} \Vdash \Gamma$ and $\mathfrak{B} \Vdash \Delta$, if there exists a sandwich $\langle \mathfrak{A}, \mathfrak{B}, \mathfrak{C} \rangle$ such that $\mathfrak{A} \Rrightarrow_{\mathcal{U}_{n-2}} \mathfrak{B}$ and $\mathfrak{B} \Rrightarrow^{\mathbf{A}}_{\mathcal{U}_{n-2}} \mathfrak{C}$, then $\mathfrak{A} \Vdash \Delta$.*

Proof Immediate from Definition 3.9, Theorem 4.18, and Lemma 4.6. $\square$

As a corollary to Theorem 4.16 and Theorem 4.18, we obtain the following intuitionistic analogue of the Keisler Sandwich Theorem for the $\mathcal{U}^*_n$ hierarchy.

Corollary 4.20 *Let $\Gamma \subseteq \Delta$ be intuitionistic theories over a first-order language $\mathcal{L}$. The following are equivalent for all $n \geq 2$:*

1. *Δ is axiomatizable by $\mathcal{U}^*_n$-sentences over Γ.*
2. *For all Kripke models $\mathfrak{A} \Vdash \Gamma$ and $\mathfrak{B} \Vdash \Delta$, if there exists a sandwich $\mathcal{F} : \langle \mathfrak{A}, \mathfrak{B}, \mathfrak{C} \rangle$ such that $\mathfrak{A} \Rrightarrow_{\mathcal{U}^*_{n-1}} \mathfrak{B}$ and $\mathcal{F} : \mathfrak{B} \Rrightarrow^{\mathbf{A}}_{\mathcal{U}^*_{n-2}} \mathfrak{C}$, then $\mathfrak{A} \Vdash \Delta$.*

For all $n \geq 3$, (1) and (2) are equivalent to the following:

3. *For all Kripke models $\mathfrak{A} \Vdash \Gamma$ and $\mathfrak{B} \Vdash \Delta$, if there exists a sandwich $\langle \mathfrak{A}, \mathfrak{B}, \mathfrak{C} \rangle$ such that $\mathfrak{A} \Rrightarrow_{\mathcal{U}^*_{n-1}} \mathfrak{B}$ and $\mathfrak{B} \Rrightarrow^{\mathbf{A}}_{\mathcal{U}^*_{n-2}} \mathfrak{C}$, then $\mathfrak{A} \Vdash \Delta$.*

Proof Immediate from Definition 3.11, Theorem 4.16, Theorem 4.18, and Lemma 4.6. $\square$

Theorem 4.11 and Theorem 4.18 can also be applied to the $\mathcal{U}^{\mathcal{S}}_n$ hierarchy.

Corollary 4.21 *Let $\Gamma \subseteq \Delta$ be intuitionistic theories over a first-order language $\mathcal{L}$. The following are equivalent for all $n \geq 1$:*

1. *Δ is axiomatizable by $\mathcal{U}_n^{\mathcal{S}}$-sentences over Γ.*
2. *For all Kripke models $\mathfrak{A} \Vdash \Gamma$ and $\mathfrak{B} \Vdash \Delta$, if $\mathfrak{A} \subseteq \mathfrak{B}$ and $\mathfrak{A} \Leftarrow_{\mathcal{E}_{n-1}^{\mathcal{S}}} \mathfrak{B}$, then $\mathfrak{A} \Vdash \Delta$.*

For all $n \geq 2$, (1) and (2) are equivalent to the following:

3. *For all Kripke models $\mathfrak{A} \Vdash \Gamma$ and $\mathfrak{B} \Vdash \Delta$, if there exists a sandwich $\langle \mathfrak{A}, \mathfrak{B}, \mathfrak{C} \rangle$ such that $\mathfrak{B} \Rrightarrow^{\mathbf{A}}_{\mathcal{U}_{n-2}^{\mathcal{S}}} \mathfrak{C}$, then $\mathfrak{A} \Vdash \Delta$.*

Proof Immediate from Definition 3.13, Theorem 4.11, Theorem 4.18, and Lemma 4.6. □

Finally, we note that if CQC $\subseteq \Gamma$, then all of the syntactic preservation theorems in this section imply the corresponding classical theorems. This is a consequence of the following lemma.

Lemma 4.22 *Let $\mathfrak{A}$ be a Kripke model over a language $\mathcal{L}$. Then $\mathfrak{A} \Vdash$ CQC if and only if for all $i \in |\mathbf{A}|$ and all sentences $\varphi \in \mathcal{L}(A_i)$, $i \Vdash^{\mathfrak{A}} \varphi \Leftrightarrow \mathfrak{A}_i \vDash \varphi$.*

Proof See [7], Appendix. □

To prove, for example, that Theorem 4.11 implies the generalized Łoś-Tarski Theorem, it suffices to prove the following.

Theorem 4.23 *Let Γ and Δ be theories such that* CQC $\subseteq \Gamma \subseteq \Delta$ *and let $\Phi, \Psi \subseteq \mathcal{L}$ such that $\mathcal{A}\mathrm{t} \subseteq \Phi$ and $\mathcal{A}\mathrm{t} \subseteq \Psi$. Let $\Theta = \{\wedge, \vee\}(\neg\Phi \cup \Psi)$. The following are equivalent:*

1. *For all Kripke models $\mathfrak{A} \Vdash \Gamma$ and $\mathfrak{B} \Vdash \Delta$, if $\mathfrak{A} \subseteq \mathfrak{B}$, $\mathfrak{A} \Lleftarrow_{\Phi} \mathfrak{B}$, and $\mathfrak{A} \Rrightarrow_{\Psi} \mathfrak{B}$, then $\mathfrak{A} \Vdash \Delta$.*
2. *For all classical models $\mathfrak{M} \vDash \Gamma$ and $\mathfrak{N} \vDash \Delta$, if $\mathfrak{M} \subseteq \mathfrak{N}$ and $\mathfrak{M} \Rrightarrow_{\Theta} \mathfrak{N}$, then $\mathfrak{M} \vDash \Delta$.*

Proof (1 $\Rightarrow$ 2) Let CQC $\subseteq \Gamma \subseteq \Delta$, and suppose that (1) holds. Let $\mathfrak{M} \vDash \Gamma$ and $\mathfrak{N} \vDash \Delta$ be classical models such that $\mathfrak{M} \subseteq \mathfrak{N}$ and $\mathfrak{M} \Rrightarrow_{\Theta} \mathfrak{N}$. Let $(\mathfrak{A}, r)$ and $(\mathfrak{B}, s)$ be one-node Kripke models with $\mathfrak{A}_r = \mathfrak{M}$ and $\mathfrak{B}_s = \mathfrak{N}$, with only the identity morphisms. Then forcing and truth in these models coincide, and we have $(\mathfrak{A}, r) \Lleftarrow_{\Phi} (\mathfrak{B}, s)$ and $(\mathfrak{A}, r) \Rrightarrow_{\Psi} (\mathfrak{B}, s)$. Thus, by (1), $(\mathfrak{A}, r) \Vdash \Delta$. So $\mathfrak{M} \vDash \Delta$.

(2 $\Rightarrow$ 1) Let CQC $\subseteq \Gamma \subseteq \Delta$, and suppose that (2) holds. Let $\mathfrak{A} \Vdash \Gamma$ and $\mathfrak{B} \Vdash \Delta$ be Kripke models such that $\mathfrak{A} \subseteq \mathfrak{B}$, $\mathfrak{A} \Lleftarrow_{\Phi} \mathfrak{B}$, and $\mathfrak{A} \Rrightarrow_{\Psi} \mathfrak{B}$. Since $\mathfrak{A} \Vdash$ CQC and $\mathfrak{B} \Vdash$ CQC, we have $\mathfrak{A}_i \vDash \Gamma$, $\mathfrak{B}_i \vDash \Delta$, and $\mathfrak{A}_i \Rrightarrow_{(\neg\Phi \cup \Psi)} \mathfrak{B}_i$ for all $i \in |\mathbf{A}|$ by Lemma 4.22. Thus, we have $\mathfrak{A}_i \subseteq \mathfrak{B}_i$ and $\mathfrak{A}_i \Rrightarrow_{\Theta} \mathfrak{B}_i$ for all $i \in |\mathbf{A}|$. Thus, by (2), $\mathfrak{A}_i \vDash \Delta$ for all $i \in |\mathbf{A}|$. So $\mathfrak{A} \Vdash \Delta$ by Lemma 4.22. □

5 Conclusion

In conclusion, we have proved four generalized syntactic preservation theorems for intuitionistic predicate logic: Theorem 4.8, Theorem 4.11, Theorem 4.16, and Theorem 4.18. From these theorems we have obtained as corollaries intuitionistic analogues of the dual of the Lyndon-Łoś-Tarski Theorem (Corollary 4.9), the Łoś-Tarski Theorem (Corollary 4.12), a sandwich theorem for sentences in $\mathcal{UE}^+$ (Corollary 4.17), and the Keisler Sandwich Theorem for sentences in the formula hierarchies $\mathcal{U}_n$, $\mathcal{U}_n^*$, and $\mathcal{U}_n^{\mathcal{S}}$ (Corollary 4.19, Corollary 4.20, and Corollary 4.21, respectively). The formula hierarchies $\langle \mathcal{U}_n, \mathcal{E}_n \rangle$ and $\langle \mathcal{U}_n^*, \mathcal{E}_n^* \rangle$ (Definition 3.9 and Definition 3.11, respectively) each contain all formulas in $\mathcal{L}$ and are equivalent over CQC to $\langle \forall_n, \exists_n \rangle$ for $\langle n \geq 0, n \geq 1 \rangle$ and $\langle n \geq 1, n \geq 2 \rangle$, respectively. The formula hierarchy $\langle \mathcal{U}_n^{\mathcal{S}}, \mathcal{E}_n^{\mathcal{S}} \rangle$ (Definition 3.13) contains all semi-positive formulas $\mathcal{S} \subseteq \mathcal{L}$ and is equivalent over CQC to $\langle \forall_n, \exists_n \rangle$ for $\langle n \geq 0, n \geq 1 \rangle$.

Two theorems that are noticeably lacking in this paper are intuitionistic analogues of the classical syntactic preservation theorems characterizing the sentences preserved under *homomorphisms* (rather than inverse images of homomorphisms) and *extensions* (rather than submodels). In fact, using the definitions adopted in this paper, the only class of sentences that are preserved under homomorphisms of Kripke models, or even under extensions of Kripke models, is the class of intuitionistic tautologies. To see this, let $\varphi \in \mathcal{L}$ be any sentence that is not an intuitionistic tautology, and let $(\mathfrak{A}, r)$ be a rooted Kripke model such that $(\mathfrak{A}, r) \Vdash \varphi$. Let $(\mathfrak{B}, s)$ be a rooted Kripke model such that $(\mathfrak{B}, s) \nVdash \varphi$. Let $(\mathfrak{C}, t)$ be the rooted Kripke model formed by adding a new node t below both r and s, with a corresponding classical model $\mathfrak{C}_t$ having only one element of which no predicate is true. Send this element to any element in $\mathfrak{A}_r$ and to any element in $\mathfrak{B}_s$. Then $(\mathfrak{A}, r) \subseteq (\mathfrak{C}, t)$ and $(\mathfrak{C}, t) \nVdash \varphi$. So φ is not preserved under extensions. Since an extension is a special kind of homomorphism, φ is not preserved under homomorphisms either.

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