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A Note on Counterexamples to the Vaught Conjecture

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Abstract If some infinitary sentence provides a counterexample to Vaught's Conjecture, then there is an infinitary sentence which also provides a counterexample but has no model of cardinality bigger than $\aleph_1$.

1 Introduction

The following theorem was shown in [2].

Theorem 1.1 *There is a countable structure $\mathcal{M}$ with S_∞ dividing* Aut*($\mathcal{M}$) and whose Scott sentence has models of size $\aleph_1$ but no higher.*

This note makes the following corollary of that theorem explicit.

Corollary 1.2 *If there is some $\sigma \in \mathcal{L}_{\omega_1\omega}$ providing a counterexample to Vaught's Conjecture for $\mathcal{L}_{\omega_1\omega}$, then there is some $\sigma' \in \mathcal{L}_{\omega_1\omega}$ which again provides a counterexample and has no model of cardinality bigger than $\aleph_1$.*

Harrington has shown that any counterexample to Vaught's Conjecture will have, necessarily uncountable, models with arbitrarily high Scott ranks below ω_2.

2 Proof

For convenience, we work under $\neg$CH. A similar argument works for the version of the Vaught Conjecture which asks for uncountably many nonisomorphic countable models without a perfect set of nonisomorphic countable models.

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Definition 2.1 S_∞ is said to *divide* a Polish group G if there is a closed subgroup $H < G$ and a continuous, onto, homomorphism

$$\pi : H \twoheadrightarrow S_\infty.$$

Notation 2.2 For $\mathcal{L}$ a countable language we let $X_{\mathcal{L}}$ be the space of all $\mathcal{L}$-structures with underlying set $\mathbb{N}$ and with the topology having basic open sets

$$\{\mathcal{N} : \mathcal{N} \models \psi(\vec{a})\}$$

for ψ quantifier free and $\vec{a}$ a finite sequence from $\mathbb{N}$.

It is well known and easily checked that $X_{\mathcal{L}}$ is a Polish space in the indicated topology. Compare [1], §2.7.

Let $\mathcal{M}$ be as in the statement of the theorem and let $\sigma \in \mathcal{L}_{\omega_1\omega}$ have exactly $\aleph_1$ many nonisomorphic models. For convenience I will assume that $\mathcal{M}$ has $\mathbb{N}$ as its underlying set. Fix H a closed subgroup of $\mathrm{Aut}(\mathcal{M})$ which maps continuously and homomorphically onto S_∞.

In particular, we have a continuous action of S_∞ on a Polish space X with exactly $\aleph_1$ many orbits. This obviously can be lifted up to a continuous action of the Polish group H with the same orbit structure, and then following [1], 2.3.5, we may lift the action of H on X up to an action of $\mathrm{Aut}(\mathcal{M})$ on a Polish space $Y \supset X$ such that, among other things, Y has the same number of orbits under $\mathrm{Aut}(\mathcal{M})$ as X does under H. That is to say, $Y/\mathrm{Aut}(\mathcal{M})$ has size $\aleph_1$.

Then by [1], 2.7.4, we can find a richer countable language $\mathcal{L}' \supset \mathcal{L}$ such that the expansions of $\mathcal{M}$ to $\mathcal{L}'$, $Y^{\mathcal{M}}_{\mathcal{L}'}$, form a universal Polish $\mathrm{Aut}(\mathcal{M})$ space. In particular, there will be an injective and Borel $\mathrm{Aut}(\mathcal{M})$-map,

$$\rho : Y \to Y^{\mathcal{M}}_{L'}.$$

Since the injective images of Borel sets are Borel, there will be an invariant Borel set

$$B \subset Y^{\mathcal{M}}_{L'}$$

which contains exactly $\aleph_1$ many orbits under the action of $\mathrm{Aut}(\mathcal{M})$.

Now let C be the set of all $\mathcal{N} \in X_{\mathcal{L}'}$ such that there exists $\mathcal{N}' \cong \mathcal{N}$ with $\mathcal{N}' \in B$. Note that C is then $\underset{\sim}{\Delta}^1_1$, and hence Borel, since, by invariance of B, we can equivalently describe C as the set of $\mathcal{N} \in X_{\mathcal{L}'}$ which satisfy $\varphi_{\mathcal{M}}$, the Scott sentence of $\mathcal{M}$, and for which any $\mathcal{N}' \in Y^{\mathcal{M}}_{\mathcal{L}'}$ isomorphic to $\mathcal{N}$ is in B.

Thus by [3], there is some $\sigma' \in \mathcal{L}'_{\omega_1\omega}$ whose models with underlying set $\mathbb{N}$ are exactly the elements of C. Since $\sigma' \Rightarrow \varphi_{\mathcal{M}}$, σ' has no model of size $\aleph_2$. By construction, σ' has exactly $\aleph_1$ many nonisomorphic models.

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