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Four Problems Concerning Recursively Saturated Models of Arithmetic

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Abstract The paper presents four open problems. One concerns a possible converse to Tarski's undefinability of truth theorem, and is of a general character. The other three are more specific. The questions are about some special ω_1-like models, initial segments of countable recursively saturated models of PA, and about extendability of automorphisms. In each case a partial answer is given. All partial solutions are based on applications of inductive satisfaction classes.

1 Introduction A partial satisfaction class for a model $\mathcal{M} \models$ PA is a subset of Form$(\mathcal{M}) \times \mathcal{M}$, satisfying Tarski's inductive definition of the satisfaction relation. Here Form$(\mathcal{M})$ is the set of formulas in the sense of $\mathcal{M}$ given by an arithmetization of the language, see Definition 2.1 below.

Although most results on satisfaction classes indicate that there is no reasonable way of defining a "nonstandard semantics" for nonstandard formulas, still one can hope to develop a kind of "nonstandard model theory." This hope is based on the possibility of approximating second order model-theoretic notions (definability, type, indiscernibility) by their natural nonstandard extensions defined in terms of inductive satisfaction classes. A good example is the proof of Theorem 3.4 below. In fact this approach to recursively saturated models of PA has been used often, however no comprehensive study has been done yet. We will not do it here either. The purpose of this note is to present four problems concerning model theory of recursively saturated models of PA. The main problems, 3.3, 4.9, and 6.2, ask about the existence of some special structures, which we call weakly Jónsson models, free cuts and absolutely nonextendible automorphisms, respectively. Partial answers to the problems, presented here, can be given in cases when techniques of "nonstandard model theory" can be applied, due to the availability of suitable inductive satisfaction classes. Moreover solutions in these cases are rather simple. On the other hand the problems in their full generality seem difficult, and standard model theoretic methods together with well-known specifically arithmetical tools (like the arithmetized completeness theorem) have not provided answers so far.

Problem 2.2 is of a much more general nature. It asks whether a certain converse to Tarski's theorem on undefinability of truth holds.

The paper is not self-contained. I will assume that the reader is familiar with concepts concerning recursively saturated models of PA; all necessary information can be found in Kaye [2]. Definitions of semiregular and strong initial segments of a model of PA are given in Kirby and Paris [5]; the best reference for Gaifman's minimal types is Gaifman's important paper [1].

2 Defining nondefinability Let $\mathcal{L}$ be the language of PA. By Q_n we will denote the closure of all Σ_n formulas of $\mathcal{L}$ under negation, conjunction, and bounded quantification. Q_∞ is the set of all formulas of $\mathcal{L}$. If $\mathcal{M}$ is a model of PA, and $e \in \mathcal{M}$ or $e = \infty$, then $Q_e(\mathcal{M})$ is the set of Q_e formulas in the sense of $\mathcal{M}$ (under a fixed arithmetization). The set of standard nonnegative integers will be denoted by $\mathbb{N}$.

Definition 2.1 Let $\mathcal{M}$ be a model of PA, and let e be an element of $\mathcal{M}$ or $e = \infty$. A subset S of $\mathcal{M}$ is a Q_e*-satisfaction class* for $\mathcal{M}$ if S consists of (codes) of pairs of the form (φ, a), where $\varphi \in Q_e(\mathcal{M})$ and a is a (code of) valuation for φ, and the following Tarski's conditions are satisfied (in this definition and later on we will identify formulas of $\mathcal{M}$ with their Gödel numbers):

1. If $\varphi = v_i + v_j = v_k$ then $(\varphi, a) \in S$ iff $a_i + a_j = a_k$, and similarly for multiplication;
2. For all $\varphi, \psi \in Q_e(\mathcal{M})$ and for all $a \in \mathcal{M}$: $(\varphi \,\&\, \psi, a) \in S$ iff $(\varphi, a) \in S$ and $(\psi, a) \in S$;
3. For all $\varphi \in Q_e(\mathcal{M})$ and for all $a \in \mathcal{M}$: $(\neg\varphi, a) \in S$ iff $(\varphi, a) \notin S$;
4. For all $\exists v_i \psi \in Q_e(\mathcal{M})$ and for all $a \in \mathcal{M}$: $(\exists v_i \psi, a) \in S$ iff $(\psi, a^\frown(i, m)) \in S$ for some $m \in \mathcal{M}$.

See [2] and [12] for a discussion and a survey of results on satisfaction classes.

The first problem I want to pose concerns satisfaction classes directly. The definition of a satisfaction class is a single sentence in the language of PA with an extra predicate symbol S and one parameter e (or no parameters if $e = \infty$). Let $\mathrm{Sat}_e(S)$ be this sentence. Thus if $\mathcal{M}$ is a model of PA and $S \subset \mathcal{M}$, then S is a Q_e-satisfaction class for $\mathcal{M}$ iff $(\mathcal{M}, S) \models \mathrm{Sat}_e(S)$.

The proof of Tarski's theorem easily gives the following: if $e > \mathbb{N}$ and $(\mathcal{M}, S) \models \mathrm{Sat}_e(S)$, then S is undefinable in $\mathcal{M}$. Our first question asks whether there is a converse to this version of Tarski's theorem.

Let PA* denote the Peano axioms in the language with extra set parameters. If $e > \mathbb{N}$ or $e = \infty$ and $(\mathcal{M}, S) \models \mathrm{Sat}_e(S) + \mathrm{PA}^*$ then we say that S is an *inductive satisfaction class* for $\mathcal{M}$.

Problem 2.2 Suppose that $\Psi(X)$ is a sentence of the language of PA with a set parameter X, such that for every model $\mathcal{M} \models \mathrm{PA}$, and every $X \subset \mathcal{M}$, if $(\mathcal{M}, X) \models \Psi(X)$ then X is undefinable in $\mathcal{M}$. Is it true then, that for every model $\mathcal{M}$ and every $X \subset \mathcal{M}$ if $(\mathcal{M}, X) \models \Psi(X) + \mathrm{PA}^*$, then there is a nonstandard $e \in \mathcal{M}$ and there is $S \subset \mathcal{M}$ such that $(\mathcal{M}, S) \models \mathrm{Sat}_e(S)$ and S is definable in $(\mathcal{M}, X)$?

Thus we ask whether Tarski's theorem is essentialy the only source of "definable" nondefinability. The assumption $(\mathcal{M}, X) \models \mathrm{PA}^*$ is necessary to eliminate trivial counterexamples like $\Psi(X)$ saying that X is a proper initial segment of $\mathcal{M}$.

Notice that $\Psi(X) = \mathrm{Sat}_\infty(X)$ satisfies the assumptions of Problem 2.2. We could have fomulated problem 2.2 to include examples of the form $\mathrm{Sat}_e(X)$, for nonstandard e. The difference seems to be of technical nature only. The same remark applies to the formulation of Theorem 2.4 and Problem 2.5 below.

There is a very general theorem of Harrington which yields the solution of the analogue of Problem 2.2, formulated for the standard model. Harrington's result is unpublished. It can be found in handwritten notes titled "MacLaughlin's Conjecture," dated September 1976.

Theorem 2.3 (Harrington) *There is a nonarithmetic arithmetical singleton $A \subseteq \mathbb{N}$ such that $0^{(\omega)}$ is not arithmetic in A.*

It is well known that if a model $\mathcal{M} \models \mathrm{PA}$ has a subset S such that $(\mathcal{M}, S) \models \mathrm{Sat}_e(S)$ for some nonstandard e, then $\mathcal{M}$ is recursively saturated. We will show that, in a certain sense, $\mathrm{Sat}_e(X)$ is the only formula with this property.

Theorem 2.4 *Suppose that $\Psi(X)$ is a theory in the language of* PA *with a set parameter X, such that for every model $\mathcal{M} \models \mathrm{PA}$ if there is $X \subset \mathcal{M}$ such that $(\mathcal{M}, X) \models \Psi(X) + \mathrm{PA}^*$, then $\mathcal{M}$ is recursively saturated. Then for every $\mathcal{M} \models \mathrm{PA}$ and every $X \subset \mathcal{M}$ such that $(\mathcal{M}, X) \models \Psi(X) + \mathrm{PA}^*$ there is a nonstandard $e \in \mathcal{M}$ and there is $S \subset \mathcal{M}$ such that $(\mathcal{M}, S) \models \mathrm{Sat}_e(S)$ and S is definable in $(\mathcal{M}, X)$.*

Proof: Suppose $(\mathcal{M}, X) \models \Psi(X) + \mathrm{PA}^*$. By the MacDowell-Specker theorem for PA*, $(\mathcal{M}, X)$ has a conservative elementary end extension, so there is $(\mathcal{N}, Y)$ such that $(\mathcal{M}, X) \prec_e (\mathcal{N}, Y)$, and every subset of $\mathcal{M}$ which is coded in $\mathcal{N}$ is definable in $(\mathcal{M}, X)$. Thus $(\mathcal{N}, Y) \models \Psi(Y)$, hence $\mathcal{N}$ is recursively saturated. We can assume that $\mathrm{cf}(\mathcal{N}) = \aleph_0$. Thus there is $S'' \subset \mathcal{N}$ such that $(\mathcal{N}, S'') \models \mathrm{Sat}_{e'}(S'') + \mathrm{PA}^*$, for some nonstandard $e' \in \mathcal{N}$, this is obvious if $\mathcal{N}$ is countable, for an argument in the case that $\mathcal{N}$ is uncountable see Kossak [7]. Let $S' = S'' \cap \mathcal{M}$. Since $(\mathcal{N}, Y)$ is a conservative extension of $(\mathcal{M}, X)$, S' is definable in $(\mathcal{M}, X)$. The problem is that S' might not be a satisfaction class for $\mathcal{M}$. The only obstacle is the existential quantifier case in Tarski's conditions. But $(\mathcal{M}, S') \models \mathrm{PA}^*$, hence we can apply overspill to the following formula $\Theta(x, A)$,

$$\forall \varphi \in Q_x \forall a \left((\exists v_i \varphi(v_i, a)) \in A \leftrightarrow \exists v\ (\varphi, a^\frown(i, v)) \in A\right).$$

For every $n \in \mathbb{N}$, S' agrees with the universal truth predicate for Q_n formulas of PA, thus we have $(\mathcal{M}, S') \models \Theta(n, S')$. Hence $(\mathcal{M}, S') \models \Theta(e, S')$ for some nonstandard $e < e'$. Now, if S is the restriction of S' to Q_e formulas, then $(\mathcal{M}, S) \models \mathrm{Sat}_e(S)$. □

Notice that Theorem 2.4 reduces Problem 2.2 to the following.

Problem 2.5 Suppose that $\Psi(X)$ is a sentence of the language of PA with a set parameter X, such that for every model $\mathcal{M} \models \mathrm{PA}$, and every $X \subset \mathcal{M}$, if $(\mathcal{M}, X) \models \Psi(X)$ then X is undefinable in $\mathcal{M}$. Is it true then, that for every model $\mathcal{M}$ if there is $X \subset \mathcal{M}$ such that $(\mathcal{M}, X) \models \Psi(X) + \mathrm{PA}^*$, then $\mathcal{M}$ is recursively saturated?

3 Weakly Jónsson models A model $\mathcal{M}$ is a *Jónsson model* if for every $K \prec \mathcal{M}$ if $\mathrm{card}(K) = \mathrm{card}(\mathcal{M})$ then $K = \mathcal{M}$. A linearly ordered model is κ-like if $\mathrm{card}(\mathcal{M}) = \kappa$ and every proper initial segment of $\mathcal{M}$ is of power smaller than κ.

It is well known (cf. Knight [6]) that there are ω_1-like Jónsson models of PA. But, as the next proposition shows, none of these models can be recursively saturated.

Proposition 3.1 *If a recursively saturated* $\mathcal{M} \models$ PA* *is of power* κ*, and* κ *is a regular cardinal, then* $\mathcal{M}$ *has proper elementary submodels of cardinality* κ*.*

Proof: First, if $\mathcal{M}$ is not κ-like, then it is easy to see that $\mathcal{M}$ has a proper elementary initial segment of cardinality κ, and the result follows.

Now, let us assume that $\mathcal{M}$ is κ-like. If $p(v)$ is a type, then by $p^{\mathcal{M}}$ we denote the set of elements realizing $p(v)$ in $\mathcal{M}$. It is easy to prove that if $\mathcal{M}$ is κ-like and $p(v)$ is realized by an element of $\mathcal{M}$ that is greater that all definable elements of $\mathcal{M}$, then $\mathrm{card}(p^{\mathcal{M}}) = \kappa$.

Let $p(v)$, $q(v)$ be independent minimal types realized in $\mathcal{M}$ (cf. [1]). Let K be the elementary submodel of $\mathcal{M}$ generated by $p^{\mathcal{M}}$. Then $q^{\mathcal{M}} \cap K = \varnothing$, and the result follows. □

Definition 3.2 We will say that a recursively saturated model $\mathcal{M}$ *weakly Jónsson* if for every recursively saturated model $K \prec \mathcal{M}$ if $\mathrm{card}(K) = \mathrm{card}(\mathcal{M})$, then $K = \mathcal{M}$.

Problem 3.3 Are there ω_1-like recursively saturated weakly Jónsson models of PA?

Suppose a model $\mathcal{M} \models$ PA*, of regular cardinality κ, has an inductive satisfaction class S. Then Proposition 3.1, applied to the structure $(\mathcal{M}, S)$, shows that $\mathcal{M}$ has a proper elementary recursively saturated submodel of power κ. Thus such models are not weakly Jónsson. The next theorem shows that the assumption on the cardinality of $\mathcal{M}$ is not necessary. The idea of the proof of this theorem is due to Kotlarski.

Theorem 3.4 *If* $\mathcal{M} \models$ PA *has an inductive satisfacion class, then* $\mathcal{M}$ *is not weakly Jónsson.*

Proof: Let $S \subset \mathcal{M}$ be such that $(\mathcal{M}, S) \models \mathrm{Sat}_e(S) + \mathrm{PA}^*$, for some nonstandard $e \in \mathcal{M}$. As in the proof of Proposition 3.1, let $p(v), q(v)$ be independent minimal types realized in $\mathcal{M}$. For $n \in \mathbb{N}$ let

$$A_n = \{a \in \mathcal{M} : \forall \varphi < n\ \mathcal{M} \models \varphi(a) \iff \varphi \in p(v)\}.$$

(Recall that we identify formulas with their Gödel numbers.) The sequence $\langle A_x : x < e\rangle$ is definable in $(\mathcal{M}, S)$. For every $n \in \mathbb{N}$, A_n is unbounded in $\mathcal{M}$, hence there is $c > \mathbb{N}$ such that, for every $c' < c$, $A_{c'}$ is unbounded in $\mathcal{M}$. Notice that if $c > \mathbb{N}$, then $A_c \subset p^{\mathcal{M}}$.

Now, for every $x \in \mathcal{M}$ let B_x be the set of (codes of) all increasing sequences of elements of A_c whose length is x. For $a \in \mathcal{M}$ and $n \in \mathbb{N}$ let

$$K_n(a) = \{t(a) : t \text{ is a Skolem term and } t < n\}.$$

The sequence $\langle K_x(a) : x < e\rangle$ is definable in $(\mathcal{M}, S)$. Notice that if $c > \mathbb{N}$, then $K(a) \subset K_c(a)$, where $K(a)$ is the Skolem closure of a in $\mathcal{M}$.

Let b be an element of $q^{\mathcal{M}}$. If $a_0, \ldots, a_i$ is a finite sequence of elements of $p^{\mathcal{M}}$ then for every $n \in \mathbb{N}$ we have $b \notin K_n(a_0, \ldots, a_i)$. Thus in $(\mathcal{M}, S)$ the following holds for every $n \in \mathbb{N}$

$$\forall s\big(\mathrm{Seq}(s) \ \& \ \mathrm{len}(s) = n \ \& \ \forall i < \mathrm{len}(s)\ (s)_i \in B_n\big) \to b \notin K_n(s).$$

Again, by overspill, there is $d > \mathbb{N}$ such that the above holds for all sequences of length $d' < d$ whose terms are in $B_{d'}$. Let $B = B_{d'}$ for some nonstandard $d' < d$. We can assume that d' is small enough so that $B_{d'}$ is unbounded in $\mathcal{M}$.

Let K be the Skolem hull of B in $\mathcal{M}$. We know that $b \notin K$, and, since B is definable in $(\mathcal{M}, S)$ and unbounded in $\mathcal{M}$, $\mathrm{card}(K) = \mathrm{card}(\mathcal{M})$. It remains to show that K is recursively saturated. To this end notice that for every $a \in \mathcal{M}$ there is $s \in B$ such that $a < (s)_0$. For $s \in B$ let us define $K_{\mathbb{N}}(s)$ to be $\{y \in K : \exists i \in \mathbb{N}\ y < (s)_i\}$. It is a routine to verify that $K_{\mathbb{N}}(s)$ is a recursively saturated structure (the crucial point is that the terms of s form an increasing sequence of elements such that $K((s)_i) < K((s)_{i+1})$, and we can use s to define in K a satisfaction class for $K_{\mathbb{N}}(s)$). Thus K is a union of its recursively saturated elementary substructures, so it is recursively saturated. □

A direct attempt to construct an ω_1-like recursively saturated weakly Jónsson model using $\Diamond$ quickly leads to the following question.

Problem 3.5 Suppose $\mathcal{M}$ is a countable recursively saturated model of PA and K is a cofinal recursively saturated elementery submodel of $\mathcal{M}$. Is there a countable recursively saturated model $\mathcal{N}$ such that $\mathcal{M} \prec_e \mathcal{N}$ and for every recursively saturated model $\mathcal{N}' \prec \mathcal{N}$, if $K \prec \mathcal{N}'$, then K is a proper submodel of $\mathcal{N}' \cap \mathcal{M}$?

I would like to conjecture that the answer to Problem 3.5 is positive, and, consequently, that there are ω_1-like recursively saturated weakly Jónsson models (at least when $\Diamond$ is available).

4 Free cuts Notion of a free cut originated from the study of automorphisms of recursively saturated models of PA.

Definition 4.1 We say that an initial segment I of $\mathcal{M} \models \mathrm{PA}$ is *free* if for all $a, b \in I$ if $\mathrm{tp}(a) = \mathrm{tp}(b)$ then $\mathrm{tp}(a, I) = \mathrm{tp}(b, I)$, where $\mathrm{tp}(x, I)$ denotes the type of x over $(\mathcal{M}, I)$ in the language of PA with an additional set parameter for I.

Some simple observations on free sets in general model-theoretic context are presented in Kossak [8].

For $I \subseteq_e \mathcal{M} \models \mathrm{PA}$, the cofinality of I in $\mathcal{M}$, $\mathrm{cf}(I)$, is the initial segment defined by:

$$\begin{aligned}\mathrm{cf}(I) \quad = \quad & \inf\{a \in \mathcal{M} : \exists b \in \mathcal{M}, \mathrm{len}(b) < a \\ & \& \ \forall i < a - 1\ (b)_i < (b)_{i+1} \ \& \ \forall x \in I \exists i < a\ x < (b)_i \in I\}.\end{aligned}$$

Notice that $\mathrm{cf}(I)$ is first-order definable in $(\mathcal{M}, I)$.

A cut I is *semiregular* in $\mathcal{M}$ if $\mathrm{cf}(I) = I$. If I is not semiregular in $\mathcal{M}$ and there are $a, b \in I$, such that $\mathrm{tp}(a) = \mathrm{tp}(b)$, and $a \in \mathrm{cf}(I) < b$, then I is not free in $\mathcal{M}$. Thus, every recursively saturated model of PA has many nonfree cuts.

Our first task will be to show that every countable recursively saturated model of PA has many free cuts. To do this we will apply the machinery of definable types. The property of the minimal types that we will need is given in Proposition 4.2. Proposition 4.2 is a straightforward consequence of basic results concerning minimal types (see [1]). Let us introduce some more notation first.

If $\mathcal{M}$ is a model of PA and $a \in \mathcal{M}$, then $K(a)$ is the Skolem closure of a in $\mathcal{M}$. Also

$$\begin{aligned} \mathcal{M}(a) &= \sup K(a) = \{x \in \mathcal{M} : \exists y \in K(a)\ x < y\}; \\ \mathcal{M}[a] &= \bigcup\{K \prec_e \mathcal{M} : K < a\}. \end{aligned}$$

$G = \mathrm{Aut}(\mathcal{M})$ is the automorphism group of $\mathcal{M}$, for $a, b \in \mathcal{M}$ $U_{a,b} = \{f \in G : f(a) = b\}$, also $G_{(a)} = U_{a,a}$. If X is a subset of $\mathcal{M}$, then $G_{\{X\}} = \mathrm{Aut}(\mathcal{M}, X)$ is the setwise stabilizer of X.

For a type $p(v)$, $p^{\mathcal{M}}$ denotes the set of elements realizing $p(v)$ in $\mathcal{M}$.

Proposition 4.2 *Let $p(v)$ be a minimal type realized in a recursively saturated model $\mathcal{M} \models$ PA. Then for all $\langle a_1, \ldots a_n\rangle, \langle a'_1, \ldots a'_n\rangle \in [p^{\mathcal{M}}]^{<\omega}$, and all $b, b' \in \mathcal{M}[\min(a_1, a'_1)]$ if $\mathrm{tp}(b) = \mathrm{tp}(b')$, then $\mathrm{tp}(b, a_1, \ldots a_n) = \mathrm{tp}(b', a'_1, \ldots a'_n)$.*

Corollary 4.3 *If $\mathcal{M}$ is a countable recursively saturated model of* PA, *and $a \in \mathcal{M}$ realizes a minimal type, then $\mathcal{M}[a]$ is free in $\mathcal{M}$.*

Proof: If $b, c \in \mathcal{M}[a]$ realize the same type, then, by Proposition 4.2, $\mathrm{tp}(b, a) = \mathrm{tp}(c, a)$. Hence, there is $f \in \mathrm{Aut}(\mathcal{M})$ such that $f(b) = c$, and $f(a) = a$. But then $f''\mathcal{M}[a] = \mathcal{M}[a]$, and the result follows. □

Let us note that for every $a \in \mathcal{M}$, if $a > \mathcal{M}(0)$, then $\mathcal{M}(a)$ is not free in $\mathcal{M}$. To prove it first show that $\mathcal{M}[a]$ is definable in $(\mathcal{M}, \mathcal{M}(a))$ (see Smoryński [13]), and then use Proposition 4.4 to get $b \in \mathcal{M}[a]$ such that $\mathrm{tp}(a) = \mathrm{tp}(b)$.

To exhibit a greater variety among free cuts we will use sequences of skies. We will say that $a \in \mathcal{M} \models$ PA codes a *sequence of skies* if for all $i, j < \mathrm{len}(a)$ if $i \neq j$, then $\mathcal{M}((a)_i) \neq \mathcal{M}((a)_j)$. Cuts determined by coded sequences of skies were introduced and studied by Smoryński [13],[14]. In particular, Smoryński considered ascending sequence of skies: $a \in \mathcal{M}$ codes an *ascending sequence of skies*, $a \in \mathrm{ASS}(\mathcal{M})$, if a codes a sequence of skies of a nonstandard length, and, for all $i < \mathrm{len}(a)$, $(a)_i < (a)_{i+1}$.

If $a \in \mathcal{M}$ codes a sequence of skies and $X \subseteq \mathbb{N}$, then

$$\begin{aligned} I_a(X) &= \bigcup\{\mathcal{M}((a)_i) : i \in X\}; \\ I^a(X) &= \bigcap\{\mathcal{M}[(a)_i] : i \in X\}. \end{aligned}$$

We will show that if $\{(a)_i : i \in X\}$ has no maximum element, then $I_a(X)$ is free in $\mathcal{M}$, and if $\{(a)_i : i \in X\}$ has no minimum element, then $I^a(X)$ is also free in $\mathcal{M}$.

The next proposition is an easy exercise in recursive saturation.

Proposition 4.4 *If $\mathcal{M} \models$ PA is recursively saturated, $a, b \in \mathcal{M}$, $\mathcal{M}(a) < b$, and $p(v)$ is an unbounded type realized in $\mathcal{M}$, then there is $c \in p^{\mathcal{M}}$ such that $\mathcal{M}(a) < c \in \mathcal{M}[b]$.*

Corollary 4.5 *Let $\mathcal{M}$ be a recursively saturated model of* PA*, and let $p(v)$ be an unbounded type realized in $\mathcal{M}$. If $a \in \mathcal{M}$ codes a sequence of skies, then there is $b \in \mathcal{M}$ such that:*

1. *For every $i \in \mathbb{N}$ $(b)_i \in p^{\mathcal{M}}$;*
2. *For every $X \subseteq \mathbb{N}$ there are $Y_1, Y_2 \subseteq \mathbb{N}$ such that if $\{(a)_i : i \in X\}$ has no maximum element, then $I_a(X) = I_b(Y_1)$, and, if $\{(a)_i : i \in X\}$ has no minimum element, then $I^a(X) = I^b(Y_2)$.*

Proof: Let $h : \mathbb{N}^2 \to \mathbb{N}$ be a recursive pairing function. Consider the type $\Gamma(a, v)$:

$$\{\varphi((v)_{h(i,j)}) : i, j \in \mathbb{N}, i \neq j, \varphi(v) \in p(v)\} \bigcup$$
$$\{(a)_i < (v)_{h(i,j)} < (a)_j \vee (a)_j < (v)_{h(i,j)} < (a)_i : i, j \in \mathbb{N}, i \neq j\}.$$

By Proposition 4.4, $\Gamma(a, v)$ is finitely realizable in $\mathcal{M}$, and since $\Gamma(w, v)$ is recursive in $p(v)$, there is $b \in \mathcal{M}$ that realizes $\Gamma(a, v)$. It is easy to verify that b has the required property. □

Theorem 4.6 *Let $\mathcal{M}$ be a countable recursively saturated model of* PA*. If $a \in \mathcal{M}$ codes a sequence of skies, and $\{(a)_i : i \in X\}$ has no maximum element, then $I_a(X)$ is free in $\mathcal{M}$. Also, if $\{(a)_i : i \in X\}$ has no minimum element, then $I^a(X)$ is free in $\mathcal{M}$.*

Proof: By Corollary 4.5, we can assume that all elements $(a)_i : i \in \mathbb{N}$ realize the same minimal type $p(v)$. Now, take $c, d \in I_a(X)$ such that $\text{tp}(c) = \text{tp}(d)$. Without loss of generality we can assume that, for all $i \in \mathbb{N}$, $c, d \in \mathcal{M}[(a)_i]$. Let h be a recursive pairing function. Consider the recursive type $\Delta(a, v)$:

$$\{(a)_i < (v)_{h(i,j)} < (a)_j \vee (a)_j < (v)_{h(i,j)} < (a)_i : i, j \in \mathbb{N}, i \neq j\} \bigcup$$
$$\{\varphi(c, v) \iff \varphi(d, v) : \varphi(w, v) \in \mathcal{L}\}.$$

By Propositions 4.2 and 4.4, $\Delta(a, v)$ is finitely realizable in $\mathcal{M}$. Let $e \in \mathcal{M}$ be a realization of $\Delta(a, v)$. Then, there is $Y \subset \mathbb{N}$ such that $I_a(X) = I_e(Y)$. Since $\text{tp}(c, e) = \text{tp}(d, e)$, there is $f \in \text{Aut}(\mathcal{M})$ such that $f(c) = d$ and $f(e) = e$. But then $f'' I_e(Y) = f'' I_e(Y)$, and the result follows. The proof for $I^a(X)$ is similar. □

Theorem 4.6 implies that for every countable recursively saturated model of PA there are continuum many nonisomorphic structures of the form $(\mathcal{M}, I)$, where I is a free elementary cut of $\mathcal{M}$. To draw this conclusion we need to know that there are continuum many nonisomorphic structures of the form $(\mathcal{M}, I_a(X))$, where a codes a sequence of skies, but this has already been done in [14], see Theorem 3.6 there.

Yet an interesting problem remains open. If I is any of the free cuts of $\mathcal{M}$ exhibited above, then I has only countably many automorphic images in $\mathcal{M}$.

Problem 4.7 Suppose $\mathcal{M} \models$ PA is countable and recursively saturated. Is there a free elementary cut $I \prec_e \mathcal{M}$ such that $\text{card}\{f''(I) : f \in \text{Aut}(\mathcal{M})\} = 2^{\aleph_0}$?

The free cuts discussed above have a property that is a priori stronger than freeness itself. Namely if K is a free cut of the form $I_a(X)$, $I^a(X)$ or $\mathcal{M}[a]$, where a realizes a minimal type in $\mathcal{M}$, and $b, c \in K$ are such that $\text{tp}(b) = \text{tp}(c)$, then there is $f \in U_{b,c}$ such that f fixes K setwise. This property can be used to prove the following (see Kossak, Kotlarski and Schmerl [10] Theorem 2.3 and Lemma 5.1).

Proposition 4.8 *If $\mathcal{M}$ is a countable recursively saturated model of* PA *and K is a free cut such that either for some a coding an infinite sequence of skies $K = I_a(X)$, or $K = I^a(X)$ or $K = \mathcal{M}[a]$ for some a realizing a minimal type in $\mathcal{M}$, then $G_{\{K\}}$ is a maximal subgroup of G.*

If K is one of the cuts mentioned in Proposition 4.8, then $G_{\{K\}}$ is an open subgroup of G, i.e., $G_{\{K\}}$ contains $G_{(a)}$ for some $a \in \mathcal{M}$. Very little is known about cuts whose setwise stabilizers are not open, therefore it would be interesting to know the answer to the next question.

Problem 4.9 Assume that $\mathcal{M} \models$ PA is countable and recursively saturated. Is there a cut $K \prec_e \mathcal{M}$ such that $(\mathcal{M}, K)$ is recursively saturated and K is free in $\mathcal{M}$?

Notice that if K is a cut of $\mathcal{M}$ and $(\mathcal{M}, K)$ is recursively saturated then $G_{\{K\}}$ in not an open subroup of G, on the other hand if K is also free in $\mathcal{M}$, then it is not difficult to prove that $G_{\{K\}}$ is maximal.

Let us note that the affirmative answer to Problem 4.9 would also provide the affirmative answer to Problem 4.7

5 Uniformity and nonuniformity

The question we will consider in this section is: what are (naturally defined) classes $\mathcal{K}$ of cuts in a recursively saturated model of PA, such that, for all $K_1, K_2 \dots, K_n,\ L_1, L_2 \dots, L_n \in \mathcal{K}$ if $K_1 < K_2 \dots < K_n,\ L_1 < L_2 < \dots < L_n$, then

$$(\mathcal{M}, K_1, K_2, \dots, K_n) \cong (\mathcal{M}, L_1, L_2, \dots, L_n).$$

We will give examples of classes that have this property and an example of a class that does not. Our examples are some special classes of free cuts considered in the previous section. The positive answer to Problem 4.9 would provide other examples promising interesting applications.

Nonuniformity in the title of this section refers to the class we are about to define. A similar example will also allow us to show that a strong combinatorial property (superstrength) does not imply freeness.

For a countable recursively saturated model $\mathcal{M} \models$ PA let,

$$\begin{aligned}
\mathcal{K}_p &= \{\mathcal{M}[a] : a \text{ realizes } p(v) \text{ in } \mathcal{M}\};\\
\mathcal{K}_1 &= \{I_a(\mathbb{N}) : a \in \mathrm{ASS}(\mathcal{M})\};\\
\mathcal{K}_2 &= \{I^a(\mathbb{N}) : a \in \mathrm{DSS}(\mathcal{M})\},
\end{aligned}$$

where, in the definition of $\mathcal{K}_2$, $\mathrm{DSS}(\mathcal{M})$ is the set of codes of *descending sequences of skies*, i.e., codes a of sequences of skies of nonstandard length such that, for all $i < \mathrm{len}(a)$, $(a)_i > (a)_{i+1}$. All cuts in $\mathcal{K}_1$ and $\mathcal{K}_2$ are free, as are all the cuts in $\mathcal{K}_p$, if $p(v)$ is a minimal type realized in $\mathcal{M}$.

The proof of Proposition 5.1 is based on Proposition 4.4 and Theorem 4.5 (and its proof), details are left to the reader.

Proposition 5.1 *Let $\mathcal{M} \models$ PA be countable and recursively saturated. If $K_1 < K_2 < \dots < K_n$, $L_1 < L_2 < \dots < L_n$ are two sequences of cuts of $\mathcal{M}$, all in one of the classes $\mathcal{K}_p$, $\mathcal{K}_1$, or $\mathcal{K}_2$, where $p(v)$ is a minimal type realized in $\mathcal{M}$, then*

$$(\mathcal{M}, K_1, K_2, \dots, K_n) \cong (\mathcal{M}, L_1, L_2, \dots, L_n).$$

Are there other naturally defined classes of cuts with the above property? A candidate for such a class is suggested by Smoryński's Theorem 2.7 in [14].

For $a \in \mathrm{ASS}(\mathcal{M})$ and a semiregular cut $I < \mathrm{len}(a)$ define

$$\mathcal{M}(I, a) = \bigcup \{\mathcal{M}((a)_i) : i \in I\}.$$

Smoryński's theorem says that if $\mathcal{M} \models \mathrm{PA}$ is countable and recursively saturated, $a, b \in \mathrm{ASS}(\mathcal{M})$, $I < \mathrm{len}(a), \mathrm{len}(b)$ is semiregular, and $I < (a)_0, (b)_0$, then

$$(\mathcal{M}, \mathcal{M}(I, a)) \cong (\mathcal{M}, \mathcal{M}(I, b)).$$

However, our Proposition 5.3 provides a counterexample to this statement. Theorem 2.7 was stated in [14] without a proof. The proof of Proposition 5.3 was born in a conversation with Schmerl.

Definition 5.2 For $\mathcal{M} \models \mathrm{PA}$ and $\mathcal{K} \prec_e \mathcal{M}$ we define

$$\mathcal{S}(\mathcal{K}) = \sup\{e \in \mathcal{M} : \exists a \in \mathcal{M}\ a \cap \mathcal{K} \text{ is a } Q_e\text{-satisfaction class for } \mathcal{M}\}.$$

Observe that $\mathcal{S}(\mathcal{K})$ is definable in $(\mathcal{M}, \mathcal{K})$ and, if $\mathcal{K}$ is not recursively saturated, then $\mathcal{S}(\mathcal{K}) = \mathbb{N}$.

Proposition 5.3 *For every countable recursively saturated model $\mathcal{M}$ of* PA *there is a semiregular cut I and $a, b \in \mathrm{ASS}(\mathcal{M})$ such that $I < (a)_0, (b)_0, \mathrm{len}(a), \mathrm{len}(b)$, and $(\mathcal{M}, \mathcal{M}(I, a)) \not\cong (\mathcal{M}, \mathcal{M}(I, b))$.*

Proof: Consider the theory T in $\mathcal{L}$ with parameters a, b and with set parameters $\mathcal{N}_1, \mathcal{N}_2, I$, axiomatized by:

1. I is semiregular, $I < (a)_0, (b)_0, \mathrm{len}(a), \mathrm{len}(b)$;
2. $\mathcal{N}_1 = \sup_{i \in I} (a)_i$, $\mathcal{N}_2 = \sup_{i \in I} (b)_i$;
3. $\forall i < \mathrm{len}(a)\, \forall j < \mathrm{len}(b)\ t((a)_i) < (a)_{i+1}\ \&\ t((b)_j) < (b)_{j+1}$, for all Skolem terms t;
4. $\mathcal{S}(\mathcal{N}_1) = I$, $\mathcal{S}(\mathcal{N}_2) > I$.

Let U be an inductive satisfaction class for $\mathcal{M}$ such that $(\mathcal{M}, U)$ is recursively saturated. Let b code an ascending sequence of skies of $(\mathcal{M}, U)$ of a nonstandard length, and let $K_2 = \mathcal{M}(\mathbb{N}, b)$. Since $(K_2, K_2 \cap U) \prec (\mathcal{M}, U)$ $K_2 \cap U$ is a nonstandard satisfaction class for K_2, thus $\mathcal{S}(K_2) > \mathbb{N}$. Let us note that since, by Proposition 5.1, $(\mathcal{M}, K_2) \cong (\mathcal{M}, \mathcal{M}(\mathbb{N}, d))$ for every $d \in \mathrm{ASS}(\mathcal{M})$, we have shown that for every such d we have $\mathcal{S}(\mathcal{M}(\mathbb{N}, d)) > \mathbb{N}$.

Let c be any nonstandard element of $\mathcal{M}$ and let $K_1 = \mathcal{M}(c)$. Then $\mathcal{S}(K_2) = \mathbb{N}$. If T_0 is a finite fragment of T, then, since only finitely many Skolem terms t are involved, we can find $a \in \mathcal{M}$ such that $(\mathcal{M}, K_1, K_2, \mathbb{N}, a, b) \models T_0$. This proves that T is consistent. By chronic resplendency of $\mathcal{M}$ (see [2] for the definition and properties) there are $\mathcal{N}_1, \mathcal{N}_2, I, a, b$ such that the structure $(\mathcal{M}, \mathcal{N}_1, \mathcal{N}_2, I, a, b)$ is a model of T and is recursively saturated. In particular $I > \mathbb{N}$. To finish the proof notice that, since I is semiregular, $I = \mathrm{cf}(\mathcal{N}_1) = \mathrm{cf}(\mathcal{N}_2)$ is definable in both $(\mathcal{M}, \mathcal{N}_1)$ and $(\mathcal{M}, \mathcal{N}_2)$, and the result follows. $\square$

The strongest known combinatorial property of a cut in a model of PA is superstrength, and it was defined in Kirby [4] as follows. If $\mathcal{M} \models \mathrm{PA}$ and $I \subseteq_e \mathcal{M}$ then I is *n-Ramsey* in $\mathcal{M}$ if every coded partition $f : [I]^n \to a,\ a \in I$ has a homogeneous set that is coded in $\mathcal{M}$ and is unbounded in $[I]^n$, where unboundedness in $[I]^n$ is defined by induction in a natural way. I is said to be *superstrong* in $\mathcal{M}$ if I has the above property with respect to all coded partitions $f : [I]^c \to a,\ a \in I$, for some nonstandard $c \in \mathcal{M}$. Thus, if $I \subseteq_e \mathcal{M}$ is n-Ramsey, for every standard n, and $(\mathcal{M}, I)$ is recursively saturated, then I is superstrong in $\mathcal{M}$. It is also shown in [4] that if I is strong in $\mathcal{M}$, then I is n-Ramsey, for every standard n.

Our last result in this section shows that superstrength does not imply freeness.

Proposition 5.4 *Let $\mathcal{M} \models \mathrm{PA}$ be countable and recursively saturated. There is $I \prec_e \mathcal{M}$ such that I is superstrong but not free in $\mathcal{M}$.*

Proof: Let S be a Q_e inductive satisfaction class for $\mathcal{M}$, for some $e > \mathbb{N}$. By taking a smaller e, if necessary, we can assume that there are a, b such that $\mathrm{tp}(a) = \mathrm{tp}(b)$ and $a < e < b < 2e$.

Using the results of Kossak and Schmerl [11] we can select S such that all elements of $\mathcal{M}$ are definable in $(\mathcal{M}, S)$. Let $(\mathcal{N}, S')$ be a conservative elementary end extension of $(\mathcal{M}, S)$. Then $\mathcal{M}$ is isomorphic to $\mathcal{N}$, $\mathcal{M}$ is strong in $\mathcal{N}$, and $\mathcal{S}(\mathcal{M}) < \mathcal{M}$ (where $\mathcal{S}(\mathcal{M})$ is computed in $\mathcal{N}$). The last claim follows from the fact that every satisfaction class for $\mathcal{M}$ coded in $\mathcal{N}$ is definable in $(\mathcal{M}, S)$, and the minimality of S implies that, if U is a $Q_{e'}$ satisfaction class for $\mathcal{M}$ definable in $(\mathcal{M}, S)$, then $e' < e + \mathbb{N}$ (see [11]).

Thus we have $\mathrm{tp}(a) = \mathrm{tp}(b)$ and $a \in \mathcal{S}(\mathcal{M}) < b \in \mathcal{M}$; hence $\mathcal{M}$ is not free in $\mathcal{N}$. The above remarks show that the folowing recursive theory $T(a, b, I)$ is consistent:

$$a, b \in I + \mathrm{tp}(a) = \mathrm{tp}(b)\ +\ a < \mathcal{S}(I) < b\ +\ I \prec_e \mathcal{M}\ +\ I \text{ i s strong.}$$

If $(\mathcal{M}, I)$ is a recursively saturated model of $T(a, b, I)$ (we are using chronic resplendency here), then I is superstrong, but not free in $\mathcal{M}$. □

6 *Absolutely nonextendible automorphisms* In Kossak and Kotlarski [9] we have considered the following question: given recursively saturated countable models $\mathcal{M}$ and $\mathcal{N}$ such that $\mathcal{M} \prec_e \mathcal{N}$, when can an automorphism f of $\mathcal{M}$ be extended to an automorphism of $\mathcal{N}$? Here I would like to discuss an "absolute" version of this problem.

Definition 6.1 Let $\mathcal{M}$ be a recursively saturated model of PA. We say that $f \in \mathrm{Aut}(\mathcal{M})$ is *absolutely nonextendible* if for every recursively saturated model $\mathcal{N}$ such that $\mathcal{M} \prec_e \mathcal{N}$, and for every $g \in \mathrm{Aut}(\mathcal{N})$, g does not extend f.

Problem 6.2 Are there absolutely nonextendible automorphisms of countable recursively saturated models of PA?

Let us note that the "dual" question: "is there an automorphism of a countable recursively saturated model of PA that can be extended to an automorphism of every countable recursively saturated elementary extension of the model?" has a strong negative answer. It is shown in [11] that every countable recursively saturated model $\mathcal{M} \models \mathrm{PA}$

has a countable recursively saturated elementary end extension $\mathcal{N}$ such that none of the nontrivial automorphisms of $\mathcal{M}$ can be extended to an automorphism of $\mathcal{N}$.

A partial answer to Problem 6.2 is given in terms of satisfaction classes.

Proposition 6.3 *Let $\mathcal{M} \models \mathrm{PA}$ be countable and recursively saturated, and let f be an automorphism of $\mathcal{M}$. If there is $S \subset \mathcal{M}$ such that $(\mathcal{M}, S) \models \mathrm{Sat}_e(S) + \mathrm{PA}^*$, for some $e > \mathbb{N}$, and $f \in \mathrm{Aut}(\mathcal{M}, S)$, then f can be extended to an automorphism of a recursively saturated elemetary end extension of $\mathcal{M}$.*

The proof of Proposition 6.3 is based on the following simple lemma concerning definable types (cf. [1]).

Lemma 6.4 *If $\mathcal{M} \models \mathrm{PA}^*$, $\mathcal{N} = \mathcal{M}(a)$ is an elementary end extension of $\mathcal{M}$ generated by $\mathcal{M}$ and an element realizing a definable type, then every automorphism of $\mathcal{M}$ can be extended to an automorphism of $\mathcal{N}$.*

Proof: The definition of a definable type says that for every fomula $\varphi(w, v)$ of the language of the theory of $\mathcal{M}$ there is a formula $\sigma_\varphi(w)$ such that for every $b \in \mathcal{M}$,

$$\mathcal{N} \models \varphi(b, a) \text{ iff } \mathcal{M} \models \sigma_\varphi(b).$$

For $c \in \mathcal{N}$ let $g(c) = t(f(b), a)$, where $t(w, v)$ is a Skolem term of the language of the theory of $\mathcal{M}$ such that $\mathcal{N} \models c = t(b, a)$.

Let $c = t(b, a)$, and suppose that $\mathcal{N} \models \varphi(c)$. Let $\psi(w, v) = \varphi(t(w, v))$. So we have: $\mathcal{N} \models \varphi(c)$ iff $\mathcal{N} \models \varphi(t(b, a))$ iff $\mathcal{M} \models \sigma_\psi(b)$ iff $\mathcal{M} \models \sigma_\psi(f(b))$ iff $\mathcal{N} \models \varphi(t(f(b), a)$ iff $\mathcal{N} \models \varphi(g(c))$. Thus g is an automorphism of $\mathcal{N}$, and, clearly, $g(c) = f(c)$ for all $c \in \mathcal{M}$. □

Now, to prove Proposition 6.3, apply the lemma to the structure $(\mathcal{M}, S)$, where S is such as in the assumptions of the proposition. Not every automorphism satisfies the assumption of Proposition 6.3; Proposition 6.5 provides an example.

Proposition 6.5 *If $\mathcal{M}$ is countable recursively saturated model of PA and $\mathbb{N}$ is strong in $\mathcal{M}$, then there is f in $\mathrm{Aut}(\mathcal{M})$ such that for every inductive satisfaction class S for $\mathcal{M}$ $f \notin \mathrm{Aut}(\mathcal{M}, S)$.*

Proof: Let a be a nonstandard element of $\mathcal{M}$. Since $\mathbb{N}$ is strong in $\mathcal{M}$ there is $f \in \mathrm{Aut}(\mathcal{M})$ such that $\mathrm{fix}(f) = K(a)$ (cf. Kaye, Kossak and Kotlarski [3], Theorem 5.3). We claim that f has the desired property. Let S be an inductive satisfaction class for $\mathcal{M}$. If $f \in \mathrm{Aut}(\mathcal{M}, S)$, then $(\mathrm{fix}(f), \mathrm{fix}(f) \cap S)$ is an elementary substructure of $(\mathcal{M}, S)$, here we consider $\mathrm{fix}(f)$ as a model with the arithmetic structure inherited from $\mathcal{M}$, thus $\mathrm{fix}(f) \cap S$ is an inductive satisfaction class for $\mathrm{fix}(f) = K(a)$. This is a contradiction since $K(a)$ is nonstandard and is not recursively saturated. □

We do not know if Proposition 6.5 is true for recursively saturated models in which $\mathbb{N}$ is not strong. The argument from the above proof cannot be repeated in this case since if $\mathbb{N}$ is not strong in $\mathcal{M}$ then for every $f \in \mathrm{Aut}(\mathcal{M})$ $\mathrm{fix}(f)$ is recursively saturated (in fact isomorphic to $\mathcal{M}$); see the proof of Proposition 5.2 in [3].

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REFERENCES

[1] Gaifman, H., "Models and types of Peano's Arithmetic," *Annals of Mathematical Logic*, vol. 9 (1976), pp. 223–306. Zbl 0332.02058 MR 53:10577 1, 3, 4, 6

[2] Kaye, R., *Models of Peano Arithmetic*, Oxford Logic Guides, Oxford University Press, Oxford, 1991. Zbl 0744.03037 MR 92k:03034 1, 2, 5

[3] Kaye, R., R. Kossak and H. Kotlarski, "Automorphisms of recursively saturated models of arithmetic," *Annals of Pure and Applied Logic*, vol. 55 (1991), pp. 67–99. Zbl 0748.03023 MR 93c:03047 6, 6

[4] Kirby, L. A. S., Ph.D. thesis, University of Manchester, 1977. 5, 5

[5] Kirby, L. A. S., and J. B. Paris, "Initial segments of models of Peano's axioms," pp. 211–226 in *Set Theory and Hierarchy Theory V*, Lecture Notes in Mathematics 619, Springer-Verlag, Berlin, 1977. Zbl 0364.02032 MR 58:10423 1

[6] Knight, J. F., "Hanf number for omitting types over particular theories," *The Journal of Symbolic Logic*, vol. 41 (1976), pp. 583–588. Zbl 0343.02039 MR 58:5170 3

[7] Kossak, R., "A certain class of models of Peano Arithmetic," *The Journal of Symbolic Logic*, vol. 48 (1983), pp. 311–320. Zbl 0514.03036 MR 84j:03076 2

[8] Kossak, R., "Remarks on free sets," *Bulletin of the Polish Academy of Sciences*, vol. 34 (1986), pp. 117–122. Zbl 0623.03056 MR 87k:03037 4

[9] Kossak, R., and H. Kotlarski, "Results on automorphisms of recursively saturated models of PA," *Fundamenta Mathematicæ*, vol. 129 (1988), pp. 9–15. Zbl 0662.03027 MR 89h:03064 6

[10] Kossak, R., H. Kotlarski and J. H. Schmerl, "On maximal subgroups of the automorphism group of a countable recursively saturated model of PA," *Annals of Pure and Applied Logic*, vol. 65 (1993), pp. 125–148. Zbl 0796.03043 MR 94m:03061 4

[11] Kossak, R., and J. H. Schmerl, "Minimal satisfaction classes with an application to rigid models of Peano Aritmetic," *Notre Dame Journal of Formal Logic*, vol. 32 (1991), pp. 392–398. Zbl 0748.03024 MR 92h:03051 5, 5, 6

[12] Kotlarski, H., "Full satisfaction classes: a survey," *Notre Dame Journal of Formal Logic*, vol. 32 (1991), pp. 573–579. Zbl 0752.03018 MR 93b:03120 2

[13] Smoryński, C., "Elementary extensions of recursively saturated models of arithmetic," *Notre Dame Journal of Formal Logic*, vol. 22 (1981), pp. 193–203. Zbl 0503.03032 MR 82g:03064 4, 4

[14] Smoryński, C., "A note on initial segment constructions in recursively saturated models of arithmetic," *Notre Dame Journal of Formal Logic*, vol. 23 (1982), pp. 393–408. Zbl 0519.03055 MR 83j:03058 4, 4, 5, 5

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