

Algebraic cycles and Todorov surfaces

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Abstract Motivated by the Bloch–Beilinson conjectures, Voisin has formulated a conjecture about 0-cycles on self-products of surfaces of geometric genus one. We verify Voisin’s conjecture for the family of Todorov surfaces with $K^2 = 2$ and fundamental group $\mathbb{Z}/2\mathbb{Z}$. As a by-product, we prove that certain Todorov surfaces have finite-dimensional motive.

1. Introduction

The Bloch–Beilinson conjectures have been hugely influential in making concrete predictions concerning the behavior of Chow groups with \mathbb{Q} -coefficients $A^*(\cdot)_{\mathbb{Q}}$ of smooth projective varieties over \mathbb{C} . (This is explained, for example, in [46], [29], and [17].) One of these concrete predictions is the following intriguing conjecture about 0-cycles on self-products of surfaces with geometric genus one.

CONJECTURE 1.1 (Voisin [42])

Let S be a smooth complex projective surface with $h^{0,2}(S) = 1$ and $q(S) = 0$. Let $a, a' \in A_{\text{hom}}^2(S)$ be two 0-cycles of degree 0. Then

$$a \times a' = a' \times a \quad \text{in } A^4(S \times S).$$

The notation $a \times a'$ is shorthand for the cycle class $(p_1)^*(a) \cdot (p_2)^*(a') \in A^4(S \times S)$, where p_1, p_2 denote projection on the first and second factors, respectively.

Conjecture 1.1 has been verified in certain cases (see [42], [23]), but is still wide open for a general K3 surface.¹

The principal aim of this article is to add some new items to the list of examples of surfaces for which Conjecture 1.1 is verified. The main result is as follows.

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¹More precisely, I am not aware of a single K3 surface with Picard number less than 9 for which Conjecture 1.1 is known.

THEOREM (=Corollary 3.2)

Let S be a Todorov surface with $K_S^2 = 2$ and $\pi_1(S) = \mathbb{Z}/2\mathbb{Z}$. Then Conjecture 1.1 is true for S .

A Todorov surface (see Definition 2.1 below for a precise definition) is a certain surface of general type for which the bicanonical map factors over a K3 surface; these surfaces have been intensively studied with the aim of providing counterexamples to local and global Torelli theorems (see [22], [36], [27], [39], [40]). There exist 11 irreducible families of Todorov surfaces (see [27]). Todorov surfaces with invariants $K_S^2 = 2$ and $\pi_1(S) = \mathbb{Z}/2\mathbb{Z}$ form one of these irreducible families, which is of dimension 12. The author [23] established the truth of Conjecture 1.1 for another irreducible family of Todorov surfaces (those with $K_S^2 = 1$, which are sometimes called *Kunev surfaces*); so now there remain nine more families to investigate.

Along the way, we obtain some other results that may be of independent interest. For example, the above result is obtained by first showing the following.

THEOREM (=Theorem 5.2)

Let S be a Todorov surface with $K_S^2 = 2$ and $\pi_1(S) = \mathbb{Z}/2\mathbb{Z}$, and let P be the K3 surface associated to S . There is an isomorphism of Chow motives

$$t_2(S) \cong t_2(P) \quad \text{in } \mathcal{M}_{\text{rat}},$$

where t_2 denotes the transcendental part of the motive (see [18]).

This has consequences for the intersection product on S (Corollary 3.7). As another consequence of Theorem 5.2, we are able to show (Corollary 5.3) that certain Todorov surfaces have finite-dimensional motive in the sense of Kimura and O’Sullivan (see [19], [1]). This provides some new examples of surfaces of general type with finite-dimensional motive. The proof of Theorem 5.2 is directly inspired by Voisin’s work in [44] and [47] on the Bloch/Hodge equivalence for complete intersections, reasoning familywise and using the technique of “spread” of algebraic cycles.

CONVENTIONS

In this article, the word *variety* will refer to a quasiprojective separated scheme of finite type over \mathbb{C} , endowed with the Zariski topology. A *subvariety* is a (possibly reducible) reduced subscheme which is equidimensional.

We will denote by $A_j(X)$ the Chow group of j -dimensional cycles on X ; for X smooth of dimension n the notations $A_j(X)$ and $A^{n-j}(X)$ will be used interchangeably. Chow groups with rational coefficients will be denoted by

$$A_j(X)_{\mathbb{Q}} := A_j(X) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

The notation $A_{\text{hom}}^j(X)$ (resp., $A_{\text{AJ}}^j(X)$) will be used to indicate the subgroups of homologically trivial (resp., Abel–Jacobi trivial) cycles. For a morphism $f: X \rightarrow Y$, we will write $\Gamma_f \in A_*(X \times Y)$ for the graph of f .

In an effort to lighten notation, we will write $H^j(X)$ (or $H_j X$) to indicate singular cohomology $H^j(X, \mathbb{Q})$ (resp., Borel–Moore homology $H_j(X, \mathbb{Q})$).

2. Todorov surfaces

This preparatory section contains the definition and basic properties of Todorov surfaces. A first result that will be crucial to us is that every Todorov surface has an associated K3 surface for which Voisin’s conjecture is known to hold (Theorem 2.5; this is work of Rito). A second crucial result is that Todorov surfaces with $K_S^2 = 2$ and $\pi_1(S) = \mathbb{Z}/2\mathbb{Z}$ can be described as quotients of certain complete intersections in a weighted projective space (Theorem 2.7; this is work of Catanese–Debarre).

DEFINITION 2.1 (see [27], [36])

A *Todorov surface* is a smooth projective surface S of general type with $p_g(S) = 1$ and $q = 0$ such that the bicanonical map ϕ_{2K_S} factors as

$$\phi_{2K_S} : S \xrightarrow{\iota} S \dashrightarrow \mathbb{P}^r,$$

where $\iota : S \rightarrow S$ is an involution for which S/ι is birational to a K3 surface (i.e., there is equality $\phi_{2K_S} \circ \iota = \phi_{2K_S}$). The K3 surface obtained by resolving the singularities of S/ι will be called the K3 surface *associated to* S .

DEFINITION 2.2 (see [27])

The *fundamental invariants* of a Todorov surface S are (α, k) , where α is such that the 2-torsion subgroup of $\text{Pic}(S)$ has order 2^α and $k = K_S^2 + 8$.

REMARK 2.3

Morrison [27, p. 335] proves that there are exactly 11 nonempty irreducible families of Todorov surfaces, corresponding to the 11 possible values of the fundamental invariants:

$$(\alpha, k) \in \{(0, 9), (0, 10), (0, 11), (1, 10), (1, 11), \\ (1, 12), (2, 12), (2, 13), (3, 14), (4, 15), (5, 16)\}.$$

Examples of surfaces belonging to each of the 11 families are given in [36]; moreover, it is shown in [36] that these surfaces provide counterexamples to the local and global Torelli theorems (see [40] for an overview on Torelli problems, and see [39], where a mixed version of the Torelli problems is proposed to remedy this failure). The family with fundamental invariants $(0, 9)$ was first described by Kunev [22]; these surfaces are sometimes called *Kunev surfaces*.

In [27], an explicit description is given of the coarse moduli space for each of the 11 families of Todorov surfaces. Lee and Polizzi [24, Theorem 4.6 and Remark 4.7] have given an alternative construction of Todorov surfaces, as deformations of product-quotient surfaces.

REMARK 2.4

The convention $k = K_S^2 + 8$ in Definition 2.2, which may appear strange at first sight, is explained as follows: the number k happens to be the number of rational double points on a so-called *distinguished partial desingularization* of S/ι . (This follows from [27, Theorem 5.2(ii)].)

We will make use of the following result.

THEOREM 2.5 (Rito [33])

Let S be a Todorov surface, and let P be the smooth minimal model of S/ι . Then there exists a generically finite degree 2 cover

$$P \rightarrow \mathbb{P}^2,$$

ramified along the union of two smooth cubics.

REMARK 2.6

For the Todorov surface with fundamental invariants $(0, 9)$ (i.e., a Kunev surface), Theorem 2.5 has already been proven by Kunev and Todorov [35].

We now restrict our attention to Todorov surfaces S with fundamental invariants $(1, 10)$. This means that $K_S^2 = 2$ and (according to [9, Theorem 2.11]) the fundamental group of S is $\mathbb{Z}/2\mathbb{Z}$. In this case, there happens to be a nice explicit description of S in terms of weighted complete intersections.

THEOREM 2.7 (Catanese–Debarre [9])

Let S be a Todorov surface with fundamental invariants $(1, 10)$. Then the canonical model of S is the quotient V/τ' , where $V \subset \mathbb{P}(1^3, 2^2)$ is a weighted complete intersection having only rational double points as singularities, given by the equations

$$\begin{cases} F = z_3^2 + cw^4 + w^2q(x_1, x_2) + Q(x_1, x_2) = 0, \\ G = z_4^2 + c'w^4 + w^2q'(x_1, x_2) + Q'(x_1, x_2) = 0. \end{cases}$$

Here $[w : x_1 : x_2 : z_3 : z_4]$ are coordinates for $\mathbb{P} := \mathbb{P}(1^3, 2^2)$, and q, q' are quadratic forms, Q, Q' are quartic forms without common factor, and c, c' are constants not both 0. The involution $\tau' : \mathbb{P} \rightarrow \mathbb{P}$ is defined as

$$[w : x_1 : x_2 : z_3 : z_4] \mapsto [-w : x_1 : x_2 : z_3 : z_4].$$

Conversely, given a weighted complete intersection $V \subset \mathbb{P}$ as above, the quotient V/τ' is the canonical model of a Todorov surface with fundamental invariants $(1, 10)$.

Proof

This is a combination of [9, Theorems 2.8 and 2.9]. □

REMARK 2.8

The focus in the paper [9] is not on Todorov surfaces as such, but rather (as the title indicates) on *all* surfaces of general type with $p_g = 1$, $q = 0$, and $K^2 = 2$. Theorem 2.7 is actually a special case of the more general [9, Theorem 2.9], which describes the canonical model of *all* surfaces with $p_g = 1$, $q = 0$, $K^2 = 2$, and $\pi_1 = \mathbb{Z}/2\mathbb{Z}$ as quotients of weighted complete intersections. As shown in [9, Theorem 2.9], such surfaces form a 16-dimensional irreducible family. The Todorov surfaces with fundamental invariants (1, 10) correspond to surfaces with these invariants and for which the bicanonical map is a Galois covering; they form a 12-dimensional subfamily inside this 16-dimensional family.

The same remark can be made about Todorov surfaces with fundamental invariants (0, 9) (i.e., Kunev surfaces): these form a 12-dimensional subfamily inside the (18-dimensional) family of all surfaces of general type with $p_g = 1$, $q = 0$, and $K^2 = 1$; this family (and the 12-dimensional subfamily of Kunev surfaces, corresponding to the bicanonical map being Galois) can also be explicitly described in terms of weighted complete intersections (see [8], [35]).

REMARK 2.9

Todorov surfaces appear as so-called *nonstandard cases* in the classification of surfaces of general type whose bicanonical map fails to be birational (see [2, Chapter 2]). The Todorov surfaces with fundamental invariants (1, 10) are covered by [2, Theorem 8(iv)], the Kunev surfaces are covered by [2, Theorem 8(iii)], and the other Todorov surfaces are covered by [2, Theorem 8(v)].

It will be convenient to rephrase Theorem 2.7 as follows.

COROLLARY 2.10

Let \mathbb{P} be the weighted projective space $\mathbb{P} := \mathbb{P}(1, 1, 1, 2, 2)$. Let

$$B \subset (\mathbb{P}H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(4)))^{\times 2}$$

denote the subspace parameterizing pairs of weighted homogeneous equations of type

$$\begin{cases} F_b = z_3^2 + cw^4 + w^2q(x_1, x_2) + Q(x_1, x_2) = 0, \\ G_b = z_4^2 + c'w^4 + w^2q'(x_1, x_2) + Q'(x_1, x_2) = 0, \end{cases}$$

where (F_b, G_b) is as in Theorem 2.7, that is, the variety

$$V_b := \{x \in \mathbb{P} \mid F_b(x) = G_b(x) = 0\}$$

has only rational double points as singularities. (Thus, B is Zariski open in a product of projective spaces $\bar{B} = \mathbb{P}^r \times \mathbb{P}^r$, parameterizing all equations of type (F_b, G_b) , without conditions on the singularities.)

Let

$$\mathcal{V} \rightarrow B$$

denote the total space of the family (i.e., the fiber over $b \in B$ is the variety $V_b \subset \mathbb{P}$), and let

$$\mathcal{S} := \mathcal{V}/\tau \rightarrow B$$

denote the family obtained by applying the (fixed-point-free) involution $\tau := \tau' \times \text{id}_B$ to $\mathcal{S} \subset \mathbb{P} \times B$. Then $\mathcal{S} \rightarrow B$ is the family of all canonical models of Todorov surfaces with fundamental invariants $(1, 10)$.

PROPOSITION 2.11

The quasiprojective varieties \mathcal{V} and \mathcal{S} defined in Corollary 2.10 are smooth.

Proof

We first establish a preparatory lemma.

LEMMA 2.12

For each point

$$x \in \mathbb{P} \setminus [0 : 0 : 0 : 1 : 0],$$

there exists a polynomial G_b as in Corollary 2.10 such that

$$x \notin (G_b = 0).$$

For each point

$$x \in \mathbb{P} \setminus [0 : 0 : 0 : 0 : 1],$$

there exists a polynomial F_b as in Corollary 2.10 such that

$$x \notin (F_b = 0).$$

Proof

If $x \in \mathbb{P}$ is different from $[0 : 0 : 0 : 1 : 0]$, consider the image of x under the projection

$$\mathbb{P} \setminus [0 : 0 : 0 : 1 : 0] \rightarrow \mathbb{P}(1, 1, 1, 2)$$

given by forgetting the z_3 -coordinate. It is easily seen that the linear system defined by G_b on $\mathbb{P}(1, 1, 1, 2)$ is basepoint-free.

Likewise, for $x \in \mathbb{P}$ different from $[0 : 0 : 0 : 0 : 1]$, consider the projection

$$\mathbb{P} \setminus [0 : 0 : 0 : 0 : 1] \rightarrow \mathbb{P}(1, 1, 1, 2)$$

given by forgetting the z_4 -coordinate. □

Consider now $\bar{B} = \mathbb{P}^r \times \mathbb{P}^r$ the projective closure of B , parameterizing complete intersections that may be badly singular. Let

$$\bar{\mathcal{V}} \subset \bar{B} \times \mathbb{P}$$

denote the incidence variety containing \mathcal{V} as an open subset, and let $\pi: \bar{\mathcal{V}} \rightarrow \mathbb{P}$ denote the morphism induced by projection. Lemma 2.12 says that, for any point

$$p \in \mathbb{P} \setminus ([0:0:0:1:0] \cup [0:0:0:0:1]),$$

the fiber over p is

$$\pi^{-1}(p) \cong \mathbb{P}^{r-1} \times \mathbb{P}^{r-1}.$$

It follows that the quasiprojective variety

$$\bar{\mathcal{V}}_{\text{reg}} := \pi^{-1}(\mathbb{P}_{\text{reg}}),$$

being a projective bundle over a projective bundle over the smooth variety \mathbb{P}_{reg} , is smooth.

But the singular locus of \mathbb{P} is exactly the line $w = x_1 = x_2 = 0$, and a direct verification shows that V_b as in Corollary 2.10 does not meet this singular line, that is, $V_b \subset \mathbb{P}_{\text{reg}}$ for each $b \in B$, and hence,

$$\mathcal{V} \subset \bar{\mathcal{V}}_{\text{reg}}.$$

This proves the smoothness of \mathcal{V} . The smoothness of \mathcal{S} now follows, since \mathcal{S} is the quotient of \mathcal{V} under a fixed-point-free involution. □

COROLLARY 2.13

The general V_b and the general S_b are smooth.

REMARK 2.14

Corollary 2.13 is also established (by a different argument) in [9, Remark 2.10].

3. Main result

In this section, the main result as announced in the introduction (Theorem 3.1) is reduced to a statement concerning the Chow group of codimension 2 cycles on the relative self-product of a family (Proposition 3.5). This reduction step is done by reasoning familywise, using the method of “spread” of algebraic cycles developed by Voisin [44], [47], [46] in her work on the Bloch/Hodge equivalence. The proof of Proposition 3.5 is postponed to Section 4.

THEOREM 3.1

Let S be a Todorov surface with $K_S^2 = 2$ and $\pi_1(S) = \mathbb{Z}/2\mathbb{Z}$, and let P be the K3 surface obtained as a resolution of singularities of S/ι . The natural correspondence from S to P induces an isomorphism

$$A_{\text{hom}}^2(S)_{\mathbb{Q}} \cong A_{\text{hom}}^2(P)_{\mathbb{Q}}.$$

Theorem 3.1 implies the truth of Voisin’s conjecture (Conjecture 1.1) for S .

COROLLARY 3.2

Let S be a Todorov surface with $K_S^2 = 2$ and $\pi_1(S) = \mathbb{Z}/2\mathbb{Z}$. Let $a, a' \in A_{\text{hom}}^2(S)$

be two 0-cycles of degree 0. Then

$$a \times a' = a' \times a \quad \text{in } A^4(S \times S).$$

Proof

Since (by Rojtman's theorem [34] and [4, Theorem 5.1]) there is no torsion in $A_{\text{hom}}^4(S \times S)$, it suffices to prove the statement with rational coefficients. Let P be the K3 surface obtained by resolving the singularities of S/ι . There is a commutative diagram

$$\begin{array}{ccc} A_{\text{hom}}^2(S)_{\mathbb{Q}} \otimes A_{\text{hom}}^2(S)_{\mathbb{Q}} & \rightarrow & A^4(S \times S)_{\mathbb{Q}} \\ \uparrow & & \uparrow \\ A_{\text{hom}}^2(P)_{\mathbb{Q}} \otimes A_{\text{hom}}^2(P)_{\mathbb{Q}} & \rightarrow & A^4(P \times P)_{\mathbb{Q}} \end{array}$$

Here the left vertical arrow is an isomorphism (Theorem 3.1). The K3 surface P admits a description as a blowup of a double cover of \mathbb{P}^2 branched along two cubics (Theorem 2.5). It follows that Voisin's conjecture is true for P , that is, any $b, b' \in A_{\text{hom}}^2 P$ satisfy

$$b \times b' = b' \times b \quad \text{in } A^4(P \times P);$$

this is proven by Voisin [42, Theorem 3.4]. This implies that Voisin's conjecture is true for S . \square

We now proceed to prove Theorem 3.1.

Proof of Theorem 3.1

This proof is directly inspired by Voisin's work in [44], [47], and [46] on the Bloch/Hodge equivalence. The work of Catanese and Debarre [9] (Theorem 2.7 in this article) implies that canonical models of Todorov surfaces with fundamental invariants $(1, 10)$ form a family

$$\mathcal{S} \rightarrow B$$

as in Corollary 2.10. Moreover, there exist morphisms of families over B

$$\mathcal{V} \rightarrow \mathcal{S} \xrightarrow{f} \mathcal{M} \rightarrow \mathcal{E} \rightarrow B,$$

where \mathcal{E} is the family of quadric cones in \mathbb{P}^3 (the quadric cone \mathcal{E}_b is the image of \mathcal{S}_b under the bicanonical map (see [9])), and \mathcal{M} is the family of K3 surfaces with rational double points. Recall from Corollary 2.10 that $\mathcal{S} = \mathcal{V}/\tau$, where τ is an involution, and $\mathcal{M} = \mathcal{S}/\iota$, where ι is an involution. As explained in [9, Remark 2.10], the family \mathcal{M} can be obtained from the family \mathcal{E} by taking a double cover with prescribed ramification, the family \mathcal{S} is obtained from \mathcal{M} by taking a double cover, and the same holds for \mathcal{V} over \mathcal{S} .

To be on the safe side, we prefer to resolve singularities and work with smooth varieties. That is, we construct a commutative diagram of families over B

$$\begin{array}{ccc}
 \tilde{\mathcal{V}} & \rightarrow & \mathcal{V} \\
 \downarrow & & \downarrow \\
 \tilde{\mathcal{S}} & \rightarrow & \mathcal{S} \\
 \downarrow \tilde{f} & & \downarrow f \\
 \tilde{\mathcal{M}} & \rightarrow & \mathcal{M} \\
 \downarrow & & \downarrow g \\
 \tilde{\mathcal{E}} & \rightarrow & \mathcal{E} \\
 & \searrow & \downarrow \\
 & & B
 \end{array}$$

where varieties in the left column are smooth. This is not harmful to the argument, thanks to the following lemma.

LEMMA 3.3

For any $b \in B$, the induced morphisms

$$\tilde{\mathcal{V}}_b \rightarrow \mathcal{V}_b, \quad \tilde{\mathcal{S}}_b \rightarrow \mathcal{S}_b, \quad \tilde{\mathcal{M}}_b \rightarrow \mathcal{M}_b, \quad \tilde{\mathcal{E}}_b \rightarrow \mathcal{E}_b$$

are birational.

Proof

As noted above (Proposition 2.11), \mathcal{S} and \mathcal{V} are smooth. It follows that the singular locus of \mathcal{M} consists of the image of the fixed locus of the involution associated to f . Likewise, the singular locus of \mathcal{E} consists of the image of the singular locus of \mathcal{M} plus the image of the fixed locus of the involution associated to g . Since the involutions associated to f and g restrict to an involution on each fiber, we have

$$\dim(\text{Sing}(\mathcal{E}) \cap \mathcal{E}_b) \leq 1 \quad \text{for all } b \in B.$$

This implies that the induced morphism

$$\tilde{\mathcal{E}}_b \rightarrow \mathcal{E}_b$$

is birational for all $b \in B$.

The variety $\tilde{\mathcal{M}}$ is obtained by resolving the singularities of the fiber product $\tilde{\mathcal{E}} \times_{\mathcal{E}} \mathcal{M}$. Since the open subset \mathcal{E}_{reg} meets every fiber \mathcal{E}_b and g restricts to a smooth morphism over \mathcal{E}_{reg} , the morphism $\tilde{\mathcal{M}} \rightarrow \mathcal{M}$ is an isomorphism over the open $g^{-1}(\mathcal{E}_{\text{reg}})$. This open subset meets all the fibers \mathcal{M}_b , and so

$$\tilde{\mathcal{M}}_b \rightarrow \mathcal{M}_b$$

is birational for all $b \in B$. The argument for \mathcal{S} and \mathcal{V} is the same. □

We will be interested in the family

$$\tilde{\mathcal{S}} \times_B \tilde{\mathcal{S}} \rightarrow B.$$

There is a relative correspondence

$$\tilde{\mathcal{D}} := 2\Delta_{\tilde{\mathcal{S}}} - ({}^t\Gamma_{\tilde{f}}) \circ \Gamma_{\tilde{f}} \in A_{s-2}(\tilde{\mathcal{S}} \times_B \tilde{\mathcal{S}}).$$

(Here s denotes the dimension of $\tilde{\mathcal{S}} \times_B \tilde{\mathcal{S}}$, $\Delta_{\tilde{\mathcal{S}}}$ is the relative diagonal, $\Gamma_{\tilde{f}}$ is the graph of \tilde{f} , and relative correspondences over B can be composed as in [10], [16], [30], [12], and [29, Section 8.1.2], since $\tilde{\mathcal{S}}, \tilde{\mathcal{M}}$ are smooth. At this point we grade the Chow group by dimension rather than codimension, since $\tilde{\mathcal{S}} \times_B \tilde{\mathcal{S}}$ may be singular.) For any $b \in B$, we have that $H^{0,2}(S_b)$ is a one-dimensional \mathbb{C} -vector space and

$$(1) \quad (f_b)^*(f_b)_* = 2 \text{id}: H^{0,2}(S_b) \rightarrow H^{0,2}(S_b).$$

We know that $H^{0,2}$ is a birational invariant for surfaces with rational singularities. (To see this, one notes that if S is a surface with rational singularities and $\tilde{S} \rightarrow S$ is a resolution of singularities, then the Leray spectral sequence implies $H^i(S, \mathcal{O}_S) \rightarrow H^i(\tilde{S}, \mathcal{O}_{\tilde{S}})$ is an isomorphism, and so $\text{Gr}_F^0 H^i(S, \mathbb{C}) \cong \text{Gr}_F^0 H^i(\tilde{S}, \mathbb{C})$, since rational singularities are Du Bois (see [20, Theorem S]).) Hence, it follows from (1) that also

$$(\tilde{f}_b)^*(\tilde{f}_b)_* = 2 \text{id}: H^{0,2}(\tilde{S}_b) \rightarrow H^{0,2}(\tilde{S}_b).$$

By using the Lefschetz (1, 1) theorem on \tilde{S}_b , this implies that, for any $b \in B$, there exist a divisor $Y_b \subset \tilde{S}_b$ and a cycle $\gamma_b \in A^2(\tilde{S}_b \times \tilde{S}_b)_{\mathbb{Q}}$ supported on $Y_b \times Y_b$ such that

$$\tilde{\mathcal{D}}|_{\tilde{S}_b \times \tilde{S}_b} = \gamma_b \quad \text{in } H^4(\tilde{S}_b \times \tilde{S}_b), \text{ for all } b \in B.$$

(Here, for any relative correspondence Γ , we use the notation $\Gamma|_{\tilde{S}_b \times \tilde{S}_b}$ to indicate the result of applying to Γ the refined Gysin homomorphism (see [14]) induced by $b \rightarrow B$.)

Thanks to Voisin’s “spreading out” result in [44, Proposition 2.7], we can find a divisor $\mathcal{Y} \subset \tilde{\mathcal{S}}$ and a cycle $\Gamma \in A_{s-2}(\tilde{\mathcal{S}} \times_B \tilde{\mathcal{S}})_{\mathbb{Q}}$ supported on $\mathcal{Y} \times_B \mathcal{Y}$ with the property that the cycle

$$\tilde{\mathcal{D}}' := \tilde{\mathcal{D}} - \Gamma \in A_{s-2}(\tilde{\mathcal{S}} \times_B \tilde{\mathcal{S}})_{\mathbb{Q}}$$

has cohomologically trivial restriction to each fiber:

$$(\tilde{\mathcal{D}}')|_{\tilde{S}_b \times \tilde{S}_b} = 0 \quad \text{in } H^4(\tilde{S}_b \times \tilde{S}_b), \text{ for all } b \in B.$$

After shrinking the base B (i.e., after replacing B by a Zariski-open $B' \subset B$), we may suppose that all the S_b ’s are smooth (Corollary 2.13), and the morphisms $\mathcal{V} \rightarrow B', \mathcal{S} \rightarrow B'$ are smooth (so, in particular, the fiber product $\mathcal{S} \times_{B'} \mathcal{S}$ is smooth). Repeating the above procedure (or simply taking the pushforward of the restriction of $\tilde{\mathcal{D}}'$), one finds a cycle

$$\mathcal{D}' \in A^2(\mathcal{S} \times_{B'} \mathcal{S})_{\mathbb{Q}}.$$

Note that there is a relation

$$\tilde{\mathcal{D}}'|_{\tilde{\mathcal{S}} \times_{B'} \tilde{\mathcal{S}}} = \phi^*(\mathcal{D}') + \gamma \in A^2(\tilde{\mathcal{S}} \times_{B'} \tilde{\mathcal{S}})_{\mathbb{Q}},$$

where

$$\phi: \tilde{\mathcal{S}} \times_{B'} \tilde{\mathcal{S}} \rightarrow \mathcal{S} \times_{B'} \mathcal{S}$$

is the birational morphism induced by the resolution morphism, and γ is a cycle supported on

$$\mathcal{Z} \times_{B'} \tilde{\mathcal{S}} \cup \tilde{\mathcal{S}} \times_{B'} \mathcal{Z},$$

for some divisor $\mathcal{Z} \subset \tilde{\mathcal{S}}$. This is because the cycles $\tilde{\mathcal{D}}'|_{\tilde{\mathcal{S}} \times_{B'} \tilde{\mathcal{S}}}$ and $\phi^*(\mathcal{D}')$ coincide outside of the exceptional locus of ϕ , which is a divisor of the form $\mathcal{Z} \times_{B'} \tilde{\mathcal{S}} \cup \tilde{\mathcal{S}} \times_{B'} \mathcal{Z}$ (and, more precisely, the extension of \mathcal{Z} to the larger family $\tilde{\mathcal{S}} \rightarrow B$ is such that $\mathcal{Z}_b \subset \tilde{\mathcal{S}}_b$ is a divisor for all $b \in B$ (see Lemma 3.3)).

Then, using a Leray spectral sequence argument as in [44, Lemma 2.12] (and also as in [45, Lemma 1.2], where the setup is exactly as in the present proof), we know that, after some further shrinking of the base B' , there exists a cycle $c \in A^2(\mathbb{P} \times \mathbb{P})_{\mathbb{Q}}$ such that

$$\mathcal{D}'' := \mathcal{D}' + (c \times B')|_{\mathcal{S} \times_{B'} \mathcal{S}} = 0 \quad \text{in } H^4(\mathcal{S} \times_{B'} \mathcal{S}).$$

But then, since

$$A^2_{\text{hom}}(\mathcal{S} \times_{B'} \mathcal{S})_{\mathbb{Q}} = 0$$

by Proposition 3.5 below, we have a rational equivalence

$$\mathcal{D}'' = 0 \quad \text{in } A^2(\mathcal{S} \times_{B'} \mathcal{S})_{\mathbb{Q}}.$$

This implies that there is also a rational equivalence

$$\tilde{\mathcal{D}}'|_{\tilde{\mathcal{S}} \times_{B'} \tilde{\mathcal{S}}} = \phi^*((c \times B')|_{\mathcal{S} \times_{B'} \mathcal{S}}) + \gamma \in A^2(\tilde{\mathcal{S}} \times_{B'} \tilde{\mathcal{S}})_{\mathbb{Q}},$$

with γ as above supported in codimension 1.

Restricting to a general $b \in B$ (such that $b \in B'$ and the divisor $\mathcal{Y} \subset \mathcal{S}$ restricts to a divisor $Y_b \subset S_b$), we now find a decomposition of the diagonal

$$\begin{aligned} 2\Delta_{\tilde{S}_b} &= {}^t\Gamma_{\tilde{f}_b} \circ \Gamma_{\tilde{f}_b} + \{\text{something supported on } Y_b \times Y_b\} \\ &\quad + \{\text{something supported on } \mathcal{Z}_b \times \tilde{\mathcal{S}}_b \cup \tilde{\mathcal{S}}_b \times \mathcal{Z}_b\} \\ &\quad + \{\text{something coming from } \mathbb{P} \times \mathbb{P}\} \quad \text{in } A^2(\tilde{\mathcal{S}}_b \times \tilde{\mathcal{S}}_b)_{\mathbb{Q}}. \end{aligned}$$

Now, by considering the action of correspondences (and noting that only the first term acts on $A^2_{\text{hom}}(\tilde{\mathcal{S}}_b)_{\mathbb{Q}} = A^2_{\text{AJ}}(\tilde{\mathcal{S}}_b)_{\mathbb{Q}}$), this decomposition implies that

$$(\tilde{f}_b)^*(\tilde{f}_b)_* = 2 \text{id}: A^2_{\text{hom}}(\tilde{\mathcal{S}}_b)_{\mathbb{Q}} \rightarrow A^2_{\text{hom}}(\tilde{\mathcal{S}}_b)_{\mathbb{Q}}, \quad \text{for general } b \in B.$$

This last equality (combined with the obvious fact that $(\tilde{f}_b)_*(\tilde{f}_b)^*$ is also twice the identity on Chow groups) proves

$$A^2_{\text{hom}}(\tilde{\mathcal{S}}_b)_{\mathbb{Q}} \cong A^2_{\text{hom}}(\tilde{M}_b)_{\mathbb{Q}}$$

for the general $\tilde{\mathcal{S}}_b$.

To extend this statement to *all* $b \in B$, one considers the cycle

$$\tilde{\mathcal{D}}' - \phi^*((c \times B)|_{\mathcal{S} \times_B \mathcal{S}}) - \bar{\gamma} \in A_{s-2}(\tilde{\mathcal{S}} \times_B \tilde{\mathcal{S}})_{\mathbb{Q}},$$

where $\bar{\gamma}$ denotes an extension of γ that is still supported on an extension of the divisor \mathcal{Z} over B (and, by abuse of language, we use the same symbol ϕ to indicate the induced morphism $\tilde{\mathcal{S}} \times_B \tilde{\mathcal{S}} \rightarrow \mathcal{S} \times_B \mathcal{S}$). For each b in the open B' , the restriction of this cycle to the fiber over b is rationally trivial. By applying Lemma 3.4 below, it follows that the restriction of this cycle to *any* fiber is rationally trivial. Next, given any $b_0 \in B$, the moving lemma ensures that the divisor $\mathcal{Y} \subset \tilde{\mathcal{S}}$ appearing in the construction may be chosen in general position with respect to \tilde{S}_{b_0} ; then, the above argument implies that

$$A^2_{\text{hom}}(\tilde{S}_{b_0})_{\mathbb{Q}} \cong A^2_{\text{hom}}(\tilde{M}_{b_0})_{\mathbb{Q}}.$$

LEMMA 3.4

Let $M \rightarrow B$ be a projective fibration, where B is a smooth variety of dimension r . Let $\Gamma \in A_i(M)_{\mathbb{Q}}$. The set of points $b \in B$ such that $\Gamma|_{M_b} = 0$ in $A_{i-r}(M_b)_{\mathbb{Q}}$ is a countable union of closed algebraic subsets of B .

Proof

Usually this is stated for smooth M , for instance, in [46, Lemma 3.2]. However, as the proof is just a Hilbert schemes argument, this still goes through for singular M . □

Let us now wrap up the proof of Theorem 3.1. Suppose S is a Todorov surface with fundamental invariants $(1, 10)$, and suppose P is a resolution of singularities of S/ι . The canonical model of S is an S_b for some $b \in B$ (Corollary 2.10). After passing to a blowup \tilde{S} of S , we get a diagram of surfaces

$$\begin{array}{ccccc} \tilde{S} & \rightarrow & S_b & \leftarrow & \tilde{S}_b \\ \downarrow & & \downarrow & & \downarrow \\ P & \rightarrow & S_b/\iota_b & \leftarrow & \tilde{M}_b \end{array}$$

where horizontal arrows are birational morphisms (Lemma 3.3), and the surfaces in the left and right columns are smooth. We conclude using the commutative diagram

$$\begin{array}{ccccc} A^2_{\text{hom}}(S)_{\mathbb{Q}} & \xrightarrow{\cong} & A^2_{\text{hom}}(\tilde{S})_{\mathbb{Q}} & \xrightarrow{\cong} & A^2_{\text{hom}}(\tilde{S}_b)_{\mathbb{Q}} \\ & & \downarrow & & \downarrow \cong \\ & & A^2_{\text{hom}}(P)_{\mathbb{Q}} & \xrightarrow{\cong} & A^2_{\text{hom}}(\tilde{M}_b)_{\mathbb{Q}} \end{array}$$

(Here horizontal arrows are isomorphisms, because A^2_{hom} is a birational invariant for smooth surfaces, and the right vertical arrow is an isomorphism as we have shown above.) The above argument relies on the following key result, the proof of which is postponed to the next section.

PROPOSITION 3.5

Let $\mathcal{V} \rightarrow B$ be the family of weighted complete intersection surfaces, and let $\mathcal{S} \rightarrow B$ be the family of Todorov surfaces as in Corollary 2.10. Suppose B is small enough

for the morphism $\mathcal{V} \rightarrow B$ to be smooth. Then

$$A_{\text{hom}}^2(\mathcal{S} \times_B \mathcal{S})_{\mathbb{Q}} = A_{\text{hom}}^2(\mathcal{V} \times_B \mathcal{V})_{\mathbb{Q}} = 0.$$

□

We now state a few corollaries of Theorem 3.1.

COROLLARY 3.6

Let S be a Todorov surface with $K_S^2 = 2$ and $\pi_1(S) = \mathbb{Z}/2\mathbb{Z}$. Then the generalized Hodge conjecture is true for the sub-Hodge structure

$$\bigwedge^2 H^2(S) \subset H^4(S \times S).$$

The Hodge conjecture is true for $(2, 2)$ -classes in $\bigwedge^2 H^2(S)$.

Proof

As already noted in [42], this follows from Corollary 3.2 using the Bloch–Srinivas method (see [7]). □

COROLLARY 3.7

Let S be a Todorov surface with $K_S^2 = 2$ and $\pi_1(S) = \mathbb{Z}/2\mathbb{Z}$. Let ι be the involution such that S/ι is birational to a K3 surface, and let $A^1(S)^\iota$ denote the ι -invariant part of $A^1(S)$. Then

$$\text{Im}(A^1 S \otimes A^1(S)^\iota \rightarrow A^2 S)$$

has dimension 1.

Proof

In view of Rojzman’s theorem, it suffices to prove the statement with rational coefficients. Let $p: S \rightarrow S/\iota$ denote the projection. Since S/ι is birational to a K3 surface (in other words, S/ι is a “K3 surface with rational double points,” in the language of [27]), there is a distinguished 0-cycle $e \in A^2(S/\iota)$ with the property that (see [3])

$$\text{Im}(A^1(S/\iota)_{\mathbb{Q}} \otimes A^1(S/\iota)_{\mathbb{Q}} \rightarrow A^2(S/\iota)_{\mathbb{Q}}) = \mathbb{Q} \cdot e.$$

Now given two divisors $D \in A^1(S)_{\mathbb{Q}}$ and $D' \in A^1(S)_{\mathbb{Q}}^\iota$, we can write $D' = p^*(F')$ for some $F' \in A^1(S/\iota)_{\mathbb{Q}}$. Using the projection formula, we find that

$$p_*(D \cdot D') = p_*(D \cdot p^*(F')) = p_*(D) \cdot F' = \text{deg}(p_*(D) \cdot F')e \quad \text{in } A^2(S/\iota)_{\mathbb{Q}}.$$

Let $e_S \in A^2(S)$ be any 0-cycle mapping to $e \in A^2(S/\iota)$. Then (as $A^2(S)_{\mathbb{Q}} \rightarrow A^2(S/\iota)_{\mathbb{Q}}$ is an isomorphism by Theorem 3.1), we have

$$D \cdot D' = \text{deg}(p_*(D) \cdot F')e_S \quad \text{in } A^2(S)_{\mathbb{Q}};$$

that is, e_S can be considered to be a distinguished 0-cycle for the intersection on S . □

REMARK 3.8

An equivalent formulation of Corollary 3.7 is as follows: for S a surface as in Corollary 3.7, there exists e_S such that, for all divisors $D_1, D_2 \in A^1(S)$, we have

$$D_1 \cdot (D_2 + \iota_*(D_2)) = ce_S \quad \text{in } A^2S,$$

for some $c \in \mathbb{Z}$.

REMARK 3.9

A result similar to (but stronger than) Theorem 3.1 was proven by Voisin for K3 surfaces. Voisin [43] proved that if X is any K3 surface and ι is a symplectic involution of X , then $A^2(X) = A^2(X)^\iota$. Theorem 3.1 is weaker than this, in the sense that we cannot prove anything for an arbitrary symplectic involution on a surface S as in Theorem 3.1; our proof only works if the involution extends to the whole family of Todorov surfaces with the given invariants.

REMARK 3.10

Below we will prove (as Theorem 5.2) the motivic version of Theorem 3.1 that was stated in the introduction. This motivic version is not necessary for the proof of Corollary 3.2 (for which the statement of Theorem 3.1 suffices), but it may have some independent interest.

4. Trivial Chow groups

This section contains the proof of Proposition 3.5, which was a key result used in the preceding section. We rely on work of Totaro [37], which is recalled in Section 4.1. Section 4.2 proves Proposition 3.5, by considering an appropriate stratification of $\widetilde{\mathbb{P} \times \mathbb{P}}$. Things work out fine, because everything is linear (i.e., all the strata and all their intersections look like affine spaces).

4.1. Weak and strong properties

DEFINITION 4.1 (Totaro [37])

For any (not necessarily smooth) quasiprojective variety X , let $A_i(X, j)$ denote Bloch's higher Chow groups. (These groups are sometimes written as $A^{n-i}(X, j)$ or $CH^{n-i}(X, j)$, where $n = \dim X$.) As explained in [37, Section 4], the relation with algebraic K -theory ensures that there are functorial cycle class maps

$$A_i(X, j)_{\mathbb{Q}} \rightarrow \mathrm{Gr}_{-2i}^W H_{2i+j}(X),$$

compatible with long exact sequences. (Here W denotes Deligne's weight filtration on Borel–Moore homology (see [32]).)

We say that X has the *weak property* if the cycle class maps induce isomorphisms

$$A_i(X)_{\mathbb{Q}} \xrightarrow{\cong} W_{-2i} H_{2i}(X)$$

for all i . We say that X has the *strong property* if X has the weak property and, in addition, the cycle class maps induce surjections

$$A_i(X, 1)_{\mathbb{Q}} \twoheadrightarrow \mathrm{Gr}_{-2i}^W H_{2i+1}(X)$$

for all i .

LEMMA 4.2

Let X be a quasiprojective variety, and let $Y \subset X$ be a closed subvariety with complement $U = X \setminus Y$. If Y and U have the strong property, then so does X .

Proof

This is the same argument as in [37, Lemma 7], which is a slightly different statement. As in [37, Lemma 7], using the localization property of higher Chow groups (see [6, [25]]), one finds a commutative diagram with exact rows

$$\begin{array}{ccccccccc} A_i(U, 1)_{\mathbb{Q}} & \rightarrow & A_i(Y)_{\mathbb{Q}} & \rightarrow & A_i(X)_{\mathbb{Q}} & \rightarrow & A_i(U)_{\mathbb{Q}} & \rightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ \mathrm{Gr}_{-2i}^W H_{2i+1}(U) & \rightarrow & \mathrm{Gr}_{-2i}^W H_{2i+1}(Y) & \rightarrow & \mathrm{Gr}_{-2i}^W H_{2i+1}(X) & \rightarrow & \mathrm{Gr}_{-2i}^W H_{2i+1}(U) & \rightarrow & 0 \end{array}$$

A diagram chase reveals that, under the assumptions of the lemma, the penultimate vertical arrow is an isomorphism.

By continuing these long exact sequences to the left, there is a commutative diagram with exact rows

$$\begin{array}{ccccccccc} A_i(Y, 1)_{\mathbb{Q}} & \rightarrow & A_i(X, 1)_{\mathbb{Q}} & \rightarrow & A_i(U, 1)_{\mathbb{Q}} & \rightarrow & A_i(Y)_{\mathbb{Q}} & \rightarrow & \\ & & \downarrow & & \downarrow & & \downarrow \cong & & \\ \mathrm{Gr}_{-2i}^W H_{2i+1}(Y) & \rightarrow & \mathrm{Gr}_{-2i}^W H_{2i+1}(X) & \rightarrow & \mathrm{Gr}_{-2i}^W H_{2i+1}(U) & \rightarrow & \mathrm{Gr}_{-2i}^W H_{2i+1}(Y) & \rightarrow & \end{array}$$

Doing another diagram chase, one learns that the second vertical arrow is a surjection. □

COROLLARY 4.3

Let X be a quasiprojective variety that admits a stratification such that each stratum is of the form $\mathbb{A}^k \setminus L$, where L is a finite union of linearly embedded affine subspaces. Then X has the strong property.

Proof

Affine space has the strong property. (This is the homotopy invariance for higher Chow groups.) The subvariety L has the weak property. By doing a diagram chase as in Lemma 4.2 (or directly applying [37, Lemma 6]), it follows that the variety $\mathbb{A}^k \setminus L$ has the strong property. The corollary now follows from Lemma 4.2. □

LEMMA 4.4

Let X be a quasiprojective variety with the strong property. Let $Y \rightarrow X$ be a projective bundle. Then Y has the strong property.

Proof

This follows from the projective bundle formula for higher Chow groups (see [5]). □

4.2. Proof of Proposition 3.5

We now proceed to prove the key proposition.

PROPOSITION 1 (=Proposition 3.5)

Let $\mathcal{V} \rightarrow B$ be the family of weighted complete intersection surfaces, and let $\mathcal{S} \rightarrow B$ be the family of Todorov surfaces as in Corollary 2.10. Let $B' \subset B$ be open such that the induced morphism $\mathcal{V}' \rightarrow B'$ is smooth. Let $\mathcal{S}' \rightarrow B'$ denote the restriction of \mathcal{S} to B' . Then

$$A_{\text{hom}}^2(\mathcal{S}' \times_{B'} \mathcal{S}')_{\mathbb{Q}} = 0,$$

$$A_{\text{hom}}^2(\mathcal{V}' \times_{B'} \mathcal{V}')_{\mathbb{Q}} = 0.$$

Proof

Since there is a finite surjective morphism

$$\mathcal{V}' \times_{B'} \mathcal{V}' \rightarrow \mathcal{S}' \times_{B'} \mathcal{S}',$$

the first statement follows from the second. To prove the second statement, we will actually prove the following.

PROPOSITION 4.5

Let $\mathcal{V}' \rightarrow B'$ be as in Proposition 3.5, and let

$$\widetilde{\mathcal{V}' \times_{B'} \mathcal{V}'} \rightarrow \mathcal{V}' \times_{B'} \mathcal{V}'$$

be the blowup along the relative diagonal. There exists a projective variety M with

$$A_*^{\text{hom}}(M)_{\mathbb{Q}} = 0$$

and such that M contains $\widetilde{\mathcal{V}' \times_{B'} \mathcal{V}'}$ as a Zariski-open set.

It is easily seen that Proposition 4.5 implies Proposition 3.5. Indeed, set

$$U := \widetilde{\mathcal{V}' \times_{B'} \mathcal{V}'}, \quad D := M \setminus U,$$

and let $m := \dim M$. Suppose $a \in A_{\text{hom}}^2(U)_{\mathbb{Q}}$. Then there exists $\bar{a} \in A_{m-2}(M)_{\mathbb{Q}}$ restricting to a and such that the class

$$[\bar{a}] \in H_{2m-4}(M)$$

maps to 0 in H^4U . Using a resolution of singularities of M , one finds that the homology class $[\bar{a}]$ comes from a Hodge class $\beta \in H^2(\tilde{D})$ (where $\tilde{D} \rightarrow D$ is a resolution of singularities of the boundary divisor D). The Lefschetz (1, 1) theorem ensures that the class β is algebraic, say, $\beta = [b]$ for some $b \in A^1(\tilde{D})_{\mathbb{Q}}$. Now

$$\bar{a}' := \bar{a} - i_*(b)$$

is a class in $A_{m-2}^{\text{hom}}(M)_{\mathbb{Q}} = 0$ restricting to a , and hence, $a = 0$. This clearly implies that also

$$A_{\text{hom}}^2(\mathcal{V}' \times_{B'} \mathcal{V}')_{\mathbb{Q}} = 0.$$

If $\phi: U \rightarrow \mathcal{V}' \times_{B'} \mathcal{V}'$ denotes the blowup, we have that $\phi_*\phi^* = \text{id}$ on $A_{\text{hom}}^2(\mathcal{V}' \times_{B'} \mathcal{V}')_{\mathbb{Q}}$. Note that the stronger statement

$$A_{\text{hom}}^i(\mathcal{V}' \times_{B'} \mathcal{V}')_{\mathbb{Q}} = 0 \quad \text{for all } i$$

is likely to be true (see Remark 4.17).

We now proceed to prove Proposition 4.5; this is a slight modification of an argument of Voisin (see [44, Proposition 2.13], [47, Lemma 1.3]; it is also explained in [46, Section 4.3]). Let

$$\bar{B} \supset B$$

denote the projective closure of B (so \bar{B} is a product of two projective spaces $\mathbb{P}^r \times \mathbb{P}^r$). Let

$$\widetilde{\mathbb{P} \times \mathbb{P}} \rightarrow \mathbb{P} \times \mathbb{P}$$

be the blowup along the diagonal. Points of $\widetilde{\mathbb{P} \times \mathbb{P}}$ correspond to the data of (x, y, z) , where x, y are points of \mathbb{P} and $z \subset X$ is a length 2 zero-dimensional subscheme with associated cycle $x + y$. Consider now the variety

$$M := \{((F_b, G_b), (x, y, z)) \mid F_b|_z = G_b|_z = 0\} \subset \bar{B} \times \widetilde{\mathbb{P} \times \mathbb{P}}.$$

Clearly M contains $\widetilde{\mathcal{V}' \times_{B'} \mathcal{V}'}$ as a Zariski-open set. We now proceed to show that M has trivial Chow groups. Note that the fiber of the projection

$$\pi: M \rightarrow \widetilde{\mathbb{P} \times \mathbb{P}}$$

over a point $(x, y, z) \in \widetilde{\mathbb{P} \times \mathbb{P}}$ is

$$\{b \in \bar{B} \mid F_b|_z = G_b|_z = 0\} \subset \bar{B},$$

which is of the form

$$\mathbb{P}^s \times \mathbb{P}^t \subset \mathbb{P}^r \times \mathbb{P}^r = \bar{B}.$$

The strategy of this proof will be to stratify $\widetilde{\mathbb{P} \times \mathbb{P}}$ such that, over each stratum, the morphism π has constant dimension.

It follows from Lemma 2.12 that, with two exceptions, every point imposes one condition on the polynomials F_b, G_b , that is, for all

$$x \in \mathbb{P} \setminus ([0:0:0:1:0] \cup [0:0:0:0:1]),$$

we have that

$$\{b \in \bar{B} \mid V_b \ni x\} \cong \mathbb{P}^{r-1} \times \mathbb{P}^{r-1} \subset \mathbb{P}^r \times \mathbb{P}^r = \bar{B}.$$

It remains to analyze what happens when we impose two points. Let us define the locus

$$\begin{aligned} \bar{Q} := f^{-1} & \left(([0:0:0:1:0] \cup [0:0:0:0:1]) \times \mathbb{P} \right. \\ & \left. \cup \mathbb{P} \times ([0:0:0:1:0] \cup [0:0:0:0:1]) \right) \subset \widetilde{\mathbb{P} \times \mathbb{P}}, \end{aligned}$$

where $f: \widetilde{\mathbb{P} \times \mathbb{P}} \rightarrow \mathbb{P} \times \mathbb{P}$ is the blowup of the diagonal. We leave aside (for later consideration) \bar{Q} and E ; that is, we write

$$P := \widetilde{\mathbb{P} \times \mathbb{P}} \setminus (E \cup \bar{Q})$$

(so P is isomorphic to an open set in $(\mathbb{P} \times \mathbb{P}) \setminus \Delta$).

We now proceed to stratify P as follows. First, we define “partial diagonals”

$$\begin{aligned} \Delta_{3,\pm,\pm} := & \{ (p, p') \in \mathbb{P} \times \mathbb{P} \mid \exists \lambda \in \mathbb{C}^* \text{ such that } p_1 = \lambda p'_1 \text{ and } p_2 = \lambda p'_2 \\ & \text{and } p_0 = \pm \lambda p'_0 \text{ and } p_4 = \pm \lambda^2 p'_4 \}, \end{aligned}$$

$$\begin{aligned} \Delta_{4,\pm,\pm} := & \{ (p, p') \in \mathbb{P} \times \mathbb{P} \mid \exists \lambda \in \mathbb{C}^* \text{ such that } p_1 = \lambda p'_1 \text{ and } p_2 = \lambda p'_2 \\ & \text{and } p_0 = \pm \lambda p'_0 \text{ and } p_3 = \pm \lambda^2 p'_3 \}. \end{aligned}$$

(Here we suppose a point $p \in \mathbb{P}$ has coordinates $p = [p_0 : p_1 : p_2 : p_3 : p_4]$.) Just to fix ideas, we have for example that $\Delta_{3,+,+} \cap \Delta_{4,+,+}$ is the diagonal of \mathbb{P} .

We define closed subvarieties $P_{1,j} \subset P$ as follows:

$$\begin{aligned} P_{1,1} &:= (\Delta_{3,+,+}) \cap P, \\ P_{1,2} &:= (\Delta_{3,+,-}) \cap P, \\ P_{1,3} &:= (\Delta_{3,-,+}) \cap P, \\ P_{1,4} &:= (\Delta_{3,-,-}) \cap P, \\ P_{1,5} &:= (\Delta_{4,+,+}) \cap P, \\ P_{1,6} &:= (\Delta_{4,+,-}) \cap P, \\ P_{1,7} &:= (\Delta_{4,-,+}) \cap P, \\ P_{1,8} &:= (\Delta_{4,-,-}) \cap P, \end{aligned}$$

and an open subvariety

$$P^0 := P \setminus \left(\bigcup_j P_{1,j} \right)$$

(i.e., P^0 is the complement in $(\mathbb{P} \times \mathbb{P}) \setminus f(\bar{Q})$ of the union of the various partial diagonals $\Delta_{3,\pm,\pm}, \Delta_{4,\pm,\pm}$). We next define closed subvarieties

$$\begin{aligned} P_{2,1} &:= P_{1,1} \cap P_{1,5}, \\ P_{2,2} &:= P_{1,1} \cap P_{1,6}, \\ P_{2,3} &:= P_{1,1} \cap P_{1,7}, \\ P_{2,4} &:= P_{1,1} \cap P_{1,8}, \end{aligned}$$

$$\begin{aligned}
 P_{2,5} &:= P_{1,2} \cap P_{1,5}, \\
 P_{2,6} &:= P_{1,2} \cap P_{1,6}, \\
 &\vdots \\
 P_{2,16} &:= P_{1,4} \cap P_{1,8}.
 \end{aligned}$$

There are open subvarieties $P_{1,j}^0 \subset P_{1,j}$ defined as

$$P_{1,j}^0 := P_{1,j} \setminus \left(\bigcup_{P_{2,k} \subset P_{1,j}} P_{2,k} \right).$$

The upshot is that we have a stratification

$$P_2 \subset P_1 \subset P = \widetilde{\mathbb{P} \times \mathbb{P}} \setminus E,$$

where $P_i := \bigcup_j P_{i,j}$ such that at each step

$$P_i \setminus P_{i+1} = \bigcup_j P_{i,j}^0.$$

(Here, by convention, we write $P = P_{0,0}$ and $P^0 = P_{0,0}^0$.)

We now return to the morphism

$$\pi: M \rightarrow \widetilde{\mathbb{P} \times \mathbb{P}}$$

defined above; by construction, each fiber F of π is of type

$$F \cong \mathbb{P}^s \times \mathbb{P}^t \subset \mathbb{P}^r \times \mathbb{P}^r = \bar{B}.$$

Let

$$M_2 := \pi^{-1}(P_2), \quad M_{1,j}^0 := \pi^{-1}(P_{1,j}^0), \quad M_0^0 := \pi^{-1}(P_0^0);$$

we thus obtain a stratification of $M_0 := M \setminus \pi^{-1}(E \cup Q)$. The point of doing this is that over each stratum the morphism π is of constant dimension.

LEMMA 4.6

Over each stratum of $M_0 \rightarrow P_0$, the morphism π restricts to a fibration with fibers $\mathbb{P}^s \times \mathbb{P}^t$. More precisely, a fiber $F = \pi^{-1}(p)$ is

$$F \cong \begin{cases} \mathbb{P}^{r-1} \times \mathbb{P}^{r-1} & \text{if } p \in P_2, \\ \mathbb{P}^{r-2} \times \mathbb{P}^{r-1} & \text{if } p \in P_{1,j}^0 \text{ with } 1 \leq j \leq 4, \\ \mathbb{P}^{r-1} \times \mathbb{P}^{r-2} & \text{if } p \in P_{1,j}^0 \text{ with } 5 \leq j \leq 8, \\ \mathbb{P}^{r-2} \times \mathbb{P}^{r-2} & \text{if } p \in P_0^0. \end{cases}$$

Proof

It is readily seen that a point p which lies on a partial diagonal $\Delta_{3,\pm,\pm}$ imposes at most one condition on the polynomials G_b of Corollary 2.10. Combined with Lemma 2.12, this observation yields that points on

$$\bigcup \Delta_{3,\pm,\pm} \setminus \left((Q \cup E) \cap \left(\bigcup \Delta_{3,\pm,\pm} \right) \right)$$

impose exactly one condition on polynomials G_b as in Corollary 2.10. On the other hand, given any point

$$p = (q, q') \in \mathbb{P} \times \mathbb{P} \setminus \left(Q \cup \bigcup \Delta_{3,\pm,\pm} \right),$$

it is readily seen that there exists G_b as in Corollary 2.10 separating the points q, q' , that is, p imposes two independent conditions on the G_b 's.

The same observation can be made concerning the partial diagonals $\Delta_{4,\pm,\pm}$: a point on a $\Delta_{4,\pm,\pm}$ and not on $Q \cup E$ imposes exactly one condition on the polynomials F_b , while points outside of $\Delta_{4,\pm,\pm} \cup Q \cup E$ impose two independent conditions on the F_b 's. Combining these two observations proves the lemma. \square

LEMMA 4.7

Each of the strata

$$P^0, \quad \bigcup_{1 \leq j \leq 4} P_{1,j}^0, \quad \bigcup_{5 \leq j \leq 8} P_{1,j}^0, \quad P_2$$

can be written as a disjoint union of varieties of type $\mathbb{A}^k \setminus L$, where L is a finite union of linearly embedded affine spaces.

Proof

First, consider $P^0 = P_{0,0}^0$. By definition, this is nothing but

$$\mathbb{P} \times \mathbb{P} \setminus \left(Q \cup \bigcup \Delta_{3,\pm,\pm} \cup \Delta_{4,\pm,\pm} \right).$$

Let $U \subset \mathbb{P}$ be the open subset ($w_0 \neq 0$). Then $P^0 \cap (U \times U)$ is isomorphic to \mathbb{A}^8 minus 8 copies of \mathbb{A}^5 that are linearly embedded. The intersection

$$P^0 \cap ((w_0 = 0) \times (w_0 \neq 0))$$

can be identified with $\mathbb{P}(1, 1, 2, 2) \times \mathbb{A}^4$. It remains to consider

$$P^0 \cap ((w_0 = 0) \times (w_0 = 0));$$

the argument is similar. (Restricting to the open ($x_1 \neq 0$) we find again a stratum of the requisite type.)

Next, consider $P_{1,j}^0$. The intersection

$$P_{1,j}^0 \cap (U \times U)$$

is isomorphic to \mathbb{A}^5 minus 4 copies of \mathbb{A}^4 that are linearly embedded. Since all intersections are linear subspaces, the assertion for the unions

$$\bigcup_{1 \leq j \leq 4} P_{1,j}^0, \quad \bigcup_{5 \leq j \leq 8} P_{1,j}^0$$

follows from this. As for P_2 , this is similar: the intersection

$$P_2 \cap (U \times U)$$

is a union of copies of \mathbb{A}^4 that are linearly embedded in \mathbb{A}^8 ; in particular, the intersections of the irreducible components are again affine spaces. \square

COROLLARY 4.8

The open $M_0 := M \setminus \pi^{-1}(E \cup Q)$ has the strong property.

Proof

It follows from Lemma 4.7, combined with Corollary 4.3, that P_2 has the strong property. Since $M_2 = \pi^{-1}(P_2)$ is a fibration over P_2 with fibers that are products of projective spaces, it follows that M_2 has the strong property (Lemma 4.4).

The strata

$$\bigcup_{1 \leq j \leq 4} M_{1,j}^0, \quad \bigcup_{5 \leq j \leq 8} M_{1,j}^0$$

are fibrations over

$$\bigcup_{1 \leq j \leq 4} P_{1,j}^0, \quad \bigcup_{5 \leq j \leq 8} P_{1,j}^0,$$

respectively, with fiber a product of projective spaces (Lemma 4.6). The base has the strong property (Lemma 4.7 and Corollary 4.3); hence, these strata of M have the strong property (Lemma 4.4). By using Lemma 4.2, it follows that M_1 has the strong property. One similarly finds that $M_0^0 = \pi^{-1}(P_{0,0}^0)$ has the strong property and, hence (by applying Lemma 4.2 again), that M^0 has the strong property. \square

We now return to the closed subset \bar{Q} that we left aside; more precisely, we consider the locally closed subset outside of the exceptional divisor

$$Q := \bar{Q} \setminus (\bar{Q} \cap E) \subset \widetilde{\mathbb{P} \times \mathbb{P}} \setminus E \quad (\cong \mathbb{P} \times \mathbb{P} \setminus \Delta).$$

We proceed to stratify Q . We define closed subvarieties:

$$\begin{aligned} Q_{1,1} &:= [0 : 0 : 0 : 1 : 0] \times \mathbb{P} \setminus ([0 : 0 : 0 : 1 : 0] \times [0 : 0 : 0 : 1 : 0]), \\ Q_{1,2} &:= [0 : 0 : 0 : 0 : 1] \times \mathbb{P} \setminus ([0 : 0 : 0 : 0 : 1] \times [0 : 0 : 0 : 0 : 1]), \\ Q_{1,3} &:= \mathbb{P} \times [0 : 0 : 0 : 1 : 0] \setminus ([0 : 0 : 0 : 1 : 0] \times [0 : 0 : 0 : 1 : 0]), \\ Q_{1,4} &:= \mathbb{P} \times [0 : 0 : 0 : 0 : 1] \setminus ([0 : 0 : 0 : 0 : 1] \times [0 : 0 : 0 : 0 : 1]), \\ Q_{2,1} &:= Q_{1,1} \cap (\Delta_{4,+,+}), \\ Q_{2,2} &:= Q_{1,2} \cap (\Delta_{3,+,+}), \\ Q_{2,3} &:= Q_{1,3} \cap (\Delta_{4,+,+}), \\ Q_{2,4} &:= Q_{1,4} \cap (\Delta_{3,+,+}), \\ Q_{2,5} &:= Q_{1,1} \cap Q_{1,4}, \\ Q_{2,6} &:= Q_{1,2} \cap Q_{1,3}. \end{aligned}$$

We also define open subvarieties

$$Q_{1,j}^0 := Q_{1,j} \setminus \left(\bigcup_{Q_{2,k} \subset Q_{1,j}} Q_{2,k} \right), \quad j = 1, \dots, 4.$$

LEMMA 4.9

The varieties $Q_{1,j}^0$ and $Q_{2,k}$ have the strong property.

Proof

This is readily deduced from Lemma 4.2. Each $Q_{1,j}$ ($j = 1, \dots, 4$) is a copy of \mathbb{P} with a point taken out. Each $Q_{2,k}$ with $1 \leq k \leq 4$ is a copy of \mathbb{P}^1 with a point taken out, and $Q_{2,5}$ and $Q_{2,6}$ are just points. Each $Q_{1,j}^0$ ($j = 1, \dots, 4$) is isomorphic to \mathbb{P} with a \mathbb{P}^1 and a point taken out. \square

Consider now the restriction of the morphism π to

$$M_Q := \pi^{-1}(Q) \rightarrow Q$$

and to the various strata of M_Q defined by the stratification of Q :

$$\begin{aligned} M_{Q_{1,j}^0} &:= \pi^{-1}(Q_{1,j}^0), & j = 1, \dots, 4, \\ M_{Q_{2,k}} &:= \pi^{-1}(Q_{2,k}), & k = 1, \dots, 6. \end{aligned}$$

LEMMA 4.10

The restriction of π has constant dimension on each $M_{Q_{1,j}^0}$ and on each $M_{Q_{2,k}}$.

Proof

Consider a point q on the stratum $Q_{1,1}^0$. (The argument for the other $Q_{1,j}^0$ is only notationally different.) We have

$$q = ([0 : 0 : 0 : 1 : 0], p)$$

for some $p \in \mathbb{P}$. Clearly all polynomials G_b as in Corollary 2.10 pass through $[0 : 0 : 0 : 1 : 0]$. Since we are outside of the partial diagonals $\Delta_{3,\pm,\pm}$ and $\Delta_{4,\pm,\pm}$, the point q imposes one condition on the polynomials G_b and two conditions on the polynomials F_b (see the proof of Lemma 4.6). It follows that the fiber is

$$\pi^{-1}(q) \cong \mathbb{P}^{r-2} \times \mathbb{P}^{r-1}.$$

The argument for the strata $Q_{2,k}$, $1 \leq k \leq 4$, is similar; consider for example a point $q \in Q_{2,1}$. Such a q imposes one condition on the G_b 's and one condition on the F_b 's, and so

$$\pi^{-1}(q) \cong \mathbb{P}^{r-1} \times \mathbb{P}^{r-1}.$$

The points

$$Q_{2,5} = ([0 : 0 : 0 : 1 : 0], [0 : 0 : 0 : 0 : 1])$$

and

$$Q_{2,6} = ([0 : 0 : 0 : 0 : 1], [0 : 0 : 0 : 1 : 0])$$

likewise impose one condition on the F_b 's and one condition on the G_b 's, and hence,

$$\pi^{-1}(Q_{2,k}) \cong \mathbb{P}^{r-1} \times \mathbb{P}^{r-1}, \quad k = 5, 6. \quad \square$$

COROLLARY 4.11

The variety M_Q has the strong property.

Proof

This is immediate from the above two lemmas, using Lemma 4.2. □

It remains to stratify the exceptional divisor E of the blowup

$$f: \widetilde{\mathbb{P} \times \mathbb{P}} \rightarrow \mathbb{P} \times \mathbb{P}$$

in a similar way. A point on E is given by the data

$$\{(x, t) \in \mathbb{P} \times \mathbb{P}^3\},$$

where (x, x) is a point on the diagonal, and L_t is a line in $\mathbb{P} \times \mathbb{P}$ passing through (x, x) and not contained in the diagonal. Consider the following loci:

$$E_{1,1} := f^{-1}([0 : 0 : 0 : 1 : 0]),$$

$$E_{1,2} := f^{-1}([0 : 0 : 0 : 0 : 1]),$$

$$E_{1,3} := E \cap \bar{\Delta}_{3,+},$$

$$E_{1,4} := E \cap \bar{\Delta}_{4,+},$$

where $\bar{\Delta}_{j,+}$ denotes the strict transform of $\Delta_{j,+}$. (That is, $E_{1,3}$ and $E_{1,4}$ parameterize lines not contained in the diagonal that remain inside $\Delta_{3,+}$ and $\Delta_{4,+}$, respectively.)

We define E^0 as the open complement

$$E^0 := E \setminus \left(\bigcup_j E_{1,j} \right).$$

We also define points

$$E_{2,1} := E_{1,1} \cap E_{1,3},$$

$$E_{2,2} := E_{1,1} \cap E_{1,4},$$

$$E_{2,3} := E_{1,2} \cap E_{1,3},$$

$$E_{2,4} := E_{1,2} \cap E_{1,4},$$

and locally closed subvarieties

$$E_{1,1}^0 := E_{1,1} \setminus (E_{2,1} \cup E_{2,2}),$$

$$E_{1,2}^0 := E_{1,2} \setminus (E_{2,3} \cup E_{2,4}),$$

$$E_{1,3}^0 := E_{1,3} \setminus (E_{2,1} \cup E_{2,3}),$$

$$E_{1,4}^0 := E_{1,4} \setminus (E_{2,2} \cup E_{2,4}).$$

LEMMA 4.12

The varieties E^0 , $E_{1,j}^0$, and $E_{2,j}$ have the strong property.

Proof

The varieties $E_{2,j}$ are just points. The varieties $E_{1,1}^0$ and $E_{1,2}^0$ are isomorphic to \mathbb{P}^3 minus two points, so this is again obvious. The varieties $E_{1,3}^0$ and $E_{1,4}^0$ are isomorphic to the diagonal of \mathbb{P} minus two points. By applying Lemma 4.2, it follows that

$$\bigcup_j E_{1,j}$$

has the strong property. As for E^0 , clearly E has the strong property; we take out $\bigcup_j E_{1,j}$, which has the strong property: the result is something with the strong property (see [37, Lemma 6]). \square

We now return to the morphism $\pi: M \rightarrow \widetilde{\mathbb{P} \times \mathbb{P}}$ defined above. The preimage

$$M_E := \pi^{-1}(E)$$

admits a stratification as a disjoint union

$$M_E = M_{E^0} \cup \bigcup_{1 \leq j \leq 4} M_{E_{1,j}^0} \cup \bigcup_{1 \leq k \leq 4} M_{E_{2,k}},$$

where M_{E^0} , $M_{E_{1,j}^0}$, and $M_{E_{2,k}}$ are defined as $\pi^{-1}(E_0)$, $\pi^{-1}(E_{1,j}^0)$, and $\pi^{-1}(E_{2,k})$.

On each stratum, the morphism π is of constant dimension.

LEMMA 4.13

The fiber $F = \pi^{-1}(p)$ of $\pi: M \rightarrow \widetilde{\mathbb{P} \times \mathbb{P}}$ is

$$F \cong \begin{cases} \mathbb{P}^{r-2} \times \mathbb{P}^r & \text{if } p = E_{2,1}, \\ \mathbb{P}^{r-1} \times \mathbb{P}^{r-1} & \text{if } p = E_{2,2}, \\ \mathbb{P}^{r-1} \times \mathbb{P}^{r-1} & \text{if } p = E_{2,3}, \\ \mathbb{P}^r \times \mathbb{P}^{r-2} & \text{if } p = E_{2,4}, \\ \mathbb{P}^{r-2} \times \mathbb{P}^{r-1} & \text{if } p \in E_{1,1}^0, \\ \mathbb{P}^{r-1} \times \mathbb{P}^{r-2} & \text{if } p \in E_{1,2}^0, \\ \mathbb{P}^{r-1} \times \mathbb{P}^{r-2} & \text{if } p \in E_{1,4}^0, \\ \mathbb{P}^{r-2} \times \mathbb{P}^{r-1} & \text{if } p \in E_{1,3}^0, \\ \mathbb{P}^{r-2} \times \mathbb{P}^{r-2} & \text{if } p \in E^0. \end{cases}$$

Proof

We consider a point on E that is given by the data

$$\{(x, t) \in \mathbb{P} \times \mathbb{P}^3\},$$

where (x, x) is a point on the diagonal and L_t is a line in \mathbb{P} passing through x . The point x imposes one condition on the G_b 's, except for the point $[0 : 0 : 0 : 1 : 0]$, which imposes no conditions on the G_b 's (see Lemma 2.12). The point x imposes one condition on the F_b 's, except for the point $[0 : 0 : 0 : 0 : 1]$, which imposes no conditions on the F_b 's.

Suppose now that the point x is not one of the exceptional points $[0 : 0 : 0 : 1 : 0]$ or $[0 : 0 : 0 : 0 : 1]$; that is, we are outside of $E_{1,1} \cup E_{1,2}$. Suppose also that the line L_t is not contained in $\Delta_{3,+,+}$; that is, we are outside of $E_{1,3}$. We consider the morphism

$$\phi : \mathbb{P} \setminus [0 : 0 : 0 : 1 : 0] \xrightarrow{\phi_1} \mathbb{P}(1, 1, 1, 2) \xrightarrow{\phi_2} \mathbb{P}' := \mathbb{P}(2, 1, 1, 4),$$

where ϕ_1 is obtained by forgetting the z_3 -coordinate, and ϕ_2 is obtained by letting groups of roots of unity act diagonally. The image $\phi(L_t)$ is a line passing through the point $\phi(x)$. Since the line bundle $\mathcal{O}_{\mathbb{P}'}(4)$ is very ample (see Lemma 4.14 below), there exists a polynomial g of weighted degree 4 such that the hypersurface $(g = 0)$ contains $\phi(x)$ and is transverse to $\phi(L_t)$. The polynomial g looks like

$$\begin{aligned} &\lambda z_4 + c'w^2 + w(\text{something quadratic in } x_1, x_2) \\ &+ (\text{something quartic in } x_1, x_2). \end{aligned}$$

The inverse image $\phi^{-1}(g = 0)$ in \mathbb{P} looks like

$$\begin{aligned} &\lambda z_4^4 + c'w^4 + w^2(\text{something quadratic in } x_1, x_2) + (\text{something quartic in } x_1, x_2) \\ &= 0; \end{aligned}$$

that is, we have found a $G_b, b \in \bar{B}$, containing x and transverse to the line L_t . This shows that a point $p \in E^0$ imposes two independent conditions on the polynomials G_b . Since the argument with respect to the F_b 's and $\Delta_{4,+,+}$ is symmetric, this proves the last line.

Suppose now $x \notin \{[0 : 0 : 0 : 1 : 0], [0 : 0 : 0 : 0 : 1]\}$ and $L_t \subset \Delta_{3,+,+}$, that is, $p \in E_{1,3}^0$. The line L_t disappears under the projection ϕ_1 , which means that all the G_b 's will be tangent to L_t ; this proves the penultimate line.

The remaining cases are similarly checked. For instance, suppose $x = [0 : 0 : 0 : 0 : 1]$ and $L_t \not\subset \Delta_{3,+,+}$, that is, the point p is in $E_{1,2}^0 \cup E_{2,4}$. The line L_t does not disappear under the projection ϕ , so (as above) the point p imposes two conditions on the G_b 's. If $L_t \not\subset \Delta_{4,+,+}$, then the point p imposes one condition on the F_b 's.² □

LEMMA 4.14

Let P' be the weighted projective space $\mathbb{P}(2, 1, 1, 4)$. Then the line bundle $\mathcal{O}_{P'}(4)$ is very ample.

²We remark that a detailed analysis of $E_{1,1}$ and $E_{1,2}$ is not absolutely necessary to our argument. An easy way out is as follows. The dimension of M is $\dim(\mathbb{P} \times \mathbb{P}) + 2(r - 2) = 2r + 4$. The dimension of $\pi^{-1}(E_{1,1} \cup E_{1,2})$ is (by what we have said above) at most $3 + r - 2 + r = 2r + 1$, so whatever happens above $E_{1,1} \cup E_{1,2}$ cannot interfere with codimension 2 cycles: we have

$$A_{s-2}(M) \cong A_{s-2}(M \setminus \pi^{-1}(E_{1,1} \cup E_{1,2}))$$

(where $s := \dim M$). That is, as long as we are only interested in codimension 2 cycles, we may just as well leave out $E_{1,1}$ and $E_{1,2}$.

Proof

The coherent sheaf $\mathcal{O}_{P'}(4)$ is locally free, because 4 is a multiple of the “weights” (see [13]). To see that this line bundle is very ample, we use the following numerical criterion.

PROPOSITION 4.15 (Delorme [11])

Let $P = \mathbb{P}(q_0, q_1, \dots, q_n)$ be a weighted projective space. Let m be the least common multiple of the q_j 's. Suppose every monomial

$$x_0^{b_0} x_1^{b_1} \dots x_n^{b_n}$$

of (weighted) degree km ($k \in \mathbb{N}^*$) is divisible by a monomial of (weighted) degree m . Then $\mathcal{O}_P(m)$ is very ample.

This is the case $E(x) = 0$ of [11, Proposition 2.3(iii)].

Lemma 4.14 is now easily established. Suppose that

$$x^b = x_0^{b_0} x_1^{b_1} x_2^{b_2} x_3^{b_3}$$

is a monomial of weighted degree $4k$, that is,

$$2b_0 + b_1 + b_2 + 4b_3 = 4k.$$

If $b_3 \geq 1$, then x_3 divides x^b and we are good. Suppose now $b_3 = 0$. If $b_0 \geq 2$, then we are good, since x_0^2 divides x^b . If $b_0 = 1$, then $b_1 + b_2 \geq 2$ and x_0 times something quadratic (x_1^2 or x_1x_2 or x_2^2) divides x^b . The remaining case $b_3 = b_0 = 0$ is obviously good. \square

COROLLARY 4.16

The variety M_E has the strong property.

Proof

This follows from Lemmas 4.4 and 4.2. \square

Now we are able to wrap up the proof. The variety M that we are interested in is a disjoint union of three strata:

$$M = M^0 \cup M_Q \cup M_E.$$

Each of these three strata has the strong property (Corollaries 4.8, 4.11, and 4.16). By applying Lemma 4.2, it follows that M has the strong property, that is,

$$A_*^{\text{hom}}(M)_{\mathbb{Q}} = 0,$$

which proves Proposition 4.5. \square

REMARK 4.17

If we assume that the “Voisin standard conjecture” (see [44, Conjecture 0.6], [46,

Conjecture 2.29]) is true, we obtain the stronger statement that

$$A_{\text{hom}}^i(\mathcal{V}' \times_{B'} \mathcal{V}')_{\mathbb{Q}} = 0$$

for all i . Since for our argument, we are only interested in (surfaces and hence) codimension 2 cycles on $\mathcal{V}' \times_{B'} \mathcal{V}'$, we have no need for this conditional stronger statement. As noted in [44], the Voisin standard conjecture is true in codimension 2, and so in this case one obtains an unconditional statement.

REMARK 4.18

We note in passing that everything we say in this section is still valid when replacing “the weak (resp., strong) property” by “the weak (resp., strong) Chow–Künneth property.” This last notion is defined on [37, p. 10] (and further studied in [38]). This implies (using [37, Proposition 2] or [15]) that the variety M satisfies

$$A^i(M) \cong \text{Hom}(A_i(M), \mathbb{Z}) \quad \text{for all } i,$$

where the left-hand side denotes Fulton and MacPherson’s operational Chow cohomology (see [14]). We do not need this statement here.

5. Motives

This section contains a motivic version of the main result, stating that, for the Todorov surfaces under consideration, the “transcendental part of the motive” (in the sense of [18]) is isomorphic to the transcendental part of the motive of the associated K3 surface (Theorem 5.2). Some consequences are given.

THEOREM 5.1 (Kahn–Murre–Pedrini [18])

Let S be any smooth projective surface, and let $h(X) \in \mathcal{M}_{\text{rat}}$ denote the Chow motive of S . There exists a self-dual Chow–Künneth decomposition $\{\pi_i\}$ of S , with the property that there is a further splitting in orthogonal idempotents

$$\pi_2 = \pi_2^{\text{alg}} + \pi_2^{\text{tr}} \quad \text{in } A^2(S \times S)_{\mathbb{Q}}.$$

The action on cohomology is

$$(\pi_2^{\text{alg}})_* H^*(S) = N^1 H^2(S), \quad (\pi_2^{\text{tr}})_* H^*(S) = H_{\text{tr}}^2(S),$$

where the transcendental cohomology $H_{\text{tr}}^2(S) \subset H^2(S)$ is defined as the orthogonal complement of $N^1 H^2(S)$ with respect to the intersection pairing. The action on Chow groups is

$$(\pi_2^{\text{alg}})_* A^*(S)_{\mathbb{Q}} = N^1 H^2(S), \quad (\pi_2^{\text{tr}})_* A^*(S) = A_{\text{AJ}}^2(S)_{\mathbb{Q}}.$$

This gives rise to a well-defined Chow motive

$$t_2(S) := (S, \pi_2^{\text{tr}}, 0) \subset h(X) \in \mathcal{M}_{\text{rat}},$$

the so-called transcendental part of the motive of S .

Proof

Let $\{\pi_i\}$ be a Chow–Künneth decomposition as in [18, Proposition 7.2.1]. The assertion then follows from [18, Proposition 7.2.3]. \square

THEOREM 5.2

Let S be a Todorov surface with $K_S^2 = 2$ and $\pi_1(S) = \mathbb{Z}/2\mathbb{Z}$, and let P be the K3 surface obtained as a resolution of singularities of S/ι . The natural correspondence from S to P induces an isomorphism of Chow motives

$$t_2(S) \cong t_2(P) \quad \text{in } \mathcal{M}_{\text{rat}}.$$

Proof

This is just a dressed-up version of the argument of Theorem 3.1. Let $\mathcal{S} \rightarrow B$ be the family of canonical models of Todorov surfaces with fundamental invariants $(1, 10)$, as in Corollary 2.10. As in the proof of Theorem 3.1, we have morphisms of families over B

$$\begin{array}{ccc} \tilde{\mathcal{S}} & \rightarrow & \mathcal{S} \\ \downarrow \tilde{f} & & \downarrow f \\ \tilde{\mathcal{M}} & \rightarrow & \mathcal{M} \\ & \searrow & \downarrow \\ & & B \end{array}$$

Here \mathcal{M} is the family of K3 surfaces with rational double points, and $\tilde{\mathcal{M}}$ is the family of desingularized K3 surfaces. Taking the graph of the morphism \tilde{f} , one gets a relative correspondence

$$\Gamma_{\tilde{f}} \in A_{s-2}(\tilde{\mathcal{S}} \times_B \tilde{\mathcal{M}}),$$

where s denotes $\dim \tilde{\mathcal{S}} \times_B \tilde{\mathcal{M}}$. The proof of Theorem 3.1, applied to the relative correspondence

$$2\Delta_{\tilde{\mathcal{S}}} - {}^t\Gamma_{\tilde{f}} \circ \Gamma_{\tilde{f}} \in A_{s-2}(\tilde{\mathcal{S}} \times_B \tilde{\mathcal{S}}),$$

gives a rational equivalence for the general (and, hence, for any) $b \in B$:

$$(2) \quad 2\Delta_{\tilde{S}_b} = {}^t\Gamma_{\tilde{f}_b} \circ \Gamma_{\tilde{f}_b} + \sum_{i,j} D_i \times D_j + \gamma \in A^2(\tilde{S}_b \times \tilde{S}_b)_{\mathbb{Q}},$$

where the $D_i \subset \tilde{S}_b$ are divisors, γ is supported on

$$E_b \times \tilde{S}_b \cup \tilde{S}_b \times E_b,$$

and $E_b \subset \tilde{S}_b$ is an exceptional divisor for the morphism $\tilde{S}_b \rightarrow S_b$. Note that γ is contained in the ideal of so-called *degenerate correspondences* $\mathcal{J}(\tilde{S}_b, \tilde{S}_b)$ (see [14, p. 309], [18, Definition 7.4.2]).

Consider S a Todorov surface with fundamental invariants $(1, 10)$, and let P be the associated K3 surface. It follows from Corollary 2.10 that the canonical model of S is an S_b for some $b \in B$, so that S is birational to the smooth surface

\widetilde{S}_b and P is birational to the smooth surface $\widetilde{\mathcal{M}}_b$. Let

$$\pi_0^S, \pi_2^S, \pi_4^S \quad \text{and} \quad \pi_0^M, \pi_2^M, \pi_4^M$$

denote a Chow–Künneth decomposition for \widetilde{S}_b and $\widetilde{\mathcal{M}}_b$, respectively, as in Theorem 5.1. Let

$$\pi_2^S = \pi_2^{S,\text{alg}} + \pi_2^{S,\text{tr}} \quad \text{and} \quad \pi_2^M = \pi_2^{M,\text{alg}} + \pi_2^{M,\text{tr}}$$

be the refined decomposition of Theorem 5.1. Since $\pi_2^{S,\text{alg}}$ is a projector on $NS(\widetilde{S}_b)_\mathbb{Q}$, we have equality

$$\sum_{i,j} D_i \times D_j = \left(\sum_{i,j} D_i \times D_j \right) \circ \pi_2^{S,\text{alg}},$$

and hence (since $\pi_2^{S,\text{tr}}$ and $\pi_2^{S,\text{alg}}$ are orthogonal), we find that

$$\left(\sum_{i,j} D_i \times D_j \right) \circ \pi_2^{S,\text{tr}} = 0 \quad \text{in } A^2(\widetilde{S}_b \times \widetilde{S}_b)_\mathbb{Q}.$$

Likewise, since $\gamma \in \mathcal{J}(\widetilde{S}_b, \widetilde{S}_b)$, it follows from [18, Theorem 7.4.3] that

$$\pi_2^{S,\text{tr}} \circ \gamma \circ \pi_2^{S,\text{tr}} = 0 \quad \text{in } A^2(\widetilde{S}_b \times \widetilde{S}_b)_\mathbb{Q}.$$

It now follows from (2) (after twice applying $\pi_2^{S,\text{tr}}$ on both sides) that

$$2\pi_2^{S,\text{tr}} = \pi_2^{S,\text{tr}} \circ {}^t\Gamma_{\widetilde{f}_b} \circ \Gamma_{\widetilde{f}_b} \circ \pi_2^{S,\text{tr}} \quad \text{in } A^2(\widetilde{S}_b \times \widetilde{S}_b)_\mathbb{Q}.$$

The next step is to remark that

$$\begin{aligned} \pi_4^M \circ \Gamma_{\widetilde{f}_b} \circ \pi_2^S &= 0, \\ \pi_2^S \circ {}^t\Gamma_{\widetilde{f}_b} \circ \pi_0^M &= 0 \end{aligned}$$

(this follows from [18, Theorem 7.3.10(i)]), and so we have

$$2\pi_2^{S,\text{tr}} \circ {}^t\Gamma_{\widetilde{f}_b} \circ \Gamma_{\widetilde{f}_b} \circ \pi_2^{S,\text{tr}} = \pi_2^{S,\text{tr}} \circ {}^t\Gamma_{\widetilde{f}_b} \circ \pi_2^M \circ \Gamma_{\widetilde{f}_b} \circ \pi_2^{S,\text{tr}}.$$

But

$$\pi_2^M \circ \Gamma \circ \pi_2^{S,\text{tr}} = \pi_2^{M,\text{tr}} \circ \Gamma \circ \pi_2^{S,\text{tr}}$$

by [18, Lemma 7.4.1], so we end up with a rational equivalence

$$2\pi_2^{S,\text{tr}} = \pi_2^{S,\text{tr}} \circ {}^t\Gamma_{\widetilde{f}_b} \circ \pi_2^{M,\text{tr}} \circ \Gamma_{\widetilde{f}_b} \circ \pi_2^{S,\text{tr}} \quad \text{in } A^2(\widetilde{S}_b \times \widetilde{S}_b)_\mathbb{Q}.$$

The fact that there is also a rational equivalence

$$2\pi_2^{M,\text{tr}} = \pi_2^{M,\text{tr}} \circ \Gamma_{\widetilde{f}_b} \circ \pi_2^{S,\text{tr}} \circ {}^t\Gamma_{\widetilde{f}_b} \circ \pi_2^{M,\text{tr}} \quad \text{in } A^2(\widetilde{\mathcal{M}}_b \times \widetilde{\mathcal{M}}_b)_\mathbb{Q}$$

is much easier: we have

$$2\Delta_{\widetilde{\mathcal{M}}_b} = \Gamma_{\widetilde{f}_b} \circ {}^t\Gamma_{\widetilde{f}_b} \quad \text{in } A^2(\widetilde{\mathcal{M}}_b \times \widetilde{\mathcal{M}}_b)_\mathbb{Q},$$

and the same argument applies.

We have now established that

$$\Gamma_{\widetilde{f}_b} : t_2(\widetilde{S}_b) \rightarrow t_2(\widetilde{\mathcal{M}}_b) \quad \text{in } \mathcal{M}_{\text{rat}}$$

is an isomorphism of motives, with inverse given by ${}^t\Gamma_{\tilde{f}_b}$. Since the surfaces S and P are birational to \tilde{S}_b and $\tilde{\mathcal{M}}_b$, respectively, and t_2 is a birational invariant among smooth surfaces, it follows that there is also an isomorphism

$$t_2(S) \cong t_2(P) \quad \text{in } \mathcal{M}_{\text{rat}}. \quad \square$$

COROLLARY 5.3

Let S be a Todorov surface with $K_S^2 = 2$ and $\pi_1(S) = \mathbb{Z}/2\mathbb{Z}$. Assume moreover that one of the following holds:

- (i) the Picard number $\rho(S)$ is at least $h^{1,1}(S) - 1$;
- (ii) the K3 surface birational to S/ι is a Kummer surface;
- (iii) the K3 surface birational to S/ι has a Shioda–Inose structure (in the sense of [26]).

Then S has finite-dimensional motive (in the sense of Kimura and O’Sullivan; see [19], [1]).

Proof

It suffices to show that the motive $t_2(S)$ is finite-dimensional and, hence (by applying Theorem 5.2), that the motive $t_2(P)$ is finite-dimensional, where P is the K3 surface birational to S/ι . In case (i), this is true since

$$H_{\text{tr}}^2(P) \cong H_{\text{tr}}^2(S)$$

has dimension at most 3, so that the Picard number $\rho(P)$ is at least 19, and K3 surfaces with Picard number at least 19 are known to have finite-dimensional motive (see [31]). In case (ii), the needed statement is obviously true. In case (iii) it follows from [23, Remark 47]. \square

REMARK 5.4

It should be noted that Todorov surfaces as in Corollary 5.3(ii) can be readily constructed; in fact, these were the first examples given by Todorov [36].

REMARK 5.5

We note in passing that surfaces S as in Corollary 5.3 not only have finite-dimensional motive; their motive is actually in the subcategory of motives of Abelian type (that is, the category of Chow motives generated by the motives of curves (see [41])). The same is true for K3 surfaces: all examples of K3 surfaces known to be finite-dimensional are actually of Abelian type. This is not surprising for the following reason: for *any* surface S with $p_g(S) = 1$, the Kuga–Satake [21] construction relates $H^2(S)$ to the cohomology of an Abelian variety. If the Kuga–Satake correspondence is algebraic (e.g., if the Hodge conjecture is true), this means that the homological motive of S is a direct factor of the motive of an Abelian variety plus a sum of curves. If S has finite-dimensional motive, then the same is true for the Chow motive of S , that is, S has motive of Abelian type.

COROLLARY 5.6

Let S, S' be two Todorov surfaces as in Corollary 5.3. Assume there exists a Hodge isometry between the transcendental lattices

$$\phi: T_S \otimes \mathbb{Q} \cong T_{S'} \otimes \mathbb{Q}$$

(i.e., ϕ is an isomorphism of Hodge structures that respects the intersection forms). Then there is an isomorphism of motives

$$t_2(S) \cong t_2(S') \quad \text{in } \mathcal{M}_{\text{rat}}.$$

Proof

Let P, P' denote the associated K3 surfaces. Then ϕ induces a Hodge isometry

$$T_P \otimes \mathbb{Q} \cong T_{P'} \otimes \mathbb{Q}$$

(since in both cases the intersection form is multiplied by 2 when going to the double cover). By Mukai [28], this Hodge isometry is induced by a cycle $\Gamma \in A^2(P \times P')_{\mathbb{Q}}$. (Note that the assumptions of Corollary 5.3 imply that P and P' have Picard number at least 17, so [28] indeed applies.) Then Γ induces an isomorphism of homological motives

$$\Gamma: t_2(P) \cong t_2(P') \quad \text{in } \mathcal{M}_{\text{hom}}$$

and, hence (using the finite dimensionality of P and P'), an isomorphism of Chow motives

$$\Gamma: t_2(P) \cong t_2(P') \quad \text{in } \mathcal{M}_{\text{rat}}.$$

The corollary now follows by combining this with Theorem 5.2. □

Another corollary is that a weak form of the relative Bloch conjecture is true for surfaces as in Corollary 5.3.

COROLLARY 5.7

Let S be a Todorov surface as in Corollary 5.3. Let $\Gamma \in A^2(S \times S)_{\mathbb{Q}}$ be a correspondence such that

$$\Gamma_* = \text{id}: H^{2,0}(S) \rightarrow H^{2,0}(S).$$

Then

$$\Gamma_*: A^2_{\text{hom}}(S)_{\mathbb{Q}} \rightarrow A^2_{\text{hom}}(S)_{\mathbb{Q}}$$

is an isomorphism.

Proof

As is well known, this holds for any surface S with finite-dimensional motive. □

6. Speculation

This final section offers some speculation about possible directions for generalization of the results in this article.

REMARK 6.1

It would be interesting to try and prove Corollary 3.2 for all surfaces S with $p_g = 1$, $K_S^2 = 2$, and $\pi_1(S) = \mathbb{Z}/2\mathbb{Z}$. Thanks to Catanese and Debarre [9] (canonical models of) these surfaces form a 16-dimensional family, explicitly described as quotients of complete intersections in $\mathbb{P}(1, 1, 1, 2, 2)$.

The problem is that, outside of the 12-dimensional Todorov locus (where there is a K3 surface over which S is a double cover), it seems difficult to exploit this fact. An argument such as that sketched by Voisin for quartic surfaces in [44, Theorem 3.10] would perhaps work to establish Corollary 3.2 for this 16-dimensional family, but this argument is conditional on knowing that (1) the generalized Hodge conjecture for

$$\bigwedge^2 H^2(S) \subset H^4(S \times S)$$

and (2) the “Voisin standard conjecture” (see [44, Conjecture 0.6], [46, Conjecture 2.29]) are true (to obtain a cycle supported on a certain subvariety).

REMARK 6.2

On the other hand, it would also be interesting to prove Corollary 3.2 for the nine other families of Todorov surfaces. Thanks to the work of Rito (see [33] or Theorem 2.5), these have an associated K3 surface for which Voisin’s conjecture is known.

The problem is in relating 0-cycles on S to 0-cycles on the associated K3 surface (i.e., in proving Theorem 3.1). In order for the “spreading out” approach of [44] and [47] to work, what is needed at the very least is that the irreducible family $\mathcal{S} \rightarrow B$ of Todorov surfaces with given fundamental invariants has to be nice enough to have the property that

$$A_{\text{hom}}^2(\mathcal{S} \times_B \mathcal{S})_{\mathbb{Q}} = 0.$$

However, in the absence of an explicit description of the family (such as that given by the weighted complete intersections of [9] in the case $K_S^2 = 2$ and $\pi_1(S) = \mathbb{Z}/2\mathbb{Z}$), this seems difficult. Can this property perhaps be proven for the total space of the deformation of [24] mentioned in Remark 2.3?

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