Theory of non-lc ideal sheaves: Basic properties

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Abstract We introduce the notion of non-lc ideal sheaves. It is an analogue of the notion of multiplier ideal sheaves. We establish the restriction theorem, which seems to be the most important property of non-lc ideal sheaves.

1. Introduction

Let X be a smooth complex algebraic variety, and let B be an effective $\mathbb{R}$-divisor on X. Then we can define the *multiplier ideal sheaf* $\mathcal{J}(X,B)$. By the definition, (X,B) is klt if and only if $\mathcal{J}(X,B)$ is trivial. There exist plenty of applications of multiplier ideal sheaves (see, e.g., the excellent book [L]). Here we introduce the notion of *non-lc ideal sheaves*. We denote it by $\mathcal{J}_{\mathrm{NLC}}(X,B)$. By the construction, the ideal sheaf $\mathcal{J}_{\mathrm{NLC}}(X,B)$ is trivial if and only if (X,B) is lc; that is, $\mathcal{J}_{\mathrm{NLC}}(X,B)$ defines the non-lc locus of the pair (X,B). So, we call $\mathcal{J}_{\mathrm{NLC}}(X,B)$ the *non-lc ideal sheaf* associated to (X,B). By the definition of $\mathcal{J}_{\mathrm{NLC}}(X,B)$ (cf. Definition 2.1), we have the inclusions

$$\mathcal{J}(X,B)\subset \mathcal{J}_{\mathrm{NLC}}(X,B)\subset \mathcal{J}\bigl(X,(1-\varepsilon)B\bigr)$$

for every $\varepsilon>0$. Although the ideal sheaf $\mathcal{J}(X,(1-\varepsilon)B)$ defines the non-lc locus of the pair (X,B) for $0<\varepsilon\ll 1$, $\mathcal{J}(X,(1-\varepsilon)B)$ does not always coincide with $\mathcal{J}_{\mathrm{NLC}}(X,B)$. This is a very important remark. In [FST] we will discuss various other ideal sheaves that define the non-lc locus of (X,B).

Let S be a smooth irreducible divisor on X such that S is not contained in the support of B. We put $B_S=B|_S$. The restriction theorem for multiplier ideal sheaves, which was obtained by Esnault and Viehweg, is one of the key results in the theory of multiplier ideal sheaves. From the analytic point of view, it is a direct consequence of the Ohsawa-Takegoshi L^2 extension theorem (see [OT]). For the details, see [Ko] and [L]. Let us recall the restriction theorem here for the reader's convenience.

Kyoto Journal of Mathematics, Vol. 50, No. 2 (2010), 225–245
DOI 10.1215/0023608X-2009-011, © 2010 by Kyoto University
Received July 21, 2009. Revised October 26, 2009. Accepted October 29, 2009.
Mathematics Subject Classification: Primary 14E15. Secondary 14E30.
Author's work partially supported by Japan Society for the Promotion of Science Grant-in-Aid for Young Scientists (A) no. 20684001 and by the Inamori Foundation.

THEOREM 1.1 (RESTRICTION THEOREM FOR MULTIPLIER IDEAL SHEAVES)
We have an inclusion

$$\mathcal{J}(S, B_S) \subseteq \mathcal{J}(X, B)|_S.$$

The main result of this article is the following restriction theorem for non-lc ideal sheaves (for the precise statement, see Theorem 2.14).

THEOREM 1.2
There is an equality

$$\mathcal{J}_{\mathrm{NLC}}(S, B_S) = \mathcal{J}_{\mathrm{NLC}}(X, S+B)|_S.$$

In particular, (S, B_S) *is lc if and only if* $(X, S+B)$ *is lc around* S.

Once we obtain this powerful restriction theorem for non-lc ideal sheaves, we can translate some results for multiplier ideal sheaves into new results for non-lc ideal sheaves. We prove, for example, a subadditivity theorem for non-lc ideal sheaves. We think that the ideal sheaf $\mathcal{J}_{\mathrm{NLC}}(X, B)$ has already appeared implicitly in some articles. However, $\mathcal{J}_{\mathrm{NLC}}(X, B)$ was thought to be useless because the Kawamata-Viehweg-Nadel vanishing theorem does not hold for lc pairs. We note that the theory of multiplier ideal sheaves heavily depends on the Kawamata-Viehweg-Nadel vanishing theorem. Fortunately, we have a new cohomological package according to Ambro's formulation, which works for lc pairs (see [F5, Chapter 2]). By this new package, we can walk around freely in the world of lc pairs. We prove vanishing theorem and global generation for non-lc ideal sheaves as applications. We hope that the notion of non-lc ideal sheaves will play important roles in various applications. In [F4], we prove the cone and contraction theorem for a pair (X, B), where X is a normal variety and B is an effective $\mathbb{R}$-divisor on X such that $K_X + B$ is $\mathbb{R}$-Cartier. In that article, we repeatedly use non-lc ideal sheaves. We note that the restriction theorem (cf. Theorem 2.14) is not necessary in [F4]. We use only the basic properties of non-lc ideal sheaves.

We summarize the contents of this article. In Section 2, we introduce the notion of non-lc ideal sheaves and give various examples. Then we prove the restriction theorem for non-lc ideal sheaves. It produces the subadditivity theorem for non-lc ideal sheaves, and so on. Our proof of the restriction theorem is quite different from the standard arguments in the theory of multiplier ideal sheaves in [L]. It also differs from the usual X-method, which was initiated by Kawamata and is the most important technique in the traditional log minimal model program. So, we explain the proof of the restriction theorem very carefully. In Section 3, we prove the vanishing theorem and the global generation for (asymptotic) non-lc ideal sheaves. The last section is an appendix, where we quickly review Kawakita's inversion of adjunction on log canonicity and the new cohomological package (cf. [F5, Chapter 2]).

Notation and conventions

We work over the complex number field $\mathbb{C}$ throughout this article. But we note that by using the Lefschetz principle, we can extend everything to the case where the base field is an algebraically closed field of characteristic zero. We closely follow the presentation of the excellent book [L] in order to make this article more accessible. We use the following notation freely.

NOTATION

(i) For an $\mathbb{R}$-Weil divisor $D = \sum_{j=1}^{r} d_j D_j$ such that D_i is a prime divisor for every i and $D_i \neq D_j$ for $i \neq j$, we define the *round-up* $\ulcorner D \urcorner = \sum_{j=1}^{r} \ulcorner d_j \urcorner D_j$ (resp., the *round-down* $\llcorner D \lrcorner = \sum_{j=1}^{r} \llcorner d_j \lrcorner D_j$), where for every real number x, $\ulcorner x \urcorner$ (resp., $\llcorner x \lrcorner$) is the integer defined by $x \leq \ulcorner x \urcorner < x+1$ (resp., $x-1 < \llcorner x \lrcorner \leq x$). The *fractional part* $\{D\}$ of D denotes $D - \llcorner D \lrcorner$. We define

$$D^{=1} = \sum_{d_j=1} D_j, \qquad D^{\leq 1} = \sum_{d_j \leq 1} d_j D_j,$$

$$D^{<1} = \sum_{d_j<1} d_j D_j, \qquad \text{and} \qquad D^{>1} = \sum_{d_j>1} d_j D_j.$$

We call D a *boundary* $\mathbb{R}$-divisor if $0 \leq d_j \leq 1$ for every j. We note that $\sim_{\mathbb{Q}}$ (resp., $\sim_{\mathbb{R}}$) denotes the $\mathbb{Q}$-linear (resp., $\mathbb{R}$-linear) equivalence of $\mathbb{Q}$-divisors (resp., $\mathbb{R}$-divisors).

(ii) For a proper birational morphism $f: X \to Y$, the *exceptional locus* $\mathrm{Exc}(f) \subset X$ is the locus where f is not an isomorphism.

(iii) Let X be a normal variety, and let B be an effective $\mathbb{R}$-divisor on X such that $K_X + B$ is $\mathbb{R}$-Cartier. Let $f: Y \to X$ be a resolution such that $\mathrm{Exc}(f) \cup f_*^{-1}B$ has a simple normal crossing support, where $f_*^{-1}B$ is the strict transform of B on Y. We write

$$K_Y = f^*(K_X + B) + \sum_i a_i E_i$$

and $a(E_i, X, B) = a_i$. We say that (X, B) is *lc* (resp., *klt*) if and only if $a_i \geq -1$ (resp., $a_i > -1$) for every i, where lc (resp., klt) stands for *log canonical* (resp., *kawamata log terminal*). Note that the *discrepancy* $a(E, X, B) \in \mathbb{R}$ can be defined for every prime divisor E *over* X. By the definition, there exists the largest Zariski open set U of X such that (X, B) is lc on U. We put $\mathrm{Nlc}(X, B) = X \setminus U$ and call it the *non-lc locus* of the pair (X, B). We sometimes simply denote $\mathrm{Nlc}(X, B)$ by X_{NLC}.

(iv) Let E be a prime divisor over X. The closure of the image of E on X is denoted by $c_X(E)$ and called the *center* of E on X.

We use the same notation as in (iii). If $a(E, X, B) = -1$ and $c_X(E)$ is not contained in $\mathrm{Nlc}(X, B)$, then $c_X(E)$ is called an *lc center* of (X, B). We note that our definition of lc centers is slightly different from the usual one.

2. Non-lc ideal sheaves

2.1. Definitions of non-lc ideal sheaves

Let us introduce the notion of *non-lc ideal sheaves*.

DEFINITION 2.1 (NON-LC IDEAL SHEAF)

Let X be a normal variety, and let Δ be an $\mathbb{R}$-divisor on X such that $K_X+\Delta$ is $\mathbb{R}$-Cartier. Let $f:Y\to X$ be a resolution with $K_Y+\Delta_Y=f^*(K_X+\Delta)$ such that $\operatorname{Supp}\Delta_Y$ is simple normal crossing. Then we put

$$\mathcal{J}_{\mathrm{NLC}}(X,\Delta)=f_*\mathcal{O}_Y\big(\ulcorner -(\Delta_Y^{<1})\urcorner-\llcorner \Delta_Y^{>1}\lrcorner\big)=f_*\mathcal{O}_Y(-\llcorner\Delta_Y\lrcorner+\Delta_Y^{=1})$$

and call it the *non-lc ideal sheaf associated to* (X,Δ).

The name comes from the following obvious lemma (see also Proposition 2.6).

LEMMA 2.2

Let X be a normal variety, and let Δ be an effective $\mathbb{R}$-divisor such that $K_X+\Delta$ is $\mathbb{R}$-Cartier. Then (X,Δ) is lc if and only if $\mathcal{J}_{\mathrm{NLC}}(X,\Delta)=\mathcal{O}_X$.

REMARK 2.3

In the same notation as in Definition 2.1, we put

$$\mathcal{J}(X,\Delta)=f_*\mathcal{O}_Y(-\llcorner\Delta_Y\lrcorner)=f_*\mathcal{O}_Y\big(K_Y-\llcorner f^*(K_X+\Delta)\lrcorner\big).$$

It is nothing but the well-known *multiplier ideal sheaf*. It is obvious that $\mathcal{J}(X,\Delta)\subseteq\mathcal{J}_{\mathrm{NLC}}(X,\Delta)$.

QUESTION 2.4

Let X be a smooth algebraic variety, and let Δ be an effective $\mathbb{R}$-divisor on X. Are there any analytic interpretations of $\mathcal{J}_{\mathrm{NLC}}(X,\Delta)^{\mathrm{an}}$? Are there any approaches to $\mathcal{J}_{\mathrm{NLC}}(X,\Delta)$ from the theory of tight closure?

DEFINITION 2.5 (NON-LC IDEAL SHEAF ASSOCIATED TO AN IDEAL SHEAF)

Let X be a normal variety, and let Δ be an $\mathbb{R}$-divisor on X such that $K_X+\Delta$ is $\mathbb{R}$-Cartier. Let $\mathfrak{a}\subseteq\mathcal{O}_X$ be a nonzero ideal sheaf on X, and let c be a real number. Let $f:Y\to X$ be a resolution such that $K_Y+\Delta_Y=f^*(K_X+\Delta)$ and $f^{-1}\mathfrak{a}=\mathcal{O}_Y(-F)$, where $\operatorname{Supp}(\Delta_Y+F)$ has a simple normal crossing support. We put

$$\mathcal{J}_{\mathrm{NLC}}\big((X,\Delta);\mathfrak{a}^c\big)=f_*\mathcal{O}_Y\big(\ulcorner -((\Delta_Y+cF)^{<1})\urcorner-\llcorner(\Delta_Y+cF)^{>1}\lrcorner\big).$$

We sometimes write $\mathcal{J}_{\mathrm{NLC}}((X,\Delta);c\cdot\mathfrak{a})=\mathcal{J}_{\mathrm{NLC}}((X,\Delta);\mathfrak{a}^c)$.

PROPOSITION 2.6

The ideal sheaves $\mathcal{J}_{\mathrm{NLC}}(X,\Delta)$ and $\mathcal{J}_{\mathrm{NLC}}((X,\Delta);\mathfrak{a}^c)$ are well defined; that is, they are independent of the resolution $f:Y\to X$. If Δ is effective and $c>0$, then $\mathcal{J}_{\mathrm{NLC}}(X,\Delta)\subseteq\mathcal{O}_X$ and $\mathcal{J}_{\mathrm{NLC}}((X,\Delta);\mathfrak{a}^c)\subseteq\mathcal{O}_X$.

This proposition follows from the next fundamental lemma.

LEMMA 2.7

Let $f: Z \to Y$ be a proper birational morphism between smooth varieties, and let B_Y be an $\mathbb{R}$-divisor on Y such that $\operatorname{Supp} B_Y$ is a simple normal crossing divisor. Assume that $K_Z + B_Z = f^*(K_Y + B_Y)$ and that $\operatorname{Supp} B_Z$ is a simple normal crossing divisor. Then we have

$$f_*\mathcal{O}_Z\big(\ulcorner -(B_Z^{<1})\urcorner - \llcorner B_Z^{>1}\lrcorner\big) \simeq \mathcal{O}_Y\big(\ulcorner -(B_Y^{<1})\urcorner - \llcorner B_Y^{>1}\lrcorner\big).$$

Proof

By $K_Z + B_Z = f^*(K_Y + B_Y)$, we obtain

$$\begin{aligned} K_Z =& f^*(K_Y + B_Y^{=1} + \{B_Y\}) \\ &+ f^*(\llcorner B_Y^{<1}\lrcorner + \llcorner B_Y^{>1}\lrcorner) - (\llcorner B_Z^{<1}\lrcorner + \llcorner B_Z^{>1}\lrcorner) - B_Z^{=1} - \{B_Z\}. \end{aligned}$$

If $a(\nu, Y, B_Y^{=1} + \{B_Y\}) = -1$ for a prime divisor ν over Y, then we can check that $a(\nu, Y, B_Y) = -1$ by using [KM, Lemma 2.45]. Since $f^*(\llcorner B_Y^{<1}\lrcorner + \llcorner B_Y^{>1}\lrcorner) - (\llcorner B_Z^{<1}\lrcorner + \llcorner B_Z^{>1}\lrcorner)$ is Cartier, we can easily see that

$$f^*(\llcorner B_Y^{<1}\lrcorner + \llcorner B_Y^{>1}\lrcorner) = \llcorner B_Z^{<1}\lrcorner + \llcorner B_Z^{>1}\lrcorner + E,$$

where E is an effective f-exceptional divisor. Thus, we obtain

$$f_*\mathcal{O}_Z\big(\ulcorner -(B_Z^{<1})\urcorner - \llcorner B_Z^{>1}\lrcorner\big) \simeq \mathcal{O}_Y\big(\ulcorner -(B_Y^{<1})\urcorner - \llcorner B_Y^{>1}\lrcorner\big).$$

We finish the proof. $\square$

Although the following lemma is not indispensable for the proof of the main theorem, it may be useful. The proof is quite nontrivial.

LEMMA 2.8

We use the same notation and assumption as in Lemma 2.7. Let S be a simple normal crossing divisor on Y such that $S \subset \operatorname{Supp} B_Y^{=1}$. Let T be the union of the irreducible components of $B_Z^{=1}$ which are mapped into S by f. Assume that $\operatorname{Supp} f_*^{-1}B_Y \cup \operatorname{Exc}(f)$ is simple normal crossing on Z. Then we have

$$f_*\mathcal{O}_T\big(\ulcorner -(B_T^{<1})\urcorner - \llcorner B_T^{>1}\lrcorner\big) \simeq \mathcal{O}_S\big(\ulcorner -(B_S^{<1})\urcorner - \llcorner B_S^{>1}\lrcorner\big),$$

where $(K_Z + B_Z)|_T = K_T + B_T$ and $(K_Y + B_Y)|_S = K_S + B_S$.

Proof

We use the same notation as in the proof of Lemma 2.7. We consider

$$\begin{aligned} 0 &\to \mathcal{O}_Z\big(\ulcorner -(B_Z^{<1})\urcorner - \llcorner B_Z^{>1}\lrcorner - T\big) \\ &\to \mathcal{O}_Z\big(\ulcorner -(B_Z^{<1})\urcorner - \llcorner B_Z^{>1}\lrcorner\big) \to \mathcal{O}_T\big(\ulcorner -(B_T^{<1})\urcorner - \llcorner B_T^{>1}\lrcorner\big) \to 0. \end{aligned}$$

Since $T = f^*S - F$, where F is an effective f-exceptional divisor, we can easily see that

$$f_*\mathcal{O}_Z\big(\ulcorner -(B_Z^{<1})\urcorner - \llcorner B_Z^{>1}\lrcorner - T\big) \simeq \mathcal{O}_Y\big(\ulcorner -(B_Y^{<1})\urcorner - \llcorner B_Y^{>1}\lrcorner - S\big).$$

We note that

$$\begin{aligned}&\big(\ulcorner -(B_Z^{<1})\urcorner - \llcorner B_Z^{>1}\lrcorner - T\big) - \big(K_Z + \{B_Z\} + (B_Z^{=1} - T)\big)\\&\quad = -f^*(K_Y + B_Y).\end{aligned}$$

Therefore, every local section of $R^1f_*\mathcal{O}_Z(\ulcorner -(B_Z^{<1})\urcorner - \llcorner B_Z^{>1}\lrcorner - T)$ contains in its support the f-image of some stratum of $(Z, \{B_Z\} + B_Z^{=1} - T)$ by Theorem A.4(1).

CLAIM

No strata of $(Z, \{B_Z\} + B_Z^{=1} - T)$ are mapped into S by f.

Proof

Assume that there is a stratum C of $(Z, \{B_Z\} + B_Z^{=1} - T)$ such that $f(C) \subset S$. Note that $\operatorname{Supp} f^*S \subset \operatorname{Supp} f_*^{-1}B_Y \cup \operatorname{Exc}(f)$ and $\operatorname{Supp} B_Z^{=1} \subset \operatorname{Supp} f_*^{-1}B_Y \cup \operatorname{Exc}(f)$. Since C is also a stratum of $(Z, B_Z^{=1})$ and $C \subset \operatorname{Supp} f^*S$, there exists an irreducible component G of $B_Z^{=1}$ such that $C \subset G \subset \operatorname{Supp} f^*S$. Therefore, by the definition of T, G is an irreducible component of T because $f(G) \subset S$ and G is an irreducible component of $B_Z^{=1}$. So, C is not a stratum of $(Z, \{B_Z\} + B_Z^{=1} - T)$. It is a contradiction. $\square$

On the other hand, $f(T) \subset S$. Therefore,

$$f_*\mathcal{O}_T\big(\ulcorner -(B_T^{<1})\urcorner - \llcorner B_T^{>1}\lrcorner\big) \to R^1f_*\mathcal{O}_Z\big(\ulcorner -(B_Z^{<1})\urcorner - \llcorner B_Z^{>1}\lrcorner - T\big)$$

is a zero-map by the above claim. Thus, we obtain

$$f_*\mathcal{O}_T\big(\ulcorner -(B_T^{<1})\urcorner - \llcorner B_T^{>1}\lrcorner\big) \simeq \mathcal{O}_S\big(\ulcorner -(B_S^{<1})\urcorner - \llcorner B_S^{>1}\lrcorner\big).$$

We finish the proof. $\square$

REMARK 2.9

Let X be an n-dimensional normal variety, and let Δ be an $\mathbb{R}$-divisor on X such that $K_X + \Delta$ is $\mathbb{R}$-Cartier. Let $f : Y \to X$ be a resolution with $K_Y + \Delta_Y = f^*(K_X + \Delta)$ such that $\operatorname{Supp} \Delta_Y$ is a simple normal crossing divisor. We put $A = \ulcorner -(\Delta_Y^{<1})\urcorner$, $N = \llcorner \Delta_Y^{>1}\lrcorner$, and $W = \Delta_Y^{=1}$. Since $R^if_*\mathcal{O}_Y(A - N - W) = 0$ for $i > 0$ by the Kawamata-Viehweg vanishing theorem, we have

$$0 \to \mathcal{J}(X, \Delta) \to \mathcal{J}_{\mathrm{NLC}}(X, \Delta) \to f_*\mathcal{O}_W(A|_W - N|_W) \to 0,$$

and

$$R^if_*\mathcal{O}_Y(A - N) \simeq R^if_*\mathcal{O}_W(A|_W - N|_W)$$

for every $i > 0$. In general, $R^if_*\mathcal{O}_Y(A - N) \neq 0$ for $1 \le i \le n - 1$.

From now on, we assume that Δ is effective. We put $F = W - E$, where E is the union of irreducible components of W which are mapped to $\operatorname{Nlc}(X, \Delta)$.

Then we have

$$f_*\mathcal{O}_Y(A-N-E)=f_*\mathcal{O}_Y(A-N)=\mathcal{J}_{\mathrm{NLC}}(X,\Delta).$$

Applying f_* to the short exact sequence

$$0\to\mathcal{O}_Y(A-N-W)\to\mathcal{O}_Y(A-N-E)\to\mathcal{O}_F(A|_F-N|_F-E|_F)\to 0,$$

we obtain

$$f_*\mathcal{O}_F(A|_F-N|_F-E|_F)=f_*\mathcal{O}_W(A|_W-N|_W).$$

In particular, $\mathcal{J}(X,\Delta)=\mathcal{J}_{\mathrm{NLC}}(X,\Delta)$ if and only if (X,Δ) has no lc centers.

2.2. Examples of non-lc ideal sheaves

Here we explain some elementary examples.

EXAMPLE 2.10

Let X be an n-dimensional smooth variety. Let $P\in X$ be a closed point, and let $\mathfrak{m}=\mathfrak{m}_P$ be the corresponding maximal ideal. Let $f:Y\to X$ be the blowup at P. Then $f^{-1}\mathfrak{m}=\mathcal{O}_Y(-E)$, where E is the exceptional divisor of f. If $c>n$, then

$$\mathcal{J}_{\mathrm{NLC}}(X;c\cdot\mathfrak{m})=f_*\mathcal{O}_Y\big(((n-1)-\llcorner c\lrcorner)E\big)=\mathcal{J}(X;c\cdot\mathfrak{m})=\mathfrak{m}^{\llcorner c\lrcorner-(n-1)}.$$

If $c<n$, then

$$\mathcal{J}_{\mathrm{NLC}}(X;c\cdot\mathfrak{m})=f_*\mathcal{O}_Y\big(((n-1)-\llcorner c\lrcorner)E\big)=\mathcal{J}(X;c\cdot\mathfrak{m})=\mathcal{O}_X.$$

When $c=n$, we note that

$$\mathcal{J}_{\mathrm{NLC}}(X;c\cdot\mathfrak{m})=f_*\mathcal{O}_Y\simeq\mathcal{O}_X\supsetneq\mathcal{J}(X;c\cdot\mathfrak{m})=f_*\mathcal{O}_Y(-E)=\mathfrak{m}.$$

EXAMPLE 2.11

Let X be a smooth variety, and let D be a smooth divisor on X. Then $\mathcal{J}_{\mathrm{NLC}}(X,D)=\mathcal{O}_X$. However,

$$\mathcal{J}_{\mathrm{NLC}}\big(X,(1+\varepsilon)D\big)=\mathcal{O}_X(-D)$$

for every $0<\varepsilon\ll 1$. On the other hand,

$$\mathcal{J}(X,D)=\mathcal{J}\big(X,(1+\varepsilon)D\big)=\mathcal{O}_X(-D)$$

for every $0<\varepsilon\ll 1$.

We note the following lemma on the *jumping numbers*, whose proof is obvious by the definitions (cf. [L, Lemma 9.3.21, Definition 9.3.22]).

LEMMA 2.12 (JUMPING NUMBERS)

Let X be a smooth variety, and let D be an effective $\mathbb{Q}$-divisor (resp., $\mathbb{R}$-divisor) on X. Let $x\in X$ be a fixed point contained in the support of D. Then there is an increasing sequence

$$0<\xi_0(D;x)<\xi_1(D;x)<\xi_2(D;x)<\cdots$$

of rational (resp., real) numbers $\xi_i = \xi_i(D;x)$ *characterized by the properties that*

$$\mathcal{J}(X, c\cdot D)_x = \mathcal{J}(X, \xi_i \cdot D)_x \quad \textit{for } c \in [\xi_i, \xi_{i+1}),$$

while $\mathcal{J}(X, \xi_{i+1}\cdot D)_x \subsetneq \mathcal{J}(X, \xi_i \cdot D)_x$ *for every* i. *The rational (resp., real) numbers* $\xi_i(D;x)$ *are called the jumping numbers of* D *at* x. *We can check the properties that*

$$\mathcal{J}_{\mathrm{NLC}}(X, c\cdot D)_x = \mathcal{J}_{\mathrm{NLC}}(X, d\cdot D)_x \quad \textit{for } c, d \in (\xi_i, \xi_{i+1}),$$

while $\mathcal{J}_{\mathrm{NLC}}(X, \xi_{i+1}\cdot D)_x \subsetneq \mathcal{J}_{\mathrm{NLC}}(X, \xi_i \cdot D)_x$ *for every* i. *Moreover,* $\mathcal{J}_{\mathrm{NLC}}(X, c\cdot D)_x = \mathcal{J}(X, c\cdot D)_x$ *for* $c \in (\xi_i, \xi_{i+1})$ *by Remark 2.9.*

EXAMPLE 2.13

Let $X = \mathbb{C}^2 = \operatorname{Spec}\mathbb{C}[z_1, z_2]$ and $D = (z_1 = 0) + (z_2 = 0) + (z_1 = z_2)$. Then we can directly check that

$$\mathcal{J}_{\mathrm{NLC}}(X, D) = \mathfrak{m}^2$$

and

$$\mathcal{J}_{\mathrm{NLC}}\big(X, (1-\varepsilon)D\big) = \mathcal{J}\big(X, (1-\varepsilon)D\big) = \mathfrak{m}$$

for $0 < \varepsilon \ll 1$, where $\mathfrak{m}$ is the maximal ideal corresponding to $0 \in \mathbb{C}^2$. On the other hand,

$$\mathcal{J}_{\mathrm{NLC}}\big(X, (1+\varepsilon)D\big) = \mathcal{J}\big(X, (1+\varepsilon)D\big) \subsetneq \mathcal{J}_{\mathrm{NLC}}(X, D)$$

for $0 < \varepsilon \ll 1$ because $D \subset \operatorname{Nlc}(X, (1+\varepsilon)D)$. Note that

$$\mathcal{J}(X, D) = \mathcal{J}\big(X, (1+\varepsilon)D\big) \subsetneq \mathcal{J}_{\mathrm{NLC}}(X, D)$$

for $0 < \varepsilon \ll 1$.

2.3. Main theorem: Restriction theorem

The following theorem is the main theorem of this article.

THEOREM 2.14 (RESTRICTION THEOREM)

Let X *be a normal variety, and let* $S + B$ *be an effective* $\mathbb{R}$*-divisor on* X *such that* S *is reduced and normal and that* S *and* B *have no common irreducible components. Assume that* $K_X + S + B$ *is* $\mathbb{R}$*-Cartier. Let* B_S *be the different on* S *such that* $K_S + B_S = (K_X + S + B)|_S$. *Then we obtain*

$$\mathcal{J}_{\mathrm{NLC}}(S, B_S) = \mathcal{J}_{\mathrm{NLC}}(X, S + B)|_S.$$

In particular, (S, B_S) *is log canonical if and only if* $(X, S + B)$ *is log canonical around* S.

REMARK 2.15

The notion of *different* was introduced by Shokurov in [S, §3]. For the definition and the basic properties, see, for example, [A, 9.2.1 Codimension one adjunction] and [F4, Section 14].

Before we start the proof of Theorem 2.14, let us see an easy example.

EXAMPLE 2.16
Let $X = \mathbb{C}^2 = \operatorname{Spec} \mathbb{C}[x, y]$, $S = (x = 0)$, and $B = (y^2 = x^3)$. We put $B_S = B|_S$. Then we have $K_S + B_S = (K_X + S + B)|_S$. By direct calculations, we obtain

$$\mathcal{J}_{\mathrm{NLC}}(S, B_S) = \mathfrak{m}^2, \qquad \mathcal{J}_{\mathrm{NLC}}(X, S + B) = \mathfrak{n}^2,$$

where $\mathfrak{m}$ (resp., $\mathfrak{n}$) is the maximal ideal corresponding to $0 \in S$ (resp., $(0,0) \in X$). Of course, we have

$$\mathcal{J}_{\mathrm{NLC}}(S, B_S) = \mathcal{J}_{\mathrm{NLC}}(X, S + B)|_S.$$

Let us start the proof of Theorem 2.14.

Proof of Theorem 2.14
We take a resolution $f : Y \to X$ with the following properties:

(i) $\operatorname{Exc}(f)$ is a simple normal crossing divisor on Y;
(ii) $f^{-1}X_{\mathrm{NLC}}$ is a simple normal crossing divisor on Y, where $X_{\mathrm{NLC}} = \mathrm{Nlc}(X, S + B)$;
(iii) $f^{-1}S$ is a simple normal crossing divisor on Y;
(iv) $f^{-1}(X_{\mathrm{NLC}} \cap S)$ is a simple normal crossing divisor on Y;
(v) $\operatorname{Exc}(f) \cup f^{-1}X_{\mathrm{NLC}} \cup f_*^{-1}B \cup f^{-1}S$ is a divisor with a simple normal crossing support.

We put $K_Y + B_Y = f^*(K_X + S + B)$. Then $\operatorname{Supp} B_Y$ is a simple normal crossing divisor by (i) and (v). Let S_Y be the strict transform of S on Y. Let T be the union of the components of $B_Y^{=1} - S_Y$ which are mapped into S by f. We can decompose $T = T_1 + T_2$ as follows.

(a) Any irreducible component of T_2 is mapped into X_{NLC} by f.
(b) No irreducible component of T_1 is mapped into X_{NLC} by f.

By (ii) and (v), any stratum of T_1 is not mapped into X_{NLC} by f.

We put $A = \ulcorner -(B_Y^{<1}) \urcorner$ and $N = \llcorner B_Y^{>1} \lrcorner$. Then A is an effective f-exceptional divisor. Moreover, $A|_{S_Y}$ is exceptional with respect to $f : S_Y \to S$. Then we have

$$\mathcal{J}_{\mathrm{NLC}}(X, S + B) = f_*\mathcal{O}_Y(A - N)$$

and

$$\mathcal{J}_{\mathrm{NLC}}(S, B_S) = f_*\mathcal{O}_{S_Y}(A - N).$$

Here we used

$$K_{S_Y} + (B_Y - S_Y)|_{S_Y} = f^*(K_S + B_S).$$

It follows from $K_Y + B_Y = f^*(K_X + S + B)$ by adjunction.

STEP 1

We consider the following short exact sequence:

$$0 \to \mathcal{O}_Y\big(A-N-(S_Y+T)\big) \to \mathcal{O}_Y(A-N) \to \mathcal{O}_{S_Y+T}(A-N) \to 0.$$

Applying $R^i f_*$, we obtain

$$\begin{aligned} 0 &\to f_*\mathcal{O}_Y\big(A-N-(S_Y+T)\big) \to f_*\mathcal{O}_Y(A-N) \\ &\to f_*\mathcal{O}_{S_Y+T}(A-N) \to R^1 f_*\mathcal{O}_Y\big(A-N-(S_Y+T)\big) \to \cdots . \end{aligned}$$

We note that

$$\begin{aligned} A-N-(S_Y+T) &- \big(K_Y+\{B_Y\}+(B_Y^{=1}-S_Y-T)\big) \\ &= -f^*(K_X+S+B) \end{aligned}$$

and that any stratum of $B_Y^{=1}-S_Y-T$ is not mapped into S by f (see conditions (iii) and (v)). Therefore, the support of every nonzero local section of $R^1 f_*\mathcal{O}_Y(A-N-(S_Y+T))$ cannot be contained in S by Theorem A.4(1). Thus, the connecting homomorphism

$$f_*\mathcal{O}_{S_Y+T}(A-N) \to R^1 f_*\mathcal{O}_Y\big(A-N-(S_Y+T)\big)$$

is a zero-map. Thus, we obtain

$$0 \to J \to \mathcal{J}_{\mathrm{NLC}}(X,S+B) \to I \to 0,$$

where $I := f_*\mathcal{O}_{S_Y+T}(A-N)$ and $J := f_*\mathcal{O}_Y(A-N-(S_Y+T))$. We note that the ideal sheaf $J = f_*\mathcal{O}_Y(A-N-(S_Y+T)) \subset \mathcal{O}_X$ defines a scheme structure on $S' = S \cup X_{\mathrm{NLC}}$. We will check that $I \subset \mathcal{O}_S$ and $I = \mathcal{J}_{\mathrm{NLC}}(X,S+B)|_S$ by $f(S_Y+T)=S$ and the following commutative diagrams:

$$\begin{tikzcd} 0 \arrow[r] & J \arrow[r] \arrow[d, "="] & \mathcal{J}_{\mathrm{NLC}}(X,S+B) \arrow[r] \arrow[d] & I \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & J \arrow[r] & \mathcal{O}_X \arrow[r] & \mathcal{O}_{S'} \arrow[r] & 0 \end{tikzcd}$$

and

$$\begin{tikzcd} 0 \arrow[r] & J \arrow[r] \arrow[d] & \mathcal{O}_X \arrow[r] \arrow[d, "="] & \mathcal{O}_{S'} \arrow[r] \arrow[d, "\alpha"] & 0 \\ 0 \arrow[r] & \mathcal{O}_X(-S) \arrow[r] & \mathcal{O}_X \arrow[r] & \mathcal{O}_S \arrow[r] & 0 \end{tikzcd}$$

It is sufficient to prove $\operatorname{Ker}\alpha \cap I = \{0\}$, where $\alpha : \mathcal{O}_{S'} \to \mathcal{O}_S$. We note that $I = \mathcal{J}_{\mathrm{NLC}}(X,S+B)/J$ and $\operatorname{Ker}\alpha = \mathcal{O}_X(-S)/J$. It is easy to see that

$$\mathcal{J}_{\mathrm{NLC}}(X,S+B) \cap \mathcal{O}_X(-S) \subset J$$

since $f(S_Y+T)=S$. Thus, $\operatorname{Ker}\alpha \cap I = \{0\}$. This means that $I \subset \mathcal{O}_S$ and $I = \mathcal{J}_{\mathrm{NLC}}(X,S+B)|_S$.

Therefore, it is enough to prove $I = \mathcal{J}_{\mathrm{NLC}}(S,B_S)$.

STEP 2

In this step, we prove that the natural inclusion

$$f_*\mathcal{O}_{S_Y+T_1}(A-N-T_2)\subset f_*\mathcal{O}_{S_Y+T}(A-N)=I$$

is an isomorphism. We consider the short exact sequence

$$\begin{aligned}0&\to\mathcal{O}_Y\bigl(A-N-(S_Y+T)\bigr)\to\mathcal{O}_Y(A-N-T_2)\\&\to\mathcal{O}_{S_Y+T_1}(A-N-T_2)\to 0.\end{aligned}$$

Applying R^if_*, we obtain

$$\begin{aligned}0&\to J\to f_*\mathcal{O}_Y(A-N-T_2)\to f_*\mathcal{O}_{S_Y+T_1}(A-N-T_2)\\&\xrightarrow{\delta} R^1f_*\mathcal{O}_Y\bigl(A-N-(S_Y+T)\bigr)\to\cdots.\end{aligned}$$

The connecting homomorphism δ is zero by exactly the same reason as in Step 1. Therefore, we obtain the following commutative diagram:

$$\begin{array}{ccccccccc}0&\longrightarrow&J&\longrightarrow&f_*\mathcal{O}_Y(A-N-T_2)&\longrightarrow&f_*\mathcal{O}_{S_Y+T_1}(A-N-T_2)&\longrightarrow&0\\&&\Big\downarrow{=}&&\Big\downarrow{\beta}&&\Big\downarrow&&\\0&\longrightarrow&J&\longrightarrow&f_*\mathcal{O}_Y(A-N)&\longrightarrow&I&\longrightarrow&0\end{array}$$

The homomorphism β is an isomorphism since $f(T_2)\subset f(N)=X_{\mathrm{NLC}}$. Therefore, we obtain

$$f_*\mathcal{O}_{S_Y+T_1}(A-N-T_2)=I\subset\mathcal{O}_S.$$

STEP 3

The inclusion

$$f_*\mathcal{O}_{S_Y}(A-N-T_2)\subset f_*\mathcal{O}_{S_Y}(A-N)=\mathcal{J}_{\mathrm{NLC}}(S,B_S)\subset\mathcal{O}_S$$

is obvious. By Kawakita's inversion of adjunction on log canonicity (cf. Corollary A.2), we obtain the opposite inclusion

$$f_*\mathcal{O}_{S_Y}(A-N)\subset f_*\mathcal{O}_{S_Y}(A-N-T_2).$$

Therefore, we obtain

$$f_*\mathcal{O}_{S_Y}(A-N-T_2)=f_*\mathcal{O}_{S_Y}(A-N)=\mathcal{J}_{\mathrm{NLC}}(S,B_S).$$

STEP 4

We consider the short exact sequence

$$\begin{aligned}0&\to\mathcal{O}_{T_1}(A-N-S_Y-T_2)\to\mathcal{O}_{S_Y+T_1}(A-N-T_2)\\&\to\mathcal{O}_{S_Y}(A-N-T_2)\to 0.\end{aligned}$$

We note that

$$f_*\mathcal{O}_{S_Y+T_1}(A-N-T_2)=I\subset\mathcal{O}_S$$

by Step 2 and

$$f_*\mathcal{O}_{S_Y}(A-N-T_2)=\mathcal{J}_{\mathrm{NLC}}(S,B_S)$$

by Step 3. By taking $R^i f_*$, we obtain

$$0\to I\to \mathcal{J}_{\mathrm{NLC}}(S,B_S)\to R^1f_*\mathcal{O}_{T_1}(A-N-S_Y-T_2)\to\cdots.$$

Here we used the fact that

$$f_*\mathcal{O}_{T_1}(A-N-S_Y-T_2)=0.$$

Note that no irreducible components of S are dominated by T_1.

Since $\mathcal{J}_{\mathrm{NLC}}(S,B_S)\subset\mathcal{O}_S$, we obtain

$$\mathcal{J}_{\mathrm{NLC}}(S,B_S)/I\subset\mathcal{O}_S/I.$$

Since

$$\begin{aligned}&A-N-(S_Y+T_2)-\big(K_Y+T_1+\{B_Y\}+(B_Y^{=1}-S_Y-T)\big)\\&\quad=-f^*(K_X+S+B),\end{aligned}$$

we have

$$\begin{aligned}&\big(A-N-(S_Y+T_2)\big)|_{T_1}-\big(K_{T_1}+(\{B_Y\}+B_Y^{=1}-S_Y-T)|_{T_1}\big)\\&\quad\sim_{\mathbb{R}}-f^*(K_X+S+B)|_{f(T_1)}.\end{aligned}$$

Therefore, the support of every nonzero local section of $R^1f_*\mathcal{O}_{T_1}(A-N-S_Y-T_2)$ cannot be contained in

$$\mathrm{Supp}(\mathcal{O}_S/I)\subset\mathrm{Supp}(\mathcal{O}_{S'}/I)=\mathrm{Supp}\big(\mathcal{O}_X/\mathcal{J}_{\mathrm{NLC}}(X,S+B)\big)=X_{\mathrm{NLC}}$$

by Theorem A.4(1). We note that no stratum of

$$\big(T_1,(\{B_Y\}+B_Y^{=1}-S_Y-T)|_{T_1}\big)$$

is mapped into X_{NLC} by f (see conditions (iv) and (v)). Thus, we obtain $I=\mathcal{J}_{\mathrm{NLC}}(S,B_S)$.

We finish the proof of the main theorem. □

In some applications, the following corollaries may play important roles.

COROLLARY 2.17

We use the notation in the proof of Theorem 2.14. We have the equalities

$$\begin{aligned}\mathcal{J}_{\mathrm{NLC}}(S,B_S)&=f_*\mathcal{O}_{S_Y}(A-N)\\&=f_*\mathcal{O}_{S_Y+T}(A-N)=f_*\mathcal{O}_{S_Y+T_1}(A-N-T_2).\end{aligned}$$

COROLLARY 2.18

We use the notation in the proof of Theorem 2.14. We obtain the short exact sequence

$$0\to J\to\mathcal{J}_{\mathrm{NLC}}(X,S+B)\to\mathcal{J}_{\mathrm{NLC}}(S,B_S)\to 0.$$

Let $\pi : X \to V$ be a projective morphism onto an algebraic variety V, and let L be a Cartier divisor on X such that $L-(K_X+S+B)$ is π-ample. Then

$$R^i\pi_*\big(J\otimes\mathcal{O}_X(L)\big)=0$$

for all $i>0$. In particular,

$$R^i\pi_*\big(\mathcal{J}_{\mathrm{NLC}}(X,S+B)\otimes\mathcal{O}_X(L)\big)\to R^i\pi_*\big(\mathcal{J}_{\mathrm{NLC}}(S,B_S)\otimes\mathcal{O}_S(L)\big)$$

is surjective for $i=0$ and is an isomorphism for every $i\geq 1$. As a corollary, we obtain

$$\pi_*\big(\mathcal{J}_{\mathrm{NLC}}(S,B_S)\otimes\mathcal{O}_S(L)\big)\subset\mathrm{Im}\big(\pi_*\mathcal{O}_X(L)\to\pi_*\mathcal{O}_S(L)\big).$$

Proof
Note that we have

$$\begin{aligned}&f^*L+A-N-(S_Y+T)-\big(K_Y+B_Y^{=1}+\{B_Y\}-(S_Y+T)\big)\\&=f^*\big(L-(K_X+S+B)\big).\end{aligned}$$

Therefore, $R^i\pi_*\big(f_*\mathcal{O}_Y(f^*L+A-N-(S_Y+T))\big)=0$ for $i>0$ by Theorem A.4(2). Thus, $R^i\pi_*(J\otimes\mathcal{O}_X(L))=0$ for all $i>0$ because $J=f_*\mathcal{O}_Y(A-N-(S_Y+T))$. □

REMARK 2.19
In Corollary 2.18, the ideal J is independent of the resolution $f:Y\to X$ by Lemma 2.8.

REMARK 2.20
In Corollary 2.18, we can weaken the assumption that $L-(K_X+S+B)$ is π-ample as follows. The $\mathbb{R}$-Cartier $\mathbb{R}$-divisor $D=L-(K_X+S+B)$ is π-nef and π-big, and $D|_C$ is π-big for every lc center C that is not contained in S (see the proof of Theorem 3.2).

2.4. Direct consequences of the restriction theorem

Let us collect some direct consequences of the restriction theorem.

PROPOSITION 2.21
Let X be a smooth variety, let D be an effective $\mathbb{R}$-divisor on X, and let $H\subset X$ be a smooth irreducible divisor that does not appear in the support of D. Then

$$\mathcal{J}_{\mathrm{NLC}}(H,D|_H)=\mathcal{J}_{\mathrm{NLC}}(X,H+D)|_H\subseteq\mathcal{J}_{\mathrm{NLC}}(X,D)|_H.$$

Proof
It is obvious. □

COROLLARY 2.22
Let $|V|$ be a free linear system, and let $H\in|V|$ be a general divisor. Then we

have

$$\mathcal{J}_{\mathrm{NLC}}(H, D|_H) = \mathcal{J}_{\mathrm{NLC}}(X, D)|_H$$

because $\mathcal{J}_{\mathrm{NLC}}(X, D) = \mathcal{J}_{\mathrm{NLC}}(X, H + D)$.

Proof
It is obvious. □

COROLLARY 2.23
Let D be an effective $\mathbb{R}$-divisor on the smooth variety X, and let $Y \subset X$ be a smooth subvariety that is not contained in the support of D. Then

$$\mathcal{J}_{\mathrm{NLC}}(Y, D_Y) \subseteq \mathcal{J}_{\mathrm{NLC}}(X, D)|_Y,$$

where $D_Y = D|_Y$.

Proof
It is obvious (see, e.g., the proof of [L, Corollary 9.5.6]). □

COROLLARY 2.24
Let $f : Y \to X$ be a morphism of smooth irreducible varieties, and let D be an effective $\mathbb{R}$-divisor on X. Assume that the support of D does not contain $f(Y)$. Then one has an inclusion

$$\mathcal{J}_{\mathrm{NLC}}(Y, f^*D) \subseteq f^{-1}\mathcal{J}_{\mathrm{NLC}}(X, D)$$

of ideal sheaves on Y.

Proof
See, for example, [L, Example 9.5.8]. □

PROPOSITION 2.25 (DIVISORS OF SMALL MULTIPLICITY)
Let D be an effective $\mathbb{R}$-divisor on a smooth variety X. Suppose that $x \in X$ is a point at which $\mathrm{mult}_x D \leq 1$. Then the ideal $\mathcal{J}_{\mathrm{NLC}}(X, D)$ is trivial at x.

Proof
It is obvious (see, e.g., [L, Proposition 9.5.13]). □

THEOREM 2.26 (GENERIC RESTRICTION)
Let X and T be smooth irreducible varieties, and let $p : X \to T$ be a smooth surjective morphism. Consider an effective $\mathbb{R}$-divisor D on X whose support does not contain any of the fibers $X_t = p^{-1}(t)$, so that for each $t \in T$, the restriction $D_t = D|_{X_t}$ is defined. Then there is a nonempty Zariski open set $U \subset T$ such that

$$\mathcal{J}_{\mathrm{NLC}}(X_t, D_t) = \mathcal{J}_{\mathrm{NLC}}(X, D)_t$$

for every $t \in U$*, where* $\mathcal{J}_{\mathrm{NLC}}(X, D)_t = \mathcal{J}_{\mathrm{NLC}}(X, D) \cdot \mathcal{O}_{X_t}$ *denotes the restriction of the indicated non-lc ideal to the fiber* X_t*. More generally, if* $t \in U$*, then*

$$\mathcal{J}_{\mathrm{NLC}}(X_t, c \cdot D_t) = \mathcal{J}_{\mathrm{NLC}}(X, c \cdot D)_t$$

for every $c > 0$*.*

Proof

We use the same notation as in the proof of [L, Theorem 9.5.35]. Let U be the nonempty Zariski open set of T which was obtained in the proof of [L, Theorem 9.5.35]. By shrinking T, we can assume that $T = U$. We take a general hypersurface H of T passing through $t \in U$. Then $\mathcal{J}_{\mathrm{NLC}}(X, c \cdot D) = \mathcal{J}_{\mathrm{NLC}}(X, X_1 + c \cdot D)$, where $X_1 = p^* H$. By Theorem 2.14,

$$\begin{aligned} \mathcal{J}_{\mathrm{NLC}}(X, c \cdot D)|_{X_1} &= \mathcal{J}_{\mathrm{NLC}}(X, X_1 + c \cdot D)|_{X_1} \\ &= \mathcal{J}_{\mathrm{NLC}}(X_1, c \cdot D|_{X_1}). \end{aligned}$$

By applying this argument $\dim T$ times, we obtain $\mathcal{J}_{\mathrm{NLC}}(X_t, c \cdot D_t) = \mathcal{J}_{\mathrm{NLC}}(X, c \cdot D)_t$. □

The following corollary is a direct consequence of Theorem 2.26.

COROLLARY 2.27 (SEMICONTINUITY)

Let $p : X \to T$ *be a smooth morphism as in Theorem 2.26, and let* D *be an effective* $\mathbb{R}$*-divisor on* X *satisfying the hypotheses of that statement. Moreover, given a section* $y : T \to X$ *of* p*, write* $y_t = y(t) \in X$*. If* $y_t \in \mathrm{Nlc}(X_t, D_t)$ *for* $t \neq 0 \in T$*, then* $y_0 \in \mathrm{Nlc}(X_0, D_0)$*.*

Proof

See the proof of [L, Corollary 9.5.39]. □

We close this subsection with the subadditivity theorem for non-lc ideal sheaves (cf. [DEL]).

THEOREM 2.28 (SUBADDITIVITY)

Let X *be a smooth variety.*

(1) *Suppose that* D_1 *and* D_2 *are any two effective* $\mathbb{R}$*-divisors on* X*. Then*

$$\mathcal{J}_{\mathrm{NLC}}(X, D_1 + D_2) \subseteq \mathcal{J}_{\mathrm{NLC}}(X, D_1) \cdot \mathcal{J}_{\mathrm{NLC}}(X, D_2).$$

(2) *If* $\mathfrak{a}, \mathfrak{b} \subseteq \mathcal{O}_X$ *are ideal sheaves, then*

$$\mathcal{J}_{\mathrm{NLC}}(X; \mathfrak{a}^c \cdot \mathfrak{b}^d) \subseteq \mathcal{J}_{\mathrm{NLC}}(X; \mathfrak{a}^c) \cdot \mathcal{J}_{\mathrm{NLC}}(X; \mathfrak{b}^d)$$

for any $c, d > 0$*. In particular,*

$$\mathcal{J}_{\mathrm{NLC}}(X; \mathfrak{a} \cdot \mathfrak{b}) \subseteq \mathcal{J}_{\mathrm{NLC}}(X; \mathfrak{a}) \cdot \mathcal{J}_{\mathrm{NLC}}(X; \mathfrak{b}).$$

Proof
The proof of the subadditivity theorem for multiplier ideal sheaves works for non-lc ideal sheaves (see, e.g., the proof of [L, Theorem 9.5.20]). We leave the details as an exercise for the reader. $\square$

3. Miscellaneous results

In this section, we collect some basic results of non-lc ideal sheaves.

3.1. Vanishing and global generation theorems

Here we state vanishing and global generation theorems explicitly. We can easily check them as applications of Theorem A.4.

THEOREM 3.1 (VANISHING THEOREM)
Let X be a smooth projective variety, let D be any $\mathbb{R}$-divisor on X, and let L be any integral divisor such that $L-D$ is ample. Then

$$H^i\big(X, \mathcal{O}_X(K_X+L)\otimes \mathcal{J}_{\mathrm{NLC}}(X,D)\big)=0$$

for $i>0$.

Proof
Let $f: Y\to X$ be a resolution with $K_Y+B_Y=f^*(K_X+D)$ such that $\operatorname{Supp} B_Y$ is a simple normal crossing divisor. Then

$$\ulcorner -(B_Y^{<1})\urcorner - \llcorner B_Y^{>1}\lrcorner + f^*(K_X+L)-(K_Y+B_Y^{=1}+\{B_Y\}) = f^*(L-D).$$

Therefore, $H^i(X, R^jf_*\mathcal{O}_Y(\ulcorner -(B_Y^{<1})\urcorner - \llcorner B_Y^{>1}\lrcorner + f^*(K_X+L)))=0$ for every $i>0$ and $j\geq 0$ by Theorem A.4(2). In particular,

$$H^i\big(X, f_*\mathcal{O}_Y(\ulcorner -(B_Y^{<1})\urcorner - \llcorner B_Y^{>1}\lrcorner + f^*(K_X+L))\big)=0$$

for $i>0$. This is the desired vanishing theorem because $\mathcal{J}_{\mathrm{NLC}}(X,D)= f_*\mathcal{O}_Y(\ulcorner -(B_Y^{<1})\urcorner - \llcorner B_Y^{>1}\lrcorner)$. $\square$

We can weaken the assumption in Theorem 3.1. However, Theorem 3.1 is sufficient for our purpose in this article. So, the reader can skip the next difficult theorem.

THEOREM 3.2
Let X be a normal variety, and let Δ be an effective $\mathbb{R}$-divisor such that $K_X+\Delta$ is $\mathbb{R}$-Cartier. Let $\pi: X\to V$ be a proper morphism onto an algebraic variety V, and let L be a Cartier divisor on X. Assume that $L-(K_X+\Delta)$ is π-nef and π-log big with respect to (X,Δ); that is, $L-(K_X+\Delta)$ is π-nef and π-big and $(L-(K_X+\Delta))|_C$ is π-big for every lc center C of the pair (X,Δ). Then we have

$$R^i\pi_*\big(\mathcal{J}_{\mathrm{NLC}}(X,\Delta)\otimes\mathcal{O}_X(L)\big)=0$$

for all $i>0$.

Proof
Let $f : Y \to X$ be a resolution with $K_Y + \Delta_Y = f^*(K_X + \Delta)$ such that $\operatorname{Supp}\Delta_Y$ is a simple normal crossing divisor. We put $F = \Delta_Y^{=1} - E$, where E is the union of irreducible components of $\Delta_Y^{=1}$ which are mapped to $X_{\mathrm{NLC}} = \mathrm{Nlc}(X, \Delta)$. If needed, we take more blowups and can assume that no strata of F are mapped to X_{NLC}. In this case, we have

$$\mathcal{J}_{\mathrm{NLC}}(X, \Delta) = f_*\mathcal{O}_Y\big(\ulcorner -(\Delta_Y^{<1})\urcorner - \llcorner \Delta_Y^{>1} \lrcorner - E\big).$$

Since

$$\begin{aligned} &\ulcorner -(\Delta_Y^{<1})\urcorner - \llcorner \Delta_Y^{>1} \lrcorner - E + f^*L - (K_Y + F + \{\Delta_Y\}) \\ &\quad = f^*\big(L - (K_X + \Delta)\big), \end{aligned}$$

we have

$$R^i\pi_* R^j f_*\mathcal{O}_Y\big(\ulcorner -(\Delta_Y^{<1})\urcorner - \llcorner \Delta_Y^{>1} \lrcorner - E + f^*L\big) = 0$$

for every $i > 0$ and $j \geq 0$ (see, e.g., [F5, Theorem 2.47]). So, we obtain

$$R^i\pi_*\big(\mathcal{J}_{\mathrm{NLC}}(X, \Delta) \otimes \mathcal{O}_X(L)\big) = 0$$

for $i > 0$. □

THEOREM 3.3 (GLOBAL GENERATION)
Let X be a smooth projective variety of dimension n. We fix a globally generated ample divisor B on X. Let D be an effective $\mathbb{R}$-divisor, and let L be an integral divisor on X such that $L - D$ is ample (or, more generally, nef and log big with respect to (X, D)). Then $\mathcal{O}_X(K_X + L + mB) \otimes \mathcal{J}_{\mathrm{NLC}}(X, D)$ is globally generated as soon as $m \geq n$.

Proof
It is obvious by Theorem 3.1 (or Theorem 3.2) and Mumford's m-regularity. □

3.2. Asymptotic non-lc ideal sheaves

Let X be a smooth variety. Let $\mathfrak{a}_\bullet = \{\mathfrak{a}_m\}$ be a graded system of ideals on X. In other words, $\mathfrak{a}_\bullet$ consists of a collection of ideal sheaves $\mathfrak{a}_k \subseteq \mathcal{O}_X$ satisfying $\mathfrak{a}_0 = \mathcal{O}_X$ and $\mathfrak{a}_m \cdot \mathfrak{a}_l \subseteq \mathfrak{a}_{m+l}$ for all $m, l \geq 1$.

DEFINITION 3.4 (NON-LC IDEAL ASSOCIATED TO A GRADED SYSTEM OF IDEALS)
The *asymptotic non-lc ideal sheaf* of $\mathfrak{a}_\bullet$ with *coefficient* or *exponent* c, written by either

$$\mathcal{J}_{\mathrm{NLC}}(X; c \cdot \mathfrak{a}_\bullet) \qquad \text{or} \qquad \mathcal{J}_{\mathrm{NLC}}(X; \mathfrak{a}_\bullet^c),$$

is defined to be the unique maximal member among the family of ideals $\{\mathcal{J}_{\mathrm{NLC}}(X; (c/p) \cdot \mathfrak{a}_p)\}$ for $p \geq 1$. Thus $\mathcal{J}_{\mathrm{NLC}}(X; c \cdot \mathfrak{a}_\bullet) = \mathcal{J}_{\mathrm{NLC}}(X; (c/p) \cdot \mathfrak{a}_p)$ for all sufficiently large and divisible integer $p \gg 0$.

EXAMPLE 3.5

Let X be a smooth projective variety, and let L be an integral divisor on X of nonnegative Iitaka dimension. We consider the base ideal $\mathfrak{b}_k = \mathfrak{b}(|kL|)$ of the complete linear system $|kL|$ for every $k \geq 0$. Let Δ be an effective $\mathbb{R}$-divisor on X such that $K_X + \Delta$ is $\mathbb{R}$-Cartier. Then $\mathfrak{b}_\bullet$ is a graded system of ideals on X. We put

$$\mathcal{J}_{\mathrm{NLC}}\big((X,\Delta),\|L\|\big) := \mathcal{J}_{\mathrm{NLC}}\big((X,\Delta);\mathfrak{b}_\bullet\big).$$

We note that $\mathcal{J}_{\mathrm{NLC}}((X,\Delta);\mathfrak{b}_\bullet)$ is the unique maximal member among the family of ideals $\{\mathcal{J}_{\mathrm{NLC}}((X,\Delta);(1/p)\cdot \mathfrak{b}_p)\}$ for $p \geq 1$.

Almost all the basic properties of asymptotic multiplier ideal sheaves in [L, 11.1, 11.2.A] can be proved for asymptotic non-lc ideal sheaves by the same arguments. Therefore, we do not repeat them here. We leave them as exercises for the reader. We state only one theorem in this subsection.

THEOREM 3.6

Let X be a smooth projective variety, let Δ be an effective Cartier divisor on X, and let L be an integral divisor on X of nonnegative Iitaka dimension. If A is an ample divisor on X, then

$$H^i\big(X,\mathcal{O}_X(K_X+\Delta+mL+A)\otimes \mathcal{J}_{\mathrm{NLC}}((X,\Delta),\|mL\|)\big)=0$$

for $i>0$. Furthermore, we assume that B is a globally generated ample divisor on X. Then for every $m \geq 1$,

$$\mathcal{O}_X(K_X+\Delta+lB+A+mL)\otimes \mathcal{J}_{\mathrm{NLC}}\big((X,\Delta),\|mL\|\big)$$

is globally generated as soon as $l \geq \dim X$.

Proof

Let $H \in |kmL|$ be a general member for a large and divisible k. Then $\mathcal{J}_{\mathrm{NLC}}((X,\Delta),\|mL\|) = \mathcal{J}_{\mathrm{NLC}}((X,\Delta),(1/k)H) = \mathcal{J}_{\mathrm{NLC}}(X,\Delta+(1/k)H)$. On the other hand, $\Delta + mL + A - (\Delta + \frac{1}{k}H) \sim_{\mathbb{Q}} A$. Thus, this theorem follows from Theorems 3.1 and 3.3. □

A. Appendix

A.1. Inversion of adjunction on log canonicity

We give some comments on the inversion of adjunction on log canonicity. The following theorem is due to Kawakita. Roughly speaking, he proved it by iterating the restriction theorem between *adjoint ideal sheaves* on X and *multiplier ideal sheaves* on S (for the proof, see [Ka]).

THEOREM A.1 (KAWAKITA)

Let X be a normal variety, let S be a reduced divisor on X, and let B be an effective $\mathbb{R}$-divisor on X such that K_X+S+B is $\mathbb{R}$-Cartier. Assume that S has *no common irreducible component with the support of* B*. Let* $\nu : S^\nu \to S$ *be the normalization, and let* B_{S^ν} *be the different on* S^ν *such that* $K_{S^\nu} + B_{S^\nu} = \nu^*((K_X + S + B)|_S)$*. Then* $(X, S + B)$ *is log canonical around* S *if and only if* (S^ν, B_{S^ν}) *is log canonical.*

By adjunction, it is obvious that (S^ν, B_{S^ν}) is log canonical if $(X, S + B)$ is log canonical around S. So the above theorem is usually called the *inversion of adjunction on log canonicity*. We need the following corollary of Theorem A.1 in the proof of Theorem 2.14. The proof is obvious.

COROLLARY A.2

Let $(X, S + B)$ *be as in Theorem A.1. Let* $P \in X$ *be a closed point such that* $(X, S + B)$ *is not log canonical at* P*. Let* $f : Y \to X$ *be a resolution such that* $K_Y + B_Y = f^*(K_X + S + B)$ *and that* $\operatorname{Supp} B_Y$ *is simple normal crossing. Then* $f^{-1}(P) \cap S_Y \cap \operatorname{Supp} N \neq \emptyset$*, where* $S_Y = f_*^{-1} S$ *and* $N = \llcorner B_Y^{>1} \lrcorner$*.*

We close this subsection with a remark on the theory of quasi-log varieties.

REMARK A.3

We use the notation in Theorem A.1. We note that $[X, K_X + S + B]$ has a natural quasi-log structure, which was introduced by Ambro (see, e.g., [F5, Chapter 3]). By adjunction, $S' = S \cup X_{\mathrm{NLC}}$ has a natural quasi-log structure induced by $[X, K_X + S + B]$. More explicitly, the defining ideal sheaf of the quasi-log variety S' is J in the proof of Theorem 2.14. In Step 1 in the proof of Theorem 2.14, we did not use the normality of S. Theorem A.1 says that $[S', (K_X + S + B)|_{S'}]$ has only qlc singularities around S if and only if (S^ν, B_{S^ν}) is lc.

A.2. New cohomological package

We quickly review Ambro's formulation of torsion-free and vanishing theorems in a simplified form (for more advanced topics and the proof, see [F5, Chapter 2]).

Let Y be a simple normal crossing divisor on a smooth variety M, and let D be an $\mathbb{R}$-divisor on M such that $\operatorname{Supp}(D + Y)$ is simple normal crossing and that D and Y have no common irreducible components. We put $B = D|_Y$ and consider the pair (Y, B). Let $\nu : Y^\nu \to Y$ be the normalization. We put $K_{Y^\nu} + \Theta = \nu^*(K_Y + B)$. A *stratum* of (Y, B) is an irreducible component of Y or the image of some lc center of $(Y^\nu, \Theta^{=1})$.

When Y is smooth and B is an $\mathbb{R}$-divisor on Y such that $\operatorname{Supp} B$ is simple normal crossing, we put $M = Y \times \mathbb{A}^1$ and $D = B \times \mathbb{A}^1$. Then $(Y, B) \simeq (Y \times \{0\}, B \times \{0\})$ satisfies the above conditions.

THEOREM A.4

Let (Y, B) *be as above. Assume that* B *is a boundary* $\mathbb{R}$*-divisor. Let* $f : Y \to X$ *be a proper morphism, and let* L *be a Cartier divisor on* Y*.*

(1) *Assume that* $H \sim_{\mathbb{R}} L - (K_Y + B)$ *is* f*-semiample. Then every nonzero local section of* $R^q f_* \mathcal{O}_Y(L)$ *contains in its support the* f*-image of some stratum of* (Y, B)*.*

(2) *Let* q *be an arbitrary nonnegative integer. Let* $\pi : X \to V$ *be a proper morphism, and assume that* $H \sim_{\mathbb{R}} f^* H'$ *for some* π*-ample* $\mathbb{R}$*-Cartier* $\mathbb{R}$*-divisor* H' *on* X*. Then,* $R^q f_* \mathcal{O}_Y(L)$ *is* π_**-acyclic; that is,* $R^p \pi_* R^q f_* \mathcal{O}_Y(L) = 0$ *for every* $p > 0$*.*

For the proof, see [F5, Theorem 2.39]. We note that [F2] is a gentle introduction to this new cohomological package. The reader can find various applications in [F1], [F3], [F6], [F5], [F7], and [F4].

Acknowledgments. I am grateful to Professor Hiromichi Takagi for his warm encouragement. Almost all the results were obtained during my short trip to Germany with the book [L] in the autumn of 2007. I thank the Max-Planck Institute for Mathematics and Mathematisches Forschungsinstitut Oberwolfach for their hospitality. I am grateful to Vladimir Lazić and Masayuki Kawakita for useful comments. I also thank Professors Mircea Mustaţă and Mihnea Popa for pointing out mistakes. I am grateful to the Mathematical Sciences Research Institute for its hospitality. I thank Shunsuke Takagi and Karl Schwede for many questions, suggestions, and useful comments.

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