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Research Article

Finite Integral Formulas Involving Multivariable Aleph-Functions

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The integrals evaluated are the products of multivariable Aleph-functions with algebraic functions, Jacobi polynomials, Legendre functions, Bessel-Maitland functions, and general class of polynomials. The main results of our paper are quite general in nature and competent at yielding a very large number of integrals involving polynomials and various special functions occurring in the problem of mathematical analysis and mathematical physics.

1. Introduction and Preliminaries

Throughout this paper, consider $\mathbb{C}, \mathbb{R}, \mathbb{R}^+, \mathbb{Z}_0^-$, and $\mathbb{N}$ to be a set of complex numbers, positive real numbers, nonpositive integers, and positive integers, respectively. The multivariable Aleph ($\aleph$) function of several complex variables generalizes the multivariable I-function, recently studied by Sharma and Ahmad [1], which itself is a generalization of G- and H-functions of multiple variables as

$$\begin{aligned}\aleph\left(z_1, z_2, \ldots, z_r\right) &= \aleph^{0,n:m_1,n_1,m_2,n_2,\ldots,m_r,n_r}_{p_i,q_i,\tau_i;R;p_{i^{(1)}},q_{i^{(1)}},\tau_{i^{(1)}};R^{(1)},\ldots,p_{i^{(r)}},q_{i^{(r)}},\tau_{i^{(r)}};R^{(r)}}\left\{\begin{matrix} z_1 \\ \vdots \\ z_r \end{matrix} \;\middle|\; \begin{matrix} A : B \\ C : D \end{matrix}\right\} \\ &= \frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\psi\left(\xi_1,\cdots,\xi_r\right)\prod_{i=1}^{r}\phi_i\left(\xi_i\right) z_i^{\xi_i}\, d\xi_1\cdots d\xi_r. \end{aligned} \tag{1}$$

where $\omega = \sqrt{-1}$,

$$\begin{aligned} A &= \left[\left(a_j;\alpha_j^{(1)},\cdots,\alpha_j^{(r)}\right)_{1,n}\right],\left[\tau_i\left(a_{ji};\alpha_{ji}^{(1)},\cdots,\alpha_{ji}^{(r)}\right)_{n+1,p_i}\right] \\ B &= \left[\left(c_j^{(1)},\gamma_j^{(1)}\right)_{1,n_1}\right],\left[\tau_{i^{(1)}}\left(c_{ji^{(1)}}^{(1)},\gamma_{ji^{(1)}}^{(1)}\right)_{n_1+1,p_i^{(1)}}\right];\cdots;\left[\left(c_j^{(r)},\gamma_j^{(r)}\right)_{1,n_r}\right],\left[\tau_{i^{(r)}}\left(c_{ji^{(1)}}^{(r)},\gamma_{ji^{(1)}}^{(r)}\right)_{n_r+1,p_i^{(r)}}\right] \\ C &= \left[\ldots,\tau_i\left(b_{ji};\beta_{ji}^{(1)},\cdots,\beta_{ji}^{(r)}\right)_{m+1,q_i}\right] \\ D &= \left[\left(d_j^{(1)},\delta_j^{(1)}\right)_{1,m_1}\right],\left[\tau_{i^{(1)}}\left(d_{ji^{(1)}}^{(1)},\delta_{ji^{(1)}}^{(1)}\right)_{m_1+1,q_i^{(1)}}\right];\cdots;\left[\left(d_j^{(r)},\delta_j^{(r)}\right)_{1,m_r}\right],\left[\tau_{i^{(r)}}\left(d_{ji^{(r)}}^{(r)},\gamma_{ji^{(r)}}^{(r)}\right)_{m_r+1,q_i^{(r)}}\right] \end{aligned} \tag{2}$$

$$\psi\left(\xi_1,\cdots,\xi_r\right) = \frac{\prod_{j=1}^{n}\Gamma\left(1-a_j+\sum_{k=1}^{r}\alpha_j^{(k)}\xi_k\right)}{\sum_{i=1}^{R}\left[\tau_i\prod_{j=n+1}^{p_i}\Gamma\left(a_{ji}-\sum_{k=1}^{r}\alpha_{ji}^{(k)}\xi_k\right)\prod_{j=1}^{q_i}\Gamma\left(1-b_{ji}+\sum_{k=1}^{r}\beta_{ji}^{(k)}\xi_k\right)\right]}, \tag{3}$$

$$\phi_k(\xi_k) = \frac{\prod_{j=1}^{m_k}\Gamma\left(d_j^{(k)} - \delta_j^{(k)}\xi_k\right)\prod_{j=1}^{n_k}\Gamma\left(1 - c_j^{(k)} + \gamma_j^{(k)}\xi_k\right)}{\sum_{i^{(k)}=1}^{R^{(k)}}\left[\tau_{i^{(k)}}\prod_{j=m_k+1}^{q_{i^{(k)}}}\Gamma\left(1 - d_{ji^{(k)}}^{(k)} + \delta_{ji^{(k)}}^{(k)}\xi_k\right)\prod_{j=n_k+1}^{p_{i^{(k)}}}\Gamma\left(c_{ji^{(k)}}^{(k)} - \gamma_{ji^{(k)}}^{(k)}\xi_k\right)\right]}, \tag{4}$$

and $a_j(j = 1, \cdots, p); b_j(j = 1, \cdots, q); c_j^{(k)}(j = 1, \cdots, n_k)$, $c_{ji^{(k)}}^{(k)}(j = n_k + 1, \cdots, p_{i^{(k)}}); d_j^{(k)}(j = 1, \cdots, m_k)\ d_{ji^{(k)}}^{(k)}(j = m_k + 1, \cdots, q_{i^{(k)}}), (k = 1, \cdots, r; i = 1, \cdots, R;$ and $i^{(k)} = 1, \cdots, R^{(k)}$ are complex numbers.

$$\begin{aligned} U_i^{(k)} = {} & \sum_{j=1}^{n}\alpha_j^{(k)} + \tau_i \sum_{j=n+1}^{p_i}\alpha_{ji}^{(k)} + \sum_{j=1}^{n_k}\gamma_j^{(k)} + \tau_{i^{(k)}}\sum_{j=n_k+1}^{p_i(k)}\gamma_{ji^{(k)}}^{(k)} \\ & - \tau_i\sum_{j=1}^{q_i}\beta_{ji^{(k)}}^{(k)} - \sum_{j=1}^{m_k}\delta_j^{(k)} - \tau_{i^{(k)}}\sum_{j=m_k+1}^{q_i(k)}\delta_{ji^{(k)}}^{(k)} \leq 0, \end{aligned} \tag{5}$$

$\tau_i(i = 1, \cdots, R), \tau_{i^{(k)}}(i^{(k)} = 1, \cdots, R^{(k)})$ are positive real numbers.

The integration path $L_{\omega\gamma\infty}$ $(\gamma \in R)$ extends from $\gamma - \omega\infty$ to $\gamma + \omega\infty$ and the poles of $\Gamma(d_j^{(k)} - \delta_j^{(k)}\xi_k)$, $j = 1, \ldots, m_k$, do not coincide with the poles of $\Gamma(1 - a_j + \sum_{i=1}^{r}\alpha_j^{(k)}\xi_k)$, $j = 1, \ldots, n$ and $\Gamma(1 - c_j^{(k)} + \gamma_j^{(k)}\xi_k)$, $j = 1, \ldots, n_k$ to the left of the contour L_k.

The existence condition for multiple Mellin-Barnes type contours (1) can be given below:

$$\left|\arg z_k\right| < \frac{1}{2}A_i^{(k)}\pi \tag{6}$$

where

$$\begin{aligned} A_i^{(k)} = {} & \sum_{j=1}^{n}\alpha_j^{(k)} - \tau_i\sum_{j=n+1}^{p_i}\alpha_{ji}^{(k)} - \tau_i\sum_{j=1}^{q_i}\beta_{ji}^{(k)} + \sum_{j=1}^{n_k}\gamma_j^{(k)} \\ & - \tau_{i^{(k)}}\sum_{j=n_k+1}^{p_{i^{(k)}}}\gamma_{ji^{(k)}}^{(k)} + \sum_{j=1}^{m_k}\delta_j^{(k)} - \tau_{i^{(k)}}\sum_{j=m_k+1}^{q_{i^{(k)}}}\delta_{ji^{(k)}}^{(k)} > 0, \end{aligned} \tag{7}$$

with $= 1, \cdots, r; i = 1, \cdots, R;$ and $i^{(k)} = 1, \cdots, R^{(k)}$.

Remark 1. By setting $\tau_i = \tau_i(k) = 1$, the multivariable Aleph-function reduces to multivariable I-function (see [1–3]]).

Remark 2. By setting $\tau_i = \tau_i(k) = 1$ $(k \in 1, \ldots, r)$ and $R = R^{(1)} =, \ldots, = R^{(r)} = 1$, the multivariable Aleph-function reduces to multivariable H-function defined by Srivastava et al. [4].

Remark 3. When we set $r = 1$, the multivariable Aleph-function reduces to Aleph-function of one variable defined by S*ü*dland [5].

For the definition of the H- function, ℵ-function, and its more generalization, the interested reader may refer to the papers [6–13].

From Rainville [14], the integral representation of the gamma function $\Gamma(x)$ is defined as

$$\Gamma(x) = \int_0^\infty e^{-t}t^{x-1}dt, \quad \Re(x) > 0. \tag{8}$$

And also the beta integral is defined as follows:

$$\begin{aligned} B(x, y) &= \int_0^1 t^{x-1}(1 - t)^{y-1}dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x + y)} \\ &= B(y, x), \quad (\Re(x), \Re(y) > 0). \end{aligned} \tag{9}$$

Further, We will use the following notations in this paper:

$$\begin{aligned} U &= m_1, n_1, m_2, n_2, \ldots, m_r, n_r, \\ V &= p_{i^{(1)}}, q_{i^{(1)}}, \tau_{i^{(1)}}; R^{(1)}, \\ W &= p_{i^{(r)}}, q_{i^{(r)}}, \tau_{i^{(r)}}; R^{(r)}. \end{aligned} \tag{10}$$

2. Integrals Involving Multivariable Aleph-Function with Algebraic Function

In this section, we evaluate integrals, the product of multivariable Aleph-functions with various algebraic functions.

$$\begin{aligned} I_1 &= \int_0^1 x^{-\rho}(1 - x)^{\rho-\sigma-1}\aleph_{p_i,q_i,\tau_i;R;V,\ldots,W}^{0,n:U}\left[z_1x^{\omega_1}, \ldots, z_rx^{\omega_r}\right]dx \\ &= \frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\psi(\xi_1, \ldots, \xi_r)\prod_{i=1}^{r}\phi_i(\xi_i)z_i^{\xi_i} \\ &\times\left\{\int_0^1 x^{1-\rho+\sum_{i=1}^{r}\omega_i\xi_i-1}(1 - x)^{\rho-\sigma-1}dx\right\}d\xi_1\ldots d\xi_r \\ &= \frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\psi(\xi_1, \ldots, \xi_r)\prod_{i=1}^{r}\phi_i(\xi_i) \\ &\cdot z_i^{\xi_i}B\left(1 - \rho + \sum_{i=1}^{r}\omega_i\xi_i, \rho - \sigma\right)d\xi_1\cdots d\xi_r = \Gamma(\rho - \sigma) \\ &\cdot\frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\psi(\xi_1, \ldots, \xi_r) \\ &\cdot\prod_{i=1}^{r}\phi_i(\xi_i)z_i^{\xi_i}\frac{\Gamma(1 - \rho + \sum_{i=1}^{r}\omega_i\xi_i)}{\Gamma(1 - \sigma + \sum_{i=1}^{r}\omega_i\xi_i)}d\xi_1\cdots d\xi_r = \Gamma(\rho - \sigma) \\ &\cdot\aleph_{p_i+1,q_i+1,\tau_i;R;V,\ldots,W}^{0,n+1:U}\left[\begin{matrix} z_1 \\ \vdots \\ z_r \end{matrix}\ \middle|\ \begin{matrix} (\rho; \omega_1, \ldots, \omega_r), A : B \\ (\sigma; \omega_1, \ldots, \omega_r), C : D \end{matrix}\right]. \end{aligned} \tag{11}$$

$$\begin{aligned} I_2 &= \int_0^1 x^{\rho-1}(1-x)^{\sigma-1} \aleph^{0,n:U}_{p_i,q_i,\tau_i;R;V,\ldots,W}\left[z_1 x^{\omega_1},\ldots,z_r x^{\omega_r}\right] dx \\ &= \frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\psi(\xi_1,\ldots,\xi_r)\prod_{i=1}^r \phi_i(\xi_i) z_i^{\xi_i} \\ &\cdot\left\{\int_0^1 x^{\rho+\sum_{i=1}^r\omega_i\xi_i-1}(1-x)^{\sigma-1}dx\right\}d\xi_1\ldots d\xi_r = \frac{1}{(2\pi\omega)^r} \\ &\cdot\int_{L_1}\cdots\int_{L_r}\psi(\xi_1,\ldots,\xi_r)\prod_{i=1}^r\phi_i(\xi_i)z_i^{\xi_i}B\left(\rho+\sum_{i=1}^r\omega_i\xi_i,\sigma\right)d\xi_1\cdots d\xi_r \\ &= \Gamma(\sigma)\frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\psi(\xi_1,\ldots,\xi_r)\prod_{i=1}^r\phi_i(\xi_i)z_i^{\xi_i} \\ &\cdot\frac{\Gamma(\rho+\sum_{i=1}^r\xi_i)}{\Gamma(\rho+\sigma+\sum_{i=1}^r\xi_i)}d\xi_1\cdots d\xi_r = \Gamma(\sigma) \\ &\cdot\aleph^{0,n+1:U}_{p_i+1,q_i+1,\tau_i;R;V,\ldots,W}\left[\begin{array}{c|c} z_1 & (1-\rho;\omega_1,\ldots,\omega_r),A:B \\ \vdots & \\ z_r & (1-\rho-\sigma;\omega_1,\ldots,\omega_r),C:D\end{array}\right]. \end{aligned} \tag{12}$$

$$\begin{aligned} I_3 &= \int_1^\infty x^{-\rho}(x-1)^{\sigma-1}\aleph^{0,n:U}_{p_i,q_i,\tau_i;R;V,\ldots,W}\left[z_1x^{\omega_1},\ldots,z_rx^{\omega_r}\right]dx \\ &= \frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\psi(\xi_1,\ldots,\xi_r)\prod_{i=1}^r\phi_i(\xi_i)z_i^{\xi_i} \\ &\cdot\left\{\int_1^\infty x^{-\rho+\sum_{i=1}^r\omega_i\xi_i}(x-1)^{\sigma-1}dx\right\}d\xi_1\ldots d\xi_r \end{aligned} \tag{13}$$

Putting $x = t+1$, we have $dx = dt$, and we use the following relation.

$$\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} = \int_0^\infty x^{\alpha-1}(x+1)^{-(\alpha+\beta)}dx = \int_0^\infty x^{\beta-1}(x+1)^{-(\alpha+\beta)}dx \tag{14}$$

$$\begin{aligned} I_3 &= \frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\psi(\xi_1,\ldots,\xi_r)\prod_{i=1}^r\phi_i(\xi_i)z_i^{\xi_i} \\ &\cdot\left\{\int_0^\infty t^{\sigma-1}(1+t)^{-(\rho-\sum_{i=1}^r\omega_i\xi_i)}dt\right\}d\xi_1\ldots d\xi_r = \Gamma(\sigma) \\ &\cdot\frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\psi(\xi_1,\ldots,\xi_r) \\ &\cdot\prod_{i=1}^r\phi_i(\xi_i)z_i^{\xi_i}\frac{\Gamma(\rho-\sigma-\sum_{i=1}^r\omega_i\xi_i)}{\Gamma(\rho-\sum_{i=1}^r\omega_i\xi_i)}d\xi_1\cdots d\xi_r = \Gamma(\sigma) \\ &\cdot\aleph^{0,n+1:U}_{p_i+1,q_i+1,\tau_i;R;V,\ldots,W}\left[\begin{array}{c|c} z_1 & (\rho;\omega_1,\ldots,\omega_r),A:B \\ \vdots & \\ z_r & (\rho-\sigma;\omega_1,\ldots,\omega_r),C:D\end{array}\right]. \end{aligned} \tag{15}$$

$$\begin{aligned} I_4 &= \int_0^\infty x^{\rho-1}(x+\beta)^{-\sigma}\aleph^{0,n:U}_{p_i,q_i,\tau_i;R;V,\ldots,W}\left[z_1x^{\omega_1},\ldots,z_rx^{\omega_r}\right]dx \\ &= \frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\psi(\xi_1,\ldots,\xi_r)\prod_{i=1}^r\phi_i(\xi_i)z_i^{\xi_i} \\ &\cdot\left\{\int_0^\infty x^{\rho+\sum_{i=1}^r\omega_i\xi_i-1}(x+\beta)^{-\sigma}dx\right\}d\xi_1\ldots d\xi_r \end{aligned} \tag{16}$$

Setting $x = \beta t$ implies $dx = \beta dt$, and then we obtain the following.

$$\begin{aligned} &= \frac{\beta^{\rho-\sigma}}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\psi(\xi_1,\ldots,\xi_r)\prod_{i=1}^r\phi_i(\xi_i)(\beta^{\omega_i}z_i)^{\xi_i} \\ &\cdot\left\{\int_0^\infty t^{\rho+\sum_{i=1}^r\omega_i\xi_i-1}(1+t)^{-\sigma}dt\right\}d\xi_1\ldots d\xi_r \end{aligned} \tag{17}$$

By applying (14), we have the following.

$$\begin{aligned} &= \frac{\beta^{\rho-\sigma}}{\Gamma(\sigma)}\frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\psi(\xi_1,\ldots,\xi_r)\prod_{i=1}^r\phi_i(\xi_i)(\beta^{\omega_i}z_i)^{\xi_i} \\ &\times\Gamma\left(\rho+\sum_{i=1}^r\omega_i\xi_i\right)\Gamma\left(\sigma-\rho-\sum_{i=1}^r\omega_i\xi_i\right)d\xi_1\ldots d\xi_r \end{aligned} \tag{18}$$

$$\begin{aligned} &= \frac{\beta^{\rho-\sigma}}{\Gamma(\sigma)} \\ &\cdot\aleph^{0,n+1:U}_{p_i+1,q_i+1,\tau_i;R;V,\ldots,W}\left[\begin{array}{c|c} \beta^{\omega_1}z_1 & (1-\rho;\omega_1,\ldots,\omega_r),A:B \\ \vdots & \\ \beta^{\omega_r}z_r & (\sigma-\rho;\omega_1,\ldots,\omega_r),C:D\end{array}\right]. \end{aligned} \tag{19}$$

$$\begin{aligned} I_5 &= \int_{-1}^1(1-x)^\rho(1+x)^\sigma \\ &\cdot\aleph^{0,n:U}_{p_i,q_i,\tau_i;R;V,\ldots,W}\left[z_1(1-x)^{\mu_1},\ldots,z_r(1-x)^{\mu_r}\right]dx \\ &= \frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\psi(\xi_1,\ldots,\xi_r)\prod_{i=1}^r\phi_i(\xi_i)z_i^{\xi_i} \\ &\cdot\left\{\int_{-1}^1(1-x)^{\rho+\sum_{i=1}^r\mu_i\xi_i}(1+x)^\sigma dx\right\}d\xi_1\ldots d\xi_r \end{aligned} \tag{20}$$

Now, we use the following formula ([14], p.261).

$$\begin{aligned} &\int_{-1}^1(1-x)^{n+\alpha}(1+x)^{n+\beta}dx \\ &\quad = 2^{2n+\alpha+\beta+1}B(1+\alpha+n,1+\beta+n) \end{aligned} \tag{21}$$

Hence, we arrive at

$$= 2^{\rho+\sigma+1}\Gamma(1+\sigma)\frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\psi(\xi_1,\ldots,\xi_r)\prod_{i=1}^r\phi_i(\xi_i)(2^{\mu_i}z_i)^{\xi_i}\times\frac{\Gamma(1+\rho+\sum_{i=1}^r\mu_i\xi_i)}{\Gamma(2+\rho+\sigma+\mu\sum_{i=1}^r\mu_i\xi_i)}d\xi_1\ldots d\xi_r \tag{22}$$

$$= 2^{\rho+\sigma+1}\Gamma(1+\sigma)\aleph^{0,n+1:U}_{p_i+1,q_i+1,\tau_i;R;V,\ldots,W}\left[\begin{array}{c|c} 2^{\mu_1}z_1 & (-\rho;\mu_1,\ldots,\mu_r),A:B \\ \vdots & \\ 2^{\mu_r}z_r & (-1-\sigma-\rho;\mu_1,\ldots,\mu_r),C:D\end{array}\right]. \tag{23}$$

3. Integrals Involving Multivariable Aleph-Function with Jacobi Polynomials

The Jacobi polynomial ([15], 4.21.2)) with parameters $\alpha, \beta \in \mathfrak{R}$ is defined by

$$p_n^{(\alpha,\beta)}(x) = \frac{(1+\alpha)_n}{n!} {}_2F_1\left[-n, 1+\alpha+\beta+n; 1+\alpha; \frac{1-x}{2}\right] \quad n \geq 1, \tag{24}$$

where ${}_2F_1[.]$ is the classical hypergeometric function. By substituting $\alpha = \beta = 0$, the Jacobi polynomial (24) reduces to Lagrange polynomial ([14], p. 157) as

$$p_n^{(\alpha,\beta)}(1) = \frac{(1+\alpha)_n}{n!}. \tag{25}$$

In this section, we derive integral formulas involving multivariable Aleph-functions multiplied by Jacobi polynomials.

$$\begin{aligned} I_6 &= \int_{-1}^{1} x^\lambda (1 - x)^\alpha (1+x)^\mu P_n^{(\alpha,\beta)}(x) \aleph^{0,n:U}_{p_i,q_i,\tau_i;R;V,\dots,W}\left[z_1(1+x)^{l_1}, \dots, z_r(1+x)^{l_r}\right] dx \\ &= \frac{1}{(2\pi\omega)^r} \int_{L_1} \cdots \int_{L_r} \psi(\xi_1,\dots,\xi_r) \prod_{i=1}^{r} \phi_i(\xi_i) \cdot z_i^{\xi_i} \times \left\{ \int_{-1}^{1} x^\lambda (1-x)^\alpha (1+x)^{\mu+\sum_{i=1}^r l_i\xi_i} \cdot P_n^{(\alpha,\beta)}(x)\, dx \right\} d\xi_1 \dots d\xi_r \end{aligned} \tag{26}$$

Next, we use the following formula:

$$\begin{aligned} &\int_{-1}^{1} x^\lambda (1-x)^\alpha (1+x)^\mu p_n^{(\alpha,\beta)}(x)\, dx \\ &= (-1)^n \frac{2^{\alpha+\mu+1}\Gamma(\mu+1)\Gamma(n+\alpha+1)\Gamma(\mu+\beta+1)}{n!\Gamma(\mu+\beta+n+1)\Gamma(\mu+\alpha+n+2)} \\ &\times {}_3F_2\left[\begin{matrix} -\lambda, \mu+\beta+1, \mu+1 \\ \mu+\beta+n+1, \mu+\alpha+n+2 \end{matrix}; 1\right], \end{aligned} \tag{27}$$

where $\alpha > -1$ and $\beta > -1$. Also, ${}_3F_2$ is the special case of generalized hypergeometric series.

Then, we have the following.

$$\begin{aligned} &= \frac{1}{(2\pi\omega)^r} \cdot \int_{L_1} \cdots \int_{L_r} \psi(\xi_1,\dots,\xi_r) \prod_{i=1}^{r} \phi_i(\xi_i) z_i^{\xi_i} (-1)^n \frac{2^{\alpha+\mu+\sum_{i=1}^r l_i\xi_i+1}}{n!} \\ &\times \frac{\Gamma(\mu+\sum_{i=1}^r l_i\xi_i+1)\Gamma(n+\alpha+1)\Gamma(\mu+\sum_{i=1}^r l_i\xi_i+\beta+1)}{\Gamma(\mu+\sum_{i=1}^r l_i\xi_i+\beta+n+1)\Gamma(\mu+\sum_{i=1}^r l_i\xi_i+\alpha+n+2)} \\ &\times {}_3F_2\left[\begin{matrix} -\lambda, \mu+\sum_{i=1}^r l_i\xi_i+\beta+1, \mu+\sum_{i=1}^r l_i\xi_i+1 \\ \mu+\sum_{i=1}^r l_i\xi_i+\beta+n+1, \mu+\sum_{i=1}^r l_i\xi_i+\alpha+n+2 \end{matrix}; 1\right] d\xi_1 \dots d\xi_r \end{aligned} \tag{28}$$

Using the definition of hypergeometric function and some simplifications, the above expression becomes

$$\begin{aligned} I_6 &= \frac{(-1)^n 2^{\alpha+\mu+1}\Gamma(\alpha+n+1)}{n!} \sum_{k=0}^{\infty} \frac{(-\lambda)_k 1^k}{k!} \frac{1}{(2\pi\omega)^r} \\ &\cdot \int_{L_1} \cdots \int_{L_r} \psi(\xi_1,\dots,\xi_r) \prod_{i=1}^{r} \phi_i(\xi_i) \times \left(2^{l_i} z_i\right)^{\xi_i} \frac{\Gamma(\mu+\sum_{i=1}^r l_i\xi_i+\beta+k+1)\Gamma(\mu+\sum_{i=1}^r l_i\xi_i+k+1)}{\Gamma(\mu+\sum_{i=1}^r l_i\xi_i+\beta+n+k+1)\Gamma(\mu+\sum_{i=1}^r l_i\xi_i+\alpha+n+k+2)} d\xi_1 \\ &\dots d\xi_r = \frac{(-1)^n 2^{\alpha+\mu+1}\Gamma(\alpha+n+1)}{n!} \sum_{k=0}^{\infty} \frac{(-\lambda)_k 1^k}{k!} \\ &\cdot \aleph^{0,n+2:U}_{p_i+2,q_i+2,\tau_i;R;V,\dots,W}\left[\begin{matrix} 2^{l_1} z_1 \\ \vdots \\ 2^{l_r} z_r \end{matrix} \middle| \begin{matrix} (-\mu-\beta-k; l_1,\dots,l_r), (-\mu-k; l_1,\dots,l_r), A : B \\ (-\mu-\beta-n-k; l_1,\dots,l_r), (-1-\mu-\alpha-n-k; l_1,\dots,l_r), C : D \end{matrix}\right]. \end{aligned} \tag{29}$$

Thus, $\alpha > -1$, $\beta > -1$, $\Re(\lambda) > -1$, and $|\arg z| < (1/2)\pi\Omega$.

$$\begin{aligned} I_7 &= \int_{-1}^{1} (1-x)^{\delta} (1+x)^{\nu} P_n^{(\mu,\nu)}(x) P_m^{(\rho,\sigma)}(x) \times \aleph_{p_i,q_i,\tau_i;R;V,\dots,W}^{0,n:U} \left[z_1 (1-x)^{l_1}, \dots, z_r (1-x)^{l_r} \right] dx \\ &= \frac{1}{(2\pi\omega)^r} \int_{L_1} \cdots \int_{L_r} \psi(\xi_1,\dots,\xi_r) \prod_{i=1}^{r} \phi_i(\xi_i) z_i^{\xi_i} \times \left\{ \int_{-1}^{1} (1-x)^{\delta+\sum_{i=1}^{r} l_i\xi_i} (1+x)^{\nu} P_n^{(\mu,\nu)}(x) \cdot P_m^{(\rho,\sigma)}(x)\, dx \right\} d\xi_1 \dots d\xi_r \\ &= \frac{\Gamma(1+\rho+m)\Gamma(1+\mu+n)}{m!n!} \cdot \sum_{k=0}^{\infty} \frac{(-m)_k(-n)_k(1+\rho+\sigma+m)_k(1+\mu+\nu+n)_k}{\Gamma(1+\rho+k)\Gamma(1+\mu+k)2^{2k}(k!)^2} \\ &\quad \times \frac{1}{(2\pi\omega)^r} \int_{L_1} \cdots \int_{L_r} \psi(\xi_1,\dots,\xi_r) \prod_{i=1}^{r} \phi_i(\xi_i) z_i^{\xi_i} \times \left\{ \int_{-1}^{1} (1-x)^{1+\delta+\sum_{i=1}^{r} l_i\xi_i+2k-1} (1+x)^{1+\nu-1} dx \right\} d\xi_1 \dots d\xi_r \end{aligned} \tag{30}$$

Now, using (21), we have the following.

$$\begin{aligned} I_7 &= \frac{\Gamma(1+\rho+m)\Gamma(1+\mu+n)}{m!n!} \cdot \sum_{k=0}^{\infty} \frac{(-m)_k(-n)_k(1+\rho+\sigma+m)_k(1+\mu+\nu+n)_k}{\Gamma(1+\rho+k)\Gamma(1+\mu+k)2^{2k}(k!)^2} \\ &\quad \times \frac{1}{(2\pi\omega)^r} \int_{L_1} \cdots \int_{L_r} \psi(\xi_1,\dots,\xi_r) \cdot \prod_{i=1}^{r} \phi_i(\xi_i) z_i^{\xi_i} 2^{\delta+\sum_{i=1}^{r} l_i\xi_i+2k+\nu+1} \times B\left(\delta + \sum_{i=1}^{r} l_i\xi_i + 2k + 1, \nu+1\right) d\xi_1 \dots d\xi_r = 2^{\delta+\nu+1} \\ &\quad \cdot \frac{\Gamma(1+\rho+m)\Gamma(1+\mu+n)\Gamma(\nu+1)}{m!n!} \times \sum_{k=0}^{\infty} \frac{(-m)_k(-n)_k(1+\rho+\sigma+m)_k(1+\mu+\nu+n)_k}{\Gamma(1+\rho+k)\Gamma(1+\mu+k)(k!)^2} \\ &\quad \times \frac{1}{(2\pi\omega)^r} \int_{L_1} \cdots \int_{L_r} \psi(\xi_1,\dots,\xi_r) \cdot \prod_{i=1}^{r} \phi_i(\xi_i) \left(2^{l_i} z_i\right)^{\xi_i} \frac{\Gamma\left(\delta + \sum_{i=1}^{r} l_i\xi_i + 2k + 1\right)}{\Gamma\left(\delta + \sum_{i=1}^{r} l_i\xi_i + 2k + \nu + 2\right)} d\xi_1 \dots d\xi_r \end{aligned} \tag{31}$$

Finally, we arrive at

$$\begin{aligned} I_7 &= 2^{\delta+\nu+1} \frac{\Gamma(1+\rho+m)\Gamma(1+\mu+n)\Gamma(\nu+1)}{m!n!} \times \sum_{k=0}^{\infty} \frac{(-m)_k(-n)_k(1+\rho+\sigma+m)_k(1+\mu+\nu+n)_k}{\Gamma(1+\rho+k)\Gamma(1+\mu+k)(k!)^2} \\ &\quad \times \aleph_{p_i+1,q_i+1,\tau_i;R;V,\dots,W}^{0,n+1:U} \left[\begin{matrix} 2^{l_1} z_1 \\ \vdots \\ 2^{l_r} z_r \end{matrix} \,\middle|\, \begin{matrix} (-\delta-2k; l_1,\dots,l_r), A : B \\ (-1-\delta-2k-\nu; l_1,\dots,l_r), C : D \end{matrix} \right]. \end{aligned} \tag{32}$$

Thus, $\delta > 0$, $\Re(\nu) > -1$, and $|\arg z| < (1/2)\pi\Omega$.

$$\begin{aligned} I_8 &= \int_{-1}^{1} (1-x)^{\rho} (1+x)^{\sigma} P_n^{(\mu,\nu)}(x) \times \aleph_{p_i,q_i,\tau_i;R;V,\dots,W}^{0,n:U} \left(z_1 (1-x)^{l_1} (1+x)^{h_1}, \dots, z_r (1-x)^{l_r} (1+x)^{h_r} \right) dx = \frac{\Gamma(1+\mu+n)}{n!} \\ &\quad \cdot \sum_{k=0}^{\infty} \frac{(-n)_k(1+\mu+\nu+n)_k}{\Gamma(1+\mu+k)2^k k!} \frac{1}{(2\pi\omega)^r} \int_{L_1} \cdots \int_{L_r} \psi(\xi_1,\dots,\xi_r) \prod_{i=1}^{r} \phi_i(\xi_i) z_i^{\xi_i} \times \left\{ \int_{-1}^{1} (1-x)^{1+\rho+\sum_{i=1}^{r} l_i\xi_i+k-1} \cdot (1+x)^{1+\sigma+\sum_{i=1}^{r} h_i\xi_i-1} dx \right\} d\xi_1 \dots d\xi_r \end{aligned} \tag{33}$$

Using (21), we have the following.

$$I_8 = 2^{\rho+\sigma+1}\frac{\Gamma(1+\mu+n)}{n!}\sum_{k=0}^{\infty}\frac{(-n)_k(1+\mu+\nu+n)_k}{\Gamma(1+\mu+k)k!} \times \frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots \cdot\int_{L_r}\psi(\xi_1,\ldots,\xi_r)\prod_{i=1}^{r}\phi_i(\xi_i)\left(2^{l_i+h_i}z_i\right)^{\xi_i} \times \frac{\Gamma(1+\rho+k+\sum_{i=1}^{r}l_i\xi_i)\Gamma(1+\sigma+\sum_{i=1}^{r}h_i\xi_i)}{\Gamma(2+\rho+\sigma+k+\sum_{i=1}^{r}(l_i+h_i)\xi_i)}d\xi_1 \ldots d\xi_r \tag{34}$$

Finally, rewriting the above equation by virtue of (1), we arrive at

$$I_8 = 2^{\rho+\sigma+1}\frac{\Gamma(1+\mu+n)}{n!}\sum_{k=0}^{\infty}\frac{(-n)_k(1+\mu+\nu+n)_k}{\Gamma(1+\mu+k)k!} \times \aleph^{0,n+2:U}_{p_i+2,q_i+1,\tau_i;R;V,\ldots,W}\left[\begin{matrix}2^{l_1+h_1}z_1\\ \vdots\\ 2^{l_r+h_r}z_r\end{matrix}\ \middle|\ \begin{matrix}(-\rho-k;l_1,\ldots,l_r),(-\sigma;h_1,\ldots,h_r),A:B\\ (-1-\rho-\sigma-k),(l_1+h_1),\ldots,(l_r+h_r)),C:D\end{matrix}\right]. \tag{35}$$

Thus, $\Re(\nu) > -1$, $\Re(\mu) > -1$, and $|\arg z| < (1/2)\pi\Omega$.

4. Integrals Involving Multivariable Aleph-Function with Legendre Function

The solution of Legendre differential equation in the form of Gauss hypergeometric type is as follows:

$$p_\nu^\mu(x) = \frac{1}{\Gamma(1-\mu)}\left(\frac{x+1}{x-1}\right)^{1/2\mu} \cdot {}_2F_1\left[-\nu,\nu+1;1-\mu;\frac{1-x}{2}\right], \quad |1-x| < 2. \tag{36}$$

Here $p_\nu^\mu(x)$ is known as the Legendre function of the first kind [16].

Next, we derive the integrals with Legendre function.

$$I_9 = \int_0^1 x^{\sigma-1}\left(1-x^2\right)^{\mu/2} p_\nu^\mu(x) \cdot \aleph^{0,n:U}_{p_i,q_i,\tau_i;R;V,\ldots,W}\left[z_1x^{\rho_1},\ldots,z_rx^{\rho_r}\right]dx = \frac{1}{(2\pi\omega)^r} \cdot\int_{L_1}\cdots\int_{L_r}\psi(\xi_1,\ldots,\xi_r)\prod_{i=1}^{r}\phi_i(\xi_i)z_i^{\xi_i} \times\left\{\int_0^1 x^{\sigma+\sum_{i=1}^{r}\rho_i\xi_i-1}\left(1-x^2\right)^{\mu/2}P_\nu^\mu(x)\,dx\right\}d\xi_1 \ldots d\xi_r \tag{37}$$

Next, we use the following formula ([16], Sec. 3.12) for $\mu \in \mathbb{N}, \Re(\sigma) > 0$

$$\int_0^1 x^{\sigma-1}\left(1-x^2\right)^{\mu/2} p_\nu^\mu(x)\,dx = \frac{(-1)^\mu 2^{-\sigma-\mu}\pi^{1/2}\Gamma(\sigma)\Gamma(1+\mu+\nu)}{\Gamma(1-\mu+\nu)\Gamma(1/2+\sigma/2+\mu/2-\nu/2)\Gamma(1+\sigma/2+\mu/2+\nu/2)}. \tag{38}$$

Now, applying (38), we obtain

$$I_9 = 2^{-\sigma-\mu}(-1)^\mu\pi^{1/2}\frac{\Gamma(1+\mu+\nu)}{\Gamma(1-\mu+\nu)}\frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\psi(\xi_1,\ldots,\xi_r)\prod_{i=1}^{r}\phi_i(\xi_i)\left(2^{-\rho_i}z_i\right)^{\xi_i}$$

$$\times \frac{\Gamma\left(\sigma+\sum_{i=1}^{r}\rho_i\xi_i\right)}{\Gamma\left(1/2+\sigma/2+\mu/2-\nu/2+\left(\sum_{i=1}^{r}\rho_i\xi_i\right)/2\right)\Gamma\left(1+\sigma/2+\mu/2+\nu/2+\left(\sum_{i=1}^{r}\rho_i\xi_i\right)/2\right)}d\xi_1\ldots d\xi_r \tag{39}$$

$$I_9 = 2^{-\sigma-\mu}(-1)^{\mu}\pi^{1/2}\frac{\Gamma(1+\mu+\nu)}{\Gamma(1-\mu+\nu)} \cdot \aleph^{0,n+1:U}_{p_i+1,q_i+2,\tau_i;R;V,\ldots,W}\left[\begin{matrix}2^{-\rho_1}z_1\\ \vdots\\ 2^{-\rho_r}z_r\end{matrix}\ \middle|\ \begin{matrix}(1-\sigma;\rho_1,\ldots,\rho_r),A:B\\ \left(\frac{1}{2}-\frac{\mu}{2}+\frac{\nu}{2}-\frac{\sigma}{2};\frac{\rho_1}{2},\ldots,\frac{\rho_r}{2}\right),\left(-\frac{\mu}{2}-\frac{\nu}{2}-\frac{\sigma}{2};\frac{\rho_1}{2},\ldots,\frac{\rho_r}{2}\right),C:D\end{matrix}\right]. \tag{40}$$

Thus, $|\arg(z)| < (1/2)\pi\Omega$, $\sigma > 0$, and $\mu \in \mathbb{N}\cup\{0\}$.

$$I_{10} = \int_0^1 x^{\sigma-1}\left(1-x^2\right)^{-\mu/2}p_\nu^\mu(x) \cdot \aleph^{0,n:U}_{p_i,q_i,\tau_i;R;V,\ldots,W}\left[z_1x^{\rho_1},\ldots,z_rx^{\rho_r}\right]dx = \frac{1}{(2\pi\omega)^r} \cdot \int_{L_1}\cdots\int_{L_r}\psi(\xi_1,\ldots,\xi_r)\prod_{i=1}^{r}\phi_i(\xi_i)z_i^{\xi_i} \times\left\{\int_0^1 x^{\sigma+\sum_{i=1}^{r}\rho_i\xi_i-1}\left(1-x^2\right)^{-\mu/2}P_\nu^\mu(x)\,dx\right\}d\xi_1\ldots d\xi_r \tag{41}$$

Next we use the following formula ([16], Sec. 3.12) for $\mu \in \mathbb{N}$, $\mathfrak{R}(\sigma) > 0$

$$\int_0^1 x^{\sigma-1}\left(1-x^2\right)^{-\mu/2}p_\nu^\mu(x)\,dx = \frac{2^{-\sigma+\mu}\pi^{1/2}\Gamma(\sigma)}{\Gamma(1/2+\sigma/2-\mu/2-\nu/2)\Gamma(1+\sigma/2-\mu/2-\nu/2)}. \tag{42}$$

Using (42), we obtain

$$I_{10} = 2^{-\sigma+\mu}\pi^{1/2}\frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\psi(\xi_1,\ldots,\xi_r)\prod_{i=1}^{r}\phi_i(\xi_i)\left(2^{-\rho_i}z_i\right)^{\xi_i} \times\frac{\Gamma\left(\sigma+\sum_{i=1}^{r}\rho_i\xi_i\right)}{\Gamma\left(1/2+\sigma/2-\mu/2-\nu/2+\left(\sum_{i=1}^{r}\rho_i\xi_i\right)/2\right)\Gamma\left(1+\sigma/2-\mu/2-\nu/2+\left(\sum_{i=1}^{r}\rho_i\xi_i\right)/2\right)}d\xi_1\ldots d\xi_r \tag{43}$$

$$I_{10} = 2^{-\sigma+\mu}\pi^{1/2}\aleph^{0,n+1:U}_{p_i+1,q_i+2,\tau_i;R;V,\ldots,W}\left[\begin{matrix}2^{-\rho_1}z_1\\ \vdots\\ 2^{-\rho_r}z_r\end{matrix}\ \middle|\ \begin{matrix}(1-\sigma;\rho_1,\ldots,\rho_r),A:B\\ \left(\frac{1}{2}+\frac{\mu}{2}+\frac{\nu}{2}-\frac{\sigma}{2};\frac{\rho_1}{2},\ldots,\frac{\rho_r}{2}\right),\left(\frac{\mu}{2}+\frac{\nu}{2}-\frac{\sigma}{2};\frac{\rho_1}{2},\ldots,\frac{\rho_r}{2}\right),C:D\end{matrix}\right]. \tag{44}$$

Thus, $|\arg(z)| < (1/2)\pi\Omega$, $\mathfrak{R}(\sigma) > 0$, and $\mathfrak{R}(\mu) \in \mathbb{N}\cup\{0\}$.

5. Integrals Involving Multivariable Aleph-Function and Bessel-Maitland Function

The Bessel-Maitland function is defined by Kiryakova [17] as follows:

$$J_\nu^\mu(x) = \phi(\mu,\nu+1:x) = \sum_{k=0}^{\infty}\frac{1}{\Gamma(\mu k+\nu+1)}\frac{(-x)^k}{k!}, \quad \mu > -1,\ x \in \mathbb{C}. \tag{45}$$

$$I_{11} = \int_0^\infty x^\rho J_\nu^\mu(x) \cdot \aleph^{0,n:U}_{p_i,q_i,\tau_i;R;V,\ldots,W}\left[z_1x^{\sigma_1},\ldots,z_rx^{\sigma_r}\right]dx\,dx = \frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\psi(\xi_1,\ldots,\xi_r)\prod_{i=1}^{r}\phi_i(\xi_i)z_i^{\xi_i} \cdot\left\{\int_0^\infty x^{\rho+\sum_{i=1}^{r}\sigma_i\xi_i}J_\nu^\mu(x)\,dx\right\}d\xi_1\ldots d\xi_r \tag{46}$$

Now, using the following formula from [18]

$$\int_0^{\infty} x^{\rho} J_{\nu}^{\mu}(x)\,dx = \frac{\Gamma(\rho+1)}{\Gamma(1+\nu-\mu-\mu\rho)} \quad (\Re(\rho) > -1, 0 < \mu < 1), \tag{47}$$

and, further, applying (47), we obtain

$$I_{14} = \frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\psi(\xi_1,\ldots,\xi_r)\prod_{i=1}^{r}\phi_i(\xi_i)\,z_i^{\xi_i}\frac{\Gamma\left(1+\rho+\sum_{i=1}^{r}\sigma_i\xi_i\right)}{\Gamma\left(1+\nu-\mu-\mu\rho-\mu\sum_{i=1}^{r}\sigma_i\xi_i\right)}d\xi_1\ldots d\xi_r \tag{48}$$

$$I_{14} = \aleph_{p_i+2,q_i,\tau_i;R;V,\ldots,W}^{0,n+1:U}\left[\begin{matrix} z_1 \\ \vdots \\ z_r\end{matrix}\ \middle|\ \begin{matrix}(-\rho;\sigma_1,\ldots,\sigma_r),(1+\nu-\mu-\mu\rho;\mu\sigma_1,\ldots,\mu\sigma_r),A:B \\ C:D\end{matrix}\right]. \tag{49}$$

Thus, $|\arg z| < (1/2)\pi\Omega, \sigma-\mu\sigma > 0, \sigma > 0, 0 < \mu < 1$, and $\Re(\rho+1) > 0$.

6. Integrals Involving Multivariable Aleph-Function and General Class of Polynomials

The general class of polynomials $S_{n_1,\ldots,n_r}^{m_1,\ldots,m_r}(x)$ defined by Srivastava [19] is as follows

$$S_{n_1,\ldots,n_r}^{m_1,\ldots,m_r}(x) = \sum_{l_i=0}^{[n_1/m_1]}\cdots\sum_{l_r=0}^{[n_r/m_r]}\prod_{i=1}^{r}\frac{(-n_i)_{m_il_i}}{l_i!}A_{n_i,l_i}x^{l_i}. \tag{50}$$

Here $n_1,n_2,\ldots,n_r \in \mathbb{N}_0$; $m_1,m_2,\ldots,m_r \in \mathbb{N}$; and the coefficients $A_{n_i,l_i}(n_i,l_i \geq 0)$ are any constant numbers. By suitable restriction of the coefficient A_{n_i,l_i} the general class of polynomials has various special cases. These include the Jacobi polynomials, the Laguerre polynomials, the Bessel polynomials, the Hermite polynomials, the Gould-Hopper polynomials, and the Brafman polynomials ([20], p. 158-161).

Next, we establish the following integral.

$$\begin{aligned} I_{12} &= \int_{-1}^{1}(1-x)^{\rho-1}(1+x)^{\sigma-1}S_{n_1,\ldots,n_{r'}}^{m_1,\ldots,m_{r'}}\left[y(1-x)^{\mu}(1+x)^{\nu}\right]\times\aleph_{p_i,q_i,\tau_i;R;V,\ldots,W}^{0,n:U}\left(z_1(1-x)^{h_1}\cdot(1+x)^{k_1},\ldots,z_r(1-x)^{h_r}(1+x)^{k_r}\right)dx \\ &= \sum_{l_1=0}^{[n_1/m_1]}\cdots\sum_{l_{r'}}^{[n_{r'}/m_{r'}]}\prod_{i=1}^{r'}\frac{(-n_j)_{m_jl_j}}{l_j!}A_{n_j,l_j}y^{l_j}\frac{1}{(2\pi\omega)^r}\cdot\int_{L_1}\cdots\int_{L_r}\psi(\xi_1\cdots\xi_r)\prod_{i=1}^{r}\phi_i(\xi_i)\,z_i^{\xi_i} \\ &\quad\times\left\{\int_{-1}^{1}(1-x)^{\rho+\sum_{i=1}^{r}h_i\xi_i+\mu\sum_{j=1}^{r'}l_j-1}\cdot(1+x)^{\sigma+\sum_{i=1}^{r}k_i\xi_i+\nu\sum_{j=1}^{r'}l_j-1}dx\right\}d\xi_1\ldots d\xi_r \end{aligned} \tag{51}$$

By applying (21), we have

$$\begin{aligned} I_{12} &= 2^{\rho+\sigma-1}\sum_{l_1=0}^{[n_1/m_1]}\cdots\sum_{l_{r'}}^{[n_{r'}/m_{r'}]}\prod_{i=1}^{r'}\frac{(-n_j)_{m_jl_j}}{l_j!}A_{n_jl_j}\left(2^{\mu+\nu}y\right)^{l_j}\times\frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots \\ &\quad\cdot\int_{L_r}\psi(\xi_1\cdots\xi_r)\prod_{i=1}^{r}\phi_i(\xi_i)\left(2^{h_i+k_i}z_i\right)^{\xi_i}\times\frac{\Gamma\left(\rho+\sum_{i=1}^{r}h_i\xi_i+\mu\sum_{j=1}^{r'}l_j\right)\Gamma\left(\sigma+\sum_{i=1}^{r}k_i\xi_i+\nu\sum_{j=1}^{r'}l_j\right)}{\Gamma\left(\rho+\sigma+\sum_{i=1}^{r}(h_i+k_i)\xi_i+(\mu+\nu)\sum_{j=1}^{r'}l_j\right)}d\xi_1\ldots d\xi_r \end{aligned} \tag{52}$$

$$\begin{aligned} I_{12} &= 2^{\rho+\sigma-1}\sum_{l_1=0}^{[n_1/m_1]}\cdots\sum_{l_{r'}}^{[n_{r'}/m_{r'}]}\prod_{i=1}^{r'}\frac{(-n_j)_{m_jl_j}}{l_j!}A_{n_jl_j}\left(2^{\mu+\nu}y\right)^{l_j} \\ &\quad\cdot\aleph_{p_i+2,q_i,\tau_i;R;V;\ldots;W}^{0,n+1:U}\left[\begin{matrix}2^{h_1+k_1}z_1 \\ \vdots \\ 2^{h_r+k_r}z_r\end{matrix}\ \middle|\ \begin{matrix}\left(1-\rho-\mu\sum_{j=1}^{r'}l_j;h_1,\ldots,h_r\right),\left(1-\sigma-\nu\sum_{j=1}^{r'}l_j;k_1,\ldots,k_r\right),A:B \\ \left(1-\rho-\sigma-(\mu+\nu)\sum_{j=1}^{r'}l_j;(h_1+k_1),\ldots,(h_r+k_r)\right),C:D\end{matrix}\right]. \end{aligned} \tag{53}$$

These converge under the following conditions:

(1) $|\arg z| < (1/2)\pi\Omega$,

(2) $\rho \geq 1, \sigma \geq 1, \mu \geq 0, \nu \geq 0, h \geq 0, k \geq 0$ (h and k are not both zero simultaneously),

(3) $\Re(\rho) + h\min[\Re(b_j/B_j)] > 0$ and $\Re(\sigma) + k\min[\Re(b_j/B_j)] > 0$.

7. Special Cases

(i) By setting δ by $\eta - 1$ and $\rho = \sigma = \mu = \nu = 0$, (32) transforms to the following integral:

$$I_{13} = 2^{\eta} \sum_{k=0}^{\infty} \frac{(-m)_k (-n)_k (1+m)_k (1+n)_k}{\Gamma(1+k)\Gamma(1+k)(k!)^2} \times \aleph^{0,n+1:U}_{p_i+1,q_i+1,\tau_i;R;V;\dots;W} \left[\begin{matrix} 2^l z_1 \\ \vdots \\ 2^l z_r \end{matrix} \,\middle|\, \begin{matrix} (1-\eta-2k; l_1,\dots,l_r), A : B \\ (-\eta-2k; l_1,\dots,l_r), C : D \end{matrix} \right]. \quad (54)$$

Thus, $|\arg z| < (1/2)\pi\Omega$.

(ii) When we substitute ρ by $\rho - 1$ and σ by $\sigma - 1$ and put $\mu = \nu = 0$, integral (35) transforms to the following:

$$I_{14} = 2^{\rho+\sigma-1} \sum_{k=0}^{\infty} \frac{(-n)_k (1+n)_k}{\Gamma(1+k)\, k!} \times \aleph^{0,n+2:U}_{p_i+2,q_i+1,\tau_i;R;V;\dots;W} \left[\begin{matrix} 2^{l_1+h_1} z_1 \\ \vdots \\ 2^{l_r+h_r} z_r \end{matrix} \,\middle|\, \begin{matrix} (1-\rho-k; l_1,\dots,l_r), (1-\sigma; h_1,\dots,h_r), A : B \\ (1-\rho-\sigma-k; (l_1+h_1),\dots,(l_r+h_r)), C : D \end{matrix} \right]. \quad (55)$$

Thus, $|\arg z| < (1/2)\pi\Omega$.

8. Concluding Remarks

In the present paper, we evaluated new integrals involving the multivariable Aleph-function with certain special functions. Certain special cases of integrals that follow Remarks 1–3 have been investigated in the literature by a number of authors [21–24]) with different arguments. Therefore, the results presented in this paper are easily converted in terms of a similar type of new interesting integrals with different arguments after some suitable parameter substitutions. In this sequel, one can obtain integral representation of more generalized special function, which has many applications in physics and engineering science.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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