Hindawi
Journal of Applied Mathematics
Volume 2017, Article ID 2451237, 23 pages
http://dx.doi.org/10.1155/2017/2451237

Research Article

Modelling of Rabies Transmission Dynamics Using Optimal Control Analysis

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Received 1 January 2017; Revised 13 March 2017; Accepted 19 April 2017; Published 16 July 2017

Academic Editor: Zhen Jin

We examine an optimal way of eradicating rabies transmission from dogs into the human population, using preexposure prophylaxis (vaccination) and postexposure prophylaxis (treatment) due to public education. We obtain the disease-free equilibrium, the endemic equilibrium, the stability, and the sensitivity analysis of the optimal control model. Using the Latin hypercube sampling (LHS), the forward-backward sweep scheme and the fourth-order Range-Kutta numerical method predict that the global alliance for rabies control's aim of working to eliminate deaths from canine rabies by 2030 is attainable through mass vaccination of susceptible dogs and continuous use of pre- and postexposure prophylaxis in humans.

1. Introduction

Rabies is an infection that mostly affects the brain of an infected animal or individual, caused by viruses belonging to the genus *Lyssavirus* of the family Rhabdoviridae and order Mononegavirales [1, 2]. This disease has become a global threat and it is also estimated that rabies occurs in more than 150 countries and territories [2]. Raccoons, skunks, bats, and foxes are the main animals that transmit the virus in the United States [2]. In Asia, Africa, and Latin America, it is known that dogs are the main source of transmission of the rabies virus into the human population [2]. When the rabies virus enters the human body or that of an animal, the infection (virus) moves rapidly along the neural pathways to the central nervous system; from there the virus continues to spread to other organs and causes injury by interrupting various nerves [2]. The symptoms of rabies are quite similar to those of encephalitis (see [3]). Due to movement of dogs in homes or the surroundings, the risk of not being infected by a rabid dog can never be guaranteed. Rabies is a major health problem in many populations dense with dogs, especially in areas where there are less or no preventive measures (vaccination and treatment) for dogs and humans. Treatment after exposure to the rabies virus is known as postexposure prophylaxis (PEP) and vaccination before exposure to the infection is known as preexposure prophylaxis.

The study of optimal control analysis in maximizing or minimizing a said target was introduced by Pontryagin and his collaborators around 1950. They developed the key idea of introducing the adjoint function to a differential equation, by forming an objective functional [4], and since then there has been a considerable study of infectious disease using optimal control analysis (see [4–12]).

Research published by Aubert [13], on the advancement of the expense of wildlife rabies in France, incorporated various variables. They follow immunization of domestic animals, the reinforcement of epidemiological reconnaissance system and the bolster given to indicative research laboratories, the costs connected with outbreaks of rabies, the clinical perception of those mammals which had bitten humans, the preventive immunization, and postexposure treatment of people. A significant percentage (72%) of the cost was the preventive immunization of local animals. In France, as in other European nations in which the red fox (Vulpes) is the

species most affected, two primary procedures for controlling rabies were assessed in [13] at the repository level to be specific: fox termination and the oral immunization of foxes. The consolidated costs and advantages of both systems were looked at and included either the expenses of fox separation or the cost of oral immunization. The total yearly costs of both techniques stayed practically identical until the fourth year, after which the oral immunization methodology turned out to be more cost effective. This estimate was made in 1988 and readjusted in 1993 and affirmed by ex-postinvestigation five years later. Accordingly, it was presumed that fox termination brought about a transient diminishment in the event of the infection while oral immunization turned out to be equipped for wiping out rabies even in circumstances in which fox population was growing. Anderson and May [14] formulated a mathematical model based on each time step dynamic which was calculated independently in every cell. Later, Bohrer et al. [15] published a paper on the viability of different rabies spatial immunization designs in a simulated host population.

The research presented by Bohrer [15] stated that, in desert environments, where host population size varies over time, nonuniform spreading of oral rabies vaccination may, under certain circumstances, be more effective than the commonly used uniform spread. The viability of a nonarbitrary spread of the immunization depends, to some extent, on the dispersal behavior of the carriers. The outcomes likewise exhibit that, in a warm domain in a few high-density regions encompassed by populations with densities below the critical threshold for the spread of the disease, the rabies infection can persist.

Levin et al. [16] also presented a model for the immune responses to rabies virus in bats. Coyne et al. [17] proposed an SEIR model, which was also used in a study predicting the local dynamics of rabies among raccoons in the United States. Childs et al. [18] also researched rabies epidemics in raccoons with a seasonal birth pulse, using optimal control of an SEIRS model which describes the population dynamics. Hampson et al. [19] also noted that rabies epidemic cycles have a period of 3–6 years in dog populations in Africa, so they built a susceptible, exposed, infectious, and vaccinate model with an intervention response variable, which showed significant synchrony.

Carroll et al. [20] also used compartmental models to describe rabies epidemiology in dog populations and explored three control methods: vaccination, vaccination pulse fertility control, and culling. An ordinary differential equation model was used to characterize the transmission dynamics of rabies between humans and dogs by [21, 22]. The work by Zinsstag et al. [23] further extended the existing models on rabies transmission between dogs to include dog-to-human transmission and concluded that human postexposure prophylaxis (PEP) with a dog vaccination campaign was the more cost effective in controlling the disease in the long run. Furthermore, Ding et al. [24] formulated an epidemic model for rabies in raccoons with discrete time and spatial features. Their goal was to analyze the strategies for optimal distribution of vaccine baits to minimize the spread of the disease and the cost of carrying out the control. Smith and Cheeseman [25] show that culling could be more effective than vaccination, given the same efficacy of control, but Tchuenche and Bauch suggest that culling could be counterproductive, for some parameter values (see [26]).

The work in [27, 28] also presented a mathematical model of rabies transmission in dogs and from the dog population to the human population in China. Their study did not consider the use optimal control analysis to the study of the rabies virus in dogs and from the dog population to the human population. Furthermore, the insightful work of Wiraningsih et al. [29] studied the stability analysis of a rabies model with vaccination effect and culling in dogs, where they introduced postexposure prophylaxis to a rabies transmission model, but the paper did not consider the noneffectiveness of the pre- and postprophylaxis on the susceptible humans and exposed humans and that of the dog population and the use of optimal control analysis. Therefore, motivated by the research predictions of the global alliance of rabies control [30] and the work mention above, we seek to adjust the model presented in [27–29], by formulating an optimal control model, so as to ascertain an optimal way of controlling rabies transmission in dogs and from the dog population to the human population taking into account the noneffectiveness (failure) of vaccination and treatment.

The paper is petition as follows. Section 2 contains the model formulation, mathematical assumptions, the mathematical flowchart, and the model equations. Section 3 contains the model analysis, invariant region, equilibrium points, basic reproduction number $\mathscr{R}_0$, and the stability analysis of the equilibria. In Section 4 we present the parameter values leading to numerical values of the basic reproduction number $\mathscr{R}_0$, the herd immunity threshold and sensitivity analysis using Latin hypercube sampling (LHS), and some numerical plots. Section 5 contains the objective functional and the optimality system of the model. Finally, Sections 6 and 7 contain discussion and conclusion, respectively.

2. Model Formulation

We present two subpopulation transmission models of rabies virus in dogs and that of the human population (see Figure 1), based on the work presented in [27–29]. The dog population has a total of four compartments. The compartments represent the susceptible dogs, $S_D(t)$, exposed dogs, E_D, infected dogs, $I_D(t)$, and partially immune dogs, $R_D(t)$. Thus, the total dog population is $N_D(t) = S_D(t) + E_D(t) + I_D(t) + R_D(t)$. The human population also has four compartments representing susceptible humans, $S_H(t)$, exposed humans, $E_H(t)$, infected humans, $I_H(t)$, and partially immune humans, $R_H(t)$. Thus, the total human population is $N_H(t) = S_H(t) + E_H(t) + I_H(t) + R_H(t)$. It is assumed that there is no human to human transmission of the rabies virus in the human submodel (see [29]). In the dog submodel, it is assumed that there is a direct transmission of the rabies virus from one dog to the other and from the infected dog compartment to the susceptible human population. It is further assumed that the susceptible dog population, $S_D(t)$, is increased by recruitment at a rate A_D and B_H is the birth or immigration rate into the susceptible human population, $S_H(t)$. It is assumed that the transmission

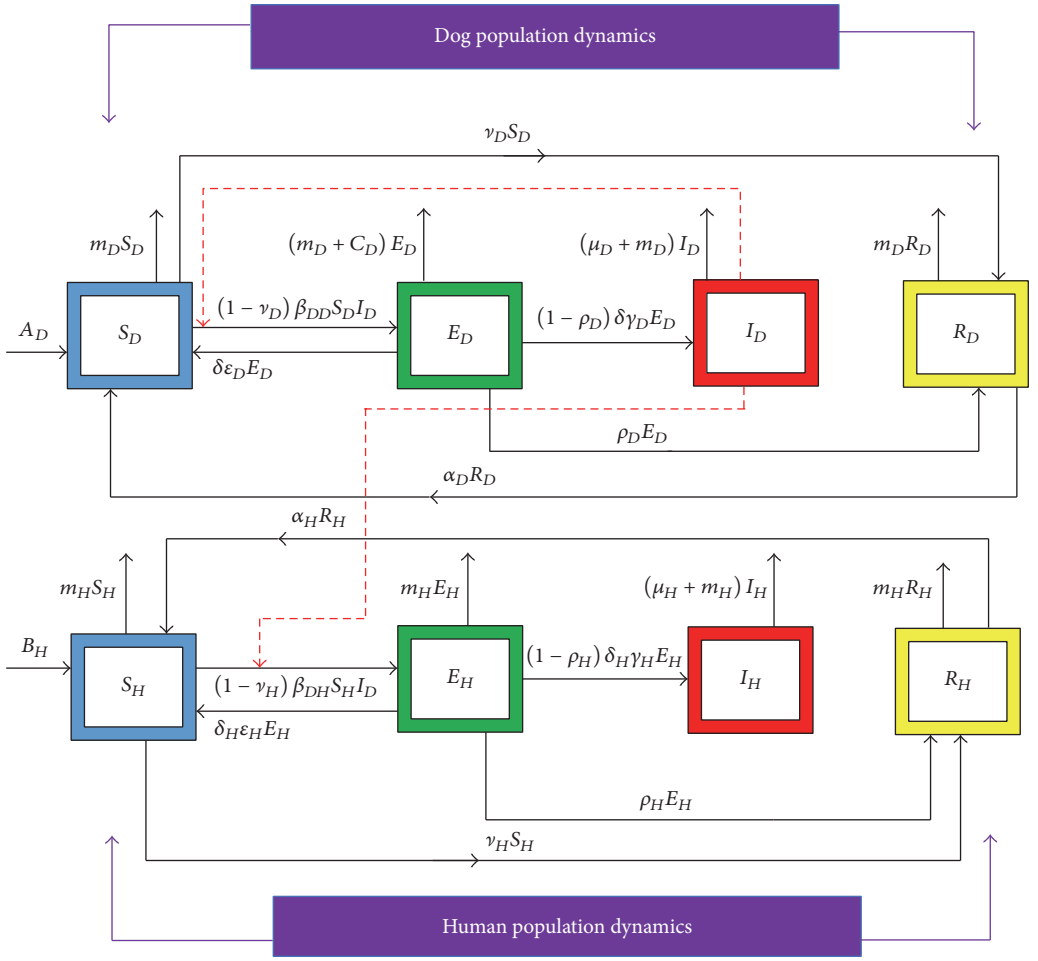


Figure 1: Optimal control model of rabies transmission dynamics.

and contact rate of the rabid dog into the dog compartment is β_{DD}. Suppose that ν_D represents the control strategy due to public education and vaccination in the dogs compartment; then the transmission dynamics become $(1 - \nu_D)\beta_{DD} S_D I_D$, where $(1 - \nu_D)$ is the noneffectiveness (failure) of the vaccine. It is also assumed that the contact rate of infectious dogs to the human population is β_{DH}. Similarly, administrating vaccination to the susceptible humans the progression rate of the susceptible humans to the exposed stage becomes $(1 - \nu_H)\beta_{DH} S_H I_D$, where ν_H is the preexposure prophylaxis (vaccination), $(1 - \nu_H)$ represents the failure of the preexposure prophylaxis in the human compartment. Furthermore, administrating postexposure prophylaxis (treatment) to affected humans at the rate ρ_H decreases the progression rate of the rabies virus, at the exposed class to the infectious class as $(1 - \rho_H)\delta_H \gamma_H E_H$, where $(1 - \rho_H)$ is the failure rate of the postexposure prophylaxis and $\delta_H \gamma_H$ represents the rate at which exposed humans progress to the infected compartment [27]. The rate of losing immunity in both compartments is represented by α_D and α_H, respectively.

The exposed humans without clinical rabies that move back to the susceptible population are denoted by the rate $\delta_H \varepsilon_H$. The natural death rate of dogs is m_D, and m_H denotes the mortality rate of humans (natural death rate), μ_D represents the death rate associated with rabies infection in dogs, and μ_H represents the disease induce death in humans. The rate at which exposed dogs die due to culling is C_D, and $\delta \varepsilon_D$ represents the rate at which exposed dogs without clinical rabies move back to the susceptible dog compartment. Subsequently, using the idea presented in [29], we assumed that the exposed dogs are treated or quarantined by their owners at the rate ρ_D; this implies that $(1 - \rho_D)\delta \gamma_D E_D$ is the progression rate of the exposed dogs to the infectious compartment, where $(1 - \rho_D)$ is the failure of the treatment or quarantined strategy, and $\delta \gamma_D E_D$ denotes those exposed dogs that develop clinical rabies [27]. Figure 1 shows the mathematical dynamics of the rabies virus in both compartments.

From Figure 1 transmission flowchart and assumptions give the disease pathways as

$$\frac{dS_D}{dt} = A_D - (1 - \nu_D)\beta_{DD} S_D I_D - (m_D + \nu_D) S_D + \delta \varepsilon_D E_D + \alpha_D R_D,$$

$$\frac{dE_D}{dt} = (1-\nu_D)\beta_{DD}S_DI_D - ((1-\rho_D)\delta\gamma_D + m_D + \rho_D + \delta\varepsilon_D + C_D)E_D,$$

$$\frac{dI_D}{dt} = (1-\rho_D)\delta\gamma_D E_D - (m_D + \mu_D)I_D,$$

$$\frac{dR_D}{dt} = \nu_D S_D + \rho_D E_D - (m_D + \alpha_D)R_D,$$

$$\frac{dS_H}{dt} = B_H - (1-\nu_H)\beta_{DH}S_HI_D - (m_H + \nu_H)S_H + \delta_H\varepsilon_H E_H + \alpha_H R_H,$$

$$\frac{dE_H}{dt} = (1-\nu_H)\beta_{DH}S_HI_D - ((1-\rho_H)\delta_H\gamma_H + m_H + \rho_H + \delta_H\varepsilon_H)E_H,$$

$$\frac{dI_H}{dt} = (1-\rho_H)\delta_H\gamma_H E_H - (m_H + \mu_H)I_H,$$

$$\frac{dR_H}{dt} = \nu_H S_H + \rho_H E_H - (m_H + \alpha_H)R_H,$$

$$\text{with } S_D(0) > 0,\ E_D(0) \geq 0,\ I_D(0) \geq 0,\ R_D(0) \geq 0,\ S_H(0) > 0,\ E_H(0) \geq 0,\ I_H(0) > 0,\ R_H(0) > 0. \tag{1}$$

3. Model Analysis

Model system (1) will be studied in a biological feasible region as outlined below. Model system (1) is basically divided into two regions; thus $\Omega = \Omega_D \times \Omega_H$.

Lemma 1. *The solution set* $\{S_D, E_D, I_D, R_D, S_H, E_H, I_H, R_H\} \in R_+^8$ *of model system (1) is contained in the feasible region* Ω.

Proof. Suppose $\{S_D, E_D, I_D, R_D, S_H, E_H, I_H, R_H\} \in R_+^8$ for all $t > 0$. We want to show that the region Ω is positively invariant, so that it becomes sufficient to look at the dynamics of model system (1), given that

$$N_D(t) = S_D(t) + E_D(t) + I_D(t) + R_D(t), \tag{2}$$

$$N_H(t) = S_H(t) + E_H(t) + I_H(t) + R_H(t), \tag{3}$$

where $N_D(t)$ is the total population of dogs at any time (t) and $N_H(t)$ is total population of humans at any time (t).

Equation (2) gives

$$\frac{dN_D}{dt} = A_D - (S_D + E_D + I_D + R_D)m_D - \mu_D I_D - C_D E_D, \tag{4}$$

which yields

$$\frac{dN_D}{dt} = A_D - m_D N_D - \mu_D I_D - C_D E_D. \tag{5}$$

Similarly (3) gives

$$\frac{dN_H}{dt} = B_H - m_H N_H - \mu_H I_H. \tag{6}$$

Now, assuming that there are no disease induced death rate and culling effect in the dogs' compartment, it implies that (5) and (6) become

$$\frac{dN_D}{dt} = A_D - m_D N_D,$$
$$\frac{dN_H}{dt} = B_D - m_H N_H. \tag{7}$$

Suppose $dN_D/dt \leq 0, dN_H/dt \leq 0, N_D \leq A_D/m_D$, and $N_H \leq B_H/m_H$, and then imposing the theorem proposed in [32] on differential inequality results in $0 \leq N_D \leq A_D/m_D$ and $0 \leq N_H \leq B_H/m_H$. Therefore (7) becomes

$$\frac{dN_D}{dt} \leq A_D - m_D N_D, \tag{8}$$

$$\frac{dN_H}{dt} \leq B_D - m_H N_H. \tag{9}$$

Solve (8) and (9) using the integrating factor (IF) method. Thus $dy/dt + p(t)y = Q,\ IF = e^{\int p(t)dt}$. After some algebraic manipulation the feasible solution of the dogs' population in model system (1) is in the region

$$\Omega_D = \left\{(S_D, E_D, I_D, R_D) \in \mathbb{R}_+^4,\ N_D \leq \frac{A_D}{m_D}\right\}. \tag{10}$$

Similarly the human population follows suit, and from (9) this implies that the feasible solution of the human population of model system (1) is in the region

$$\Omega_H = \left\{(S_H, E_H, I_H, R_H) \in \mathbb{R}_+^4,\ N_H \leq \frac{B_H}{m_H}\right\}. \tag{11}$$

Therefore, the feasible solutions are contained in Ω. Thus $\Omega = \Omega_D \times \Omega_H$. From the standard comparison theorem used on differential inequality in [33], it implies that

$$\begin{aligned} N_D(t) &\le N_D(0)\, e^{-(m_D)t} + \frac{A_D}{m_D}\left(1 - e^{-(m_D)t}\right), \\ N_H(t) &\le N_H(0)\, e^{-(m_H)t} + \frac{B_H}{m_H}\left(1 - e^{-(m_H)t}\right). \end{aligned} \tag{12}$$

Hence, the total dog population size $N_D(t) \to A_D/m_D$ as $t \to \infty$. Similarly, the total human population size $N_H(t) \to B_H/m_H$ as $t \to \infty$. This means that the infected state variables (E_D, I_D, E_H, I_H) of the two populations tend to zero as time goes to infinity. Therefore, the region Ω is pulling (attracting) all the solutions in $\mathbb{R}^8_+$. This gives the feasible solution set of model system (1) as

$$\begin{pmatrix} S_D \\ E_D \\ I_D \\ R_D \\ S_H \\ E_H \\ I_H \\ R_H \end{pmatrix} \in \mathbb{R}^8_+ \mid \begin{pmatrix} S_D > 0 \\ E_D \ge 0 \\ I_D \ge 0 \\ I_D \ge 0 \\ R_D \ge 0 \\ S_H > 0 \\ E_H \ge 0 \\ I_H \ge 0 \\ R_H \ge 0 \\ N_D \le \dfrac{A_D}{m_D} \\ N_H \le \dfrac{B_H}{m_H} \end{pmatrix}. \tag{13}$$

□

Hence, (1) is mathematically well posed and epidemiologically meaningful.

3.1. Disease-Free Equilibrium $\mathscr{E}_0$. Suppose there is no infection of rabies in both compartments; then ($E_D = 0$, $I_D = 0$, $E_H = 0$, $I_H = 0$). Incorporating this into (1) leads to

$$\begin{aligned} A_D - (m_D + \nu_D) S_D + \alpha_D R_D &= 0, \\ \nu_D S_D - (m_D + \alpha_D) R_D &= 0, \\ B_H - (m_H + \nu_H) S_H + \alpha_H R_H &= 0, \\ \nu_H S_H - (m_H + \alpha_H) R_H &= 0. \end{aligned} \tag{14}$$

After some algebraic manipulation of (14), the disease-free equilibrium point becomes $\mathscr{E}_0 = (S_D^0, E_D^0, I_D^0, R_D^0, S_H^0, E_H^0, I_H^0, R_H^0)$ with

$$\begin{aligned} \mathscr{E}_0 = \Bigg(& \frac{A_D(m_D + \alpha_D)}{m_D(m_D + \alpha_D + \nu_D)}, 0, 0, \\ & \frac{A_D \nu_D}{m_D(m_D + \alpha_D + \nu_D)}, \frac{B_H(m_H + \alpha_H)}{m_H(m_H + \alpha_H + \nu_H)}, 0, 0, \\ & \frac{B_H \nu_H}{m_H(m_H + \alpha_H + \nu_H)} \Bigg). \end{aligned} \tag{15}$$

3.2. Basic Reproduction Number $\mathscr{R}_0$. Here, the basic reproduction number ($\mathscr{R}_0$) measures the average number of new infections produced by one infected dog in a completely susceptible (dog and human) population (see also [34]). Now taking E_D, I_D, E_H, and I_H as our infected compartments gives

$$\begin{aligned} f_1 &= (1 - \nu_D)\beta_{DD} S_D I_D \\ &\quad - \left((1 - \rho_D)\delta\gamma_D + m_D + \rho_D + \delta\varepsilon_D + C_D\right) E_D, \\ f_2 &= (1 - \rho_D)\delta\gamma_D E_D - (m_D + \mu_D) I_D, \\ f_3 &= (1 - \nu_H)\beta_{DH} S_H I_D \\ &\quad - \left((1 - \rho_H)\delta_H\gamma_H + m_H + \rho_H + \delta_H\varepsilon_H\right) E_H, \\ f_4 &= (1 - \rho_H)\delta_H\gamma_H E_H - (m_H + \mu_H) I_H, \end{aligned} \tag{16}$$

where $f_1 = dE_D/dt$, $f_2 = dI_D/dt$, $f_3 = dE_H/dt$, and $f_4 = dI_H/dt$.

Now, using the next generation matrix operator $G = FV^{-1}$ and the Jacobian matrix

$$J = \begin{pmatrix} \dfrac{\partial f_1}{\partial E_D} & \dfrac{\partial f_1}{\partial I_D} & \dfrac{\partial f_1}{\partial E_H} & \dfrac{\partial f_1}{\partial I_H} \\ \dfrac{\partial f_2}{\partial E_D} & \dfrac{\partial f_2}{\partial I_D} & \dfrac{\partial f_2}{\partial E_H} & \dfrac{\partial f_2}{\partial I_H} \\ \dfrac{\partial f_3}{\partial E_D} & \dfrac{\partial f_3}{\partial I_D} & \dfrac{\partial f_3}{\partial E_H} & \dfrac{\partial f_3}{\partial I_H} \\ \dfrac{\partial f_4}{\partial E_D} & \dfrac{\partial f_4}{\partial I_D} & \dfrac{\partial f_4}{\partial E_H} & \dfrac{\partial f_4}{\partial I_H} \end{pmatrix}, \tag{17}$$

as described in [34], results in

$$J = \begin{pmatrix} -\left((1 - \rho_D)\delta\gamma_D + m_D + \rho_D + \delta\varepsilon_D + C_D\right) & (1 - \nu_D)\beta_{DD} S_D & 0 & 0 \\ (1 - \rho_D)\delta\gamma_D & -(m_D + \mu_D) & 0 & 0 \\ 0 & (1 - \nu_H)\beta_{DH} S_H & -\left((1 - \rho_H)\delta_H\gamma_H + m_H + \rho_H + \delta_H\varepsilon_H\right) & 0 \\ 0 & 0 & (1 - \rho_H)\delta_H\gamma_H & -(m_H + \mu_H) \end{pmatrix}. \tag{18}$$

Using the fact that $J = F - V$ gives F and V evaluated at $\mathscr{E}_0$ as

$$F(\mathscr{E}_0) = \begin{pmatrix} 0 & \frac{(1-\nu_D)\beta_{DD}A_D(m_D+\alpha_D)}{m_D(m_D+\nu_D+\alpha_D)} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \frac{(1-\nu_H)\beta_{DH}(m_H+\alpha_H)B_H}{m_H(m_H+\nu_H+\alpha_H)} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$V(\mathscr{E}_0) = \begin{pmatrix} ((1-\rho_D)\delta\gamma_D + m_D + \rho_D + \delta\varepsilon_D + C_D) & 0 & 0 & 0 \\ -(1-\rho_D)\delta\gamma_D & (m_D+\mu_D) & 0 & 0 \\ 0 & 0 & ((1-\rho_H)\delta_H\gamma_H + m_H + \rho_H + \delta_H\varepsilon_H) & 0 \\ 0 & 0 & -(1-\rho_H)\delta_H\gamma_H & (m_H+\mu_H) \end{pmatrix}, \tag{19}$$

where the element in matrix F constitutes the new infection terms, while that of matrix V constitutes the new transfer of infection terms from one compartment to another.

Now, splitting matrix V into four 2×2 submatrices and finding its corresponding inverses result in $G = FV^{-1}$, given by

$$G = \begin{pmatrix} \frac{(1-\rho_D)(1-\nu_D)\delta\gamma_D\beta_{DD}A_D(m_D+\alpha_D)}{((1-\rho_D)\delta\gamma_D + m_D + \rho_D + \delta\varepsilon_D + C_D)(m_D+\mu_D)m_D(m_D+\nu_D+\alpha_D)} & \frac{(1-\nu_D)\beta_{DD}A_D(m_D+\alpha_D)}{m_D(m_D+\nu_D+\alpha_D)} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{(1-\rho_D)\delta\gamma_D(1-\nu_H)\beta_{DH}B_H(m_H+\alpha_H)}{((1-\rho_D)\delta\gamma_D + m_D + \rho_D + \delta\varepsilon_D + C_D)(m_D+\mu_D)m_H(m_H+\nu_H+\alpha_H)} & \frac{(1-\nu_H)\beta_{DH}(m_H+\alpha_H)B_H}{(m_D+\nu_D)m_H(m_H+\nu_H+\alpha_H)} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{20}$$

Letting

$$\begin{aligned} a &= \frac{(1-\rho_D)(1-\nu_D)\delta\gamma_D\beta_{DD}A_D(m_D+\alpha_D)}{((1-\rho_D)\delta\gamma_D + m_D + \rho_D + \delta\varepsilon_D + C_D)(m_D+\mu_D)m_D(m_D+\nu_D+\alpha_D)}, \\ b &= \frac{(1-\nu_D)\beta_{DD}A_D(m_D+\alpha_D)}{m_D(m_D+\nu_D+\alpha_D)}, \\ c &= \frac{(1-\rho_D)(1-\nu_H)\delta\gamma_D\beta_{DH}(m_H+\alpha_H)}{((1-\rho_D)\delta\gamma_D + m_D + \rho_D + \delta\varepsilon_D + C_D)(m_D+\mu_D)m_H(m_H+\nu_H+\alpha_H)}, \\ d &= \frac{(1-\nu_H)\beta_{DH}(m_H+\alpha_H)B_H}{(m_D+\nu_D)m_H(m_H+\nu_H+\alpha_H)} \end{aligned} \tag{21}$$

implies

$$G = \begin{pmatrix} a & b & 0 & 0 \\ 0 & 0 & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{22}$$

Finding the matrix determinant of (22) and denoting it by D give the expression $D = |G - \mathbb{I}\lambda|$, where $\mathbb{I}$ is the identity matrix of a 4×4 matrix; thus

$$D = \begin{vmatrix} a-\lambda & b & 0 & 0 \\ 0 & -\lambda & 0 & 0 \\ c & d & -\lambda & 0 \\ 0 & 0 & 0 & -\lambda \end{vmatrix} = 0. \tag{23}$$

This gives a characteristic equation of the form $\lambda^3(a - \lambda) = 0$; solving the characteristic polynomial results in the following eigenvalues: $\lambda_i = [0, 0, 0, a]$. The basic reproduction number $\mathscr{R}_0$ is the spectral radius (largest eigenvalue) $\rho(FV^{-1})$, also defined as the dominant eigenvalue of FV^{-1}.

Therefore,

$$\mathscr{R}_0 = \frac{(1-\rho_D)(1-\nu_D)\delta\gamma_D\beta_{DD}A_D(m_D+\alpha_D)}{((1-\rho_D)\delta\gamma_D + m_D + \rho_D + \delta\varepsilon_D + C_D)(m_D+\mu_D)m_D(m_D+\nu_D+\alpha_D)}. \tag{24}$$

Remark 2. $\mathscr{R}_0$ contains the secondary infection produced by the infectious compartment of dogs (in the presence of preexposure prophylaxis (vaccination), postexposure prophylaxis (treatment/quarantine), and culling of exposed dogs). When $\mathscr{R}_0 < 1$, the infection gradually leaves the dog compartment, but when $\mathscr{R}_0 > 1$, the rabies virus remains in the dog compartments for a longer time, thereby increasing the rate at which the susceptible dogs and humans get infected by a rabid dog.

3.3. Endemic Equilibrium $\mathscr{E}_1$. The endemic equilibrium is given as

$$\begin{aligned}
S_D^* &= \frac{A_D(m_D+\alpha_D)}{m_D(m_D+\nu_D+\alpha_D)\mathscr{R}_0}, \\
E_D^* &= \frac{(m_D+\mu_D)}{(1-\rho_D)\delta\gamma_D}I_D^*, \\
I_D^* &= \frac{[(1-\rho_D)\delta\gamma_D + m_D + \rho_D + \delta\gamma_D](m_D+\mu_D)m_D(m_D+\nu_D+\alpha_D)(\mathscr{R}_0-1)}{(m_D+\alpha_D)(1-\nu_D)\beta_{DD}[(1-\rho_D)\delta\gamma_D + m_D + C_D] + m_D(1-\nu_D)\beta_{DD}\rho_D}, \\
R_D^* &= \frac{A_D\nu_D(1-\nu_D)\beta_{DD}(1-\rho_D)\delta\gamma_D(m_D+\alpha_D) + (1-\nu_D)\beta_{DD}\rho_D(m_D+\mu_D)I_D^*}{m_D(m_D+\nu_D+\alpha_D)\mathscr{R}_0(1-\nu_D)\beta_{DD}(1-\rho_D)\delta\gamma_D(m_D+\alpha_D)}, \\
S_H^* &= \frac{B_H(m_H+\alpha_H) + [\delta_H\varepsilon_H + \alpha_H\rho_H]E_H^*}{[(1-\nu_H)(m_H+\alpha_H)\beta_{DH}I_D^* + m_H(m_H+\alpha_H+\nu_H)]}, \\
E_H^* &= \frac{(1-\nu_H)B_H(m_H+\alpha_H)\beta_{DH}I_D^*}{(m_H+\alpha_H)[(1-\nu_H)\beta_{DH}I_D^*((1-\rho_D)\delta_H\gamma_H + m_H + \rho_H) + (m_H+\nu_H)((1-\rho_D)\delta_H\gamma_H + m_H + \rho_H + \delta_H\varepsilon_H)] - (1-\nu_H)\beta_{DH}I_D^*\alpha_H\rho_H}, \\
I_H^* &= \frac{(1-\rho_H)\delta_H\gamma_H}{m_H+\mu_H}E_H^*, \\
R_H^* &= \frac{B_H\nu_H(m_H+\nu_H) + [(\nu_H\delta_H\varepsilon_H + \nu_H\alpha_H\rho_H) + \rho_H(1-\nu_H)(m_H+\alpha_H)\beta_{DH}I_D^* + \rho_H m_H(m_H+\alpha_H+\nu_H)]E_H^*}{[(1-\nu_H)(m_H+\alpha_H)^2\beta_{DH}I_D^* + (m_H+\alpha_H)m_H(m_H+\alpha_H+\nu_H)]}.
\end{aligned} \tag{25}$$

Note that if $\mathscr{R}_0 = 1$, it results in the disease-free equilibrium; if $\mathscr{R}_0 > 1$, then there exists a unique endemic equilibrium; if $\mathscr{R}_0 < 1$, then there exist two endemic equilibriums.

3.4. Stability Analysis of $\mathscr{E}_0$. Linearizing (1) at $\mathscr{E}_0$ and subtracting eigenvalue λ along the main diagonal yield

$$\mathbb{J}(\mathscr{E}_0) = \begin{pmatrix} b_1-\lambda & b_7 & a_1 & \alpha_D & 0 & 0 & 0 & 0 \\ 0 & a_2-\lambda & a_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & b_{10} & b_2-\lambda & 0 & 0 & 0 & 0 & 0 \\ \nu_D & \rho_D & 0 & b_3-\lambda & 0 & 0 & 0 & 0 \\ 0 & 0 & a_4 & 0 & b_4-\lambda & b_5 & 0 & \alpha_H \\ 0 & 0 & a_5 & 0 & 0 & a_6-\lambda & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & b_9 & b_6-\lambda & 0 \\ 0 & 0 & 0 & 0 & \nu_H & \rho_H & 0 & b_8-\lambda \end{pmatrix}, \tag{26}$$

where

$$\begin{aligned} a_1 &= \frac{-(1-\nu_D)\beta_{DD}(m_D+\alpha_D)A_D}{m_D(m_D+\nu_D+\alpha_D)}, \\ a_2 &= -((1-\rho_D)\delta\gamma_D + m_D + \rho_D + \delta\varepsilon_D + C_D), \\ a_3 &= \frac{(1-\nu_D)\beta_{DD}(m_D+\alpha_D)A_D}{m_D(m_D+\nu_D+\alpha_D)}, \\ a_4 &= \frac{-(1-\nu_H)\beta_{DH}(m_H+\alpha_H)}{m_H(m_H+\nu_H+\alpha_H)}, \\ a_5 &= \frac{(1-\nu_H)\beta_{DH}(m_H+\alpha_H)}{m_H(m_H+\nu_H+\alpha_H)}, \\ a_6 &= -((1-\rho_H)\delta_H\gamma_H + m_H + \rho_H + \delta_H\varepsilon_H), \\ b_1 &= -(m_D+\nu_D), \\ b_2 &= -(m_D+\mu_D), \\ b_3 &= -(m_D+\alpha_D), \\ b_4 &= -(\nu_H+m_H), \\ b_5 &= \delta_H\varepsilon_H, \\ b_6 &= -(m_H+\mu_H), \\ b_7 &= \delta\varepsilon_D, \\ b_8 &= -(m_H+\alpha_H), \\ b_9 &= (1-\rho_H)\delta_H\gamma_H, \\ b_{10} &= (1-\rho_D)\delta\gamma_D. \end{aligned} \tag{27}$$

Simplifying matrix $\mathbb{J}(\mathscr{E}_0)$ gives

$$\begin{aligned} &(b_6-\lambda)(a_6-\lambda)(b_4-\lambda)(b_8-\lambda) \\ &\cdot\left[\lambda^4 + a_{11}\lambda^3 + a_{12}\lambda^2 + a_{13}\lambda + a_{14}\right] = 0, \end{aligned} \tag{28}$$

where

$$\begin{aligned} a_{11} &= (-b_2 - a_2 - b_1 - b_3), \\ a_{12} &= \nu_D\alpha_D + a_2b_3 + a_2b_1 + b_2b_3 + b_2b_1 + b_3b_1 + b_2a_2 \\ &\quad - (1-\rho_D)\delta\gamma_D a_3, \\ a_{13} &= -a_2\nu_D\alpha_D - b_2\nu_D\alpha_D + a_3(1-\rho_D)\delta\gamma_D b_2 + a_3(1 \\ &\quad - \rho_D)\delta\gamma_D b_1 - a_2b_2b_3 - b_2a_2b_1 - a_2b_3b_1 - b_2b_3b_1, \\ a_{14} &= (b_1b_2b_3a_2 + (1-\rho_D)\delta\gamma_D a_3b_3b_1 \\ &\quad + (1-\rho_D)\delta\gamma_D a_3\nu_D + \nu_D\alpha_D b_2a_2). \end{aligned} \tag{29}$$

From (28) the four characteristic factors that are negative are

$$\begin{aligned} \lambda_1 &= b_6, \\ \lambda_2 &= a_6, \\ \lambda_3 &= b_4, \\ \lambda_4 &= b_8, \end{aligned} \tag{30}$$

where $a_6 = -((1-\rho_H)\delta_H\gamma_H + m_H + \rho_H + \delta_H + \delta_H\varepsilon_H)$, $b_6 = -(m_H+\mu_H)$, $b_4 = -(\nu_H+m_H)$, and $b_8 = -(m_H+\alpha_H)$. The other four characteristic factors can be obtained using the Routh-Hurwitz criterion. Routh-Hurwitz stability criterion is a test to ascertain the nature of the eigenvalues. If the roots of the polynomial are all positive, then the polynomial has a negative real part [35, 36]. The remaining four characteristic eigenvalues are obtained as follows:

$$\lambda^4 + a_{11}\lambda^3 + a_{12}\lambda^2 + a_{13}\lambda + a_{14} = 0. \tag{31}$$

Hence, simplifying the coefficient of the above characteristic polynomial in (31) yields

$$\begin{aligned} a_{11} &= ((1-\rho_D)\delta\gamma_D + m_D + \rho_D + \delta\varepsilon_D + C_D) \\ &\quad + (m_D+\mu_D) + (m_D+\alpha_D) + (m_D+\nu_D), \end{aligned}$$

$$
\begin{aligned}
a_{12} &= \nu_D\alpha_D + \left(\left(1-\rho_D\right)\delta\gamma_D + m_D + \rho_D + \delta\varepsilon_D + C_D\right)\\
&\quad\cdot\left[\left(m_D+\alpha_D\right)+\left(m_D+\nu_D\right)\right]+\left(m_D+\mu_D\right)\\
&\quad\cdot\left[\left(m_D+\alpha_D\right)+\left(m_D+\nu_D\right)\right]+\left(m_D+\alpha_D\right)\\
&\quad\cdot\left(m_D+\nu_D\right)+\left(\left(1-\rho_D\right)\delta\gamma_D + m_D + \rho_D + C_D\right)\\
&\quad\cdot\left(m_D+\mu_D\right)\left(1-\mathscr{R}_0\right),\\
a_{13} &= \left(\left(1-\rho_D\right)\delta\gamma_D + m_D + \rho_D + \delta\varepsilon_D + C_D\right)\nu_D\alpha_D\\
&\quad+\left(m_D+\mu_D\right)\nu_D\alpha_D+\left(m_D+\mu_D\right)\left(m_D+\alpha_D\right)\\
&\quad\cdot\left(m_D+\nu_D\right)\\
&\quad+\left(\left(1-\rho_D\right)\delta\gamma_D + m_D + \rho_D + \delta\varepsilon_D + C_D\right)\\
&\quad\cdot\left(m_D+\alpha_D\right)\left(m_D+\mu_D\right)\left[1-\frac{\mathscr{R}_0\left(m_D+\mu_D\right)}{m_D+\alpha_D}\right]\\
&\quad+\left(\left(1-\rho_D\right)\delta\gamma_D + m_D + \rho_D + \delta\varepsilon_D + C_D\right)\\
&\quad\cdot\left(m_D+\alpha_D\right)\left(m_D+\nu_D\right)\left[1-\frac{\mathscr{R}_0}{m+\alpha}\right],\\
a_{14} &= \nu_D\alpha_D\left(m_D+\nu_D\right)\\
&\quad\cdot\left(\left(1-\rho_D\right)\delta\gamma_D + m_D + \rho_D + \delta\varepsilon_D + C_D\right)\\
&\quad+\frac{\left(1-\rho_D\right)\delta\gamma_D\left(1-\nu_D\right)\beta_{DD}\left(m_D+\alpha_D\right)A_D\nu_D}{m_D\left(m_D+m_D+\nu_D+\alpha_D\right)}\\
&\quad+\left(m_D+\nu_D\right)\left(m_D+\mu_D\right)\left(m_D+\alpha_D\right)\\
&\quad\cdot\left(\left(1-\rho_D\right)\delta_D\gamma_D + m_D + \rho_D + \delta\varepsilon_D + C_D\right)\\
&\quad\cdot\left(1-\mathscr{R}_0\right).
\end{aligned}
\tag{32}
$$

Therefore, from the Routh-Hurwitz criterion of order four, it implies that the conditions, $a_{11} > 0$, $a_{12} > 0$, $a_{13} > 0$, $a_{14} > 0$, and $a_{11}a_{12}a_{13} > a_{13}^2 + a_{11}^2a_{14}$, are satisfied if $\mathscr{R}_0 < 1$. Hence, the disease-free equilibrium $\mathscr{E}_0$ is locally asymptotically stable when $\mathscr{R}_0 < 1$ (see [37]).

3.4.1. Global Stability of $\mathscr{E}_0$

Theorem 3. *The disease-free equilibrium $\mathscr{E}_0$ of model (1) is globally asymptotically stable if $\mathscr{R}_0 \le 1$ and unstable if $\mathscr{R}_0 > 1$.*

Proof. Let $\mathscr{V}$ be a Lyapunov function with positive constants $\mathscr{K}_1$, $\mathscr{K}_2$, $\mathscr{K}_3$, and $\mathscr{K}_4$ such that

$$
\begin{aligned}
\mathscr{V} &= \left(S_D - S_D^0 - S_D^0\ln\frac{S_D}{S_D^0}\right)+\mathscr{K}_1E_D+\mathscr{K}_2I_D\\
&\quad+\left(R_D-R_D^0-R_D^0\ln\frac{R_D}{R_D^0}\right)\\
&\quad+\left(S_H-S_H^0-S_H^0\ln\frac{S_H}{S_H^0}\right)+\mathscr{K}_3E_H+\mathscr{K}_4I_H\\
&\quad+\left(R_H-R_H^0-R_H^0\ln\frac{R_H}{R_H^0}\right).
\end{aligned}
\tag{33}
$$

Taken the derivative of the Lyapunov function with respect to time gives

$$
\begin{aligned}
\frac{d\mathscr{V}}{dt} &= \left(1-\frac{S_D^0}{S_D}\right)\frac{dS_D}{dt}+\mathscr{K}_1\frac{dE_D}{dt}+\mathscr{K}_2\frac{dI_D}{dt}\\
&\quad+\left(1-\frac{R_D^0}{R_D}\right)\frac{dR_D}{dt}+\left(1-\frac{S_H^0}{S_H}\right)\frac{dS_H}{dt}\\
&\quad+\mathscr{K}_3\frac{dE_H}{dt}+\mathscr{K}_4\frac{dI_H}{dt}+\left(1-\frac{R_H^0}{R_H}\right)\frac{dR_H}{dt}.
\end{aligned}
\tag{34}
$$

Plugging (1) into (34) results in

$$
\begin{aligned}
\frac{d\mathscr{V}}{dt} &= \left(1-\frac{S_D^0}{S_D}\right)\left[A_D-\left(1-\nu_D\right)\beta_{DD}S_DI_D\right.\\
&\quad\left.-\left(m_D+\nu_D\right)S_D+\delta\varepsilon_DE_D+\alpha R_D\right]\\
&\quad+\mathscr{K}_1\left[\left(1-\nu_D\right)\beta_{DD}S_DI_D\right.\\
&\quad\left.-\left(\left(1-\rho_D\right)\delta\gamma_D+m_D+\rho_D+\delta\varepsilon_D+C_D\right)E_D\right]\\
&\quad+\mathscr{K}_2\left[\left(1-\rho_D\right)\delta\gamma_DE_D-\left(m_D+\mu_D\right)I_D\right]+\left(1\right.\\
&\quad\left.-\frac{R_D^0}{R_D}\right)\left[\nu_DS_D+\rho_DE_D-\left(m_D+\alpha_D\right)R_D\right]+\left(1\right.\\
&\quad\left.-\frac{S_H^0}{S_H}\right)\left[B_H-\left(1-\nu_H\right)\beta_{DH}S_HI_D-\left(m_H+\nu_H\right)S_H\right.\\
&\quad\left.+\delta_H\varepsilon_HE_H+\alpha_HR_H\right]+\mathscr{K}_3\left[\left(1-\nu_H\right)\beta_{DH}S_HI_D\right.\\
&\quad\left.-\left(\left(1-\rho_H\right)\delta_H\gamma_H+m_H+\rho_H+\delta_H\varepsilon_H\right)E_H\right]\\
&\quad+\mathscr{K}_4\left[\left(1-\rho_H\right)\delta_H\gamma_HE_H-\left(m_H+\mu_H\right)I_H\right]+\left(1\right.\\
&\quad\left.-\frac{R_H^0}{R_H}\right)\left[\nu_HS_H+\rho_HE_H-\left(m_H+\alpha_H\right)R_H\right].
\end{aligned}
\tag{35}
$$

Now, after forming the Lyapunov function $\mathscr{V}$ on the space of the eight state variables, thus $(S_D, E_D, I_D, R_D, S_H, E_H, I_H, R_H)$, and introducing the idea from [37], it is clear that if $E_D(t)$, $I_D(t)$, $E_H(t)$, and $I_H(t)$ at the disease-free equilibrium are globally stable (thus, $E_D = 0$, $I_D = 0$, $E_H = 0$, and $I_H = 0$), then $S_D(t) \to A_D(m_D+\alpha_D)/m_D(m_D+\alpha_D+\nu_D)$, $R_D(t) \to A_D\nu_D/m_D(m_D+\alpha_D+\nu_D)$, $S_H(t) \to B_H(m_H+\alpha_H)/m_H(m_H+\alpha_H+nu_H)$, and $R_H(t) \to B_H\nu_H/m_H(m_H+\alpha_H+H+\nu_H)$ as $t \to \infty$.

Therefore, it can be assumed that

$$\begin{aligned} S_D &\le S_D^0 = \frac{A_D(m_D+\alpha_D)}{m_D(m_D+\alpha_D+\nu_D)}, \\ R_D &\le R_D^0 = \frac{A_D\nu_D}{m_D(m_D+\alpha_D+\nu_D)}, \\ S_H &\le S_H^0 = \frac{B_H(m_H+\alpha_H)}{m_H(m_H+\alpha_H+nu_H)}, \\ R_H &\le R_H^0 = \frac{B_H\nu_H}{m_H(m_H+\alpha_H+H+\nu_H)}, \end{aligned} \tag{36}$$

(see [38]) and replacing it into (35) yields

$$\begin{aligned} \frac{d\mathscr{V}}{dt} \le{}& \mathscr{K}_1\left[\frac{(1-\nu_D)\beta_{DD}A_D(m_D+\alpha_D)}{m_D(m_D+\alpha_D+\nu_D)}I_D\right. \\ &\left.-\left((1-\rho_D)\delta\gamma_D+m_D+\rho_D+\delta\varepsilon_D+C_D\right)E_D\right] \\ &+\mathscr{K}_2\left[(1-\rho_D)\delta\gamma_D E_D-(m_D+\mu_D)I_D\right] \\ &+\mathscr{K}_3\left[\frac{(1-\nu_H)\beta_{DH}B_H(mH+\alpha_H)}{m_H(m_H+\alpha_H+\nu_H)}I_d\right. \\ &\left.-\left((1-\rho_H)\delta_H\gamma_H+m_H+\rho_H+\delta_H\varepsilon_H\right)E_H\right] \\ &+\mathscr{K}_4\left[(1-\rho_H)\delta_H\gamma_H E_H-(m_H+\mu_H)I_H\right], \end{aligned} \tag{37}$$

This implies that

$$\begin{aligned} \frac{d\mathscr{V}}{dt} \le{}& \left[\frac{\mathscr{K}_1(1-\nu_D)\beta_{DD}A_D(m_D+\alpha_D)}{m_D(m_D+\alpha_D+\nu_D)}\right. \\ &-\mathscr{K}_2(m_D+\mu_D) \\ &\left.+\frac{\mathscr{K}_3(1-\nu_H)\beta_{DH}B_H(m_H+\alpha_H)}{m_H(m_H+\alpha_H+\nu_H)}\right]I_D \\ &+\left[\mathscr{K}_2(1-\rho_D)\delta\gamma_D\right. \\ &\left.-\mathscr{K}_1\left((1-\rho_D)\delta\gamma_D+m_D+\rho_D+\delta\varepsilon_D+C_D\right)\right]E_D \\ &+\left[\mathscr{K}_4(1-\rho_H)\delta_H\gamma_H\right. \\ &\left.-\mathscr{K}_3\left((1-\rho_H)\delta_H\gamma_H+m_H+\delta_H\varepsilon_H\right)\right]E_H \\ &-\mathscr{K}_4(m_H+\mu_H). \end{aligned} \tag{38}$$

Equating the coefficient of I_D, E_D, I_H, and E_H in (38) to zero gives

$$\begin{aligned} &\mathscr{K}_4=\mathscr{K}_3=0, \\ &\mathscr{K}_2=\left((1-\rho_D)\delta\gamma_D+m_D+\rho_D+\delta\varepsilon_D+C_D\right), \\ &\mathscr{K}_1=(1-\rho_D)\delta\gamma_D, \end{aligned} \tag{39}$$

and we obtain

$$\begin{aligned} \frac{d\mathscr{V}}{dt} &\le \left((1-\rho_D)\delta\gamma_D+m_D+\rho_D+\delta\varepsilon_D+C_D\right) \\ &\quad\cdot(m_D+\mu_D)(\mathscr{R}_0-1)I_D, \\ &\le 0, \quad \text{if } \mathscr{R}_0\le 1. \end{aligned} \tag{40}$$

Additionally $d\mathscr{V}/dt = 0$ if and only if $I_D = 0$. Therefore, for $E_D = I_D = E_H = I_H = 0$ it shows that $S_D(t) \to A_D(m_D + \alpha_D)/m_D(m_D + \alpha_D + \nu_D)$, $R_D(t) \to A_D\nu_D/m_D(m_D + \alpha_D + \nu_D)$, $S_H(t) \to B_H(m_H + \alpha_H)/m_H(m_H + \alpha_H + \nu_H)$, and $R_H(t) \to B_H\nu_H/m_H(m_H + \alpha_H + \nu_H)$ as $t \to \infty$. Hence, the largest compact invariant set in $\{(S_D, E_D, I_D, R_D, S_H, E_H, I_H, R_H) \in \Omega : d\mathscr{V}/dt \le 0\}$ is the singleton set $\{\mathscr{E}_0\}$. Therefore, from La Salle's invariance principle, we conclude that $\mathscr{E}_0$ is globally asymptotically stable in Ω if $\mathscr{R}_0 \le 1$ (see also [38, 39]). □

3.5. Global Stability of Endemic Equilibrium $\mathscr{E}_1$

Theorem 4. *The endemic equilibrium $\mathscr{E}_1$ of model (1) is globally asymptotically stable whenever $\mathscr{R}_0 > 1$.*

Proof. Suppose $\mathscr{R}_0 > 1$; then the existence of the endemic equilibrium point is assured. Using the common quadratic Lyapunov function

$$V(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n}\frac{c_i}{2}\left(x_i - x_i^*\right)^2, \tag{41}$$

as illustrated in [40], we consider a Lyapunov function with the following candidate:

$$\begin{aligned} \mathscr{V}(S_D, E_D, I_D, R_D, S_H, E_H, I_H, R_H) ={}& \frac{1}{2}\left[(S_D - S_D^*)\right. \\ &\left.+(E_D - E_D^*)+(I_D - I_D^*)+(R_D - R_D^*)\right]^2 \\ &+\frac{1}{2}\left[(S_H - S_H^*)+(E_H - E_H^*)+(I_H - I_H^*)\right. \\ &\left.+(R_H - R_H^*)\right]^2. \end{aligned} \tag{42}$$

Now, differentiating (42) along the solution curve of (1) gives

$$\begin{aligned} \frac{d\mathscr{V}}{dt} ={}& \left[(S_D - S_D^*)+(E_D - E_D^*)+(I_D - I_D^*)\right. \\ &\left.+(R_D - R_D^*)\right]\frac{d(S_D+E_D+I_D+R_D)}{dt} \\ &+\left[(S_H - S_H^*)+(E_H - E_H^*)+(I_H - I_H^*)\right. \\ &\left.+(R_H - R_H^*)\right]\frac{d(S_H+E_H+I_H+R_H)}{dt}. \end{aligned} \tag{43}$$

From (1) it implies that $d(S_D + E_D + I_D + R_D)/dt = A_D - m_D(S_D + E_D + I_D + R_D) - C_D E_D - \mu_D I_D$ and $d(S_H + E_H + I_H + R_H)/dt = B - m(S_H + E_H + I_H + R_H) - \mu_H I_H$, which when plugged into (43) gives

$$\begin{aligned}\frac{d\mathscr{V}}{dt} &= \left[\left(S_D - S_D^*\right) + \left(E_D - E_D^*\right) + \left(I_D - I_D^*\right)\right. \\ &\quad \left. + \left(R_D - R_D^*\right)\right]\left(A_D - m_D\left(S_D + E_D + I_D + R_D\right)\right. \\ &\quad \left. - C_D E_D - \mu_D I_D\right) + \left[\left(S_H - S_H^*\right) + \left(E_H - E_H^*\right)\right. \\ &\quad \left. + \left(I_H - I_H^*\right) + \left(R_H - R_H^*\right)\right]\left(B_H\right. \\ &\quad \left. - m\left(S_H + E_H + I_H + R_H\right) - \mu_H I_H\right).\end{aligned} \tag{44}$$

Now assuming

$$\begin{aligned}A_D &= m_D\left(S_D^* + E_D^* + I_D^* + R_D^*\right) + C_D E_D^* + \mu_D I_D^*, \\ B_H &= m_H\left(S_H^* + E_H^* + I_H^* + R_H^*\right) + \mu_H I_H^*\end{aligned} \tag{45}$$

and substituting it into (44), we have

$$\begin{aligned}\frac{d\mathscr{V}}{dt} &= \left[\left(S_D - S_D^*\right) + \left(E_D - E_D^*\right) + \left(I_D - I_D^*\right) + \left(R_D\right.\right. \\ &\quad \left.\left. - R_D^*\right)\right]\left[m_D\left(S_D^* + E_D^* + I_D^* + R_D^*\right) + C_D E_D^* + \mu_D I_D^*\right. \\ &\quad \left. - m_D\left(S_D + E_D + I_D + R_D\right) - C_D E_D - \mu_D I_D\right] \\ &\quad + \left[\left(S_H - S_H^*\right) + \left(E_H - E_H^*\right) + \left(I_H - I_H^*\right) + \left(R_H\right.\right. \\ &\quad \left.\left. - R_H^*\right)\right]\left[m_H\left(S_H^* + E_H^* + I_H^* + R_H^*\right) + \mu_H I_H^*\right. \\ &\quad \left. - m\left(S_H + E_H + I_H + R_H\right) - \mu_H I_H\right], \\ \frac{d\mathscr{V}}{dt} &= \left[\left(S_D - S_D^*\right) + \left(E_D - E_D^*\right) + \left(I_D - I_D^*\right) + \left(R_D\right.\right. \\ &\quad \left.\left. - R_D^*\right)\right]\left[\left(-m_D\left(S_D - S_D^*\right) - m_D\left(E_D - E_D^*\right)\right.\right. \\ &\quad \left. - m_D\left(I_D - I_D^*\right) - m_D\left(R_D - R_D^*\right) - C_D\left(E_D - E_D^*\right)\right. \\ &\quad \left.\left. - \mu_D\left(I_D - I_D^*\right)\right)\right] + \left[\left(S_H - S_H^*\right) + \left(E_H - E_H^*\right)\right. \\ &\quad \left. + \left(I_H - I_H^*\right) + \left(R_H - R_H^*\right)\right]\left[\left(-m_H\left(S_H - S_H^*\right)\right.\right. \\ &\quad \left. - m_H\left(E_H - E_H^*\right) - m_H\left(I_H - I_H^*\right)\right. \\ &\quad \left.\left. - m_H\left(R_H - R_H^*\right) - \mu_H\left(I_H - I_H^*\right)\right)\right].\end{aligned} \tag{46}$$

This also implies that

$$\begin{aligned}\frac{d\mathscr{V}}{dt} &= -m_D\left(S_D - S_D^*\right)^2 - \left(C_D + m_D\right)\left(E_D - E_D^*\right)^2 \\ &\quad - \left(m_D + \mu_D\right)\left(I_D - I_D^*\right)^2 - m_D\left(R_D - R_D^*\right)^2 - \left(2m_D\right. \\ &\quad \left. + C_D\right)\left(S_D - S_D^*\right)\left(E_D - E_D^*\right) - \left(2m_D + \mu_D\right)\left(S_D\right. \\ &\quad \left. - S_D^*\right)\left(I_D - I_D^*\right) - \left(2m_D + \mu_D + C_D\right)\left(E_D - E_D^*\right)\left(I_D\right. \\ &\quad \left. - I_D^*\right) - 2m_D\left(R_D - R_D^*\right)\left(I_D - I_D^*\right) - \left(2m_D + \mu_D\right. \\ &\quad \left. + C_D\right)\left(R_D - R_D^*\right)\left(I_D - I_D^*\right) - m_H\left(S_H - S_H^*\right)^2 \\ &\quad - m_H\left(E_H - E_H^*\right)^2 - \left(m_H - \mu_H\right)\left(I_H - I_H^*\right)^2 \\ &\quad - m_H\left(R_H - R_H^*\right)^2 - 2m_H\left(S_H - S_H^*\right)\left(E_H - E_H^*\right) \\ &\quad - \left(2m_H - \mu_H\right)\left(S_H - S_H^*\right)\left(I_H - I_H^*\right) - \left(2m_H + \mu_H\right) \\ &\quad \cdot \left(E_H - E_H^*\right)\left(I_H - I_H^*\right) \\ &\quad - m_H\left[\left(I_H - I_H^*\right)\left(R_H - R_H^*\right)\right. \\ &\quad \left. + \left(S_H - S_H^*\right)\left(R_H - R_H^*\right)\right].\end{aligned} \tag{47}$$

This shows that $d\mathscr{V}/dt$ is negative and $d\mathscr{V}/dt = 0$, if and only if $S_D = S_D^*$, $E_D = E_D^*$, $I_D = I_D^*$, $R_D = R_D^*$, $S_H = S_H^*$, $E_H = E_H^*$, $I_H = I_H^*$, $R_H = R_H^*$. Additionally every solution of (1) with the initial conditions approaches $\mathscr{E}_1$ as $t \to \infty$ (see [38, 39]); therefore, the largest compact invariant set in $\{(S_D, E_D, I_D, R_D, S_H, E_H, I_H, R_H) \in \Omega : d\mathscr{V}/dt \leq 0\}$ is the singleton set $\{\mathscr{E}_1\}$. Therefore, from Lasalle's invariant principle [41], it implies that the endemic equilibrium $\mathscr{E}_1$ is globally asymptotically stable in Ω whenever $\mathscr{R}_0 > 1$. □

4. Numerical Analysis

Considering the parameter values in Table 1, we will ascertain the numerical importance of our analysis.

4.1. Different Scenarios of the Basic Reproduction Number $\mathscr{R}_0$. We shall denote $\mathscr{R}_0$ without pre- and postexposure prophylaxis (treatment) as $\mathscr{R}_0^*$ and $\mathscr{R}_0$ without preexposure prophylaxis and culling as $\mathscr{R}_0^{**}$ and the $\mathscr{R}_0$ without postexposure prophylaxis (treatment) and culling as $\mathscr{R}_0^{***}$. Therefore, using the parameter values in Table 1, $\mathscr{R}_0^*$, $\mathscr{R}_0^{**}$, and $\mathscr{R}_0^{***}$ are given as follows:

$$\begin{aligned}\mathscr{R}_0^* &= \frac{\beta_{DD} A_D \delta \gamma_D}{\left(\delta\gamma_D + m_D + \delta\varepsilon_D + C_D\right) \times \left(m_D + \mu_D\right) m_D}, \\ \mathscr{R}_0^* &= 3.027, \\ \mathscr{R}_0^{**} &= \frac{\left(1 - \rho_D\right) \delta \gamma_D \beta_{DD} A_D}{\left(\left(1 - \rho_D\right) \delta\gamma_D + m_D + \rho_D + \delta\varepsilon_D\right)\left(m_D + \mu_D\right) m_D}, \\ \mathscr{R}_0^{**} &= 2.181, \\ \mathscr{R}_0^{***} &= \frac{\left(1 - \nu_D\right) \delta \gamma_D \beta_{DD} A_D \left(m_D + \alpha_D\right)}{\left(\delta\gamma_D + m_D + \delta\varepsilon_D\right)\left(m_D + \mu_D\right) m_D \left(m_D + \nu_D + \alpha_D\right)}, \\ \mathscr{R}_0^{***} &= 1.914.\end{aligned} \tag{48}$$

Therefore, from the above calculations it indicates that the best way in reducing or minimizing the rabies virus in the dogs compartment is to use more of preexposure prophylaxis (vaccination).

Table 1: Parameter values.

Parameter	Description	Standard value	Source
A_D	Recruitment rate of dogs	$3 \times 10^6 y^{-1}$	[27]
α_D	Loss of immunity in dogs	$1y^{-1}$	[27]
C_D	Death rate of dogs due to culling	$0.3y^{-1}$	Assumed
m_D	Natural death rate of dogs	$0.056y^{-1}$	[27]
μ_D	Disease induced mortality in dogs	$1y^{-1}$	[27]
ν_D	Preexposure prophylaxis for dogs	$0.25y^{-1}$	Assumed
ρ_D	Postexposure prophylaxis for dogs	$0.2y^{-1}$	[27]
β_{DD}	Transmission rate in dogs	$1.58 \times 10^{-7} y^{-1}$	[27]
γ_D	Latency period in dogs	$(2.37/6)\, y^{-1}$	[27]
$\delta\varepsilon_D$	Rate of no clinical rabies	$0.4y^{-1}$	[27]
B_H	Birth rate (humans)	$0.0314y^{-1}$	[31]
β_{DH}	Transmission rate (dog-humans)	$2.29 \times 10^{-12} y^{-1}$	[27]
α_H	Loss of immunity (humans)	$1y^{-1}$	[27]
m_H	Natural death rate (humans)	$0.0074y^{-1}$	[31]
μ_H	Disease induced mortality (humans)	$1y^{-1}$	[27]
ν_H	Preexposure prophylaxis for humans	$0.54y^{-1}$	Assumed
ρ_H	Postexposure prophylaxis for humans	$0.1y^{-1}$	[27]
γ_H	Latency rate (humans)	$(1/6)\, y^{-1}$	[27]
$\gamma_H\varepsilon_H$	Rate of no clinical rabies (humans)	$2.4y^{-1}$	[27]

4.1.1. Herd Immunity Threshold H_1. Therefore, from the above numerical values, we are motivated to know the number of humans or dogs that should be vaccinated when $\mathcal{R}_0^* = 3.027$.

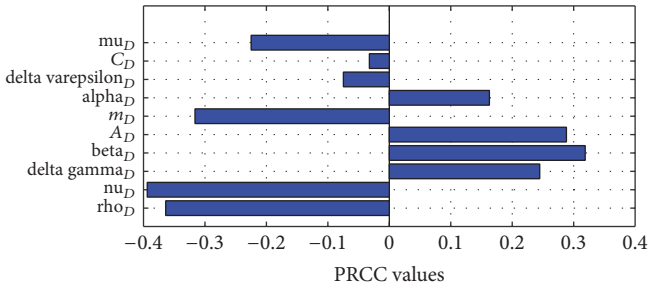
$$H_1 := 1 - \frac{1}{\mathcal{R}_0^*} = 0.66. \tag{49}$$

This shows that if $\mathcal{R}_0^* = 3.027$, then 66% of individuals and dogs should receive vaccination.

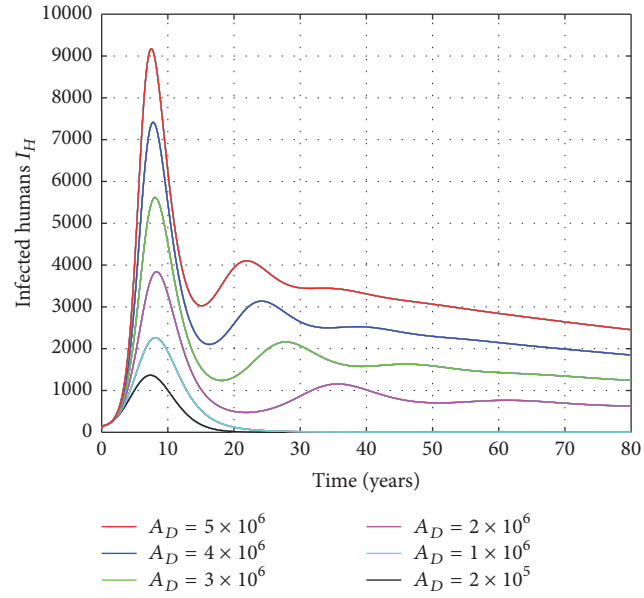
4.2. Sensitivity Analysis. To determine parameters that contribute most to the rabies transmission, we used two sensitivity analysis approach: the normalised forward sensitivity index as presented in [37] and the Latin hypercube sampling as described in [42]. To determine the dependence of parameters in $\mathcal{R}_0$, using a sampling size, $n = 1000$, the partial rank correction coefficients (PRCC) value of the ten parameters in $\mathcal{R}_0$ are shown in Figure 2(a). The longer the bar in Figure 2(a) suggests that the statistical influence of those parameters to changes in $\mathcal{R}_0$ is high. Also, using the normalised forward sensitivity index gives the following values and the nature of their signs in Table 2, based on the parameter value given in Table 1. The plus sign or minus sign signifies that the influence is positive or negative, respectively [42],

$$\begin{aligned}
\Gamma_{\mathcal{R}_0}^{\beta_{DD}} &= \frac{\partial \mathcal{R}_0}{\partial \beta_D} \frac{\beta_{DD}}{\mathcal{R}_0} = 1,\\
\Gamma_{\mathcal{R}_0}^{A_D} &= \frac{\partial \mathcal{R}_0}{\partial A_D} \frac{A_D}{\mathcal{R}_0} = 1,\\
\Gamma_{\mathcal{R}_0}^{\mu_D} &= \frac{\partial \mathcal{R}_0}{\partial \mu_D} \frac{\mu_D}{\mathcal{R}_0} = \frac{-\mu_D}{(m_D + \mu_D)} = -0.95,\\
\Gamma_{\mathcal{R}_0}^{\delta\varepsilon_D} &= \frac{\partial \mathcal{R}_0}{\partial \delta\varepsilon_D} \frac{\delta\varepsilon_D}{\mathcal{R}_0}\\
&= \frac{\delta\varepsilon_D}{((1-\rho_D)\,\delta\gamma_D - \delta\varepsilon_D - C_D - m_D - \rho_D)}\\
&= -1.61,\\
\Gamma_{\mathcal{R}_0}^{C_D} &= \frac{\partial \mathcal{R}_0}{\partial C_D} \frac{C_D}{\mathcal{R}_0}\\
&= \frac{C_D}{((1-\rho_D)\,\delta\gamma_D - \delta\varepsilon_D - C_D - m_D - \rho_D)}\\
&= -0.45,\\
\Gamma_{\mathcal{R}_0}^{\alpha_D} &= \frac{\partial \mathcal{R}_0}{\partial \alpha_D} \frac{\alpha_D}{\mathcal{R}_0} = 0.28,\\
\Gamma_{\mathcal{R}_0}^{m_D} &= \frac{\partial \mathcal{R}_0}{\partial m_D} \frac{m_D}{\mathcal{R}_0} = -1.64,\\
\Gamma_{\mathcal{R}_0}^{\delta\gamma_D} &= \frac{\partial \mathcal{R}_0}{\partial \delta\gamma_D} \frac{\delta\gamma_D}{\mathcal{R}_0} = 1.33,\\
\Gamma_{\mathcal{R}_0}^{\rho_D} &= \frac{\partial \mathcal{R}_0}{\partial \rho_D} \frac{\rho_D}{\mathcal{R}_0} = -0.5,\\
\Gamma_{\mathcal{R}_0}^{\nu_D} &= \frac{\partial \mathcal{R}_0}{\partial \nu_D} \frac{\nu_D}{\mathcal{R}_0} = -0.52.
\end{aligned} \tag{50}$$

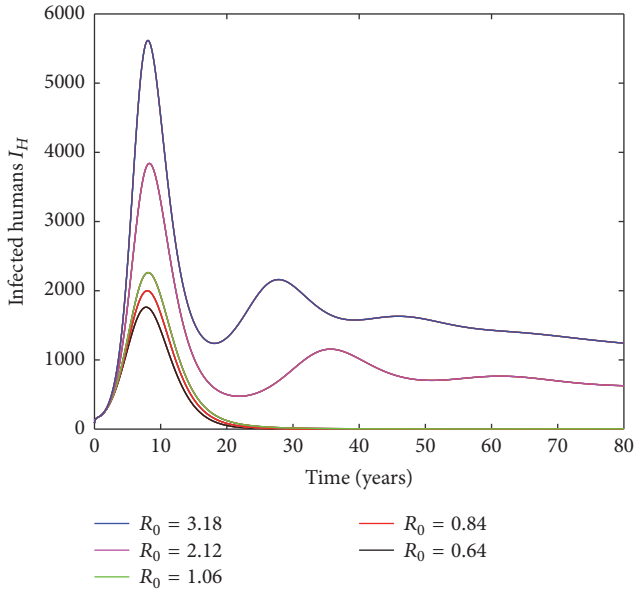
Therefore, from Table 2 it shows that an addition or a reduction in the values of β_{DD}, α_D, $\delta\gamma_D$, and A_D will have an increase or a decrease in the spread of the rabies virus. For



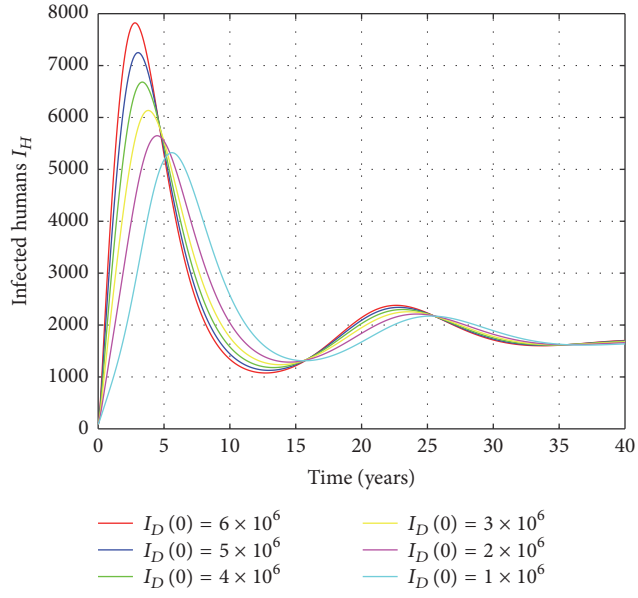
(a) LHS plot for the parameters in $\mathscr{R}_0$



(b) Effect of varying recruitment rate on the infected humans



(c) Effect of an increase in $\mathscr{R}_0$ on the infected humans



(d) Effect of varying the initial infected dog population size on the infected humans

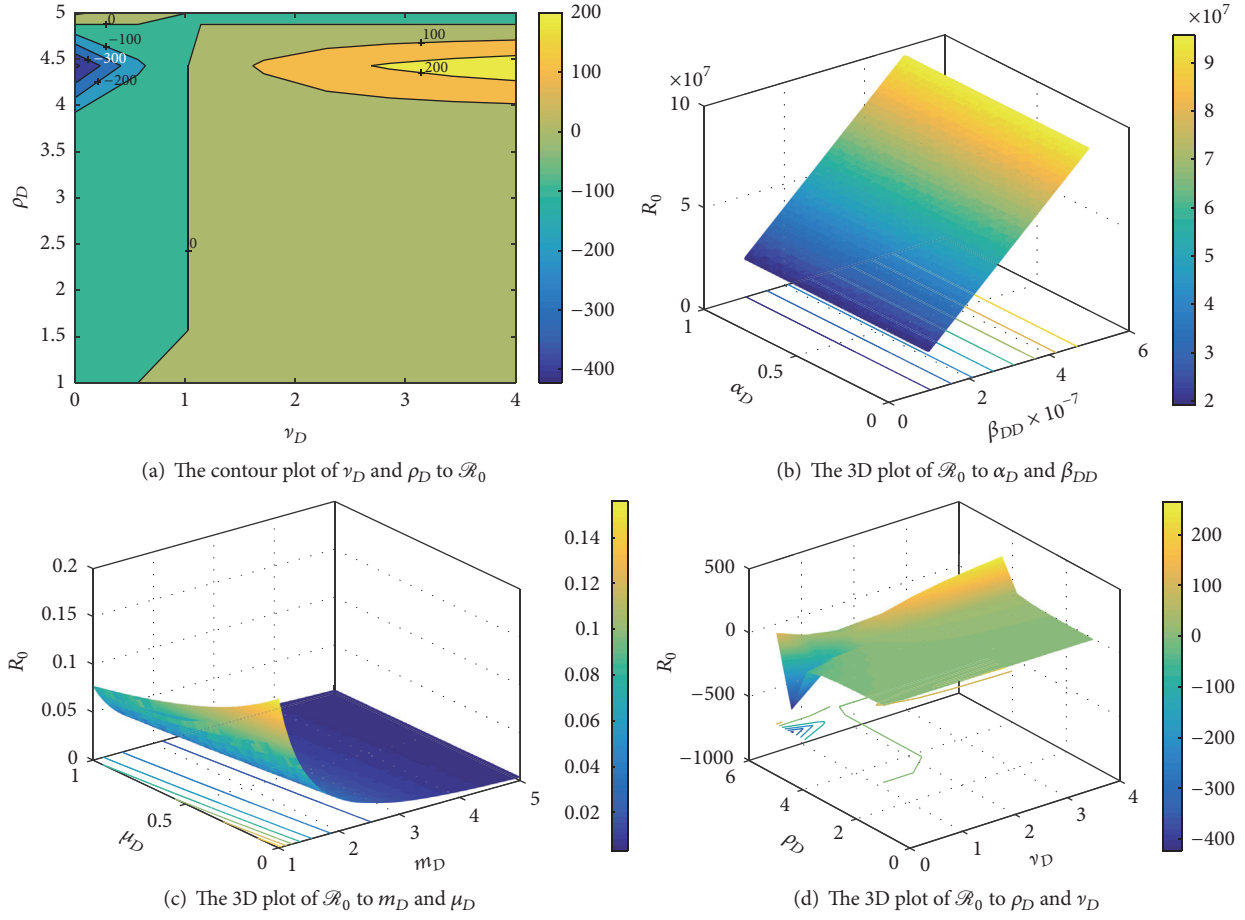
FIGURE 2: The graphical representation of some parameters in $\mathscr{R}_0$ and the effect of varying some initial state values on the model.

example, $\Gamma_{\mathscr{R}_0}^{\beta_D} = 1$ indicates that increasing or reducing the transmission rate by 5% may increase or reduce the number of secondary infection by 5%. The negative sign in Table 2 will have a reduction in the basic reproduction number, $\mathscr{R}_0$, when the values of those parameters are increased, and a reduction in the values of ρ_D, ν_D, μ_D, m_D, and $\delta\varepsilon_D$ will lead to an increase in the number of secondary infections.

The Latin hypercube sampling (LHS) in Figure 2(a) shows that μ_D, C_D, α_D, and $\delta\gamma_D$ have a minimal influence on the rate at which the rabies virus is spread. The Latin hypercube sampling (LHS) plots for the ten parameters in $\mathscr{R}_0$ show that culling of exposed dogs does not actually minimize the spread of rabies as compared to vaccination of susceptible dogs. Figure 2(a) also shows that the most influential parameter in spreading the infection is β_{DD} followed by A_D. Figure 2(c) shows that an increase in the basic reproduction number will contribute to a high level of secondary infection in the human population. Similarly, Figure 2(a) shows that vaccination of dogs ν_D is the most effective way of controlling the rabies virus in the dog population as compared to the treatment/quarantine of exposed dogs, ρ_D. Figure 3(a) gives the contour nature of ν_D and ρ_D, which shows a more saturated effect on the basic reproduction number. Figure 3(b) shows that β_{DD} and α_D have a positive relation with the basic

Table 2: Sensitivity signs of $\mathcal{R}_0$ to the parameters in (24).

Parameter	Description	Sensitivity sign
β_{DD}	Transmission rate of dogs	+ve
A_D	Recruitment rate of dogs	+ve
μ_D	Disease induce death rate of dogs	−ve
$\delta\varepsilon_D$	Rate of no clinical rabies	−ve
C_D	Culling of exposed dogs	−ve
α_D	Loss of immunity in dogs	+ve
m_D	Natural death rate of dogs	−ve
$\delta\gamma_D$	Rate at which exposed dogs become infective (infective rate)	+ve
ρ_D	Postexposure prophylaxis (treatment/quarantined)	−ve
ν_D	Preexposure prophylaxis (vaccination)	−ve



(a) The contour plot of ν_D and ρ_D to $\mathcal{R}_0$

(b) The 3D plot of $\mathcal{R}_0$ to α_D and β_{DD}

(c) The 3D plot of $\mathcal{R}_0$ to m_D and μ_D

(d) The 3D plot of $\mathcal{R}_0$ to ρ_D and ν_D

Figure 3: The graphical representation of some parameters in $\mathcal{R}_0$.

reproduction number $\mathcal{R}_0$. Therefore, an increase in β_{DD} and α_D will have a direct increase in the spread of the rabies virus. Figure 2(b) indicates that with a high number of recruitment of dogs into the susceptible dog's compartment will have a corresponding high increase in the number of infected humans. Figure 2(d) demonstrates that a high number of infected dogs in the compartment will lead to an increase in the number of infected humans. Figure 3(c) shows that a high increase in the number of disease induce death rate and natural death rate will have a negative reflection on $\mathcal{R}_0$; biologically, we would not recommend this approach in minimizing the spread of the disease, since an increase in both μ_D and m_D may result in a high rate of the disease in the human population, even though μ_D and m_D naturally

reduce the number of susceptible and infected dogs in the population. Finally, Figure 3(d) shows the 3D plot of Figure 3(a).

5. Objective Functional

Given that $y(t) \in Y \in \mathbb{R}^n$ is a state variable of model system (1) and $u(t) \in U \in \mathbb{R}^n$ are the control variables at any time (t) with $t_{(0)} \leq t \leq t_{(f)}$, then an optimal control problem consists of finding a piecewise continuous control $u(t)$ and its corresponding state $y(t)$. This optimizes the cost functional $J[y(t), u(t)]$ using Pontryagin's maximum principle [43]. Therefore we set the following likelihood control strategies:

(1) $u_1 = \nu_D$ is the control effort aimed at increasing the immunity of susceptible dogs (preexposed prophylaxis).

(2) $u_2 = \rho_D$ is the control effort aimed at treating the exposed dogs (postexposed prophylaxis).

(3) $u_3 = \nu_H$ is the control effort aimed at increasing the immunity of susceptible humans (preexposure prophylaxis).

(4) $u_4 = \rho_H$ is the control effort aimed at treating the exposed humans (postexposed prophylaxis).

Our goal is to seek optimal controls such as ν_D^*, ρ_D^*, ν_H^*, and ρ_H^* that minimize the objective functional:

$$J = \min \int_{t_0}^{t_f} \left[A_1 E_D + A_2 E_H + A_3 I_D + A_4 I_H + \frac{B1}{2}\nu_D^2 + \frac{B2}{2}\rho_D^2 + \frac{B3}{2}\nu_H^2 + \frac{B4}{2}\rho_H^2 \right] dt. \tag{51}$$

Therefore, (51) is subject to

$$\begin{aligned}
\frac{dS_d}{dt} &= A_D - (1-\nu_D)\beta_{DD} S_D I_D - (m_D + \nu_D) S_D + \delta\varepsilon_D E_D + \alpha_D R_D,\\
\frac{dE_d}{dt} &= (1-\nu_D)\beta_{DD} S_D I_D - ((1-\rho_D)\delta\gamma_D + m_D + \rho_D + \delta\varepsilon_D + C_D) E_D,\\
\frac{dI_D}{dt} &= (1-\rho_D)\delta\gamma_D E_D - (m_D + \mu_D) I_D,\\
\frac{dR_D}{dt} &= \nu_D S_D + \rho_D E_D - (m_D + \alpha_D) R_D,\\
\frac{dS_H}{dt} &= B_H - (1-\nu_H)\beta_{DH} S_H I_d - (m_H + \nu_H) S_H + \delta_H\varepsilon_H E_H + \alpha_H R_H,\\
\frac{dE_H}{dt} &= (1-\nu_H)\beta_{DH} S_H I_d - ((1-\rho_H)\delta_H\gamma_H + m_H + \rho_H + \delta_H\varepsilon_H) E_H,\\
\frac{dI_H}{dt} &= (1-\rho_H)\delta_H\gamma_H E_H - (m_H + \mu_H) I_H,\\
\frac{dR_H}{dt} &= \nu_H S_H + \rho_H E_H - (m_H + \alpha_H) R_H,\\
S_D &> 0,\ E_D \geq 0,\ I_D \geq 0,\ R_D \geq 0,\ S_H > 0,\ E_H \geq 0,\ I_H \geq 0,\ R_H \geq 0.
\end{aligned} \tag{52}$$

From (51) the quantities A_1 and A_2 denote the weight constants of the exposed classes and A_3 and A_4 are the weight of the infectious classes, respectively. $B1, B2, B3, B4$ are the weight constants for the dog and human controls. $B1\nu_D^2$, $B2\rho_D^2$, $B3\nu_H^2$, $B4\rho_H^2$ describe the cost associated with rabies vaccination and treatment. The square of the control variables shows the severity of the side effects of the vaccination and treatment. Employing Pontryagin's maximum principle, we form the Hamiltonian equation with state variables $S_D = S_D^*$, $E_D = E_D^*$, $I_D = I_D^*$, R_D^* and $S_H = S_H^*$, $E_H = E_H^*$, $I_H = I_H^*$, R_H^* as

$$\begin{aligned}
H = {} & A_1 E_D^* + A_2 E_H^* + A_3 I_D^* + A_4 I_H^* + \frac{B1}{2}\nu_D^2 + \frac{B2}{2}\\
& \cdot \rho_D^2 + \frac{B3}{2}\nu_H^2 + \frac{B4}{2}\rho_H^2 + \lambda_1 [A_D\\
& - (1-\nu_D)\beta_{DD} S_D^* I_D^* - (m_D + \nu_D) S_D^* + \delta\varepsilon E_D^*
\end{aligned}$$

$$
\begin{aligned}
&+ \alpha_D R_D^*] + \lambda_2 [(1-\nu_D)\beta_{DD} S_D^* I_D^* \\
&- ((1-\rho_D)\delta\gamma_D + m_D + \rho_D + \delta\varepsilon_D + C_D) E_D^*] \\
&+ \lambda_3 [(1-\rho_D)\delta\gamma_D E_D^* - (m_D + \mu_D) I_D^*] \\
&+ \lambda_4 [\nu_D S_D^* + \rho_D E_D^* - (m_D + \alpha_D) R_D^*] + \lambda_5 [B_H \\
&- (1-\nu_H)\beta_{DH} S_H^* I_D^* - (m_H + \nu_H) S_H^* + \delta_H \varepsilon_H E_H^* \\
&+ \alpha_H R_H^*] + \lambda_6 [(1-\nu_H)\beta_{DH} S_H^* I_D^* \\
&- ((1-\rho_H)\delta_H\gamma_H + m_H + \rho_H + \delta_H\varepsilon_H) E_H^*] \\
&+ \lambda_7 [(1-\rho_H)\delta_H\gamma_H E_H^* - (m_H + \mu_H) I_H^*] \\
&+ \lambda_8 [\nu_H S_H^* + \rho_H E_H^* - (m_H + \alpha_H) R_H^*].
\end{aligned} \tag{53}
$$

Considering the existence of adjoint functions λ_i, $i = 1, 2, \ldots, 8$, satisfying

$$
\begin{aligned}
\frac{d\lambda_1}{dt} &= -\frac{\partial H}{\partial S_D^*} \\
&= \lambda_1 ((1-\nu_D)\beta_{DD} I_D^* + m_D + \nu_D) \\
&\quad - \lambda_2 (1-\nu_D)\beta_{DD} I_D^* - \lambda_4 \nu_D, \\
\frac{d\lambda_2}{dt} &= -\frac{\partial H}{\partial E_D^*} \\
&= \lambda_2 ((1-\rho_D)\delta\gamma_D + m_D + \rho_D + \delta\varepsilon_D + C_D) \\
&\quad - \lambda_1 \delta\varepsilon_D - \lambda_3 (1-\rho_D)\delta\gamma_D - \lambda_4 \rho_D - A_1, \\
\frac{d\lambda_3}{dt} &= -\frac{\partial H}{\partial I_D^*} \\
&= \lambda_3 (m_D + \mu_D) + \lambda_1 (1-\nu_D)\beta_{DD} S_D^* \\
&\quad + \lambda_5 (1-\nu_H)\beta_{DH} S_H^* - \lambda_2 (1-\nu_D)\beta_D S_D^* \\
&\quad - \lambda_6 (1-\nu_H)\beta_{DH} S_H^* - A_3, \\
\frac{d\lambda_4}{dt} &= -\frac{\partial H}{\partial R_D^*} = \lambda_4 (m_D + \alpha_D) - \lambda_1 \alpha_D, \\
\frac{d\lambda_5}{dt} &= -\frac{\partial H}{\partial S_H^*} \\
&= \lambda_5 ((1-\nu_H)\beta_{dH} I_D^* + m_H + \nu_H) \\
&\quad - \lambda_6 (1-\nu_H)\beta_{DH} I_D^* - \lambda_8 \nu_H, \\
\frac{d\lambda_6}{dt} &= -\frac{\partial H}{\partial E_H^*} \\
&= \lambda_6 ((1-\rho_H)\delta_H\gamma_H + m_H + \rho_H + \delta_H\varepsilon_H) \\
&\quad - \lambda_5 \delta_H \varepsilon_H - \lambda_7 (1-\rho_H)\delta_H\gamma_H - \lambda_8 \rho_H - A_2, \\
\frac{d\lambda_7}{dt} &= -\frac{\partial H}{\partial I_H^*} = \lambda_7 (m_H + \mu_H) - A_4, \\
\frac{d\lambda_8}{dt} &= -\frac{\partial H}{\partial R_H^*} = \lambda_8 (m_H + \alpha_H) - \lambda_5 \alpha_H,
\end{aligned} \tag{54}
$$

with transversality condition $\lambda_i(t_f) = 0$ for $i = 1, \ldots, 8$ for the control set u_i, hence we have

$$
\begin{aligned}
\frac{\partial H}{\partial u_i} &= 0, \quad \text{where } i = 1, 2, 3, 4, \\
\left.\frac{\partial H}{\partial \nu_D}\right|_{\nu_D = \nu_D^*} &:= B1\nu_D^* - \lambda_1 S_D^* + \lambda_4 S_D^* + \lambda_1 \beta_{DD} S_D^* I_D^* \\
&\quad - \lambda_2 \beta_D S_D^* I_D^* = 0, \\
\nu_D^* &= \frac{(\lambda_1 S_D^* - \lambda_4 S_D^*) + (\lambda_2 - \lambda_1)\beta_{DD} I_D^* S_D^*}{B1}, \\
\left.\frac{\partial H}{\partial \rho_D}\right|_{\rho_D = \rho_D^*} &:= B2\rho_D^* - \lambda_1 E_D^* + \lambda_4 E_D^* + \lambda_2 E_D \delta\gamma_D E_D^* \\
&\quad - \lambda_3 \delta\gamma_D E_D^* = 0, \\
\rho_D^* &= \frac{(\lambda_2 E_D^* - \lambda_4 E_D^*) + (\lambda_3 - \lambda_2)\delta\gamma_D E_D^*}{B2}, \\
\left.\frac{\partial H}{\partial \nu_H}\right|_{\nu_H = \nu_H^*} &:= B3\nu_H^* - \lambda_5 S_H + \lambda_8 S_H + \lambda_5 \beta_{DH} S_H I_D \\
&\quad - \lambda_6 \beta_{DH} S_H^* I_D^* = 0, \\
\nu_H^* &= \frac{(\lambda_5 S_H^* - \lambda_8 S_H^*) + (\lambda_6 - \lambda_5)\beta_{DH} S_H^* I_D^*}{B3}, \\
\left.\frac{\partial H}{\partial \rho_H}\right|_{\rho_H = \rho_H^*} &:= B4\rho^* - \lambda_6 E_H^* + \lambda_8 E_H^* + \lambda_6 \delta_H \gamma_H E_H^* \\
&\quad - \lambda_7 \delta_H \gamma_H E_H^* = 0, \\
\rho_H^* &= \frac{(\lambda_6 E_H^* - \lambda_8 E_H^*) + (\lambda_7 - \lambda_6)\delta_H\gamma_H E_H^*}{B4}.
\end{aligned} \tag{55}
$$

Now, using an appropriate variation argument and taking the bounds into account, the optimal control strategies are given as

$$
\nu_D^* = \min \left\{ \max \left(0, \frac{(\lambda_1 - \lambda_4) S_D^* + (\lambda_2 - \lambda_1)\beta_{DD} I_D^* S_D^*}{B1} \right), \nu_{D\max} \right\}, \tag{56}
$$

$$
\rho_D^* = \min \left\{ \max \left(0, \frac{(\lambda_2 - \lambda_4) E_D^* + (\lambda_3 - \lambda_2)\delta\gamma_D E_D^*}{B2} \right), \rho_{D\max} \right\}, \tag{57}
$$

$$\nu_H^* = \min\left\{\max\left(0, \frac{(\lambda_5-\lambda_8)S_H^* + (\lambda_6-\lambda_5)\beta_{DH}S_H^*I_D^*}{B3}\right), \nu_{H\max}\right\}, \tag{58}$$

$$\rho_H^* = \min\left\{\max\left(0, \frac{(\lambda_6-\lambda_8)E_H^* + (\lambda_7-\lambda_6)\delta_H\gamma_H E_H^*}{B4}\right), \rho_{H\max}\right\}. \tag{59}$$

Optimality System. Substituting the representation of the optimal vaccination and treatment control with corresponding adjoint function, we have the optimality system as

$$\begin{aligned}
\frac{dS_D}{dt} &= A_D - \left(1 - \min\left\{\max\left(0, \frac{(\lambda_1-\lambda_4)S_D^* + (\lambda_2-\lambda_1)\beta_{DD}I_D^*S_D^*}{B1}\right), \nu_{D\max}\right\}\right)\\
&\quad\cdot \beta_{DD}S_DI_D - m_DS_D - \min\left\{\max\left(0, \frac{(\lambda_1-\lambda_4)S_D^* + (\lambda_2-\lambda_1)\beta_{DD}I_D^*S_D^*}{B1}\right), \nu_{D\max}\right\}S_D\\
&\quad + \delta\varepsilon_DE_D + \alpha_DR_D,\\
\frac{dE_D}{dt} &= \left(1 - \min\left\{\max\left(0, \frac{(\lambda_1-\lambda_4)S_D^* + (\lambda_2-\lambda_1)\beta_{DD}I_D^*S_D^*}{B1}\right), \nu_{D\max}\right\}\right)\\
&\quad\cdot \beta_{DD}S_DI_D - \left(\left(1 - \min\left\{\max\left(0, \frac{(\lambda_2-\lambda_4)E_D^* + (\lambda_3-\lambda_2)\delta\gamma_DE_D^*}{B2}\right), \rho_{\max}\right\}\right)\delta\gamma_D + m_D + \delta\varepsilon_D + C_D\right)E_D\\
&\quad - \min\left\{\max\left(0, \frac{(\lambda_2-\lambda_4)E_D^* + (\lambda_3-\lambda_2)\delta\gamma_DE_D^*}{B2}\right), \rho_{D\max}\right\}E_D,\\
\frac{dI_D}{dt} &= \delta\gamma_DE_D - (m_D+\mu_D)I_D,\\
\frac{dR_D}{dt} &= \min\left\{\max\left(0, \frac{(\lambda_1-\lambda_4)S_D^* + (\lambda_2-\lambda_1)\beta_{DD}I_D^*S_D^*}{B1}\right), \nu_{D\max}\right\}S_D\\
&\quad - (m_D+\alpha_D)R_D + \min\left\{\max\left(0, \frac{(\lambda_2-\lambda_4)E_D^* + (\lambda_3-\lambda_2)\delta\gamma_DE_D^*}{B2}\right), \rho_{D\max}\right\}E_D,\\
\frac{dS_H}{dt} &= B_H - \left(1 - \min\left\{\max\left(0, \frac{(\lambda_5-\lambda_8)S_H^* + (\lambda_6-\lambda_5)\beta_{DH}S_H^*I_D^*}{B3}\right), \nu_{H\max}\right\}\right)\\
&\quad\cdot \beta_{DH}S_HI_D - m_HS_H - \min\left\{\max\left(0, \frac{(\lambda_5-\lambda_8)S_H^* + (\lambda_6-\lambda_5)\beta_{DH}S_H^*I_D^*}{B3}\right), \nu_{H\max}\right\}S_H\\
&\quad + \delta_H\varepsilon_HE_H + \alpha_HR_H,\\
\frac{dE_H}{dt} &= \left(1 - \min\left\{\max\left(0, \frac{(\lambda_6-\lambda_8)E_H^* + (\lambda_7-\lambda_6)\delta_H\gamma_HE_H^*}{B4}\right), \rho_{H\max}\right\}\right)\\
&\quad\cdot \beta_{DH}S_HI_D - (\delta_H\gamma_H + m_H + \delta_H\varepsilon_H)E_H\\
&\quad - \min\left\{\max\left(0, \frac{(\lambda_6-\lambda_8)E_H^* + (\lambda_7-\lambda_6)\delta_H\gamma_HE_H^*}{B4}\right), \rho_{H\max}\right\}E_H,\\
\frac{dI_H}{dt} &= \delta_H\gamma_HE_H - (m_H+\mu_H)I_H,\\
\frac{dR_H}{dt} &= \min\left\{\max\left(0, \frac{(\lambda_5-\lambda_8)S_H^* + (\lambda_6-\lambda_5)\beta_{DH}S_H^*I_D^*}{B3}\right), \nu_{H\max}\right\}S_H\\
&\quad - (m_H+\alpha_H)R_H + \min\left\{\max\left(0, \frac{(\lambda_6-\lambda_8)E_H^* + (\lambda_7-\lambda_6)\delta_H\gamma_HE_H^*}{B4}\right), \rho_{H\max}\right\}E_H,\\
&\frac{d\lambda_1}{dt}, \frac{d\lambda_2}{dt}, \frac{d\lambda_3}{dt}, \frac{d\lambda_4}{dt}, \frac{d\lambda_5}{dt}, \frac{d\lambda_6}{dt}, \frac{d\lambda_7}{dt}, \frac{d\lambda_8}{dt},\\
&\qquad \text{with, } \lambda_i\left(t_f\right) = 0,\ i = 1,2,3,4,5,6,7,8.
\end{aligned} \tag{60}$$

5.1. Numerical Simulations of the Optimality System. To determine the control strategies ν_D, ρ_D, ν_H, and ρ_H, as given in the objective functional, we began an iteration of the model until convergence is achieved. The results of the simulation of the control strategies are displayed below. We consider equal weights of ($A_1 = 1,\ A_2 = 1,\ A_3 = 1,\ A_4 = 1$) for both exposed and infected classes. We varied the cost associated with the objective functional, which indicate that, with low cost of vaccination, the rate at which individuals will seek for vaccination of their susceptible dogs will increase, and this could result in low transmission of rabies in a heterogeneous population. We consider the various cost of preexposure prophylaxis and postexposure prophylaxis to be ($B1 = 1$,

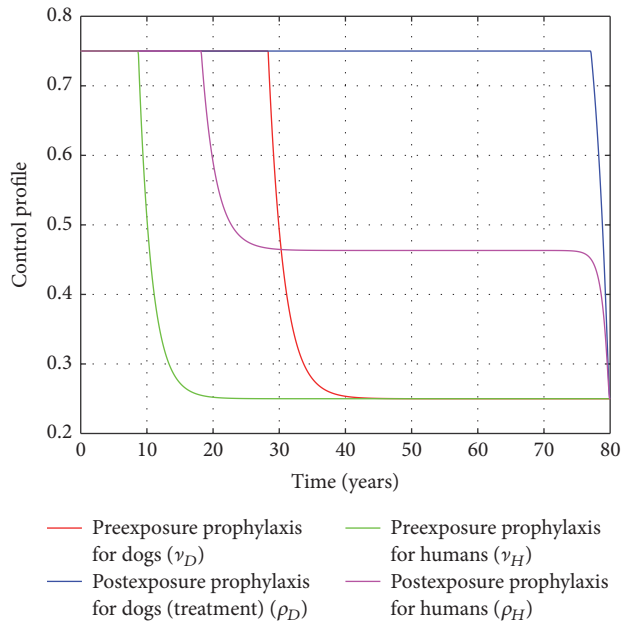
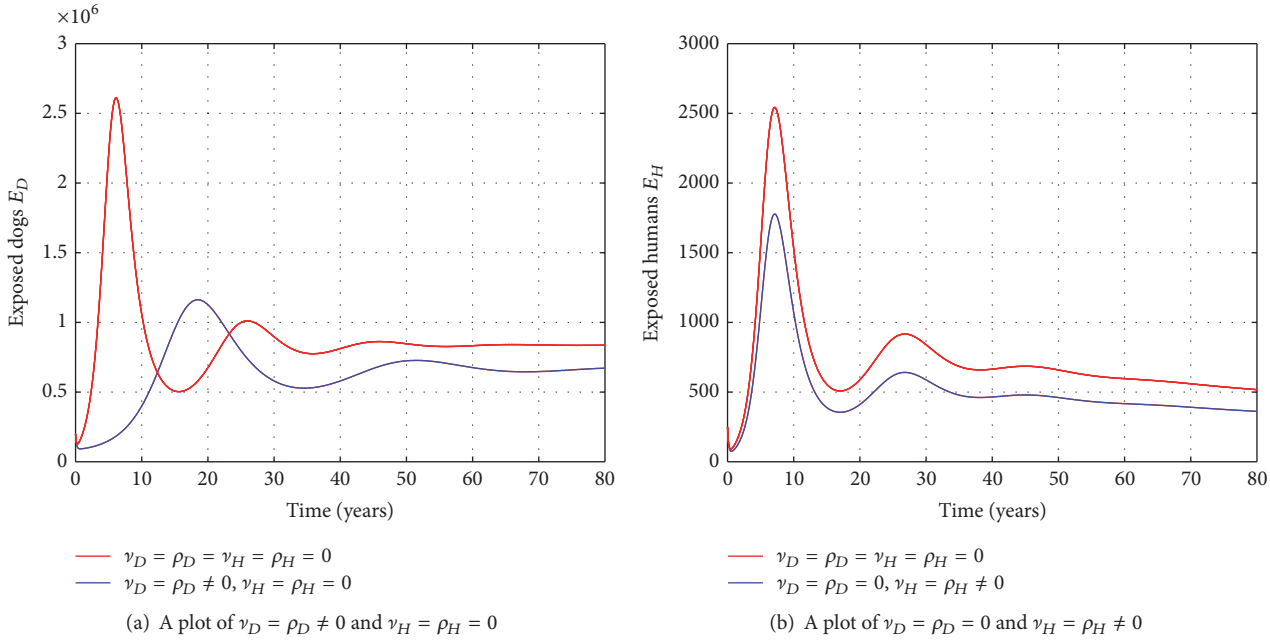


Figure 4: The simulation effect of the controls.

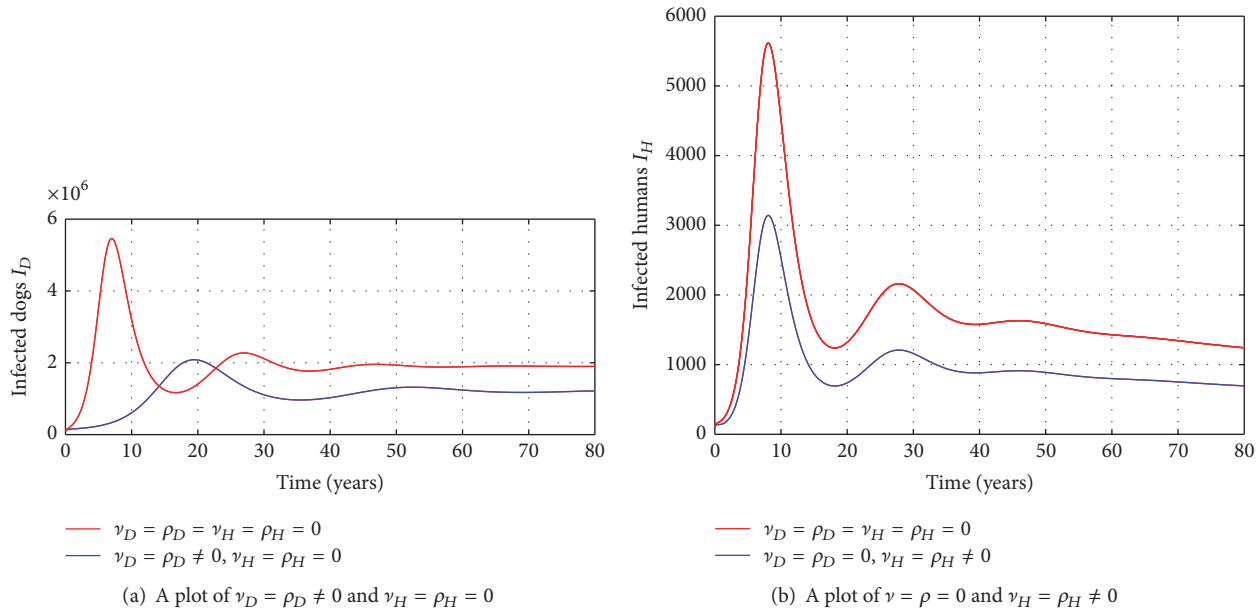


(a) A plot of $\nu_D = \rho_D \neq 0$ and $\nu_H = \rho_H = 0$

(b) A plot of $\nu_D = \rho_D = 0$ and $\nu_H = \rho_H \neq 0$

Figure 5: The trajectories of the model with and without pre- and postexposure prophylaxis on exposed humans and that of the exposed dogs.

$B2 = 4$, $B3 = 1$, $B4 = 4$). We found that the optimal time in controlling the infection using preexposure prophylaxis in dogs is much better than using postexposure prophylaxis in dogs, as shown by the trajectories of the red line and blue line in Figure 4, respectively. The blue line in Figure 4 indicates that applying postexposure prophylaxis will considerably take a longer time in controlling of rabies in dogs. The green line in Figure 4 signifies that preexposure prophylaxis in humans increases the immunity levels of humans and hence reduces the rate at which individuals move to the infected stage. Figures 5 and 6 show the effect of using only one control strategy on the model. Therefore, Figure 5(a) shows that applying only postexposure prophylaxis (treatment or quarantine) of dogs has a low positive impact on the model. Figure 5(b) shows that sticking to the use of pre- and postexposure prophylaxis in human without administering pre- and postexposure



(a) A plot of $\nu_D = \rho_D \neq 0$ and $\nu_H = \rho_H = 0$

(b) A plot of $\nu = \rho = 0$ and $\nu_H = \rho_H \neq 0$

FIGURE 6: The trajectories of the model with and without pre- and postexposure prophylaxis on infected humans and that of the infected dogs.

prophylaxis in the dog population will result in a high of the rabies infection in the human population. Figure 6(a) also shows that combining pre- and postexposure prophylaxis (vaccination and treatment/quarantine) in the dog compartment will reduce the spread of the rabies virus, thereby reducing the using of pre- and postexposure prophylaxis (vaccination and treatment) in humans. Figure 6(b) indicates that a rapid use of pre- and postexposure prophylaxis in the human population will reduce the number of rabies deaths in the human population. Figure 7 shows the simulation effects of applying both controls on the model. Figure 7(a) shows that, with the use of the optimal control strategies, the rate of the infection in the susceptible dogs will reduce significantly. Figures 7(b) and 7(c) show that there is a proportional decrease in the number of exposed and infected dogs when the control measures are applied. Similarly, Figures 7(e) and 7(f) show a significant decrease in the number of infected and exposed humans when the control measures are applied. Figure 7(d) shows that there is a proportional increase in the number of recovered dogs when the control measures are applied. Finally, Figures 8(a)–8(h) show the simulation effect of corresponding adjoint functions.

6. Discussion

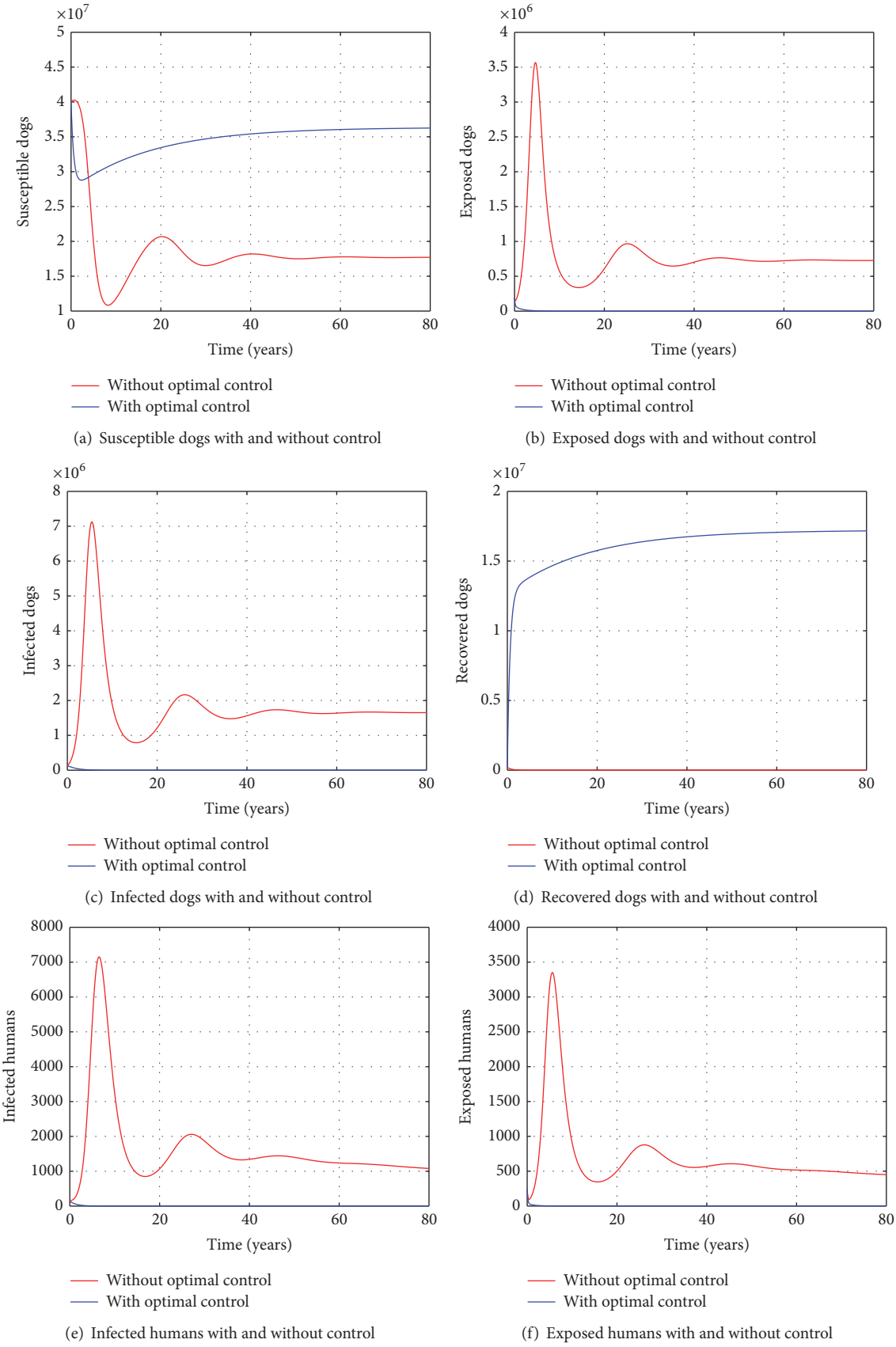
The numerical simulations of the resulting optimality system show that, during the case where it is more expensive to vaccinate than treatment, more resources should be invested in treating affected individuals until the disease prevalence begins to fall. This option, however, does not reduce the number of individuals expose to the disease quickly enough, thus resulting in an overall increase in the infected human population. On the other hand, if it is more expensive to treat than to vaccinate, then more susceptible dogs should be vaccinated, so as to lower the rate at which newborn dogs get infected. Nevertheless, in the case where both measures are equally expensive, the simulation shows that the optimal way to drive the epidemic towards eradication within any specified period is to use more preexposure prophylaxis in both compartments.

7. Conclusion

We studied an optimal control model of rabies transmission dynamics in dogs and the best way of reducing death rate of rabies in humans. The stability analysis shows that the disease-free equilibrium is locally and globally asymptotically stable. We also obtained an optimal control solution for the model which predicts that the optimal way of eliminating deaths from canine rabies as projected by the global alliance for rabies control [30] is using more of preexposure prophylaxis in both dogs and humans and public education; however, the results show that the effective and optimal consideration of preexposure prophylaxis and postexposure prophylaxis in humans without an optimal use of vaccination in the dog population is not beneficial if total elimination of the disease is desirable in Africa and Asia. Any combination strategy which involves vaccination in the dogs' population gives a better result and hence it may be beneficial in eliminating the disease in Asia, Africa, and Latin America.

Disclosure

The authors fully acknowledge that this paper was developed as a result of the first author's thesis work submitted to the Department of Mathematics, Kwame Nkrumah University of



(a) Susceptible dogs with and without control

(b) Exposed dogs with and without control

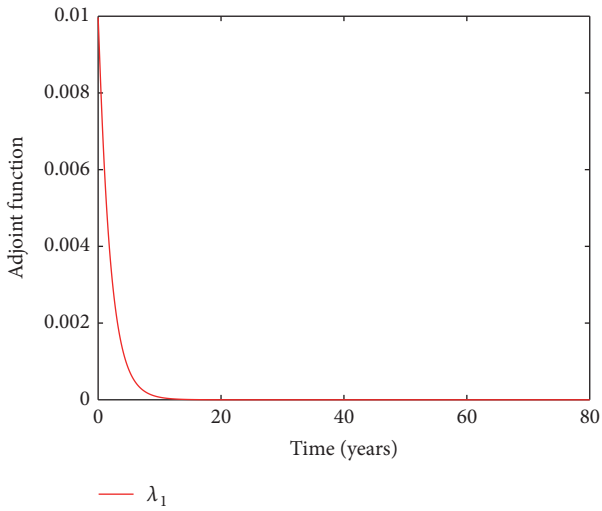
(c) Infected dogs with and without control

(d) Recovered dogs with and without control

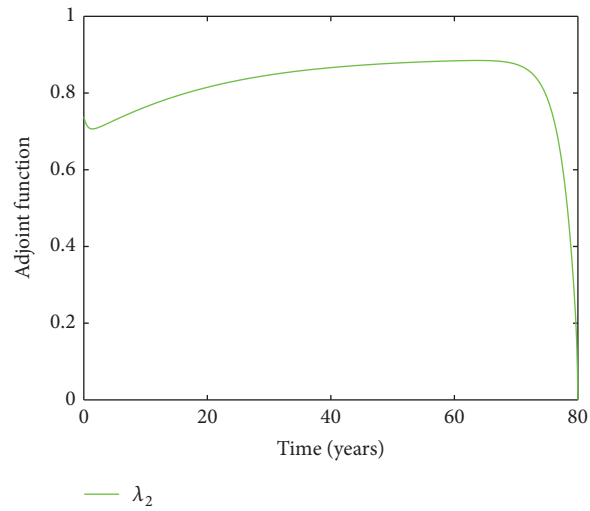
(e) Infected humans with and without control

(f) Exposed humans with and without control

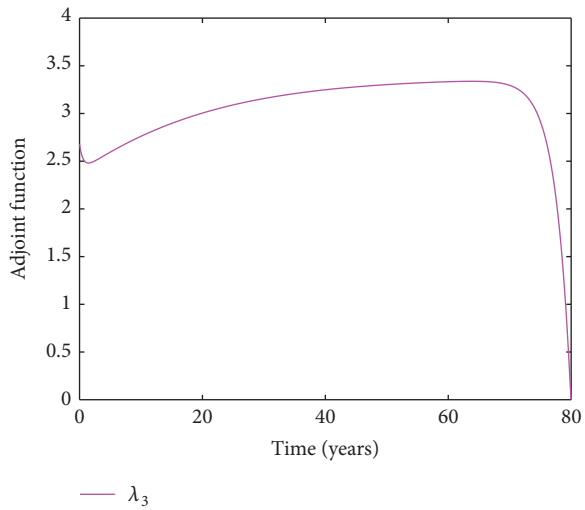
Figure 7: The trajectories of the model with and without optimal control on individual compartments.



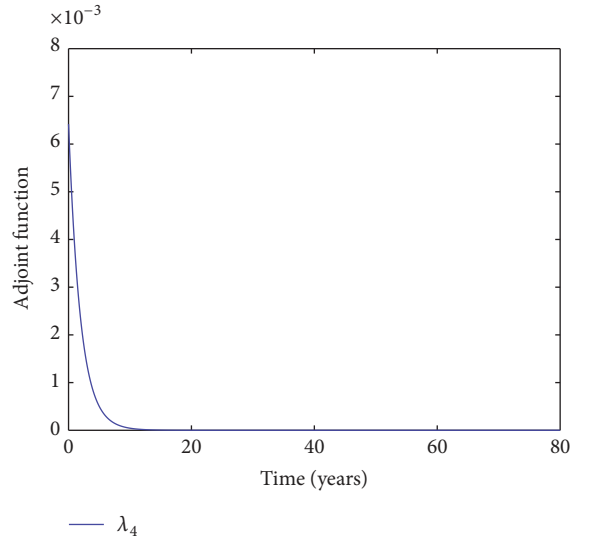
(a) The cost function λ_1 for $A_1 = A_2 = A_3 = A_4 = 1$ and $B1 = 1$, $B2 = 4$, $B3 = 1$, $B4 = 4$



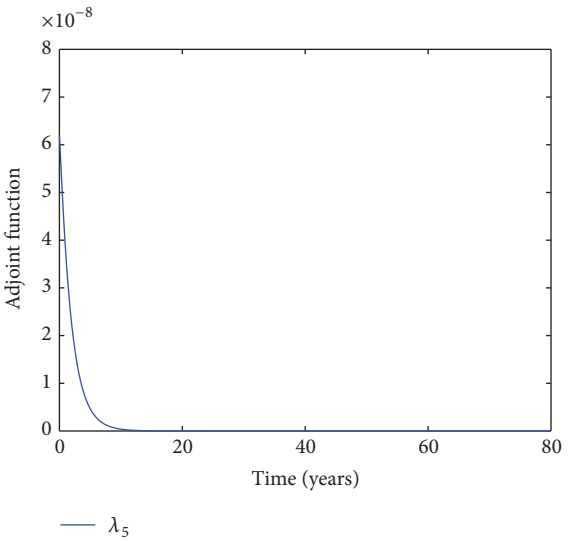
(b) The cost function λ_2 for $A_1 = A_2 = A_3 = A_4 = 1$ and $B1 = 1$, $B2 = 4$, $B3 = 1$, $B4 = 4$



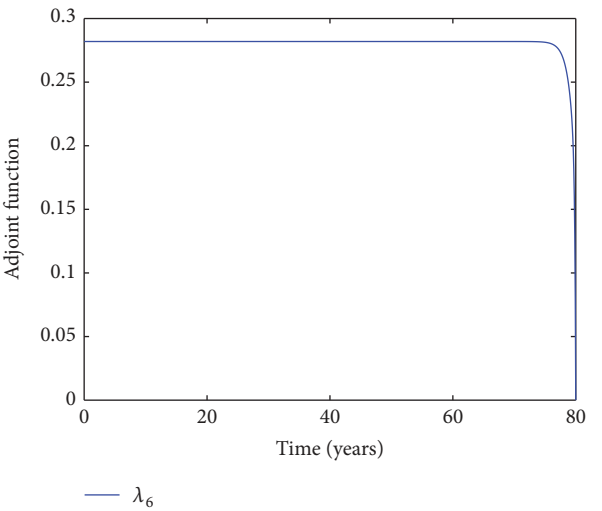
(c) The cost function λ_3 for $A_1 = A_2 = A_3 = A_4 = 1$ and $B1 = 1$, $B2 = 4$, $B3 = 1$, $B4 = 4$



(d) The cost function λ_4 for $A_1 = A_2 = A_3 = A_4 = 1$ and $B1 = 1$, $B2 = 4$, $B3 = 1$, $B4 = 4$

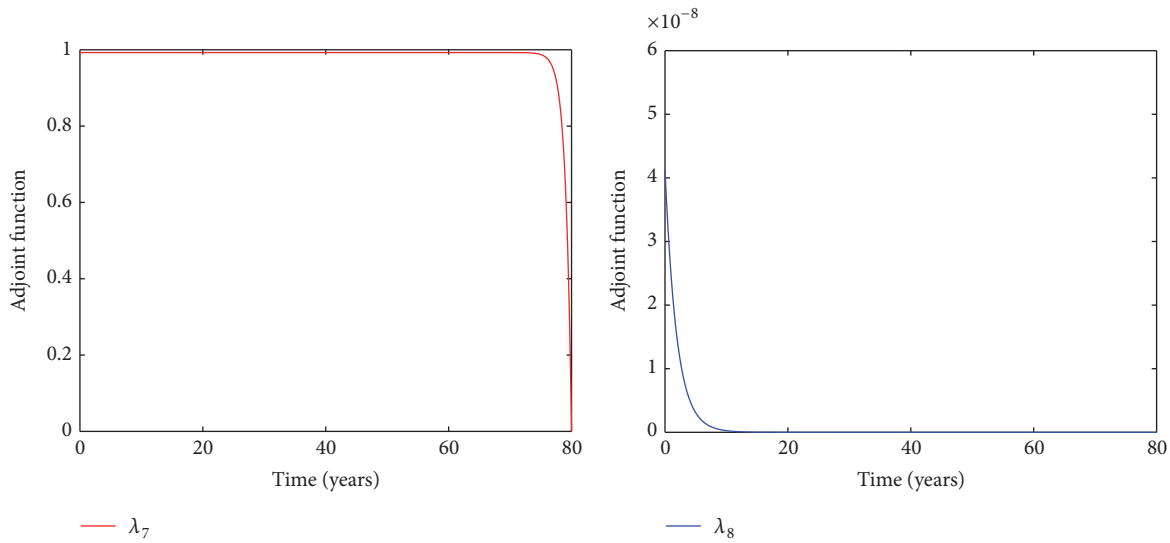


(e) The cost function λ_5 for $A_1 = A_2 = A_3 = A_4 = 1$ and $B1 = 1$, $B2 = 4$, $B3 = 1$, $B4 = 4$



(f) The cost function λ_6 for $A_1 = A_2 = A_3 = A_4 = 1$ and $B1 = 1$, $B2 = 4$, $B3 = 1$, $B4 = 4$

Figure 8: Continued.



(g) The cost function λ_7 for $A_1 = A_2 = A_3 = A_4 = 1$ and $B1 = 1$, $B2 = 4$, $B3 = 1$, $B4 = 4$

(h) The cost function λ_8 for $A_1 = A_2 = A_3 = A_4 = 1$ and $B1 = 1$, $B2 = 4$, $B3 = 1$, $B4 = 4$

FIGURE 8: The trajectories of the model with and without optimal control on individual compartments and corresponding adjoint function.

Science and Technology (see http://ir.knust.edu.gh/xmlui/handle/123456789/10053).

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

The first author's work (thesis) was partly supported by the Government of Ghana Annual Research Grant for Postgraduate Studies.

References

[1] D. T. S. Hayman, N. Johnson, D. L. Horton et al., "Evolutionary history of rabies in Ghana," *PLoS Neglected Tropical Diseases*, vol. 5, no. 4, Article ID e1001, 2011.

[2] "Rabies fact sheet," 2016, http://www.who.int/mediacentre/factsheets/fs099/en/.

[3] C. E. Rupprecht, D. Briggs, C. M. Brown, R. Franka, S. L. Katz, H. D. Kerr et al., *Use of A Reduced (4-Dose) Vaccine Schedule for Postexposure Prophylaxis to Prevent Human Rabies: Recommendations of The Advisory Committee on Immunization Practices*, Department of Health and Human Services, Centers for Disease Control and Prevention, 2010.

[4] R. L. M. Neilan, *Optimal control applied to population and disease models [P.hD. thesis]*, University of Tennessee, Knoxville, Tenn, USA, 2009.

[5] A. O. Isere, J. E. Osemwenkhae, and D. Okuonghae, "Optimal control model for the outbreak of cholera in Nigeria," *African Journal of Mathematics and Computer Science Research*, vol. 7, no. 2, pp. 24–30, 2014.

[6] H. S. Rodrigues, "Optimal control and numerical optimization applied to epidemiological models," 2014, https://arxiv.org/abs/1401.7390?context=math.

[7] A. A. Lashari, *Mathematical Modeling and Optimal Control of A Vector*, National University of Modern Languages, Islamabad, Pakistan, 2012.

[8] A. Abdelrazec, *Modeling, Dynamics and Optimal Control of West Nile Virus with Seasonality*, York University, Toronto, Canada, 2014.

[9] D. Greenhalgh, "Age-structured models and optimal control in mathematical equidemiology: a survey," 2010.

[10] A. Stashko, *The Effects of Prevention and Treatment Interventions in A Microeconomic Model of HIV Transmission*, Duke University, Durham, NC, USA, 2012.

[11] B. Seidu and O. D. Makinde, "Optimal control of HIV/AIDS in the workplace in the presence of careless individuals," *Computational and Mathematical Methods in Medicine*, vol. 2014, Article ID 831506, 19 pages, 2014.

[12] S. D. D. Njankou and F. Nyabadza, "An optimal control model for Ebola virus disease," *Journal of Biological Systems*, vol. 24, no. 1, pp. 29–49, 2016.

[13] M. Aubert, "Costs and benefits of rabies control in wildlife in France," *Revue Scientifique et Technique de l'OIE*, vol. 18, no. 2, pp. 533–543, 1999.

[14] R. M. Anderson and R. M. May, "Population biology of infectious diseases," in *Proceedings of the Dahlem Workshop*, Springer, Berlin, Germany, March 1982.

[15] G. Bohrer, S. Shem-Tov, E. Summer, K. Or, and D. Saltz, "The effectiveness of various rabies spatial vaccination patterns in a simulated host population with clumped distribution," *Ecological Modelling*, vol. 152, no. 2-3, pp. 205–211, 2002.

[16] S. A. Levin, T. G. Hallam, and L. J. Gross, Eds., *Applied Mathematical Ecology*, vol. 18 of *Biomathematics*, Springer, Berlin, Germany, 2012.

[17] M. J. Coyne, G. Smith, and F. E. McAllister, "Mathematic model for the population biology of rabies in raccoons in the mid-Atlantic states," *American journal of veterinary research*, vol. 50, no. 12, pp. 2148–2154, 1989.

[18] J. E. Childs, A. T. Curns, M. E. Dey et al., "Predicting the local dynamics of epizootic rabies among raccoons in the United

States," *Proceedings of the National Academy of Sciences of the United States of America*, vol. 97, no. 25, pp. 13666–13671, 2000.

[19] K. Hampson, J. Dushoff, J. Bingham, G. Brückner, Y. H. Ali, and A. Dobson, "Synchronous cycles of domestic dog rabies in sub-Saharan Africa and the impact of control efforts," *Proceedings of the National Academy of Sciences of the United States of America*, vol. 104, no. 18, pp. 7717–7722, 2007.

[20] M. J. Carroll, A. Singer, G. C. Smith, D. P. Cowan, and G. Massei, "The use of immunocontraception to improve rabies eradication in urban dog populations," *Wildlife Research*, vol. 37, no. 8, pp. 676–687, 2010.

[21] X. Wang and J. Lou, "Two dynamic models about rabies between dogs and human," *Journal of Biological Systems*, vol. 16, no. 4, pp. 519–529, 2008.

[22] W. Yang and J. Lou, "The dynamics of an interactional model of rabies transmitted between human and dogs," *Bollettino della Unione Matematica Italiana. Serie 9*, vol. 2, no. 3, pp. 591–605, 2009.

[23] J. Zinsstag, S. Dürr, M. A. Penny et al., "Transmission dynamics and economics of rabies control in dogs and humans in an African city," *Proceedings of the National Academy of Sciences of the United States of America*, vol. 106, no. 35, pp. 14996–15001, 2009.

[24] W. Ding, L. J. Gross, K. Langston, S. Lenhart, and L. A. Real, "Rabies in raccoons: optimal control for a discrete time model on a spatial grid," *Journal of Biological Dynamics*, vol. 1, no. 4, pp. 379–393, 2007.

[25] G. C. Smith and C. L. Cheeseman, "A mathematical model for the control of diseases in wildlife populations: culling, vaccination and fertility control," *Ecological Modelling*, vol. 150, no. 1-2, pp. 45–53, 2002.

[26] J. M. Tchuenche and C. T. Bauch, "Can culling to prevent monkeypox infection be counter-productive? Scenarios from a theoretical model," *Journal of Biological Systems*, vol. 20, no. 3, pp. 259–283, 2012.

[27] J. Zhang, Z. Jin, G.-Q. Sun, T. Zhou, and S. Ruan, "Analysis of rabies in China: transmission dynamics and control," *PLoS ONE*, vol. 6, no. 7, p. e20891, 2011.

[28] O. A. Grace, "Modelling of the spread of rabies with pre-exposure vaccination of humans," *Mathematical Theory and Modeling*, vol. 4, no. 8, 2014.

[29] E. D. Wiraningsih, F. Agusto, L. Aryati et al., "Stability analysis of rabies model with vaccination effect and culling in dogs," *Applied Mathematical Sciences*, vol. 9, no. 77-80, pp. 3805–3817, 2015.

[30] "Global alliance for rabies control," 2016, https://rabiesalliance.org/.

[31] CIA world factbook and other sources, 2016, http://www.theodora.com/wfbcurrent/ghana/ghana_people.html.

[32] G. Birkhoff and G. Rota, *Ordinary differential equations*, Ginn & Co, New York, NY, USA, 1962.

[33] V. Lakshmikantham, S. Leela, and S. Kaul, "Comparison principle for impulsive differential equations with variable times and stability theory," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 22, no. 4, pp. 499–503, 1994.

[34] O. Diekmann, J. A. Heesterbeek, and J. A. Metz, "On the definition and the computation of the basic reproduction ratio R_0 in models for infectious diseases in heterogeneous populations," *Journal of Mathematical Biology*, vol. 28, no. 4, pp. 365–382, 1990.

[35] E. C. Pielou, *An introduction to Mathematical Ecology*, Wiley-InterScience, Hoboken, NJ, USA, 1969.

[36] R. C. May, "Stable limit cycles in pre-predator population. A comment," *Science*, p. 1074, 1973.

[37] M. Martcheva, *An Introduction to Mathematical Epidemiology*, vol. 61 of *Texts in Applied Mathematics*, Springer, New York, NY, USA, 2015.

[38] T. T. Yusuf and F. Benyah, "Optimal control of vaccination and treatment for an SIR epidemiological model," *World Journal of Modelling and Simulation*, vol. 8, no. 3, pp. 194–204, 2012.

[39] S. D. Hove-Musekwa, F. Nyabadza, and H. Mambili-Mamboundou, "Modelling hospitalization, home-based care, and individual withdrawal for people living with HIV/AIDS in high prevalence settings," *Bulletin of Mathematical Biology*, vol. 73, no. 12, pp. 2888–2915, 2011.

[40] C. V. De León, "Constructions of Lyapunov functions for classics SIS, SIR and SIRS epidemic model with variable population size," *Foro-Red-Mat: Revista Electrónica De Contenido Matemático*, vol. 26, no. 5, article 1, 2009.

[41] J. LaSalle, "The stability of dynamical systems," in *Proceedings of the CBMS-NSF Regional Conference Series in Applied Mathematics 25*, SIAM, Philadelphia, Pa, USA, 1976.

[42] T. Zhang, K. Wang, X. Zhang, and Z. Jin, "Modeling and analyzing the transmission dynamics of HBV epidemic in Xinjiang, China," *PLoS ONE*, vol. 10, no. 9, Article ID e138765, 2015.

[43] O. Sharomi and T. Malik, "Optimal control in epidemiology," *Annals of Operations Research*, vol. 251, no. 1-2, pp. 55–71, 2017.