Research Article

Bound for the 2-Page Fixed Linear Crossing Number of Hypercube Graph via SDP Relaxation

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The crossing number of graph G is the minimum number of edges crossing in any drawing of G in a plane. In this paper we describe a method of finding the bound of 2-page fixed linear crossing number of G. We consider a conflict graph G' of G. Then, instead of minimizing the crossing number of G, we show that it is equivalent to maximize the weight of a cut of G'. We formulate the original problem into the MAXCUT problem. We consider a semidefinite relaxation of the MAXCUT problem. An example of a case where G is hypercube is explicitly shown to obtain an upper bound. The numerical results confirm the effectiveness of the approximation.

1. Introduction

Let G be a simple connected graph with a vertex-set $V(G) = \{v_1, v_2, v_3, \ldots, v_n\}$ and an edge-set $E(G) = \{e_1, e_2, e_3, \ldots, e_m\}$. The crossing number of graph G, denoted $\mathrm{cr}(G)$, is the minimum number of pairwise intersections of edge crossing on the plane drawing of graph G. Clearly, $\mathrm{cr}(G) = 0$ if and only if G is planar. It is known that the exact crossing numbers of any graphs are very difficult to compute. In 1973, Erdös and Guy [1] wrote, "*Almost all questions that one can ask about crossing numbers remain unsolved.*" In fact, Garey and Johnson [2] prove that computing the crossing number is NP-complete.

A 2-page drawing of G is a representation of G on the plane such that its vertices are placed on a straight horizontal line L according to fixed vertex ordering and its edges are drawn as a semicircle above or below L but never cross L.

The *n-cube* or *n-dimensional hypercube* Q_n is recursively defined in terms of the Cartesian products. The one-dimension cube Q_1 is simply K_2 where K_2 is a complete graph with 2 vertices. For $n \geq 2$, Q_n is defined recursively as $Q_{n-1} \times K_2$. The order of Q_n is $|V(Q_n)| = 2^n$ and its size is $|E(Q_n)| = n2^{n-1}$. Since Q_n is planar for $n = 1, 2, 3$, so $\mathrm{cr}(Q_n) = 0$ for each such n. Eggleton and Guy [3] showed that $\mathrm{cr}(Q_4) = 8$ but $\mathrm{cr}(Q_n)$ is unknown for $n \geq 5$.

It was declared by Eggleton and Guy [3] that the crossing numbers of the hypercube Q_n (non-2-page) for $n \geq 3$ was

$$\mathrm{cr}(Q_n) \leq \frac{5}{32}4^n - \left\lfloor \frac{n^2+1}{2} \right\rfloor 2^{n-2}. \tag{1}$$

Then, in 1973, Erdös and Guy [1] conjectured equality in (1). In 1993, a lower bound of $\mathrm{cr}(Q_n)$ was proved by Sýkora and Vrťo [4]:

$$\mathrm{cr}(Q_n) \geq \frac{1}{20}4^n + O\left(n^2 2^n\right). \tag{2}$$

In 2008, Faria et al. [5] constructed a new drawing of Q_n in the plane which led to the conjectured number of crossings

$$\frac{5}{32}4^n - \left\lfloor \frac{n^2+1}{2} \right\rfloor 2^{n-2}. \tag{3}$$

To the best of our knowledge, the fixed linear crossing number for Q_n has not been established. In this paper, we discuss a method to obtain an approximation for fixed linear crossing number for hypercube graph.

2. 2-Page Drawings of Hypercube Graph Q_n

Throughout this paper, we consider the ordering of hypercube graph Q_n. Since Q_n is defined recursively as $Q_{n-1} \times K_2$, for
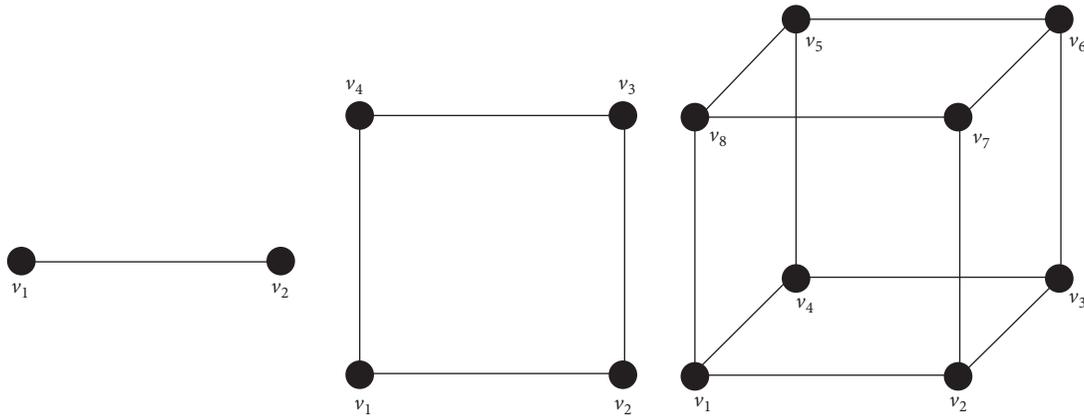
Figure 1: n-Cube graphs with fixed vertex ordering for $n = 1, 2, 3$.
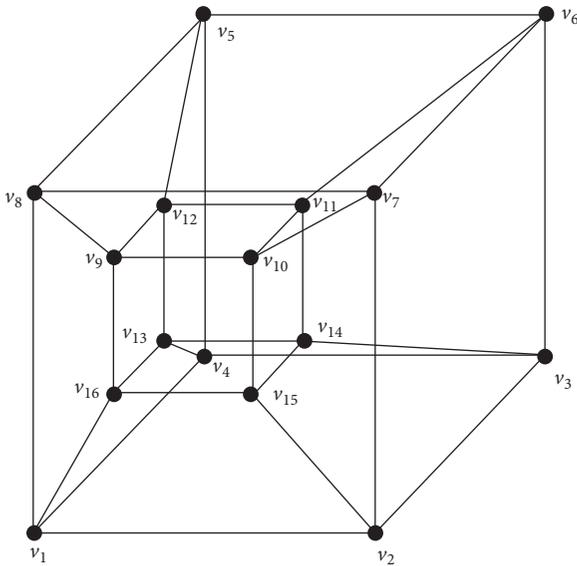
Figure 2: n-Cube graph with fixed vertex ordering for $n = 4$.
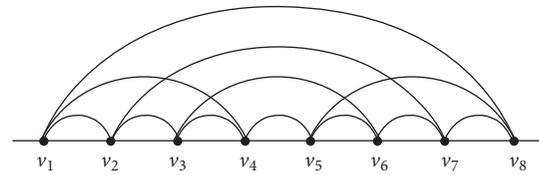
Figure 3: The 2-page drawing of Q_3 with fixed vertex ordering.

other, no two edges cross more than once, and no three edges cross in a point.

For a given 2-page drawing of Q_n with the fixed vertex ordering, a pair of edges $e_{ij} = (v_i, v_j)$ and $e_{lk} = (v_l, v_k)$ are potential crossing if e_{ij} and e_{lk} cross each other when routed on the same side of L. Clearly, e_{ij} and e_{lk} are potential crossing if and only if $v_i < v_l < v_j < v_k$ or $v_l < v_i < v_k < v_j$.

Next we give the definition of conflict graph G' of graph G.

Definition 1. Given a graph G. We define an associated conflict graph $G' = (V', E')$ of a graph $G = (V, E)$. There is corresponding one-to-one and onto mapping between the set of $V'(G')$ and $E(G)$. Two vertices of G' are adjacent if any two edges in G are potential crossing.

For example, according to the given fixed vertex ordering of Q_3 (see Figure 3), Q'_3 is a graph of $n2^{n-1}$ nodes, $V'(Q'_3) = \{v'_{12}, v'_{23}, v'_{34}, v'_{45}, v'_{56}, v'_{67}, v'_{78}, v'_{14}, v'_{36}, v'_{58}, v'_{27}, v'_{18}\}$. v'_{24} and v'_{35} are adjacent in Q'_3 because e_{24} and e_{35} are potential crossing in a 2-page drawing of Q_3. A fixed vertex ordering of Q_4 and its potential crossing can be seen in Figure 4.

In this paper, we are interested only in fixed linear embeddings of Q_n. There is a crossing between e_{ij} and e_{lk} if and only if e_{ij} and e_{lk} are potential crossing and embedded on the same side of L. We can see that the number of edge crossings depends on the order of vertices and on the sides to which the edges are assigned.

The 2-page linear crossing number of Q_n, denoted by $v_2(Q_n)$, is the minimum number of pairwise intersections of edges crossings determined by a 2-page drawing of Q_n. The 2-page fixed linear crossing number of Q_n is the minimum

$n = 2, \ldots$, where Q_1 is a simple graph with 2 vertices together with a single edge incident to both vertices, Q_n has 2 copies of Q_{n-1} with edges connecting between them. Given a fixed ordering on Q_{n-1}, the vertices of the first Q_{n-1} are labeled $1, 2, 3, \ldots, 2^{n-1}$ and the vertices of the second Q_{n-1} are labeled $2^{n-1}+1, 2^{n-1}+2, \ldots, 2^{n-1}+2^{n-1} = 2^n$. The two vertices between the first Q_{n-1} and the second Q_{n-1} are adjacent if and only if the sum of the labeled is $2^n + 1$. Figures 1 and 2 present the ordering of Q_1, Q_2, Q_3 and Q_4 which we consider throughout this paper. Notice that our method is independent on vertex ordering; therefore, for a fixed n, we can apply the method $(2^n)!$ times so as to obtain the 2-page linear crossing number.

The 2-page drawing of Q_n can be represented by drawing the vertices of Q_n on a straight horizontal line L according a fixed vertex ordering. Each edge fully contained one of the two half-planes (pages) as a semicircle and never cross L. Notice that no edge crosses itself, no adjacent edges cross each

FIGURE 4: The 2-page drawing of Q_4 with fixed vertex ordering.

number of pairwise intersections of edges crossings determined by a 2-page drawing of Q_n with fixed vertex ordering of Q_n. It is known that $\nu_2(Q_n) = 0$ for $n = 1, 2, 3$, $\nu_2(Q_n) > 0$ for $n \geq 4$.

3. Reduction to MAXCUT Problem

In this section, we show that the problem can be reduced to the maximum cut problem. Next, we reduce the fixed linear crossing number problem to the maximum cut problem (MAXCUT). The MAXCUT problem is as follows.

Maximum Cut Problem (MAXCUT). Given an undirected graph $G' = (V', E')$ the edge e_{ij} of the graph is associated with nonnegative weights a_{ij}. The problem is to find a cut of the largest possible weight, that is, to partition the set of V' into disjoint sets V_1 and V_2 such that the total weight of all edges linking V_1 and V_2 (i.e., with one incident node in V_1 and the other one in V_2) is as large as possible.

In the MAXCUT problem, we may assume that the weights $a_{ij} = a_{ji} \geq 0$ are defined for every pair i, j of indices: it suffices to set $a_{ij} = 0$ for pairs i, j of nonadjacent nodes. For the unweighted graph, we assume that $a_{ij} = 1$ for $i, j = 1, 2, \ldots, n$.

Let G be a graph with a fixed vertex permutation. Given a vertex partition (V_1, V_2) of its conflict graph G', the associated cut embedding is the fixed linear embedding of G where edges corresponding to V_1 and V_2 are embedded to the half spaces above and below the vertex line, respectively.

Lemma 2 (see [6]).

$$\nu_2(G) = |E'| - \mathrm{MC}(G'), \tag{4}$$

where $|E'|$ is a number of potential crossing of 2-page drawing of G, which is the number of edges of G'. $\mathrm{MC}(G')$ is the size of the maxcut of G'.

Proof. Given a 2-page (circle) drawing of G, define $W \subset V_n$ as the chords that are drawn inside the circle. The edges of E' with precisely one endpoint in W now correspond to edges of G that do not cross in the drawing. □

Theorem 3 (see [7]). *Consider a partition (V_1, V_2) of V'. Then the corresponding cut embedding is a fixed linear embedding of G with a minimum number of crossings if and only if (V_1, V_2) is a maximum cut in G'.*

Proof. Let F' be the set of edges in G' with one endpoint in V_1 and one endpoint in V_2, that is, the cut given by (V_1, V_2). By definition of G', we know that every crossing in the cut embedding associated with (V_1, V_2) corresponds to an edge in G' such that either both its endpoint belong to V_1 or both belong to V_2, that is, to an edge in $E' \setminus F'$. Thus, the number of crossings is $|E'| - |F'|$. As $|E'|$ is constant for a fixed vertex permutation, the result follows. □

Theorem 3 reduces the fixed linear crossing number problem to the maximum cut problem (MAXCUT). In the next section, we describe the relaxation of the MAXCUT problem which leads to semidefinite programming.

3.1. Formulating MAXCUT by Semidefinite Relaxation. In this section, we show that 2-page crossing number of hypercube graph problem can be obtained by computing a semidefinite relaxation of MAXCUT.

First of all, we introduce the adjacency matrix of G denoted $\mathrm{Adj}(G)$ as we know it is an $n \times n$ matrix with the property

$$\mathrm{Adj}(G) \equiv \left[a_{ij} \right],$$

$$a_{ij} \in \{0, 1\}, \quad a_{ij} = \begin{cases} 1, & \text{if } v_i \text{ and } v_j \text{ are adjacent;} \\ 0, & \text{otherwise.} \end{cases} \tag{5}$$

From $\mathrm{Adj}(G)$ we construct the conflict graph of G denoted G'. Finally, we perform MAXCUT on graph G'. We use semidefinite relaxation to approximate the optimal value solution to the MAXCUT problem. Obviously the approximation is larger than the actual MAXCUT optimal value. The feasibility of the relaxation set is strictly larger than the original ones.

According to [2], the MAXCUT problem can be formulated as follows:

$$\max \quad \frac{1}{4} \sum_{i,j=1}^{n} a_{ij} \left(1 - x_i x_j \right)$$

$$\text{s.t.} \quad x_i^2 = 1 \tag{6}$$

$$i = 1, \ldots, n.$$
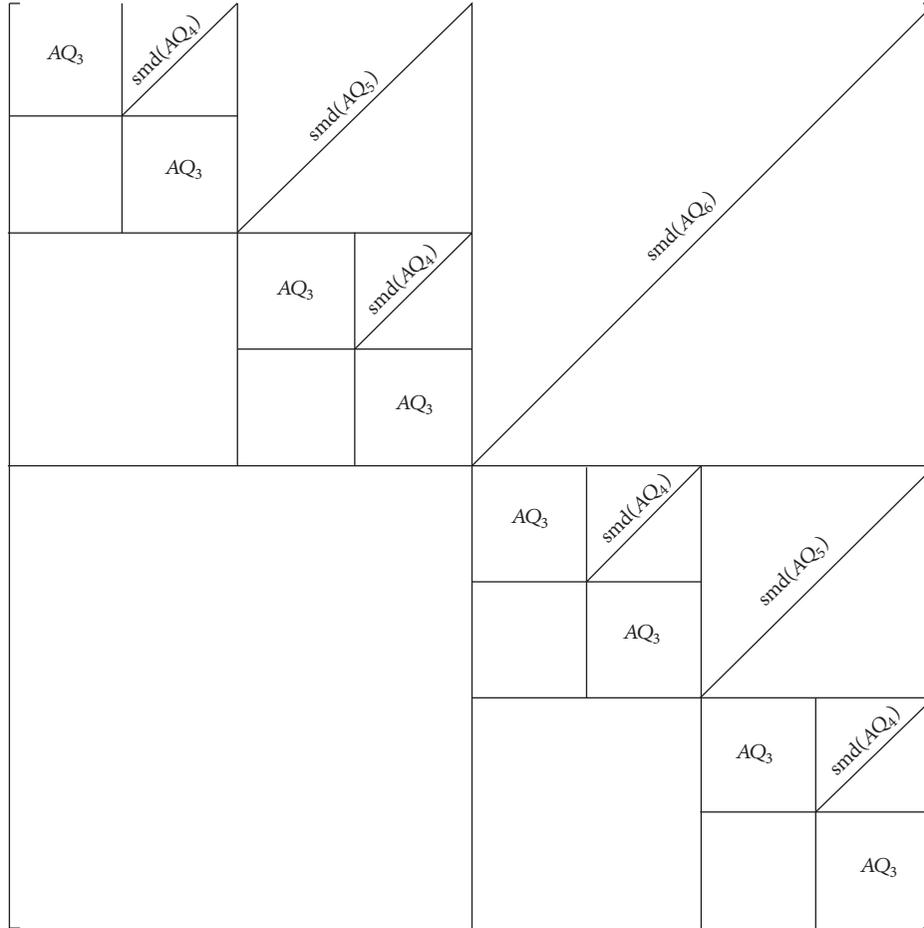
FIGURE 5: The adjacency matrix of size $2^6 \times 2^6$ of Q_6.

We call the optimal value of (6) as "OPT." Then, the relaxation of (6) can be rewritten as

$$\max \quad \frac{1}{4} \sum_{i,j=1}^{n} a_{ij} \left(1 - X_{ij} \right)$$

$$\text{s.t.} \quad X = \left[X_{ij} \right]_{i,j=1}^{n} = X^T \succeq 0, \tag{7}$$

$$X_{ii} = 1, \quad i = 1, \ldots, n,$$

where $A = [a_{ij}]$ is an adjacency matrix of G' and $X = [X_{ij}]$ is a feasible solution to the semidefinite relaxation. The problem (7) is equivalent to

$$\min \quad \frac{1}{4} \operatorname{tr} (AX)$$

$$\text{s.t.} \quad X = \left[X_{ij} \right]_{i,j=1}^{n} = X^T \succeq 0, \tag{8}$$

$$X_{ii} = 1, \quad i = 1, \ldots, n,$$

where A is a given adjacency matrix of G' and $X = [X_{ij}]$ is a feasible solution to the semidefinite relaxation. We call the optimal value of (8) as "SDP."

As we have seen from the relation (4), we let $|E'_n|$ be the number of potential crossing of 2-page drawing of Q_n with our fixed vertex ordering (i.e., $|E'_n|$ is the number of edges of Q'_n).

We can determine $|E'_n|$ by considering the upper half of the main diagonal of the adjacency matrix of Q_n.

Definition 4. Let $AG = [a_{ij}]$ be the $n \times n$ adjacency matrix of G. The element a_{ij} where $i + j = n + 1$ is called minor diagonal of adjacency matrix of G and the element a_{ij} where $i + j = n + 1, i < j$ is called semiminor diagonal of adjacency matrix of G, denoted by smd(AG).

For simplicity, the size of smd(AG) is a number of elements in smd(AG). Let AQ_n be the adjacency matrix of graph Q_n. Therefore AQ_n is $2^n \times 2^n$ symmetric matrix. It is clear that the size of smd(AQ_n) is 2^{n-1}.

Let AG_1 and AG_2 be adjacency matrices of graphs G_1 and G_2, respectively; we say that the number of potential crossing between AG_1 and AG_2, denoted by PC[AG_1, AG_2], is simply the number of potential crossing between 2-page drawing of graph G_1 and G_2. The adjacency matrix of size $2^6 \times 2^6$ of Q_6 with respect to our ordering is presented in Figure 5.

Lemma 5. *For any integer $n \geq 5$,*

$$
\begin{aligned}
&\mathrm{PC}\left[\mathrm{smd}\left(AQ_{n-1}\right), \mathrm{smd}\left(AQ_n\right)\right] \\
&\quad = \mathrm{PC}\left[\mathrm{smd}\left(AQ_{n-1}\right), \mathrm{smd}\left(AQ_{n+i}\right)\right],
\end{aligned}
\tag{9}
$$

where $i = 1, 2, \ldots, n$.

Lemma 6. *For every adjacency matrix of Q_n, AQ_n, where $n \geq 4$, there exists adjacency matrix of Q_3, AQ_3, which is a submatrix embedding in AQ_n. The number of submatrix AQ_3 embedding in AQ_n is 2^{n-3}.*

Lemma 7. *For any integer $n \geq 5$,*

$$
\mathrm{PC}\left[AQ_3, \mathrm{smd}\left(AQ_n\right)\right] = \mathrm{PC}\left[AQ_3, \mathrm{smd}\left(AQ_{n+i}\right)\right],
\tag{10}
$$

where $i = 1, 2, \ldots, n$.

Lemma 8. *For any integer $n \geq 5$,*

$$
\begin{aligned}
&\mathrm{PC}\left[AQ_{n-1}, \mathrm{smd}\left(AQ_n\right)\right] \\
&\quad = \mathrm{PC}\left[AQ_{n-1}, \mathrm{smd}\left(AQ_{n+i}\right)\right],
\end{aligned}
\tag{11}
$$

where $i = 1, 2, \ldots, n$.

Lemma 9. *For $n \geq 4$, the number of block $\mathrm{smd}(AQ_i)$ embedding in AQ_n equals 2^{n-i}, $4 \leq i \leq n$.*

Lemmas 5–9 follow directly from the definitions.

Lemma 10. *For any integer $n \geq 4$,*

$$
\mathrm{PC}\left[\mathrm{smd}\left(AQ_n\right), \mathrm{smd}\left(AQ_{n+1}\right)\right] = \left(\frac{2^{n-1}}{2}\right)\left(2^n - 2\right).
\tag{12}
$$

Proof. The number of potential crossing between $\mathrm{smd}(AQ_n)$ and $\mathrm{smd}(AQ_{n+1})$ is a result of the number of potential crossing between all of the element $a_{1,2^n}, a_{2,2^n-1}, \ldots, a_{2^{n-1},2^{n-1}+1}$ in AQ_n and $\mathrm{smd}(AQ_{n+1})$. That is,

$$
\begin{aligned}
&\mathrm{PC}\left[\mathrm{smd}\left(AQ_n\right), \mathrm{smd}\left(AQ_{n+1}\right)\right] \\
&\quad = \left(2^n - 2\right) + \left(2^n - 4\right) + \cdots + 2 + 0 \\
&\quad = \left(\frac{2^{n-1}}{2}\right)\left(2^n - 2\right).
\end{aligned}
\tag{13}
$$

$\square$

Theorem 11. *For any integer $n \geq 5$,*

$$
\left|E_n'\right| = 2\left|E_{n-1}'\right| + 2^{n+1} + 2\sum_{k=5}^{n}\left(2^{n-k}\right)\left(\frac{2^k}{4} - 1\right)\left(\frac{2^k}{4}\right),
\tag{14}
$$

$$
\left|E_4'\right| = 40,
$$

where $|E_n'|$ is the number of potential crossing of 2-page drawing of Q_n with our fixed vertex ordering.

Proof. We prove this lemma by considering the number of potential crossing of 2-page drawing of Q_n with our fixed vertex ordering. Since Q_n has 2 copies of Q_{n-1} with some edges connecting between them, the number of potential crossing of AQ_n is a result of twice of the number of potential crossing within AQ_{n-1} together with $\mathrm{PC}[AQ_3, \mathrm{smd}(AQ_n)]$ and $\mathrm{PC}[\mathrm{smd}(AQ_i), \mathrm{smd}(AQ_n)]$, $i = 4, 5, \ldots, n-1$.

Hence, it is enough to show that the number of potential crossing between 2-page drawing of Q_{n-1} and Q_{n-1} is equal to $\gamma(n)$, where

$$
\gamma(n) = 2^{n+1} + 2\sum_{k=5}^{n}\left(2^{n-k}\right)\left(\frac{2^k}{4} - 1\right)\left(\frac{2^k}{4}\right).
\tag{15}
$$

Note $\gamma(n)$ is the number of potential crossing between all of submatrices AQ_3 and $\mathrm{smd}(AQ_n)$ and also between $\mathrm{smd}(AQ_i)$ and $\mathrm{smd}(AQ_n)$, $i = 4, 5, \ldots, n-1$. By Lemmas 6 and 9,

$$
\begin{aligned}
\gamma(n) &= 2^{n-3} \cdot \mathrm{PC}\left[AQ_3, \mathrm{smd}\left(AQ_n\right)\right] + 2^{n-4} \\
&\quad \cdot \mathrm{PC}\left[\mathrm{smd}\left(AQ_4\right), \mathrm{smd}\left(AQ_n\right)\right] + 2^{n-5} \\
&\quad \cdot \mathrm{PC}\left[\mathrm{smd}\left(AQ_5\right), \mathrm{smd}\left(AQ_n\right)\right] + \cdots \\
&\quad + 2^{n-(n-2)} \cdot \mathrm{PC}\left[\mathrm{smd}\left(AQ_{n-2}\right), \mathrm{smd}\left(AQ_n\right)\right] \\
&\quad + 2^{n-(n-1)} \cdot \mathrm{PC}\left[\mathrm{smd}\left(AQ_{n-1}\right), \mathrm{smd}\left(AQ_n\right)\right].
\end{aligned}
\tag{16}
$$

We precede by mathematical induction on n. For $n = 5$, it can be easily seen that $\gamma(5) = 176$ by counting. Assuming (15) holds true, now we consider $\gamma(n+1)$ as a number of potential crossing between all of the submatrices AQ_3 and $\mathrm{smd}(AQ_{n+1})$ and also between $\mathrm{smd}(AQ_i)$ and $\mathrm{smd}(AQ_{n+1})$, $i = 4, 5, \ldots, n$. By the Lemmas 5, 6, 7, 9, 8, and 10,

$$
\begin{aligned}
\gamma(n+1) &= 2^{(n+1)-3} \cdot \mathrm{PC}\left[AQ_3, \mathrm{smd}\left(AQ_{n+1}\right)\right] \\
&\quad + 2^{(n+1)-4} \cdot \mathrm{PC}\left[\mathrm{smd}\left(AQ_4\right), \mathrm{smd}\left(AQ_{n+1}\right)\right] \\
&\quad + 2^{(n+1)-5} \cdot \mathrm{PC}\left[\mathrm{smd}\left(AQ_5\right), \mathrm{smd}\left(AQ_{n+1}\right)\right] + \cdots \\
&\quad + 2^{(n+1)-(n-2)} \cdot \mathrm{PC}\left[\mathrm{smd}\left(AQ_{n-2}\right), \mathrm{smd}\left(AQ_{n+1}\right)\right] \\
&\quad + 2^{(n+1)-(n-1)} \cdot \mathrm{PC}\left[\mathrm{smd}\left(AQ_{n-1}\right), \mathrm{smd}\left(AQ_{n+1}\right)\right] \\
&\quad + 2^{(n+1)-n} \cdot \mathrm{PC}\left[\mathrm{smd}\left(AQ_n\right), \mathrm{smd}\left(AQ_{n+1}\right)\right] \\
&= 2\left(2^{n-3} \cdot \mathrm{PC}\left[AQ_3, \mathrm{smd}\left(AQ_n\right)\right] + 2^{n-4}\right. \\
&\quad \cdot \mathrm{PC}\left[\mathrm{smd}\left(AQ_4\right), \mathrm{smd}\left(AQ_n\right)\right] + 2^{n-5} \\
&\quad \cdot \mathrm{PC}\left[\mathrm{smd}\left(AQ_5\right), \mathrm{smd}\left(AQ_n\right)\right] + \cdots + 2^{n-(n-2)} \\
&\quad \cdot \mathrm{PC}\left[\mathrm{smd}\left(AQ_{n-2}\right), \mathrm{smd}\left(AQ_n\right)\right] + 2^{n-(n-1)} \\
&\quad \left. \cdot \mathrm{PC}\left[\mathrm{smd}\left(AQ_{n-1}\right), \mathrm{smd}\left(AQ_n\right)\right]\right) + 2^{(n+1)-n} \\
&\quad \cdot \mathrm{PC}\left[\mathrm{smd}\left(AQ_n\right), \mathrm{smd}\left(AQ_{n+1}\right)\right] = 2\gamma(n)
\end{aligned}
$$

$$+ 2^{(n+1)-n} \cdot \mathrm{PC}\left[\mathrm{smd}\left(AQ_n\right), \mathrm{smd}\left(AQ_{n+1}\right)\right]$$

$$= 2\gamma(n) + 2^{(n+1)-n} \cdot \left(\frac{2^{n-1}}{2}\right)\left(2^n - 2\right) = 2\gamma(n)$$

$$+ (2)\left(2^{n-1}\right)\left(2^{n-1} - 1\right) = 2\gamma(n) + 2\left(\frac{2^{n+1}}{4} - 1\right)$$

$$\cdot \left(\frac{2^{n+1}}{4}\right) = 2\left[2^{n+1}\right.$$

$$+ 2\sum_{k=5}^{n}\left(2^{n-k}\right)\left(\frac{2^k}{4} - 1\right)\left(\frac{2^k}{4}\right)\right] + 2\left(\frac{2^{n+1}}{4} - 1\right)$$

$$\cdot \left(\frac{2^{n+1}}{4}\right) = 2 \cdot 2^{n+1} + 2$$

$$\cdot \sum_{k=5}^{n}\left(2^{n+1-k}\right)\left(\frac{2^k}{4} - 1\right)\left(\frac{2^k}{4}\right) + 2\left(\frac{2^{n+1}}{4} - 1\right)$$

$$\cdot \left(\frac{2^{n+1}}{4}\right) = 2^{(n+1)+1}$$

$$+ 2\left[\sum_{k=5}^{n}\left(2^{n+1-k}\right)\left(\frac{2^k}{4} - 1\right)\left(\frac{2^k}{4}\right)\right.$$

$$+ \left(2^{(n+1)-(n+1)}\right)\left(\frac{2^{n+1}}{4} - 1\right)\left(\frac{2^{n+1}}{4}\right)\right] = 2^{(n+1)+1}$$

$$+ 2\left[\sum_{k=5}^{n+1}\left(2^{(n+1)-k}\right)\left(\frac{2^k}{4} - 1\right)\left(\frac{2^k}{4}\right)\right].$$

$$(17)$$

□

The next theorem shows how effective the relaxation is.

Theorem 12 (see [8]). *Let OPT be the optimal value of the MAXCUT problem and SDP be the optimal value of the semidefinite relaxation. Then*

$$1 \geq \frac{OPT}{SDP} \geq 0.87856\cdots \qquad (18)$$

Theorem 12 guarantees that the optimal value of the MAXCUT is close to the optimal value of the semidefinite relaxation. From (4), we have

$$\mathrm{AP}\left(\nu_2\left(Q_n\right)\right) = \left|E_n'\right| - \mathrm{AP}\left(\mathrm{MC}\left(Q_n'\right)\right), \qquad (19)$$

where $\left|E_n'\right|$ is a number of potential crossing of 2-page drawing of Q_n. $\mathrm{AP}(\nu_2(Q_n))$ is an approximation of 2-page fixed linear crossing number of Q_n and $\mathrm{AP}(\mathrm{MC}(Q_n'))$ is an approximation of $\mathrm{MC}(Q_n')$.

Corollary 13. *Let $\mathrm{AP}(\nu_2(Q_n))$ be an approximation of $\nu_2(Q_n)$. Then we have*

$$\mathrm{AP}\left(\nu_2\left(Q_n\right)\right) \leq \nu_2\left(Q_n\right) \leq k\left(n\right)\mathrm{AP}\left(\nu_2\left(Q_n\right)\right), \qquad (20)$$

where $k(n)$ is a computable quantity depending on n.

Table 1: The numerical results of the approximation of 2-page fixed linear crossing number of Q_n, $\mathrm{AP}(\nu_2(Q_n))$ for $n = 4, 5, 6$.

n	$\mathrm{AP}(\mathrm{MC}(Q_n'))$	$\left|E_n'\right|$	$\mathrm{AP}(\nu_2(Q_n))$
4	35	40	5
5	207	256	48
6	1034	1344	310

Table 2: The computable quantity $k(n)$, the bound of $\nu_2(Q_n)$, and the upper bound of $\mathrm{cr}(Q_n)$ for $n = 4, 5, 6$.

n	$k(n)$	$\mathrm{AP}(\nu_2(Q_n))$	$k(n)\mathrm{AP}(\nu_2(G))$	Upper bound of $\mathrm{cr}(Q_n)$
4	1.91	5	9.55	8
5	1.55	48	74.36	56
6	1.43	310	444.42	352

Proof. From (4), (18), and (19), we have

$$\frac{\nu_2\left(Q_n\right)}{\mathrm{AP}\left(\nu_2\left(Q_n\right)\right)} = \frac{\left|E_n'\right| - \mathrm{MC}\left(Q_n'\right)}{\left|E_n'\right| - \mathrm{AP}\left(\mathrm{MC}\left(Q_n'\right)\right)}$$

$$\leq \frac{\left|E_n'\right| - 0.87856\mathrm{AP}\left(\mathrm{MC}\left(Q_n'\right)\right)}{\left|E_n'\right| - \mathrm{AP}\left(\mathrm{MC}\left(Q_n'\right)\right)}. \qquad (21)$$

Let $k(n) = (\left|E_n'\right| - 0.87856\mathrm{AP}(\mathrm{MC}(Q_n')))/(\left|E_n'\right| - \mathrm{AP}(\mathrm{MC}(Q_n')))$ be the computable quantity depending on n. Then,

$$\frac{\nu_2\left(Q_n\right)}{\mathrm{AP}\left(\nu_2\left(Q_n\right)\right)} \leq k\left(n\right)$$

$$\nu_2\left(Q_n\right) \leq k\left(n\right) \cdot \mathrm{AP}\left(\nu_2\left(Q_n\right)\right). \qquad (22)$$

□

Corollary 13 shows that the upper bound of $\nu_2(Q_n)$ is $k(n) \cdot \mathrm{AP}(\nu_2(Q_n))$, where $k(n)$ is the computable quantity depending on n.

3.2. Experimental Results. In this section, we consider the hypercube graph Q_n for $n = 4, 5, 6$. Then, we give some examples for approximating the problems of the semidefinite relaxation in the form (8). We approximate this problem via MATLAB program together with an optimization toolbox called "SeDuMi." The SeDuMi is a package for solving optimization problems with linear, quadratic, and semidefinite constraints.

In Table 1, the second column shows numerical results for the approximation of the MAXCUT on the associated conflict graph Q_n' by using the semidefinite relaxation. It is well known that this problem can be solved in a polynomial time. The third column displays the numbers of potential crossing of 2-page drawing of Q_n referring to our fixed vertex ordering that we evaluate from (14). Notice that this potential crossing of 2-page drawing of Q_n is the exact value. From (19), we calculate the approximation of 2-page fixed linear crossing number of Q_n for $n = 4, 5, 6$. The results are shown in the last column.

In Table 2, we present the lower bound of $\nu_2(Q_n)$, $\mathrm{AP}(\nu_2(Q_n))$ and the upper bound of $\nu_2(Q_n)$, $k(n)\mathrm{AP}(\nu_2(G))$.

The second column shows the values of $k(n)$ for $n = 4, 5, 6$. We see that as n get larger the values of $k(n)$ tend to decline continuously. It is interesting to study the behavior of $k(n)$ as $n \to \infty$. It does not surprise to see that our approximation is strictly larger than the upper bound of $\mathrm{cr}(Q_n)$ (14) since the latter one does not have a restriction that all vertices must be placed on a line. However, it is surprising to see that these numbers are not so different from each other.

4. Concluding Remarks

In this paper, given graph G, we show how the associating conflict graph G' is constructed. We recharacterize the problem of finding the crossing number of graph G to the MAXCUT problem of G'. We approximate the MAXCUT problem by the semidefinite relaxation which can be solved easily by a standard optimization package; in this case, we use SeDuMi 1.02. The numerical results show reasonable outcome. Clearly, another relaxation method can be explored. Moreover, it would be quite interesting to see the behavior of $k(n)$ as n get larger. One can further study how to estimate $k(n)$ for a larger n.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Acknowledgments

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