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Research Article

On Hermite-Hadamard Type Inequalities for *s*-Convex Functions on the Coordinates via Riemann-Liouville Fractional Integrals

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We obtain some Hermite-Hadamard type inequalities for s-convex functions on the coordinates via Riemann-Liouville integrals. Some integral inequalities with the right-hand side of the fractional Hermite-Hadamard type inequality are also established.

1. Introduction

If $f : I \to R$ is a convex function on the interval I, then, for any $a, b \in I$ with $a \neq b$, we have the following double inequality:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(t)\,dt \leq \frac{f(a)+f(b)}{2}. \qquad (1)$$

This remarkable result is well known in the literature as the Hermite-Hadamard inequality.

Since then, some refinements of the Hermite-Hadamard inequality on convex functions have been extensively investigated by a number of authors (e.g., [1–7]).

Definition 1 (see [8]). Let $s \in (0, 1]$ be a fixed real number. A function $f : I \subset [0, \infty) \to [0, \infty)$ is said to be s-convex in the second sense, or that f belongs to the class K_s^2, if the inequality,

$$f(\alpha x + (1-\alpha)y) \leq \alpha^s f(x) + (1-\alpha)^s f(y), \qquad (2)$$

holds for all $x, y \in I$ and $\alpha \in [0, 1]$.

It can be easily seen that, for $s = 1$, s-convexity reduces to ordinary convexity of functions defined on $[0, \infty)$.

In [4], Dragomir defined convex functions on the coordinates as follows.

Let us consider the bidimensional interval $\Delta := [a, b] \times [c, d]$ in $\mathbb{R}^2$ with $a < b$ and $c < d$; a mapping $f : \Delta \to \mathbb{R}$ is said to be convex on Δ if the inequality,

$$\begin{aligned} &f(\lambda x + (1-\lambda)z, \lambda y + (1-\lambda)w) \\ &\quad \leq \lambda f(x, y) + (1-\lambda) f(z, w), \end{aligned} \qquad (3)$$

holds for all $(x, y), (z, w) \in \Delta$ and $\lambda \in [0, 1]$.

A function $f : \Delta \to \mathbb{R}$ is said to be coordinated convex on Δ if the partial mappings $f_y : [a, b] \to \mathbb{R}$, $f_y(u) = f(u, y)$ and $f_x : [c, d] \to \mathbb{R}$, $f_x(v) = f(x, v)$ are convex for all $y \in [c, d]$ and $x \in [a, b]$.

A formal definition for convex functions on the coordinates may be stated as follows.

Definition 2. A function $f : \Delta \to \mathbb{R}$ is said to be convex on coordinates on Δ if the inequality,

$$\begin{aligned} &f(\lambda x + (1-\lambda)z, ty + (1-t)w) \\ &\quad \leq \lambda t f(x, y) + \lambda(1-t) f(x, w) + (1-\lambda) t f(z, y) \\ &\qquad + (1-t)(1-\lambda) f(z, w), \end{aligned} \qquad (4)$$

holds for all $(x, y), (z, y), (x, w), (z, w) \in \Delta$, and $t, \lambda \in [0, 1]$.

In [4], Dragomir established the following Hadamard-type inequalities for convex functions on the coordinates in a rectangle from the plane $\mathbb{R}^2$.

Theorem 3 (see [4]). *Suppose that $f : \Delta = [a,b]\times[c,d] \to \mathbb{R}$ is convex on the coordinates on Δ. Then one has the inequalities:*

$$f\left(\frac{a+b}{2},\frac{c+d}{2}\right) \le \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dy\,dx \le \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}. \tag{5}$$

The concept of s-convex functions on the coordinates in the second sense was introduced by Alomari and Darus in [9].

Let us consider the bidimensional interval $\Delta := [a,b]\times[c,d]$ in $\mathbb{R}^2$ with $a<b$ and $c<d$; a mapping $f:\Delta\to\mathbb{R}$ is s-convex on Δ if the inequality,

$$f(\lambda x+(1-\lambda)z,\lambda y+(1-\lambda)w) \le \lambda^s f(x,y)+(1-\lambda)^s f(z,w), \tag{6}$$

holds for all $(x,y),(z,w)\in\Delta$ with $\lambda\in[0,1]$ and for some fixed $s\in(0,1]$.

A function $f:\Delta\to\mathbb{R}$ is said to be s-convex on the coordinates on Δ in the second sense if the partial mappings $f_y:[a,b]\to\mathbb{R}$, $f_y(u)=f(u,y)$, and $f_x:[c,d]\to\mathbb{R}$, $f_x(v)=f(x,v)$ are s-convex in the second sense for all $y\in[c,d]$ and $x\in[a,b]$ with some fixed $s\in(0,1]$.

A formal definition for convex functions on the coordinates in the second sense may be stated as follows.

Definition 4. A function $f:\Delta\to\mathbb{R}$ is said to be s-convex on coordinates in the second sense on Δ if the inequality,

$$\begin{aligned} f(\lambda x+(1-\lambda)z,ty+(1-t)w) &\le \lambda^s t^s f(x,y)+\lambda^s(1-t)^s f(x,w) \\ &\quad + (1-\lambda)^s t^s f(z,y)+(1-t)^s(1-\lambda)^s f(z,w), \end{aligned} \tag{7}$$

holds for all $(x,y),(z,y),(x,w),(z,w)\in\Delta$ with $t,\lambda\in[0,1]$ and some fixed $s\in(0,1]$.

In [10], Alomari and Darus proved the following inequalities based on the above definition.

Theorem 5 (see [10]). *Suppose that $f:\Delta=[a,b]\times[c,d]\subseteq[0,\infty)^2\to[0,\infty)$ is s-convex on the coordinates in the second sense on Δ. Then one has the inequalities:*

$$\begin{aligned} 4^{s-1}f\left(\frac{a+b}{2},\frac{c+d}{2}\right) &\le \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dy\,dx \\ &\le \big(f(a,c)+f(a,d)+f(b,c) \\ &\quad +f(b,d)\big)\left((s+1)^2\right)^{-1}. \end{aligned} \tag{8}$$

It is remarkable that Sarikaya et al. [11] proved the following interesting inequalities of Hermite-Hadamard type involving Riemann-Liouville fractional integrals.

Theorem 6 (see [11]). *Let $f:[a,b]\to R$ be a positive function with $a<b$ and $f\in L_1[a,b]$. If f is a convex function on $[a,b]$, then the following inequalities for fractional integrals hold*

$$f\left(\frac{a+b}{2}\right) \le \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J_{a^+}^\alpha f(b)+J_{b^-}^\alpha f(a)\right] \le \frac{f(a)+f(b)}{2}, \tag{9}$$

with $\alpha>0$.

We remark that the symbol $J_{a^+}^\alpha$ and $J_{b^-}^\alpha f$ denote the left-sided and right-sided Riemann-Liouville fractional integrals of the order $\alpha\ge0$ with $a\ge0$ which are defined by

$$\begin{aligned} J_{a^+}^\alpha f(x) &= \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1} f(t)\,dt, \quad x>a, \\ J_{b^-}^\alpha f(x) &= \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1} f(t)\,dt, \quad x<b, \end{aligned} \tag{10}$$

respectively. Here, $\Gamma(\alpha)$ is the Gamma function defined by $\Gamma(\alpha)=\int_0^\infty e^{-t}t^{\alpha-1}dt$.

Definition 7 (see [12]). Let $f\in L_1[a,b]\times[c,d]$. The Riemann-Liouville fractional integrals $J_{a^+,c^+}^{\alpha,\beta}$, $J_{a^+,d^-}^{\alpha,\beta}$, $J_{b^-,c^+}^{\alpha,\beta}$ and $J_{b^-,d^-}^{\alpha,\beta}$ of order $\alpha,\beta>0$ with $a,c\ge0$ are defined by

$$\begin{aligned} J_{a^+,c^+}^{\alpha,\beta} f(x,y) &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^x\int_c^y (x-t)^{\alpha-1}(y-s)^{\beta-1} f(t,s)\,ds\,dt, \\ &\qquad x>a, \quad y>c, \\ J_{a^+,d^-}^{\alpha,\beta} f(x,y) &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^x\int_y^d (x-t)^{\alpha-1}(s-y)^{\beta-1} f(t,s)\,ds\,dt, \\ &\qquad x>a, \quad y<d, \\ J_{b^-,c^+}^{\alpha,\beta} f(x,y) &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_x^b\int_c^y (t-x)^{\alpha-1}(y-s)^{\beta-1} f(t,s)\,ds\,dt, \\ &\qquad x<b, \quad y>c, \\ J_{b^-,d^-}^{\alpha,\beta} f(x,y) &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_x^b\int_y^d (t-x)^{\alpha-1}(s-y)^{\beta-1} f(t,s)\,ds\,dt, \\ &\qquad x<b, \quad ,y<d. \end{aligned} \tag{11}$$

In [12], Sarıkaya proposed the following Hermite-Hadamard-type inequalities for Riemann-Liouville fractional integrals by using convex functions of two variables on the coordinates.

Theorem 8 (see [12]). *Let $f : \Delta \subseteq \mathbb{R}^2 \to \mathbb{R}$ be convex functions on the coordinates on $\Delta = [a,b] \times [c,d]$ in $\mathbb{R}^2$ with $0 \leq a < b$, $0 \leq c < d$, and $f \in L_1(\Delta)$. Then one has the inequalities:*

$$\begin{aligned} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) &\leq \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(b-a)^\alpha (d-c)^\beta} \\ &\quad \times \left[J^{\alpha,\beta}_{a^+,c^+} f(b,d) + J^{\alpha,\beta}_{a^+,d^-} f(b,c) \right.\\ &\quad \left. + J^{\alpha,\beta}_{b^-,c^+} f(a,d) + J^{\alpha,\beta}_{b^-,d^-} f(a,c)\right] \\ &\leq \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}. \end{aligned} \tag{12}$$

In [12], Sarıkaya established some Hermite-Hadamard inequalities for convex functions on the coordinates in the second sense via fractional integrals based on the following lemma.

Lemma 9 (see [12]). *Let $f : \Delta \subseteq \mathbb{R}^2 \to \mathbb{R}$ be a partial differentiable mapping on $\Delta := [a,b] \times [c,d]$ in $\mathbb{R}^2$ with $0 \leq a < b$, $0 \leq c < d$. If $\partial^2 f/\partial t \partial s \in L(\Delta)$, then the following equality holds:*

$$\begin{aligned} &\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4} + \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(b-a)^\alpha (d-c)^\beta} \\ &\quad \times \left[J^{\alpha,\beta}_{a^+,c^+} f(b,d) + J^{\alpha,\beta}_{a^+,d^-} f(b,c) + J^{\alpha,\beta}_{b^-,c^+} f(a,d) \right.\\ &\quad \left. + J^{\alpha,\beta}_{b^-,d^-} f(a,c)\right] - A \\ &= \frac{(b-a)(d-c)}{4} \\ &\quad \times \left\{ \iint_0^1 t^\alpha s^\beta \frac{\partial^2 f}{\partial t \partial s}(ta+(1-t)b, sc+(1-s)d)\,ds\,dt \right.\\ &\quad - \iint_0^1 (1-t)^\alpha s^\beta \frac{\partial^2 f}{\partial t \partial s} \\ &\qquad \times (ta+(1-t)b, sc+(1-s)d)\,ds\,dt \\ &\quad - \iint_0^1 t^\alpha (1-s)^\beta \frac{\partial^2 f}{\partial t \partial s} \\ &\qquad \times (ta+(1-t)b, sc+(1-s)d)\,ds\,dt \\ &\quad + \iint_0^1 (1-t)^\alpha (1-s)^\beta \frac{\partial^2 f}{\partial t \partial s} \\ &\qquad \left. \times (ta+(1-t)b, sc+(1-s)d)\,ds\,dt \right\}, \end{aligned} \tag{13}$$

where

$$\begin{aligned} A &= \frac{\Gamma(\beta+1)}{4(d-c)^\beta}\left[J^\beta_{c^+} f(a,d) + J^\beta_{c^+} f(b,d) + J^\beta_{d^-} f(a,c) \right.\\ &\qquad \left. + J^\beta_{d^-} f(b,c)\right] \\ &\quad + \frac{\Gamma(\alpha+1)}{4(b-a)^\alpha}\left[J^\alpha_{a^+} f(b,c) + J^\alpha_{a^+} f(b,d) + J^\alpha_{b^-} f(a,c) \right.\\ &\qquad \left. + J^\alpha_{b^-} f(a,d)\right]. \end{aligned} \tag{14}$$

In this paper, we establish some Hermite-Hadamard type inequalities for s-convex functions on the coordinates functions via Riemann-Liouville integrals. Some integral inequalities with the right-hand side of the fractional Hermite-Hadamard type inequality are also given.

2. Fractional Inequalities for s-Convex Functions on the Coordinates

Theorem 10. *Suppose that $f : \Delta = [a,b] \times [c,d] \subseteq [0,\infty)^2 \to [0,\infty)$ is s-convex function on the coordinates in the second sense on Δ. Then one has the inequalities:*

$$\begin{aligned} &4^{s-1} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \\ &\quad \leq \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(b-a)^\alpha (d-c)^\beta} \\ &\qquad \times \left[J^{\alpha,\beta}_{a^+,c^+} f(b,d) + J^{\alpha,\beta}_{a^+,d^-} f(b,c) + J^{\alpha,\beta}_{b^-,c^+} f(a,d) \right.\\ &\qquad \left. + J^{\alpha,\beta}_{b^-,d^-} f(a,c)\right] \\ &\quad \leq \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4} \\ &\qquad \times \left(\frac{1}{(\alpha+s)(\beta+s)} + \frac{1}{\alpha+1} B(\beta, s+1) + \frac{1}{\beta+1} \right.\\ &\qquad \left. \times B(\alpha, s+1) + B(\beta, s+1) B(\alpha, s+1)\right). \end{aligned} \tag{15}$$

Proof. From (7), with $x = t_1 a + (1-t_1)b$, $y = (1-t_1)a + t_1 b$, $z = t_2 c + (1-t_2)d$, $w = (1-t_2)c + t_2 d$, and $t = \lambda = 1/2$, we get

$$\begin{aligned} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) &\leq \frac{1}{4^s}\left[f(t_1 a + (1-t_1)b, t_2 c + (1-t_2)d) \right.\\ &\quad + f(t_1 a + (1-t_1)b, (1-t_2)c + t_2 d) \\ &\quad + f((1-t_1)a + t_1 b, t_2 c + (1-t_2)d) \\ &\quad \left. + f((1-t_1)a + t_1 b, (1-t_2)c + t_2 d)\right]. \end{aligned} \tag{16}$$

Multiplying both sides of above inequality by $t_1^{\alpha-1}t_2^{\beta-1}$ then integrating the resulting inequality with respect to t_1, t_2 over $[0,1]\times[0,1]$, we obtain

$$\begin{aligned}
&\frac{1}{\alpha\beta} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \\
&\quad\le \frac{1}{4^s}\left[\iint_0^1 t_1^{\alpha-1}t_2^{\beta-1} f\left(t_1 a + (1-t_1) b, t_2 c + (1-t_2) d\right) dt_2\, dt_1\right. \\
&\qquad + \iint_0^1 t_1^{\alpha-1}t_2^{\beta-1} f\left(t_1 a + (1-t_1) b, (1-t_2) c + t_2 d\right) dt_2\, dt_1 \\
&\qquad + \iint_0^1 t_1^{\alpha-1}t_2^{\beta-1} f\left((1-t_1) a + t_1 b, t_2 c + (1-t_2) d\right) dt_2\, dt_1 \\
&\qquad \left.+ \iint_0^1 t_1^{\alpha-1}t_2^{\beta-1} f\left((1-t_1) a + t_1 b, (1-t_2) c + t_2 d\right) dt_2\, dt_1\right].
\end{aligned} \tag{17}$$

Using the change of the variable, we get

$$\begin{aligned}
&\frac{1}{\alpha\beta} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \\
&\le \frac{1}{4^s}\left[\int_a^b\int_c^d (b-x)^{\alpha-1}(d-y)^{\beta-1} f(x,y)\, dy\, dx\right. \\
&\quad + \int_a^b\int_c^d (b-x)^{\alpha-1}(y-c)^{\beta-1} f(x,y)\, dy\, dx \\
&\quad + \int_a^b\int_c^d (x-a)^{\alpha-1}(d-y)^{\beta-1} f(x,y)\, dy\, dx \\
&\quad \left.+ \int_a^b\int_c^d (x-a)^{\alpha-1}(y-c)^{\beta-1} f(x,y)\, dy\, dx\right],
\end{aligned} \tag{18}$$

by which the first inequality is proved. For the proof of the second inequality, we note that f is s-convex on coordinates, then

$$\begin{aligned}
&f\left(t_1 a + (1-t_1) b, t_2 c + (1-t_2) d\right) \\
&\quad \le t_1^s t_2^s f(a,c) + t_1^s (1-t_2)^s f(a,d) \\
&\qquad + (1-t_1)^s t_2^s f(b,c) + (1-t_1)^s (1-t_2)^s f(b,d), \\
&f\left(t_1 a + (1-t_1) b, (1-t_2) c + t_2 d\right) \\
&\quad \le t_1^s (1-t_2)^s f(a,c) + t_1^s t_2^s f(a,d) \\
&\qquad + (1-t_1)^s (1-t_2)^s f(b,c) + (1-t_1)^s t_2^s f(b,d), \\
&f\left((1-t_1) a + t_1 b, t_2 c + (1-t_2) d\right) \\
&\quad \le (1-t_1)^s t_2^s f(a,c) + (1-t_1)^s (1-t_2)^s f(a,d) \\
&\qquad + t_1^s t_2^s f(b,c) + t_1^s (1-t_2)^s f(b,d), \\
&f\left((1-t_1) a + t_1 b, (1-t_2) c + t_2 d\right) \\
&\quad \le (1-t_1)^s (1-t_2)^s f(a,c) + (1-t_1)^s t_2^s f(a,d) \\
&\qquad + t_1^s (1-t_2)^s f(b,c) + t_1^s t_2^s f(b,d).
\end{aligned} \tag{19}$$

Multiplying both sides of above inequalities by $t_1^{\alpha-1}t_2^{\beta-1}$, then integrating the resulting inequality with respect to t_1, t_2 over $[0,1]\times[0,1]$, we obtain

$$\begin{aligned}
&\iint_0^1 t_1^{\alpha-1}t_2^{\beta-1}\left[f\left(t_1 a + (1-t_1) b, t_2 c + (1-t_2) d\right)\right. \\
&\qquad + f\left(t_1 a + (1-t_1) b, (1-t_2) c + t_2 d\right) \\
&\qquad + f\left((1-t_1) a + t_1 b, t_2 c + (1-t_2) d\right) \\
&\qquad \left.+ f\left((1-t_1) a + t_1 b, (1-t_2) c + t_2 d\right)\right] dt_2\, dt_1 \\
&\quad \le \left[f(a,c) + f(a,d) + f(b,c) + f(b,d)\right] \\
&\qquad \times \iint_0^1 t_1^{\alpha-1}t_2^{\beta-1}\left[t_1^s t_2^s + t_1^s (1-t_2)^s + (1-t_1)^s t_2^s + (1-t_1)^s (1-t_2)^s\right] dt_2\, dt_1 \\
&\quad = \left[f(a,c) + f(a,d) + f(b,c) + f(b,d)\right] \\
&\qquad \times \left(\frac{1}{(\alpha+s)(\beta+s)} + \frac{1}{\alpha+1} B(\beta, s+1) + \frac{1}{\beta+1} B(\alpha, s+1) + B(\beta, s+1) B(\alpha, s+1)\right).
\end{aligned} \tag{20}$$

Here, using the change of the variable, we have

$$\begin{aligned}
&\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{(b-a)^\alpha (d-c)^\beta}\left[J_{a^+,c^+}^{\alpha,\beta} f(b,d) + J_{a^+,d^-}^{\alpha,\beta} f(b,c) + J_{b^-,c^+}^{\alpha,\beta} f(a,d) + J_{b^-,d^-}^{\alpha,\beta} f(a,c)\right] \\
&\quad \le \left[f(a,c) + f(a,d) + f(b,c) + f(b,d)\right] \\
&\qquad \times \left(\frac{1}{(\alpha+s)(\beta+s)} + \frac{1}{\alpha+1} B(\beta, s+1) + \frac{1}{\beta+1} B(\alpha, s+1) + B(\beta, s+1) B(\alpha, s+1)\right).
\end{aligned} \tag{21}$$

The proof is completed. □

Remark 11. Applying Theorem 10 for $s = 1$, we get Theorem 8.

We note that the Beta functions is defined by

$$B(x,y) = \int_0^1 (1-t)^{y-1} t^{x-1} dt, \quad x > 0, \ y > 0. \tag{22}$$

3. Inequalities for Differentiable Functions

Theorem 12. *Let $f : \Delta = [a,b] \times [c,d] \subseteq [0,\infty)^2 \to [0,\infty)$ be a partial differentiable mapping with $0 \le a < b$, $0 \le c < d$. If $|\partial^2 f/\partial t_1 \partial t_2|$ is s-convex on the coordinates in the second sense on Δ for some fixed $s \in (0,1]$. Then one has the inequalities:*

$$\begin{aligned}
&\Bigg| \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4} + \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(b-a)^{\alpha}(d-c)^{\beta}} \\
&\quad \times \left[J^{\alpha,\beta}_{a^+,c^+} f(b,d) + J^{\alpha,\beta}_{a^+,d^-} f(b,c) \right. \\
&\qquad \left. + J^{\alpha,\beta}_{b^-,c^+} f(a,d) + J^{\alpha,\beta}_{b^-,d^-} f(a,c) \right] - A \Bigg| \\
&= \frac{(b-a)(d-c)}{4} \left[\frac{1}{\alpha+s+1} + B(s+1,\alpha+1) \right] \\
&\quad \times \left[\frac{1}{\beta+s+1} + B(s+1,\beta+1) \right] \\
&\quad \times \left[\left| \frac{\partial^2 f}{\partial t_1 \partial t_2}(a,c) \right| + \left| \frac{\partial^2 f}{\partial t_1 \partial t_2}(b,c) \right| \right. \\
&\qquad \left. + \left| \frac{\partial^2 f}{\partial t_1 \partial t_2}(a,d) \right| + \left| \frac{\partial^2 f}{\partial t_1 \partial t_2}(b,d) \right| \right],
\end{aligned} \tag{23}$$

where A is as given in Lemma 9.

Proof. From Lemma 9, we obtain

$$\begin{aligned}
&\Bigg| \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4} \\
&\quad + \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(b-a)^{\alpha}(d-c)^{\beta}} \left[J^{\alpha,\beta}_{a^+,c^+} f(b,d) + J^{\alpha,\beta}_{a^+,d^-} f(b,c) \right. \\
&\qquad + J^{\alpha,\beta}_{b^-,c^+} f(a,d) \\
&\qquad \left. + J^{\alpha,\beta}_{b^-,d^-} f(a,c) \right] - A \Bigg| \\
&\le \frac{(b-a)(d-c)}{4} \\
&\quad \times \left\{ \iint_0^1 t_1^{\alpha} t_2^{\beta} \left| \frac{\partial^2 f}{\partial t_1 \partial t_2} (t_1 a + (1-t_1) b, t_2 c \right. \right. \\
&\qquad \left. + (1-t_2) d) \right| dt_2\, dt_1 \\
&\quad + \iint_0^1 (1-t_1)^{\alpha} t_2^{\beta} \\
&\qquad \times \left| \frac{\partial^2 f}{\partial t_1 \partial t_2} \right. \\
&\qquad \left. \times (t_1 a + (1-t_1) b, t_2 c + (1-t_2) d) \right| dt_2\, dt_1 \\
&\quad + \iint_0^1 t_1^{\alpha} (1-t_2)^{\beta} \\
&\qquad \times \left| \frac{\partial^2 f}{\partial t_1 \partial t_2} \right. \\
&\qquad \left. \times (t_1 a + (1-t_1) b, t_2 c + (1-t_2) d) \right| dt_2\, dt_1 \\
&\quad + \iint_0^1 (1-t_1)^{\alpha} (1-t_2)^{\beta} \\
&\qquad \times \left| \frac{\partial^2 f}{\partial t_1 \partial t_2} (t_1 a + (1-t_1) b, t_2 c \right. \\
&\qquad \left. \left. + (1-t_2) d) \right| dt_2\, dt_1 \right\}.
\end{aligned} \tag{24}$$

Because $|\partial^2 f/\partial u \partial v|$ is s-convex function on the coordinates on Δ, we obtain

$$\begin{aligned}
&\Bigg| \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4} \\
&\quad + \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(b-a)^{\alpha}(d-c)^{\beta}} \left[J^{\alpha,\beta}_{a^+,c^+} f(b,d) + J^{\alpha,\beta}_{a^+,d^-} f(b,c) \right. \\
&\qquad \left. + J^{\alpha,\beta}_{b^-,c^+} f(a,d) + J^{\alpha,\beta}_{b^-,d^-} f(a,c) \right] - A \Bigg| \\
&\le \frac{(b-a)(d-c)}{4} \\
&\quad \times \iint_0^1 \left[t_1^{\alpha} t_2^{\beta} + (1-t_1)^{\alpha} t_2^{\beta} + t_1^{\alpha} (1-t_2)^{\beta} \right. \\
&\qquad \left. + (1-t_1)^{\alpha} (1-t_2)^{\beta} \right] \\
&\qquad \times \left| \frac{\partial^2 f}{\partial t_1 \partial t_2} (t_1 a + (1-t_1) b, t_2 c + (1-t_2) d) \right| dt_2\, dt_1 \\
&\le \frac{(b-a)(d-c)}{4} \\
&\quad \times \iint_0^1 \left[t_1^{\alpha} t_2^{\beta} + (1-t_1)^{\alpha} t_2^{\beta} + t_1^{\alpha} (1-t_2)^{\beta} \right. \\
&\qquad \left. + (1-t_1)^{\alpha} (1-t_2)^{\beta} \right]
\end{aligned}$$

$$\times \left\{ t_1^s t_2^s \left| \frac{\partial^2 f}{\partial t_1 \partial t_2}(a,c) \right| + (1-t_1)^s t_2^s \left| \frac{\partial^2 f}{\partial t_1 \partial t_2}(b,c) \right| + t_1^s (1-t_2)^s \left| \frac{\partial^2 f}{\partial t_1 \partial t_2}(a,d) \right| + (1-t_1)^s (1-t_2)^s \left| \frac{\partial^2 f}{\partial t_1 \partial t_2}(b,d) \right| \right\} dt_2\, dt_1$$

$$= \frac{(b-a)(d-c)}{4} \left[\frac{1}{\alpha+s+1} + B(s+1,\alpha+1) \right] \times \left[\frac{1}{\beta+s+1} + B(s+1,\beta+1) \right]$$

$$\times \left[\left| \frac{\partial^2 f}{\partial t_1 \partial t_2}(a,c) \right| + \left| \frac{\partial^2 f}{\partial t_1 \partial t_2}(b,c) \right| + \left| \frac{\partial^2 f}{\partial t_1 \partial t_2}(a,d) \right| + \left| \frac{\partial^2 f}{\partial t_1 \partial t_2}(b,d) \right| \right], \tag{25}$$

which completes the proof. □

Theorem 13. *Let $f : \Delta = [a,b] \times [c,d] \subseteq [0,\infty)^2 \to [0,\infty)$ be a partial differentiable mapping with $0 \le a < b$, $0 \le c < d$. If $|\partial^2 f/\partial u \partial v|^q$ is s-convex on the coordinates in the second sense on Δ for some fixed $s \in (0,1]$ and $q > 1$. Then, one has the inequalities:*

$$\begin{aligned} &\left| \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4} + \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(b-a)^\alpha (d-c)^\beta} \right. \\ &\quad \times \left[J^{\alpha,\beta}_{a^+,c^+} f(b,d) + J^{\alpha,\beta}_{a^+,d^-} f(b,c) + J^{\alpha,\beta}_{b^-,c^+} f(a,d) \right. \\ &\quad \left.\left. + J^{\alpha,\beta}_{b^-,d^-} f(a,c) \right] - A \right| \\ &\le \frac{(b-a)(d-c)}{[(\alpha p+1)(\beta p+1)]^{1/p}} \\ &\quad \times \left(\frac{\left|(\partial^2 f/\partial t_1 \partial t_2)(a,c)\right|^q + \left|(\partial^2 f/\partial t_1 \partial t_2)(b,c)\right|^q + \left|(\partial^2 f/\partial t_1 \partial t_2)(a,d)\right|^q + \left|(\partial^2 f/\partial t_1 \partial t_2)(b,d)\right|^q}{(s+1)^2} \right)^{1/q}, \end{aligned} \tag{26}$$

where $(1/p) + (1/q) = 1$ and A are as given in Lemma 9.

Proof. From Lemma 9, we obtain

$$\begin{aligned} &\left| \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4} + \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(b-a)^\alpha (d-c)^\beta} \right. \\ &\quad \times \left[J^{\alpha,\beta}_{a^+,c^+} f(b,d) + J^{\alpha,\beta}_{a^+,d^-} f(b,c) + J^{\alpha,\beta}_{b^-,c^+} f(a,d) \right. \\ &\quad \left.\left. + J^{\alpha,\beta}_{b^-,d^-} f(a,c) \right] - A \right| \\ &\le \frac{(b-a)(d-c)}{4} \\ &\quad \times \left\{ \iint_0^1 t_1^\alpha t_2^\beta \times \left| \frac{\partial^2 f}{\partial t_1 \partial t_2}(t_1 a + (1-t_1) b, t_2 c + (1-t_2) d) \right| dt_2\, dt_1 \right. \\ &\quad + \iint_0^1 (1-t_1)^\alpha t_2^\beta \times \left| \frac{\partial^2 f}{\partial t_1 \partial t_2} c(t_1 a + (1-t_1) b, t_2 c + (1-t_2) d) \right| dt_2\, dt_1 \\ &\quad + \iint_0^1 t_1^\alpha (1-t_2)^\beta \times \left| \frac{\partial^2 f}{\partial t_1 \partial t_2}(t_1 a + (1-t_1) b, t_2 c (1-t_2) d) \right| dt_2\, dt_1 \\ &\quad + \iint_0^1 (1-t_1)^\alpha (1-t_2)^\beta \times \left| \frac{\partial^2 f}{\partial t_1 \partial t_2} \right. \\ &\quad \left.\left. \times (t_1 a + (1-t_1) b, t_2 c + (1-t_2) d) \right| dt_2\, dt_1 \right\}. \end{aligned} \tag{27}$$

By using the well-known Hölder's inequality for double integrals, then one has

$$\left| \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4} + \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(b-a)^\alpha (d-c)^\beta} \right.$$

$$\times\left[J^{\alpha,\beta}_{a^+,c^+}f(b,d)+J^{\alpha,\beta}_{a^+,d^-}f(b,c)+J^{\alpha,\beta}_{b^-,c^+}f(a,d)+J^{\alpha,\beta}_{b^-,d^-}f(a,c)\right]-A\Bigg|$$
$$\le\frac{(b-a)(d-c)}{4}\left\{\left(\iint_0^1 t_1^{p\alpha}t_2^{p\beta}dt_2\,dt_1\right)^{1/p}+\left(\iint_0^1(1-t_1)^{p\alpha}t_2^{p\beta}dt_2\,dt_1\right)^{1/p}+\left(\iint_0^1 t_1^{p\alpha}(1-t_2)^{p\beta}dt_2\,dt_1\right)^{1/p}+\left(\iint_0^1(1-t)^{p\alpha}(1-s)^{p\beta}dt_2\,dt_1\right)^{1/p}\right\}$$
$$\times\left(\iint_0^1\left|\frac{\partial^2 f}{\partial t_1\partial t_2}(t_1a+(1-t_1)b,t_2c+(1-t_2)d)\right|^q dt_2\,dt_1\right)^{1/q}. \tag{28}$$

Because $|\partial^2 f/\partial t_1\partial t_2|$ is s-convex function on the coordinates on Δ, by (8), we have

$$\iint_0^1\left|\frac{\partial^2 f}{\partial t_1\partial t_2}(t_1a+(1-t_1)b,t_2c+(1-t_2)d)\right|^q dt_2\,dt_1 \le\frac{\left|(\partial^2 f/\partial t_1\partial t_2)(a,c)\right|^q+\left|(\partial^2 f/\partial t_1\partial t_2)(b,c)\right|^q+\left|(\partial^2 f/\partial t_1\partial t_2)(a,d)\right|^q+\left|(\partial^2 f/\partial t_1\partial t_2)(b,d)\right|^q}{(s+1)^2}. \tag{29}$$

So

$$\Bigg|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}+\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(b-a)^\alpha(d-c)^\beta}\times\left[J^{\alpha,\beta}_{a^+,c^+}f(b,d)+J^{\alpha,\beta}_{a^+,d^-}f(b,c)+J^{\alpha,\beta}_{b^-,c^+}f(a,d)+J^{\alpha,\beta}_{b^-,d^-}f(a,c)\right]-A\Bigg|$$
$$\le\frac{(b-a)(d-c)}{[(\alpha p+1)(\beta p+1)]^{1/p}}\times\left(\frac{\left|(\partial^2 f/\partial t_1\partial t_2)(a,c)\right|^q+\left|(\partial^2 f/\partial t_1\partial t_2)(b,c)\right|^q+\left|(\partial^2 f/\partial t_1\partial t_2)(a,d)\right|^q+\left|(\partial^2 f/\partial t_1\partial t_2)(b,d)\right|^q}{(s+1)^2}\right)^{1/q}, \tag{30}$$

which completes the proof. □

Theorem 14. *Let $f:\Delta=[a,b]\times[c,d]\subseteq[0,\infty)^2\to[0,\infty)$ be a partial differentiable mapping with $0\le a<b$, $0\le c<d$. If $|\partial^2 f/\partial u\partial v|^q$ is s-concave on the coordinates in the second sense on Δ for some fixed $s\in(0,1]$ and $q>1$, then one has the inequalities:*

$$\Bigg|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}+\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(b-a)^\alpha(d-c)^\beta}\times\left[J^{\alpha,\beta}_{a^+,c^+}f(b,d)+J^{\alpha,\beta}_{a^+,d^-}f(b,c)+J^{\alpha,\beta}_{b^-,c^+}f(a,d)+J^{\alpha,\beta}_{b^-,d^-}f(a,c)\right]-A\Bigg|$$
$$\le\frac{(b-a)(d-c)}{[(\alpha p+1)(\beta p+1)]^{1/p}}\times 4^{(s-1)/q}\left|\frac{\partial^2 f}{\partial t_1\partial t_2}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|, \tag{31}$$

where $(1/p)+(1/q)=1$ and A is as given in Lemma 9.

Proof. Similarly as in Theorem 13, we obtain

$$\Bigg|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}+\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(b-a)^\alpha(d-c)^\beta}\times\left[J^{\alpha,\beta}_{a^+,c^+}f(b,d)+J^{\alpha,\beta}_{a^+,d^-}f(b,c)+J^{\alpha,\beta}_{b^-,c^+}f(a,d)+J^{\alpha,\beta}_{b^-,d^-}f(a,c)\right]-A\Bigg|$$
$$\le\frac{(b-a)(d-c)}{4}\times\left\{\left(\iint_0^1 t_1^{p\alpha}t_2^{p\beta}dt_2\,dt_1\right)^{1/p}+\left(\iint_0^1(1-t_1)^{p\alpha}t_2^{p\beta}dt_2\,dt_1\right)^{1/p}\right.$$

$$+\left(\iint_0^1 t_1^{p\alpha}(1-t_2)^{p\beta}dt_2\,dt_1\right)^{1/p}$$
$$+\left(\iint_0^1 (1-t)^{p\alpha}(1-s)^{p\beta}dt_2\,dt_1\right)^{1/p}\Bigg\}$$
$$\times\left(\iint_0^1\left|\frac{\partial^2 f}{\partial t_1\partial t_2}\left(t_1a+(1-t_1)b,t_2c+(1-t_2)d\right)\right|^q dt_2\,dt_1\right)^{1/q}. \tag{32}$$

Because $|\partial^2 f/\partial t_1\partial t_2|$ is s-concave function on the coordinates on Δ, by the reversed direction of (8), then one has

$$\begin{aligned}&\iint_0^1\left|\frac{\partial^2 f}{\partial t_1\partial t_2}\left(t_1a+(1-t_1)b,t_2c+(1-t_2)d\right)\right|^q dt_2\,dt_1\\ &\quad\le 4^{s-1}\left|\frac{\partial^2 f}{\partial t_1\partial t_2}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|^q.\end{aligned} \tag{33}$$

Then

$$\begin{aligned}&\Bigg|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}+\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(b-a)^\alpha(d-c)^\beta}\\ &\quad\times\left[J^{\alpha,\beta}_{a^+,c^+}f(b,d)+J^{\alpha,\beta}_{a^+,d^-}f(b,c)\right.\\ &\qquad\left.+J^{\alpha,\beta}_{b^-,c^+}f(a,d)+J^{\alpha,\beta}_{b^-,d^-}f(a,c)\right]-A\Bigg|\\ &\quad\le\frac{(b-a)(d-c)}{\left[(\alpha p+1)(\beta p+1)\right]^{1/p}}\\ &\qquad\times 4^{(s-1)/q}\left|\frac{\partial^2 f}{\partial t_1\partial t_2}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|.\end{aligned} \tag{34}$$

We get the desired results. □

4. Conclusion

In this paper, we obtain some Hermite-Hadamard type inequalities for coordinated s-convex functions via Riemann-Liouville integrals. An interesting topic is whether we can use the methods in this paper to establish the Hermite-Hadamard inequalities for other kinds of convex functions on the coordinates via Riemann-Liouville integrals.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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