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Research Article

A Class of Iterative Nonlinear Difference Inequality with Weakly Singularity

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We discuss a class of new nonlinear weakly singular difference inequality, which is solved by change of variable, discrete Hölder inequality, discrete Jensen inequality, the mean-value theorem for integrals and amplification method, and Gamma function. Explicit bound for the unknown function is given clearly. Moreover, an example is presented to show the usefulness of our results.

1. Introduction

Being an important tool in the study of qualitative properties of solutions of differential equations and integral equations, various generalizations of Gronwall inequalities and their applications have attracted great interests of many mathematicians (such as [1–21]). Gronwall-Bellman inequality [22, 23] can be stated as follows: if u and f are nonnegative and continuous functions on an interval $[a, b]$ satisfying

$$u(t) \le c + \int_a^t f(s)\, u(s)\, ds, \quad t \in [a, b], \tag{1}$$

for some constant $c \ge 0$, then

$$u(t) \le c \exp\left(\int_a^t f(s)\, ds\right), \quad t \in [a, b]. \tag{2}$$

In 1981, Henry [2] discussed the following linear singular integral inequality:

$$u(t) \le a + b\int_0^t (t-s)^{\beta-1} u(s)\, ds. \tag{3}$$

In 2007, Ye et al. [20] discussed linear singular integral inequality:

$$u(t) \le a(t) + b(t)\int_0^t (t-s)^{\beta-1} u(s)\, ds. \tag{4}$$

In 2011, Abdeldaim and Yakout [21] studied a new integral inequality of Gronwall-Bellman-Pachpatte type

$$\begin{aligned} u(t) \le u_0 + \int_{t_0}^t & f(s)\, u(s) \\ & \times \left[u(s) + \int_{t_0}^s h(\tau) \right. \\ & \left. \times \left[u(\tau) + \int_{t_0}^\tau g(\xi)\, u(\xi)\, d\xi\right] d\tau\right] ds. \end{aligned} \tag{5}$$

On the other hand, many physical problems arising in a wide variety of applications are governed by finite difference equations. The theory of difference equations has been developed as a natural discrete analogue of corresponding theory of differential equations. Difference inequalities which give explicit bounds on unknown functions provide a very useful and important tool in the study of many qualitative as well as quantitative properties of solutions of nonlinear difference equations (such as [24–32]). Sugiyama [26] established the most precise and complete discrete analogue of the Gronwall inequality in the following form:

$$u(n) \le u_0 + \sum_{s=n_0}^{n-1} f(s)\, u(s). \tag{6}$$

For instance, Pachpatte [27] considered the following discrete inequality:

$$u(n) \le u_0 + \sum_{s=n_0}^{n-1} f(s)[u(s) + h(s)] + \sum_{s=n_0}^{n-1} f(s)\left(\sum_{\tau=n_0}^{s-1} g(\tau)u(\tau)\right), \quad \forall n \in N_0. \tag{7}$$

In 2006, Cheung and Ren [29] studied

$$u^p(m,n) \le c + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} a(s,t)u^q(s,t) + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} b(s,t)u^q(s,t)w(u(s,t)). \tag{8}$$

Later, Zheng et al. [31] discussed the following discrete inequality:

$$u(n) \le a(n) + \sum_{i=1}^{k}\sum_{s=0}^{n-1} f_i(n,s)w_i(u(s)). \tag{9}$$

Motivated by the results given in [2, 20, 21, 32], in this paper, we discuss a new linear singular integral inequality

$$\begin{aligned} u(n) \le{}& a(n) + b(n) \\ &\times \sum_{s=0}^{n-1}(t_n - t_s)^{\beta-1}\tau_s w_1(u(s)) \\ &\times \left[u(s) + h(s) + \sum_{\tau=0}^{s-1}(t_s - t_\tau)^{\beta-1}\tau_\tau w_2(u(\tau))\right], \end{aligned} \tag{10}$$

where $t_0 = 0$, $\tau_s = t_{s+1} - t_s > 0$, $\sup_{s\in\mathbf{N},0\le s\le n-1}\{\tau_s, s \in \mathbf{N}\} = \tau$, and $\lim_{n\to\infty} t_n = \infty$.

For the reader's convenience, we present some necessary lemmas.

Lemma 1 (discrete Jensen inequality [28]). *Let $A_1, A_2, \ldots, A_n$ be nonnegative real numbers, $r > 1$ is a real number, and n is a natural number. Then*

$$(A_1 + A_2 + \cdots + A_n)^r \le n^{r-1}(A_1^r + A_2^r + \cdots + A_n^r). \tag{11}$$

Lemma 2 (discrete Hölder inequality [30]). *Let a_i, b_i ($i = 1, 2, \ldots, n$) be nonnegative real numbers, and let p, q be positive numbers such that $(1/q) + (1/p) = 1$; then*

$$\sum_{s=0}^{n-1} a_i b_i \le \left(\sum_{s=0}^{n-1} a_i^p\right)^{1/p}\left(\sum_{s=0}^{n-1} b_i^q\right)^{1/q}. \tag{12}$$

Lemma 3. *Let $t_0 = 0$, $\tau_s = t_{s+1} - t_s > 0$, $\sup_{s\in\mathbf{N},0\le s\le n-1}\{\tau_s, s \in \mathbf{N}\} = \tau$, and $\lim_{n\to\infty} t_n = \infty$. If $\beta \in [0, 1/2]$, $0 < p < 1/(1-\beta)$, then*

$$\sum_{s=0}^{n-1}(t_n - t_s)^{p(\beta-1)}e^{pt_s}\tau_s \le \frac{e^{pt_n}}{p^{1+p(\beta-1)}}\Gamma(1 + p(\beta - 1)), \tag{13}$$

where $1 + p(\beta-1) > 0$, $\Gamma(\beta) := \int_0^\infty \tau^{\beta-1}e^{-\tau}d\tau$ is the well-known Γ-function.

Proof. By the definition of integration and the conditions in Lemma 3, we have

$$\sum_{s=0}^{n-1}(t_n - t_s)^{p\beta-p}e^{pt_s}\tau_s \le \int_0^{t_n}(t_n - s)^{p\beta-p}e^{ps}ds. \tag{14}$$

Using a change of variables $\tau = t_n - s$ and $\xi = p\tau$, we have the estimation

$$\begin{aligned} \int_0^{t_n}(t_n - s)^{p(\beta-1)}e^{ps}ds &= -\int_{t_n}^0 \tau^{p(\beta-1)}e^{pt_n - p\tau}d\tau \\ &= e^{pt_n}\int_0^{t_n}\tau^{p(\beta-1)}e^{-p\tau}d\tau \\ &= \frac{e^{pt_n}}{p^{1+p(\beta-1)}}\int_0^{pt_n}\xi^{p(\beta-1)}e^{-\xi}d\xi \\ &\le \frac{e^{pt_n}}{p^{1+p(\beta-1)}}\Gamma(1 + p(\beta - 1)). \end{aligned} \tag{15}$$

Since $0 < \beta \le 1/2$, $p < 1/(1-\beta)$, $1 + p(\beta - 1) > 0$, and $\Gamma(1 + p(\beta - 1)) \in R^+$, from (14) and (15), we have the relation (13). □

2. Main Result

In this section, we give the estimation of unknown function in (10). Let $\mathbf{N} := \{0, 1, 2, \ldots\}$. For function $z(n)$, its difference is defined by $\Delta z = z(n+1) - z(n)$. Obviously, the linear difference equation $\triangle z(n) = b(n)$ with the initial condition $z(n_0) = 0$ has the solution $z(n) = \sum_{s=0}^{n-1} b(s)$. For convenience, in the sequel we complementarily define that $\sum_{s=0}^{-1} b(s) = 0$.

Theorem 4. *Suppose that $0 < \beta \le 1/2$ is a constant, $a(n), b(n)$ are nonnegative and nondecreasing functions defined on $\mathbf{N}$, $w_1(u), w_2(u)/w_1(u), w_2^q(u^{1/q})/(uw_1^q(u^{1/q}))$ are nonnegative, nondecreasing, and continuous functions defined on $\mathbf{R}_+$, $t_0 = 0$, $\tau_s = t_{s+1} - t_s > 0$, $\sup_{s\in\mathbf{N},0\le s\le n-1}\{\tau_s, s \in \mathbf{N}\} = \tau$, and $\lim_{n\to\infty} t_n = \infty$. If $u(t)$ satisfies (10), then*

$$u(n) \le \left\{H_1^{-1}\left\{H_2^{-1}\left[H_3^{-1}(U(n))\right]\right\}\right\}^{1/q}, \quad \forall n \in \mathbf{N}_1, \tag{16}$$

where

$$U(n) := H_3\left(H_2\left(H_1\left(2^{q-1}a(n)\right)+f(n)\sum_{s=0}^{n-1}h^q(s)e^{-qt_s}\right.\right.$$
$$\left.\left.+f(n)\sum_{s=0}^{n-1}e^{-qt_s}\right)+g(n)\sum_{s=0}^{n-1}e^{-qt_s}, \tag{17}$$

$$H_1(u) := \int_{c_1}^{u}\frac{ds}{w_1^q(s^{1/q})}, \quad u>0,\ c_1>0, \tag{18}$$

$$H_2(u) := \int_{c_2}^{u}\frac{ds}{H_1^{-1}(s)}, \quad u>0,\ c_2>0, \tag{19}$$

$$H_3(u) := \int_{c_3}^{u}\frac{H_2^{-1}\left(H_1^{-1}(s)\right)w_1^q\left(\left(H_2^{-1}\left(H_1^{-1}(s)\right)\right)^{1/q}\right)}{w_2^q\left(\left(H_2^{-1}\left(H_1^{-1}(s)\right)\right)^{1/q}\right)}, \quad u>0,\ c_3>0, \tag{20}$$

$$f(n) := 6^{q-1}(b(n))^q\tau^{q(p-1)/p}\times\left[\frac{e^{pt_n}}{p^{1+p(\beta-1)}}\Gamma\left(1+p(\beta-1)\right)\right]^{q/p}, \tag{21}$$

$$g(n) := \tau^{q(p-1)/p}\left(\frac{e^{pt_n}}{p^{1+p(\beta-1)}}\Gamma\left(1+p(\beta-1)\right)\right)^{q/p}, \tag{22}$$

and $p=1+\beta$, $q=1+1/\beta$, $\mathbf{N}_1 := \{0,1,2,\ldots,K_1\}$, K_1 *is the largest integer number such that*

$$U(K_1)\in \operatorname{Dom}\left(H_3^{-1}\right), \qquad H_3^{-1}\left(U(K_1)\right)\in \operatorname{Dom}\left(H_2^{-1}\right),$$
$$H_2^{-1}\left(H_3^{-1}\left(U(K_1)\right)\right)\in \operatorname{Dom}\left(H_1^{-1}\right). \tag{23}$$

Proof. From (10), we have

$$u(n) \le a(n)+b(n)\times\sum_{s=0}^{n-1}(t_n-t_s)^{\beta-1}e^{t_s}e^{-t_s}\tau_s w_1(u(s))\times\left[u(s)+h(s)+\sum_{\sigma=0}^{s-1}(t_s-t_\sigma)^{\beta-1}\tau_\sigma w_2(u(\sigma))\right], \quad \forall n\in\mathbf{N}. \tag{24}$$

Applying Lemma 2 with $p=1+\beta$, $q=1+1/\beta$ to (24), we obtain that

$$u(n) \le a(n)+b(n)\tau^{(p-1)/p}\left[\sum_{s=0}^{n-1}(t_n-t_s)^{p\beta-p}e^{pt_s}\tau_s\right]^{1/p}\times\left[\sum_{s=0}^{n-1}e^{-qt_s}w_1^q(u(s))\times\left[u(s)+h(s)+\sum_{\sigma=0}^{s-1}(t_s-t_\sigma)^{\beta-1}\tau_\sigma w_2(u(\sigma))\right]^q\right]^{1/q}, \tag{25}$$

where $\tau_s<\tau$ is used. Applying Lemma 3, we have

$$u(n) \le a(n)+b(n)\tau^{(p-1)/p}\left[\frac{e^{pt_n}}{p^{1+p(\beta-1)}}\Gamma\left(1+p(\beta-1)\right)\right]^{1/p}\times\left[\sum_{s=0}^{n-1}e^{-qt_s}w_1^q(u(s))\times\left[u(s)+h(s)+\sum_{\sigma=0}^{s-1}(t_s-t_\sigma)^{\beta-1}\tau_\sigma w_2(u(\sigma))\right]^q\right]^{1/q}. \tag{26}$$

By discrete Jensen inequality (11) with $n=2$, $r=q$, from (26) we obtain that

$$u^q(n) \le 2^{q-1}(a(n))^q+2^{q-1}(b(n))^q\tau^{q(p-1)/p}\times\left[\frac{e^{pt_n}}{p^{1+p(\beta-1)}}\Gamma\left(1+p(\beta-1)\right)\right]^{q/p}\times\sum_{s=0}^{n-1}e^{-qt_s}w_1^q(u(s))\times\left[u(s)+h(s)+\sum_{\sigma=0}^{s-1}(t_s-t_\sigma)^{\beta-1}\tau_\sigma w_2(u(\sigma))\right]^q. \tag{27}$$

Again using discrete Jensen inequality (11) with $n = 3$, $r = q$, from (27) we obtain that

$$\begin{aligned} u^q(n) \le\ & 2^{q-1}(a(n))^q + 2^{q-1}(b(n))^q \tau^{q(p-1)/p} \\ & \times \left[\frac{e^{pt_n}}{p^{1+p(\beta-1)}}\Gamma(1+p(\beta-1))\right]^{q/p} \\ & \times \sum_{s=0}^{n-1} e^{-qt_s} w_1^q(u(s)) \\ & \times \left[3^{q-1}u^q(s) + 3^{q-1}h^q(s) + 3^{q-1} \right. \\ & \left. \times \left[\sum_{\sigma=0}^{s-1}(t_s - t_\sigma)^{\beta-1}\tau_\sigma w_2(u(\sigma))\right]^q\right], \\ & \forall n \in \mathbf{N}. \end{aligned} \tag{28}$$

For $[\sum_{\sigma=0}^{s-1}(t_s - t_\sigma)^{\beta-1}\tau_\sigma w_2(u(\sigma))]^q$ in (28), applying Lemma 2 with $p = 1 + \beta$, $q = 1 + 1/\beta$, we obtain that

$$\begin{aligned} & \left[\sum_{\sigma=0}^{s-1}(t_s - t_\sigma)^{\beta-1}\tau_\sigma w_2(u(\sigma))\right]^q \\ & \le \tau^{q(p-1)/p}\left(\sum_{\sigma=0}^{s-1}(t_s - t_\sigma)^{p\beta-p} e^{pt_\sigma}\tau_\sigma\right)^{q/p} \\ & \quad \times \sum_{\sigma=0}^{s-1} e^{-qt_\sigma} w_2^q(u(\sigma)) \\ & \le \tau^{q(p-1)/p}\left(\frac{e^{pt_s}}{p^{1+p(\beta-1)}}\Gamma(1+p(\beta-1))\right)^{q/p} \\ & \quad \times \sum_{\sigma=0}^{s-1} e^{-qt_\sigma} w_2^q(u(\sigma)); \end{aligned} \tag{29}$$

here Lemma 3 is used. Substituting (29) into (28), we have

$$\begin{aligned} u^q(n) \le\ & 2^{q-1}(a(n))^q + 2^{q-1}(b(n))^q \tau^{q(p-1)/p} \\ & \times \left[\frac{e^{pt_n}}{p^{1+p(\beta-1)}}\Gamma(1+p(\beta-1))\right]^{q/p} \sum_{s=0}^{n-1} e^{-qt_s} w_1^q(u(s)) \\ & \times \left[3^{q-1}u^q(s) + 3^{q-1}h^q(s) + 3^{q-1}\tau^{q(p-1)/p}\right. \\ & \times \left(\frac{e^{pt_s}}{p^{1+p(\beta-1)}}\Gamma(1+p(\beta-1))\right)^{q/p} \\ & \left. \times \sum_{\sigma=0}^{s-1} e^{-qt_\sigma} w_2^q(u(\sigma))\right] \\ =\ & 2^{q-1}(a(n))^q + f(n)\sum_{s=0}^{n-1} e^{-qt_s} w_1^q(u(s)) \\ & \times \left[u^q(s) + h^q(s) + g(s)\sum_{\sigma=0}^{s-1} e^{-qt_\sigma} w_2^q(u(\sigma))\right] \\ =\ & 2^{q-1}(a(n))^q + f(n)\sum_{s=0}^{n-1} h^q(s) e^{-qt_s} w_1^q(u(s)) \\ & + f(n)\sum_{s=0}^{n-1} e^{-qt_s} w_1^q(u(s)) \\ & \times \left[u^q(s) + g(s)\sum_{\sigma=0}^{s-1} e^{-qt_\sigma} w_2^q(u(\sigma))\right], \\ & \forall n \in \mathbf{N}, \end{aligned} \tag{30}$$

where $f(n)$ and $g(n)$ are defined by (21) and (22), respectively. Let $v(n) := u^q(n)$; from (30) we have

$$\begin{aligned} v(n) \le\ & 2^{q-1}(a(n))^q + f(n)\sum_{s=0}^{n-1} h^q(s) e^{-qt_s} w_1^q\left(v^{1/q}(s)\right) \\ & + f(n)\sum_{s=0}^{n-1} e^{-qt_s} w_1^q\left(v^{1/q}(s)\right) \\ & \times \left[v(s) + g(s)\sum_{\sigma=0}^{s-1} e^{-qt_\sigma} w_2^q\left(v^{1/q}(\sigma)\right)\right], \\ & \forall n \in \mathbf{N}. \end{aligned} \tag{31}$$

Since $f(n)$, $g(n)$ are nondecreasing functions, from (31) we have

$$\begin{aligned} v(n) \le\ & 2^{q-1}(a(K))^q + f(K)\sum_{s=0}^{n-1} h^q(s) e^{-qt_s} w_1^q\left(v^{1/q}(s)\right) \\ & + f(K)\sum_{s=0}^{n-1} e^{-qt_s} w_1^q\left(v^{1/q}(s)\right) \\ & \times \left[v(s) + g(K)\sum_{\sigma=0}^{s-1} e^{-qt_\sigma} w_2^q\left(v^{1/q}(\sigma)\right)\right], \\ & \forall n \in [0, K] \cap \mathbf{N}, \end{aligned} \tag{32}$$

where $K \in \mathbf{N}$, $K \le K_1$ is chosen arbitrarily.

Let $z_1(t)$ denote the function on the right-hand side of (32), which is a positive and nondecreasing function on $[0, K] \cap \mathbf{N}$. From (32), we have

$$z_1(0) = 2^{q-1}(a(K))^q, \qquad v(n) \le z_1(n), \quad \forall n \in [0, K] \cap \mathbf{N}. \tag{33}$$

Using $\Delta z_1(n) = z_1(n+1) - z_1(n)$ and (33), we obtain

$$\begin{aligned}
\Delta z_1(n) &= f(K) h^q(n) e^{-qt_n} w_1^q\left(v^{1/q}(n)\right) \\
&\quad + f(K) e^{-qt_n} w_1^q\left(v^{1/q}(n)\right) \\
&\quad \times \left[v(n) + g(K) \sum_{\sigma=0}^{n-1} e^{-qt_\sigma} w_2^q\left(v^{1/q}(\sigma)\right)\right] \\
&\le f(K) h^q(n) e^{-qt_n} w_1^q\left(z_1^{1/q}(n)\right) \\
&\quad + f(K) e^{-qt_n} w_1^q\left(z_1^{1/q}(n)\right) \\
&\quad \times \left[z_1(n) + g(K) \sum_{\sigma=0}^{n-1} e^{-qt_\sigma} w_2^q\left(z_1^{1/q}(\sigma)\right)\right],
\end{aligned} \tag{34}$$

for all $n \in [0, K] \cap \mathbf{N}$.

Let

$$y(n) = z_1(n) + g(K) \sum_{\sigma=0}^{n-1} e^{-qt_\sigma} w_2^q\left(z_1^{1/q}(\sigma)\right), \quad \forall n \in [0, K] \cap \mathbf{N}. \tag{35}$$

Then

$$y(0) = z_1(0), \qquad z_1(n) \le y(n), \quad \forall n \in [0, K] \cap \mathbf{N}. \tag{36}$$

From (35), we have

$$\begin{aligned}
\Delta y(n) &= \Delta z_1(n) + g(K) e^{-qt_n} w_2^q\left(z_1^{1/q}(n)\right) \\
&\le f(K) h^q(n) e^{-qt_n} w_1^q\left(z_1^{1/q}(n)\right) \\
&\quad + f(K) e^{-qt_n} w_1^q\left(z_1^{1/q}(n)\right) y(n) \\
&\quad + g(K) e^{-qt_n} w_2^q\left(z_1^{1/q}(n)\right) \\
&\le f(K) h^q(n) e^{-qt_n} w_1^q\left(y^{1/q}(n)\right) \\
&\quad + f(K) e^{-qt_n} w_1^q\left(y^{1/q}(n)\right) y(n) \\
&\quad + g(K) e^{-qt_n} w_2^q\left(y^{1/q}(n)\right).
\end{aligned} \tag{37}$$

It implies that, for all $n \in [0, K] \cap \mathbf{N}$,

$$\begin{aligned}
\frac{\Delta y(n)}{w_1^q\left(y^{1/q}(n)\right)} &\le f(K) h^q(n) e^{-qt_n} + f(K) e^{-qt_n} y(n) \\
&\quad + g(K) e^{-qt_n} \frac{w_2^q\left(y^{1/q}(n)\right)}{w_1^q\left(y^{1/q}(n)\right)}.
\end{aligned} \tag{38}$$

On the other hand, by the mean-value theorem for integrals, for arbitrarily given integers $n, n+1 \in [0, K] \cap \mathbf{N}$, there exists ξ in the open interval $(y(n), y(n+1))$ such that

$$\begin{aligned}
H_1(y(n+1)) - H_1(y(n)) &= \int_{y(n)}^{y(n+1)} \frac{ds}{w_1^q\left(s^{1/q}\right)} \\
&= \frac{\Delta y(n)}{w_1^q\left(\xi^{1/q}\right)} \le \frac{\Delta y(n)}{w_1^q\left(y^{1/q}(n)\right)},
\end{aligned} \tag{39}$$

for all $n \in [0, K] \cap \mathbf{N}$, where H_1 is defined by (18). From (38) and (39), we have

$$\begin{aligned}
H_1(y(n+1)) - H_1(y(n)) &\le f(K) h^q(n) e^{-qt_n} \\
&\quad + f(K) e^{-qt_n} y(n) \\
&\quad + g(K) e^{-qt_n} \frac{w_2^q\left(y^{1/q}(n)\right)}{w_1^q\left(y^{1/q}(n)\right)}.
\end{aligned} \tag{40}$$

Taking $n = s$ in (40) and summing up over s from 0 to $n-1$, from (40) we obtain

$$\begin{aligned}
H_1(y(n)) &\le H_1(y(0)) + \sum_{s=0}^{n-1} f(K) h^q(s) e^{-qt_s} \\
&\quad + \sum_{s=0}^{n-1} f(K) e^{-qt_s} y(s) \\
&\quad + \sum_{s=0}^{n-1} g(K) e^{-qt_s} \frac{w_2^q\left(y^{1/q}(s)\right)}{w_1^q\left(y^{1/q}(s)\right)} \\
&\le H_1(y(0)) + \sum_{s=0}^{K-1} f(K) h^q(s) e^{-qt_s} \\
&\quad + \sum_{s=0}^{n-1} f(K) e^{-qt_s} y(s) \\
&\quad + \sum_{s=0}^{n-1} g(K) e^{-qt_s} \frac{w_2^q\left(y^{1/q}(s)\right)}{w_1^q\left(y^{1/q}(s)\right)}, \\
&\qquad \forall n \in [0, K] \cap \mathbf{N}.
\end{aligned} \tag{41}$$

Let $z_2(t)$ denote the function on the right-hand side of (41), which is a positive and nondecreasing function on $[0, K] \cap \mathbf{N}$. From (41), we have

$$\begin{aligned}
z_2(0) &= H_1(y(0)) + \sum_{s=0}^{K-1} f(K) h^q(s) e^{-qt_s}, \\
y(n) &\le H_1^{-1}(z_2(n)), \\
&\quad \forall n \in [0, K] \cap \mathbf{N}.
\end{aligned} \tag{42}$$

Using $\Delta z_2(n) = z_2(n+1) - z_2(n)$ and (42), we obtain

$$\begin{aligned}
\Delta z_2(n) &= f(K) e^{-qt_n} y(n) + g(K) e^{-qt_n} \frac{w_2^q\left(y^{1/q}(n)\right)}{w_1^q\left(y^{1/q}(n)\right)} \\
&\le f(K) e^{-qt_n} H_1^{-1}(z_2(n)) \\
&\quad + g(K) e^{-qt_n} \frac{w_2^q\left(\left(H_1^{-1}(z_2(n))\right)^{1/q}\right)}{w_1^q\left(\left(H_1^{-1}(z_2(n))\right)^{1/q}\right)}, \\
&\qquad \forall n \in [0, K] \cap \mathbf{N}.
\end{aligned} \tag{43}$$

From (43), we have

$$\frac{\Delta z_2(n)}{H_1^{-1}(z_2(n))} \le f(K)e^{-qt_n} + g(K)e^{-qt_n}\frac{w_2^q\left(\left(H_1^{-1}(z_2(n))\right)^{1/q}\right)}{H_1^{-1}(z_2(n))\,w_1^q\left(\left(H_1^{-1}(z_2(n))\right)^{1/q}\right)}, \tag{44}$$

for all $n \in [0, K] \cap \mathbf{N}$. Again by the mean-value theorem for integrals, for arbitrarily given integers $n, n+1 \in [0, K] \cap \mathbf{N}$, there exists ξ in the open interval $(z_2(n), z_2(n+1))$ such that

$$\begin{aligned} H_2(z_2(n+1)) - H_2(z_2(n)) &= \int_{z_2(n)}^{z_2(n+1)} \frac{ds}{H_1^{-1}(s)} \\ &= \frac{\Delta z_2(n)}{H_1^{-1}(\xi)} \le \frac{\Delta z_2(n)}{H_1^{-1}(z_2(n))}, \end{aligned} \tag{45}$$

where H_2 is defined by (19). From (44) and (45), we have

$$\begin{aligned} &H_2(z_2(n+1)) - H_2(z_2(n)) \\ &\quad\le f(K)e^{-qt_n} + g(K)e^{-qt_n} \\ &\qquad\times \frac{w_2^q\left(\left(H_1^{-1}(z_2(n))\right)^{1/q}\right)}{H_1^{-1}(z_2(n))\,w_1^q\left(\left(H_1^{-1}(z_2(n))\right)^{1/q}\right)}. \end{aligned} \tag{46}$$

Taking $n = s$ in (46) and summing up over s from 0 to $n-1$, from (46) we obtain

$$\begin{aligned} &H_2(z_2(n)) \\ &\quad\le H_2(z_2(0)) + \sum_{s=0}^{n-1} f(K)e^{-qt_s} \\ &\qquad+ \sum_{s=0}^{n-1} g(K)e^{-qt_s}\frac{w_2^q\left(\left(H_1^{-1}(z_2(n))\right)^{1/q}\right)}{H_1^{-1}(z_2(s))\,w_1^q\left(\left(H_1^{-1}(z_2(s))\right)^{1/q}\right)} \\ &\quad\le H_2(z_2(0)) + \sum_{s=0}^{K-1} f(K)e^{-qt_s} \\ &\qquad+ \sum_{s=0}^{n-1} g(K)e^{-qt_s}\frac{w_2^q\left(\left(H_1^{-1}(z_2(n))\right)^{1/q}\right)}{H_1^{-1}(z_2(s))\,w_1^q\left(\left(H_1^{-1}(z_2(s))\right)^{1/q}\right)}, \end{aligned} \tag{47}$$

for all $n \in [0, K] \cap \mathbf{N}$. Let

$$\begin{aligned} z_3(n) &= H_2(z_2(0)) + \sum_{s=0}^{K-1} f(K)e^{-qt_s} \\ &\quad+ \sum_{s=0}^{n-1} g(K)e^{-qt_s}\frac{w_2^q\left(\left(H_1^{-1}(z_2(n))\right)^{1/q}\right)}{H_1^{-1}(z_2(s))\,w_1^q\left(\left(H_1^{-1}(z_2(s))\right)^{1/q}\right)}. \end{aligned} \tag{48}$$

Then

$$\begin{gathered} z_3(0) = H_2(z_2(0)) + \sum_{s=0}^{K-1} f(K)e^{-qt_s}, \\ z_2(n) \le H_2^{-1}(z_3(n)). \end{gathered} \tag{49}$$

From (48) and (49), we have

$$\begin{aligned} &\frac{H_1^{-1}\left(H_2^{-1}(z_3(s))\right)w_1^q\left(\left(H_1^{-1}\left(H_2^{-1}(z_3(s))\right)\right)^{1/q}\right)\Delta z_3(n)}{w_2^q\left(\left(H_1^{-1}\left(H_2^{-1}(z_3(n))\right)\right)^{1/q}\right)} \\ &\quad\le g(K)e^{-qt_n}. \end{aligned} \tag{50}$$

Using the mean-value theorem for integrals, from (50) we have

$$H_3(z_3(n)) \le H_3(z_3(0)) + \sum_{s=0}^{n-1} g(K)e^{-qt_s}, \tag{51}$$

where H_3 is defined by (20). From (36), (42), (49), and (51), we have

$$\begin{aligned} z_1(n) &\le y(n) \le H_1^{-1}(z_2(n)) \le H_1^{-1}\left(H_2^{-1}(z_3(n))\right) \\ &\le H_1^{-1}\left\{H_2^{-1}\left[H_3^{-1}\left(H_3(z_3(0)) + \sum_{s=0}^{n-1} g(K)e^{-qt_s}\right)\right]\right\} \\ &= H_1^{-1}\left\{H_2^{-1}\left[H_3^{-1}\left(H_3\left(H_2(z_2(0)) + \sum_{s=0}^{K-1} f(K)e^{-qt_s}\right)\right.\right.\right. \\ &\qquad\left.\left.\left. + \sum_{s=0}^{n-1} g(K)e^{-qt_s}\right)\right]\right\} \end{aligned}$$

$$\begin{aligned}&= H_1^{-1}\\&\quad\times\left\{H_2^{-1}\left[H_3^{-1}\left(H_3\left(H_2\left(H_1\left(z_1\left(0\right)\right)\right.\right.\right.\right.\right.\\&\qquad\left.+\sum_{s=0}^{K-1} f\left(K\right) h^q\left(s\right) e^{-qt_s}\right)\\&\qquad\left.+\sum_{s=0}^{K-1} f\left(K\right) e^{-qt_s}\right)\\&\qquad\left.\left.\left.+\sum_{s=0}^{n-1} g\left(K\right) e^{-qt_s}\right)\right]\right\},\\&\qquad\forall n\in[0,K]\cap\mathbf{N}.\end{aligned}\tag{52}$$

Using $v(n) := u^q(n)$ and (33), from (52) we obtain that

$$\begin{aligned}&u\left(n\right)\\&= v^{1/q}\left(n\right)\ \le z_1^{1/q}\left(n\right)\\&\le\left\{H_1^{-1}\left\{H_2^{-1}\left[H_3^{-1}\left(H_3\left(H_2\left(H_1\left(z_1\left(0\right)\right)\right.\right.\right.\right.\right.\right.\\&\qquad\left.+\sum_{s=0}^{K-1} f\left(K\right) h^q\left(s\right) e^{-qt_s}\right)\\&\qquad\left.+\sum_{s=0}^{K-1} f\left(K\right) e^{-qt_s}\right)\\&\qquad\left.\left.\left.\left.+\sum_{s=0}^{n-1} g(K)e^{-qt_s}\right)\right]\right\}\right\}^{1/q},\\&\qquad\forall n\in[0,K]\cap\mathbf{N}.\end{aligned}\tag{53}$$

Since K is chosen arbitrarily, from (53) we have

$$\begin{aligned}&u\left(n\right)\\&\le\left\{H_1^{-1}\left\{H_2^{-1}\left[H_3^{-1}\left(H_3\left(H_2\left(H_1\left(z_1\left(0\right)\right)\right.\right.\right.\right.\right.\right.\\&\qquad\left.+\sum_{s=0}^{n-1} f\left(n\right) h^q\left(s\right) e^{-qt_s}\right)\\&\qquad\left.+\sum_{s=0}^{n-1} f\left(n\right) e^{-qt_s}\right)\\&\qquad\left.\left.\left.\left.+\sum_{s=0}^{n-1} g(n)e^{-qt_s}\right)\right]\right\}\right\}^{1/q},\\&\qquad\forall n\in\mathbf{N}_1.\end{aligned}\tag{54}$$

This is our required estimation (16) of unknown function in (10). □

3. Application

In this section, we apply our results to discuss the boundedness of solutions of an iterative difference equation with a weakly singular kernel.

Example 5. Suppose that $u(n)$ satisfies the difference equation

$$\begin{aligned}x\left(n\right) &= 1+\sum_{s=0}^{n-1}\left(t_n-t_s\right)^{-2/3}\tau_s\\&\quad\times\left[x\left(s\right)+\sum_{\tau=0}^{s-1}\left(t_s-t_\tau\right)^{-2/3}\tau_\tau x\left(\tau\right)\left(\ln\left|x\left(\tau\right)\right|^4\right)^{1/4}\right],\\&\qquad\forall n\in\mathbf{N},\end{aligned}\tag{55}$$

where $t_0 = 0$, $\tau_s = t_{s+1}-t_s > 0$, $\sup_{s\in\mathbf{N},0\le s\le n-1}\{\tau_s, s\in\mathbf{N}\} = \tau$, and $\lim_{n\to\infty}t_n = \infty$. Then we have

$$\begin{aligned}\left|x\left(n\right)\right| &\le 1+\sum_{s=0}^{n-1}\left(t_n-t_s\right)^{-2/3}\tau_s\\&\quad\times\left[\left|x\left(s\right)\right|+\sum_{\tau=0}^{s-1}\left(t_s-t_\tau\right)^{-2/3}\right.\\&\qquad\left.\times\tau_\tau\left|x\left(\tau\right)\right|\left(\ln\left|x\left(\tau\right)\right|^4\right)^{1/4}\right],\end{aligned}\tag{56}$$

for all $n\in\mathbf{N}$. Let $a(n)\equiv b(n)\equiv 1$, $h(n)\equiv 0$, $\beta = 1/3$, $p = 4/3$, $q = 4$, $w_1(u)\equiv 1$, $w_2(u) = u(\ln u^4)^{1/4}$. From (18) to (20) we obtain that

$$\begin{aligned}&H_1\left(u\right) := \int_0^u ds = u,\quad u>0,\qquad H_1\left(\infty\right)=\infty,\\&H_1^{-1}\left(u\right) = u,\\&H_2\left(u\right) := \int_1^u\frac{ds}{s} = \ln u,\quad u>0,\qquad H_2\left(\infty\right)=\infty,\\&H_2^{-1}\left(u\right) = e^u,\\&H_3\left(u\right) := \int_1^u\frac{ds}{s} = \ln u,\quad u>0,\qquad H_3\left(\infty\right)=\infty,\\&H_3^{-1}\left(u\right) = e^u,\\&f\left(n\right) := 6^3\tau\left[\frac{e^{4t_n/3}}{\left(4/3\right)^{1/9}}\Gamma\left(\frac{1}{9}\right)\right]^3,\\&g\left(n\right) := \tau\left(\frac{e^{4t_n/3}}{\left(4/3\right)^{1/9}}\Gamma\left(\frac{1}{9}\right)\right)^3.\end{aligned}\tag{57}$$

Using Theorem 4, we get

$$
\begin{aligned}
u(n) &\leq \left\{ \exp\left[\exp\left(\ln\left(\ln 8 + 6^3 \tau \left[\frac{e^{4t_n/3}}{(4/3)^{1/9}} \Gamma\left(\frac{1}{9}\right) \right]^3 \right.\right.\right.\right. \\
&\qquad \left. \times \sum_{s=0}^{n-1} e^{-qt_s} \right) \\
&\qquad + \tau \left(\frac{e^{4t_n/3}}{(4/3)^{1/9}} \Gamma\left(\frac{1}{9}\right) \right)^3 \\
&\qquad \left.\left.\left.\left. \times \sum_{s=0}^{n-1} e^{-qt_s} \right) \right] \right\}^{1/q}, \quad \forall n \in \mathbf{N},
\end{aligned}
\tag{58}
$$

which is an upper bound of $|x(n)|$ in (55).

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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