Research Article

Some Upper Matrix Bounds for the Solution of the Continuous Algebraic Riccati Matrix Equation

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We propose diverse upper bounds for the solution matrix of the continuous algebraic Riccati matrix equation (CARE) by building the equivalent form of the CARE and using some matrix inequalities and linear algebraic techniques. Finally, numerical example is given to demonstrate the effectiveness of the obtained results in this work as compared with some existing results in the literature. These new bounds are less restrictive and provide more efficient results in some cases.

1. Introduction

In many areas of optimal control [1–3], robust control [1], robust stability [4], filter design [5], stability theory and analysis [6–8], control design [9] in control theory [10, 11] including optimization stability theory, and transient performance performance nonlinear systems [12], the algebraic Riccati and Lyapunov matrix equations play an important role.

For example, consider the following linear system such that $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $x_0 \in \mathbb{R}^n$ [13]:

$$\dot{x}(t) = Ax(t) + Bu(t),$$
$$x(0) = x_0, \tag{1}$$

with the state feedback control

$$u(t) = -Kx(t), \quad K = B^T P, \tag{2}$$

and the performance index

$$J = \int_0^\infty \left(x^T Q x + u^T u \right) dt, \tag{3}$$

where $R = BB^T \in \mathbb{R}^{n \times n}$, $Q \in \mathbb{R}^{n \times n}$ is positive semidefinite matrix, and P is the positive semidefinite solution to the continuous algebraic Riccati matrix equation (CARE)

$$AP + PA^T - PRP + Q = 0. \tag{4}$$

When $B = 0$ and A is stable matrix, the CARE (4) becomes the continuous algebraic Lyapunov matrix equation (CALE)

$$A^T P + PA = -Q. \tag{5}$$

It is assumed that the pair $(A, R^{1/2})$ is stabilizable. Then the CARE (4) has a unique symmetric positive semidefinite stabilizing solution if the pair $(A, Q^{1/2})$ is observable.

The problem of estimating solution bounds for the algebraic Riccati and Lyapunov matrix equations has widely been considered in the recent years, since these equations are widely used in many fields of control system analysis and design. A number of works have reported numerical algorithms to get the exact solution of the mentioned equations [7]. However, we should note that the analytical solution of these equations has some complications and computational burdens, specially, when the dimensions of the system matrices increase. Thus, for some applications such as stability analysis [8], it is the only preferred solution matrix bounds for the exact solution that can be obtained without hard and complicated burdens. Moreover, as mentioned in [12], in practice, the solution matrix bounds can also be used as approximations of the exact solution or initial guesses in the numerical algorithms for the exact solution [10].

The existing results obtained during 1974–1994 have been summarized by Kwon et al. [14] only including all eigenvalue bounds such as the extreme eigenvalues, the summation,

the trace, majorization inequalities, the product, and the determinant. Unfortunately, by this time, the upper matrix bounds for the solution of the CARE (4) have not been proposed in the literature. However, Lee in [15] has proposed upper and lower matrix bounds for the CARE (4) and henceforth many reports have been presented for the upper [16–20] and lower [18, 19, 21] bounds for the solution of the CARE (4). As matrix bounds include all eigenvalue bounds [14, 22, 23] particularly the minimum and maximum eigenvalues, trace [10, 24, 25], determinant [14], and norm [26] bounds, it is seen that they are the most general and useful. Therefore, this paper presents upper matrix bounds for the solution of the CARE (4) by utilizing various matrix identities and matrix inequalities.

Let $\mathbb{R}^{n \times m}$ be the set of $n \times m$ real matrices. In this paper, we denote the eigenvalues of an $n \times n$ real matrix by $\lambda_i(X)$; if $X \in \mathbb{R}^{n \times n}$ is a symmetric matrix, then its eigenvalues are arranged in the nonincreasing order $\lambda_1(X) \geq \lambda_2(X) \geq \cdots \geq \lambda_n(X)$. For $X \in \mathbb{R}^{n \times m}$, suppose that the singular values of X are ordered in nonincreasing form; that is, $s_1(X) \geq s_2(X) \geq \cdots \geq s_n(X)$. Also, let $\text{tr}(X)$, X^T, X^{-1}, and $\det(X)$ denote the trace, transpose, inverse, determinant, respectively. Additionally, the spectral condition number of any matrix X is defined by $\kappa_2(X) = s_1(X) s_1(X^{-1})$. Write $X \geq (>)0$, if X is a positive semidefinite (positive definite) matrix. For the symmetric matrices of the same size X and Y, if $X - Y$ is positive semidefinite, we write $X \geq Y$ or $Y \leq X$. Then, Weyl's monotonicity principle means that $Y \leq X$ leads to $\lambda_i(Y) \leq \lambda_i(X)$, $i = 1, 2, \ldots, n$. The identity matrix in $\mathbb{R}^{n \times n}$ is shown by I.

The following lemmas are used to prove the main result of this paper.

Lemma 1 (see [27, 28]). *Let $X \in \mathbb{R}^{n \times n}$ be symmetric matrix. Then the following inequality holds:*

$$\lambda_n(X) I \leq X \leq \lambda_1(X) I. \tag{6}$$

Lemma 2 (see [27, 28]). *For any matrix $A \in \mathbb{R}^{n \times m}$ and any positive semidefinite matrices $X, Y \in \mathbb{R}^{n \times n}$ such that $X \geq Y > (\geq)0$, it holds that $A^T X A \geq A^T Y A$, with strict inequality if X and Y are positive definite and A is of full rank.*

Lemma 3 (see [27, 28]). *For any symmetric matrices $X, Y \in \mathbb{R}^{n \times n}$, the following inequality holds:*

$$\lambda_{i+j-1}(X + Y) \leq \lambda_j(X) + \lambda_i(Y), \tag{7}$$
$$i + j \leq n + 1, \quad 1 \leq i, \ j \leq n.$$

Lemma 4 (see [28]). *Let $X, Y \in \mathbb{R}^{n \times n}$, for $i = 1, 2, \ldots, n$, one has*

$$s_i(X + Y) \leq s_i(X) + s_1(Y). \tag{8}$$

Lemma 5 (see [28]). *Let $X \in \mathbb{R}^{m \times n}$, $Y \in \mathbb{R}^{n \times m}$, for $i = 1, 2, \ldots, n$, one has*

$$s_i(XY) \leq s_i(X) s_1(Y). \tag{9}$$

Lemma 6 (see [29]). *Let $X \in \mathbb{R}^{n \times n}$, for $i = 1, 2, \ldots, n$, then*

$$\left| \lambda_i \left(\frac{A + A^T}{2} \right) \right| \leq s_i(A). \tag{10}$$

Lemma 7 (see [30]). *The following matrix inequality:*

$$\begin{pmatrix} W & S \\ S^T & V \end{pmatrix} > 0, \tag{11}$$

where $W = W^T$ and $V = V^T$, is equivalent to either

$$V > 0, \quad W - SV^{-1}S^T > 0, \tag{12}$$

or

$$W > 0, \quad V - S^T W^{-1} S > 0. \tag{13}$$

Lemma 8 (see [17]). *The positive semidefinite solution P of the CARE (4) has the following upper bound on its maximal eigenvalue:*

$$\lambda_1(P) \leq \lambda_1\left(D^T D\right) \frac{\lambda_1\left[\left(Q + K^T K\right) D^T D\right]}{\lambda_n\left(Z D^T D\right)} \equiv \eta, \tag{14}$$

where K is any matrix stabilizing $A + BK$ (i.e., $\text{Re}(\lambda_i(A + BK)) < 0$ for all i) and the nonsingular matrix D and positive definite matrix Z are chosen to yield the LMI

$$(A + BK)^T D^T D + D^T D (A + BK) \leq -Z. \tag{15}$$

This eigenvalue upper bound (14) is always calculated if there exists a unique positive semidefinite solution of the CARE (4).

2. Main Results

Zhang and Liu in [19] obtained the lower and upper bounds for the solution of the CARE (4) which improve the results in [21]. Also, Lee in [18] proposed upper and lower bounds for the solution of the CARE (4) by considering a different approach. In this section, we will present diverse upper matrix bounds for the solution matrix of the CARE (4) in the light of the reported results in [18, 19], by utilizing the above lemmas and generating some matrix identities.

Theorem 9. *Assume that Q is symmetric positive definite and there exists a unique symmetric positive semidefinite solution P to the CARE (4). Then P satisfies the following inequality:*

$$P \leq \left\{ \frac{1}{1 - \lambda_1(I - \varepsilon R)} \right.$$
$$\left. \times \left\{ M + 2\varphi_1 \varepsilon \lambda_1(I - \varepsilon R) s_1\left[A(I - \varepsilon R)^{-1}\right] I \right\} \right\}^{1/2}$$
$$\equiv P_{u1}(\varepsilon, A, Q, R),$$

$$\tag{16}$$

where the positive semidefinite matrix M_1 and the positive constant δ are defined by

$$M = \varepsilon Q - \varepsilon^2 A (I - \varepsilon R)^{-1} A^T$$
$$+ \varepsilon^2 \lambda_1 (I - \varepsilon R) A (I - \varepsilon R)^{-2} A^T, \tag{17}$$

$$\delta = \frac{\varepsilon \lambda_1 (I - \varepsilon R) s_1 \left[A(I - \varepsilon R)^{-1} \right]}{1 - \lambda_1 (I - \varepsilon R)},$$

where ε is any positive constant such that

$$0 < \varepsilon < \left\| R + A^T Q^{-1} A \right\|^{-1}, \tag{18}$$

and positive constant φ_1 is defined by

$$\varphi_1 \equiv \frac{1}{1 - \lambda_1 (I - \varepsilon R)}$$
$$\times \left\{ \left\{ [1 - \lambda_1 (I - \varepsilon R)] \lambda_1 (M) + \delta^2 \right\}^{1/2} + \delta \right\}. \tag{19}$$

Proof. By adding and subtracting $(1/\varepsilon)PP + A((1/\varepsilon)(I - R))^{-1} A^T$ from (4), one gets

$$\left[P + A\left(\frac{1}{\varepsilon}I - R\right)^{-1} \right] \left(\frac{1}{\varepsilon}I - R\right) \left[P + A\left(\frac{1}{\varepsilon}I - R\right)^{-1} \right]^T$$
$$- \frac{1}{\varepsilon}PP - A\left(\frac{1}{\varepsilon}I - R\right)^{-1} A^T + Q = 0, \tag{20}$$

therefore,

$$\left[P + A\left(\frac{1}{\varepsilon}I - R\right)^{-1} \right] \left(\frac{1}{\varepsilon}I - R\right) \left[P + A\left(\frac{1}{\varepsilon}I - R\right)^{-1} \right]^T$$
$$= \frac{1}{\varepsilon}PP + A\left(\frac{1}{\varepsilon}I - R\right)^{-1} A^T - Q. \tag{21}$$

Applying Lemmas 1 and 2 to (21) gives

$$\frac{1}{\varepsilon}PP + A\left(\frac{1}{\varepsilon}I - R\right)^{-1} A^T - Q$$
$$= \left[P + A\left(\frac{1}{\varepsilon}I - R\right)^{-1} \right] \left(\frac{1}{\varepsilon}I - R\right)$$
$$\times \left[P + A\left(\frac{1}{\varepsilon}I - R\right)^{-1} \right]^T$$
$$\leq \lambda_1 \left(\frac{1}{\varepsilon}I - R\right) \left[P + A\left(\frac{1}{\varepsilon}I - R\right)^{-1} \right]$$
$$\times \left[P + A\left(\frac{1}{\varepsilon}I - R\right)^{-1} \right]^T$$
$$\leq \lambda_1 \left(\frac{1}{\varepsilon}I - R\right) \left[P^2 + A\left(\frac{1}{\varepsilon}I - R\right)^{-2} A^T \right.$$
$$\left. + P\left(\frac{1}{\varepsilon}I - R\right)^{-1} A^T + A\left(\frac{1}{\varepsilon}I - R\right)^{-1} P \right]. \tag{22}$$

For the part $P((1/\varepsilon)I - R)^{-1} A^T + A((1/\varepsilon)I - R)^{-1} P$ of (22), applying Lemmas 1, 6, and 5, respectively, shows that

$$P\left(\frac{1}{\varepsilon}I - R\right)^{-1} A^T + A\left(\frac{1}{\varepsilon}I - R\right)^{-1} P$$
$$\leq \lambda_1 \left(P\left(\frac{1}{\varepsilon}I - R\right)^{-1} A^T + A\left(\frac{1}{\varepsilon}I - R\right)^{-1} P \right) I$$
$$\leq \left| \lambda_1 \left(P\left(\frac{1}{\varepsilon}I - R\right)^{-1} A^T + A\left(\frac{1}{\varepsilon}I - R\right)^{-1} P \right) I \right| \tag{23}$$
$$\leq 2 s_1 \left(P\left(\frac{1}{\varepsilon}I - R\right)^{-1} A^T \right) I$$
$$\leq 2 \lambda_1 (P) s_1 \left(A\left(\frac{1}{\varepsilon}I - R\right)^{-1} \right) I.$$

Thus, in light of the fact (23), (22) becomes

$$\frac{1}{\varepsilon}PP + A\left(\frac{1}{\varepsilon}I - R\right)^{-1} A^T - Q$$
$$\leq \lambda_1 \left(\frac{1}{\varepsilon}I - R\right) \left[P^2 + A\left(\frac{1}{\varepsilon}I - R\right)^{-2} A^T \right.$$
$$\left. + 2 \lambda_1 (P) s_1 \left(A\left(\frac{1}{\varepsilon}I - R\right)^{-1} \right) I \right]. \tag{24}$$

If $Q > 0$ and ε satisfies (18), then

$$I - \varepsilon R - \varepsilon A^T Q^{-1} A > 0, \quad \varepsilon Q > 0. \tag{25}$$

By the application of the Schur complement formula of Lemma 7 to (25), we can say that the above inequalities are satisfied if and only if

$$\begin{pmatrix} \varepsilon Q & \varepsilon A \\ \varepsilon A^T & I - \varepsilon R \end{pmatrix} > 0, \tag{26}$$

which means that

$$\varepsilon Q - \varepsilon^2 A (I - \varepsilon R) A^T > 0, \quad I - \varepsilon R > 0. \tag{27}$$

Therefore, we say that (24) is equivalent to

$$P^2 \leq \lambda_1 (I - \varepsilon R) \left[P^2 + A\left(\frac{1}{\varepsilon}I - R\right)^{-2} A^T \right.$$
$$\left. + 2 \lambda_1 (P) s_1 \left(A\left(\frac{1}{\varepsilon}I - R\right)^{-1} \right) I \right] \tag{28}$$
$$+ \varepsilon Q - \varepsilon^2 A (I - \varepsilon R)^{-1} A^T.$$

Since $1 - \lambda_1 (I - \varepsilon R) > 0$, (28) can be rewritten as

$$P^2 \leq \frac{1}{1 - \lambda_1 (I - \varepsilon R)}$$
$$\times \left\{ \varepsilon Q - \varepsilon^2 A (I - \varepsilon R)^{-1} A^T \right.$$
$$+ \varepsilon^2 \lambda_1 (I - \varepsilon R) A (I - \varepsilon R)^{-2} A^T \tag{29}$$
$$\left. + 2 \lambda_1 (P) \varepsilon \lambda_1 (I - \varepsilon R) s_1 \left[A(I - \varepsilon R)^{-1} \right] I \right\}.$$

Utilizing the relations in Lemmas 1 and 3, (29) becomes

$$\lambda_1 \left(P^2 \right)$$

$$\leq \frac{1}{1 - \lambda_1 \left(I - \varepsilon R \right)} \lambda_1$$

$$\times \left\{ \left[\varepsilon Q - \varepsilon^2 A (I - \varepsilon R)^{-1} A^T \right. \right.$$

$$+ \varepsilon^2 \lambda_1 \left(I - \varepsilon R \right) A (I - \varepsilon R)^{-2} A^T \right]$$

$$+ 2\lambda_1 \left(P \right) \varepsilon \lambda_1 \left(I - \varepsilon R \right) s_1 \left[A (I - \varepsilon R)^{-1} \right] \right\} \quad (30)$$

$$\leq \frac{1}{1 - \lambda_1 \left(I - \varepsilon R \right)}$$

$$\times \left\{ \lambda_1 \left[\varepsilon Q - \varepsilon^2 A (I - \varepsilon R)^{-1} A^T \right. \right.$$

$$+ \varepsilon^2 \lambda_1 \left(I - \varepsilon R \right) A (I - \varepsilon R)^{-2} A^T \right]$$

$$+ 2\lambda_1 \left(P \right) \varepsilon \lambda_1 \left(I - \varepsilon R \right) s_1 \left[A (I - \varepsilon R)^{-1} \right] \right\}.$$

Solving (30) according to $\lambda_1(P)$ gives

$$\lambda_1 \left(P \right) \leq \left\{ \frac{1}{1 - \lambda_1 \left(I - \varepsilon R \right)} \lambda_1 \right.$$

$$\times \left[\varepsilon Q - \varepsilon^2 A (I - \varepsilon R)^{-1} A^T \right.$$

$$+ \varepsilon^2 \lambda_1 \left(I - \varepsilon R \right) A (I - \varepsilon R)^{-2} A^T \right]$$

$$+ \left[\frac{\varepsilon \lambda_1 \left(I - \varepsilon R \right) s_1 \left[A (I - \varepsilon R)^{-1} \right]}{1 - \lambda_1 \left(I - \varepsilon R \right)} \right]^2 \right\}^{1/2} \quad (31)$$

$$+ \frac{\varepsilon \lambda_1 \left(I - \varepsilon R \right) s_1 \left[A (I - \varepsilon R)^{-1} \right]}{1 - \lambda_1 \left(I - \varepsilon R \right)} \equiv \varphi_1.$$

Substituting (31) into (29) results in the upper bound

$$P \leq \left\{ \frac{1}{1 - \lambda_1 \left(I - \varepsilon R \right)} \right.$$

$$\times \left\{ \varepsilon Q - \varepsilon^2 A (I - \varepsilon R)^{-1} A^T \right.$$

$$+ \varepsilon^2 \lambda_1 \left(I - \varepsilon R \right) A (I - \varepsilon R)^{-2} A^T \quad (32)$$

$$+ 2\varphi_1 \varepsilon \lambda_1 \left(I - \varepsilon R \right) s_1 \left[A (I - \varepsilon R)^{-1} \right] I \right\}^{1/2}.$$

This completes the proof. □

Remark 10. The inequality (3.5) in [19] is clearly as follows:

$$P^2 \leq \lambda_1 \left(I - \varepsilon R \right) \left\{ \lambda_1^2 \left(P \right) + 2\lambda_1 \left(P \right) \varepsilon s_1 \left[A (I - \varepsilon R)^{-1} \right] \right.$$

$$+ \left. \varepsilon^2 s_1^2 \left[A (I - \varepsilon R)^{-1} \right] \right\} \quad (33)$$

$$+ \varepsilon Q - \varepsilon^2 A (I - \varepsilon R)^{-1} A^T.$$

Thus, when the inequality (28) is considered, from the facts

$$P^2 \leq \lambda_1^2 \left(P \right),$$

$$\left[A (I - \varepsilon R)^{-1} \right] \left[A (I - \varepsilon R)^{-1} \right]^T \leq s_1^2 \left[A (I - \varepsilon R)^{-1} \right], \quad (34)$$

it is seen that the upper bound in Theorem 9 is always sharper than the result given by Theorem 3.1 in [19].

Remark 11. It is well known that most of the studies in the literature have focused to derive the bounds for the maximum and minimum eigenvalues, the trace, and the determinant for the solution of the CARE (4); however, the matrix solution bounds are quite restriction. Among the mentioned bounds, the matrix solution bounds are the most useful and efficient because other bounds that are dependent on eigenvalue can be derived directly from matrix solution bounds via monotonicity.

By using Theorem 9, we can derive the following result immediately.

Corollary 12. *Assume that Q is symmetric positive definite and there exists a unique symmetric positive semidefinite solution P to the CARE (4). Then P satisfies the following upper eigenvalue bounds:*

$$\lambda_i \left(P \right) \leq \min_\varepsilon \lambda_i \left(P_{u1} \left(\varepsilon, A, Q, R \right) \right) = P_{u1}^1$$

$$\leq \lambda_i \left(P_{u1} \left(\varepsilon, A, Q, R \right) \right),$$

$$\mathrm{tr} \left(P \right) \leq \sum_{i=1}^n P_{u1}^1 \leq \min_\varepsilon \mathrm{tr} \left(P_{u1} \left(\varepsilon, A, Q, R \right) \right)$$

$$\leq \mathrm{tr} \left(P_{u1} \left(\varepsilon, A, Q, R \right) \right), \quad (35)$$

$$\det \left(P \right) \leq \prod_{i=1}^n P_{u1}^1 \leq \min_\varepsilon \det \left(P_{u1} \left(\varepsilon, A, Q, R \right) \right)$$

$$\leq \det \left(P_{u1} \left(\varepsilon, A, Q, R \right) \right),$$

where ε satisfies (18) and P_{u1} is defined by (16).

By establishing the more general form than the matrix identity used in Theorem 9 for the CARE (4), one gets the following upper bounds.

Theorem 13. *Let X be any symmetric positive definite matrix. Then the unique symmetric positive semidefinite solution P to the CARE (4) has the following upper bound*

$$P \leq M_1^{-1/2} \left\{ M_1^{1/2} \left[Q - X_1 + \lambda_1 \left(X_1^{-1} \right) \right. \right.$$

$$\times \left[\eta s_1 \left(A \right) + \lambda_1 \left(X_1 \right) \right]^2 I \right] \quad (36)$$

$$\times M_1^{1/2} \right\}^{1/2} M_1^{-1/2}$$

$$\equiv P_{u2} \left(\eta, X_1, A, Q, R \right),$$

where the positive definite matrix X_1 is chosen so that

$$M_1 \equiv A^T X_1^{-1} A + R > 0, \quad (37)$$

and η is defined by (14).

Proof. By adding and subtracting $X_1 + PA^T X_1^{-1} AP$ to the CARE (4), we can get

$$\left(PA^T + X_1\right) X_1^{-1} \left(PA^T + X_1\right)^T - PRP \tag{38}$$
$$+ Q - X_1 - PA^T X_1^{-1} AP = 0$$

which is equivalent to

$$0 < P\left(R + A^T X_1^{-1} A\right) P \tag{39}$$
$$= Q - X_1 + \left(PA^T + X_1\right) X_1^{-1} \left(PA^T + X_1\right)^T.$$

Introducing Lemmas 1, 2, 4, 5, and 8, respectively, to (39) gives

$$0 < P\left(R + A^T X_1^{-1} A\right) P$$
$$= Q - X_1 + \left(PA^T + X_1\right) X_1^{-1} \left(PA^T + X_1\right)^T$$
$$\leq Q - X_1 + \lambda_1\left(X_1^{-1}\right) \left[\left(PA^T + X_1\right)\left(PA^T + X_1\right)^T\right]$$
$$\leq Q - X_1 + \lambda_1\left(X_1^{-1}\right) \left[s_1^2\left(PA^T + X_1\right)\right] I \tag{40}$$
$$\leq Q - X_1 + \lambda_1\left(X_1^{-1}\right) \left[s_1\left(PA^T\right) + \lambda_1\left(X_1\right)\right]^2 I$$
$$\leq Q - X_1 + \lambda_1\left(X_1^{-1}\right) \left[\lambda_1\left(P\right) s_1\left(A\right) + \lambda_1\left(X_1\right)\right]^2 I$$
$$\leq Q - X_1 + \lambda_1\left(X_1^{-1}\right) \left[\eta s_1\left(A\right) + \lambda_1\left(X_1\right)\right]^2 I.$$

By the definition (37) of M_1 and pre- and postmultiplying M_1 to (40) yields

$$\left(M_1^{1/2} P M_1^{1/2}\right)^2 = M_1^{1/2} P M_1 P M_1^{1/2}$$
$$\leq M_1^{1/2} \left[Q - X_1 + \lambda_1\left(X_1^{-1}\right) \left[\eta s_1\left(A\right) + \lambda_1\left(X_1\right)\right]^2 I\right] M_1^{1/2}. \tag{41}$$

Solving this inequality for P shows the upper bound (36).

This builds the proof. □

Remark 14. Note that for the upper bound (36), the matrices R and Q don not have to be nonsingular. This means that the upper bound proposed by Theorem 13 can always be computed without any condition for positive definite matrix X_1 which arbitrarily is selected.

From Theorem 13, we have the following corollaries.

Corollary 15. *The positive semidefinite solution P to the CARE (4) has*

$$P \leq \left\{\frac{1}{\lambda_n\left(M_1\right)} \left[Q - X_1\right.\right.$$
$$\left.\left.+ \lambda_1\left(X_1^{-1}\right) \left[\eta s_1\left(A\right) + \lambda_1\left(X_1\right)\right]^2 I\right]\right\}^{1/2}$$
$$\equiv P_{u2}^1,$$
$$\tag{42}$$

where η and M_1 for the positive definite matrix X_1 are defined by (14) and (37), respectively.

Proof. Applying Lemma 1 to the right side of (41) and solving it with regard to P give the upper bound P_{u2}^1. □

Corollary 16. *The solution P to the CARE (4) satisfies the following upper eigenvalue bounds:*

$$\lambda_i\left(P\right) \leq \min \lambda_i\left(P_{u2}\left(\eta, X_1, A, Q, R\right)\right) = P_{u2}^*$$
$$\leq \lambda_i\left(P_{u2}\left(\eta, X_1, A, Q, R\right)\right),$$
$$\text{tr}\left(P\right) \leq \sum_{i=1}^n P_{u2}^* \leq \min \text{tr}\left(P_{u2}\left(\eta, X_1, A, Q, R\right)\right)$$
$$\leq \text{tr}\left(P_{u2}\left(\eta, X_1, A, Q, R\right)\right), \tag{43}$$
$$\det\left(P\right) \leq \prod_{i=1}^n P_{u2}^* \leq \min \det\left(P_{u2}\left(\eta, X_1, A, Q, R\right)\right)$$
$$\leq \det\left(P_{u2}\left(\eta, X_1, A, Q, R\right)\right),$$

where η is defined by (14) and the positive matrix X_1 is selected so as to satisfy the definition (37), respectively.

Theorem 17. *Let P be the positive semidefinite solution of the CARE (4). Then P has the upper bound*

$$P \leq M_2^{-1/2} \left\{M_2^{1/2} \left[Q - X_2 + \lambda_1\left(X_2^{-1}\right) X_2^2\right.\right.$$
$$\left.\left. + 2\kappa_2\left(X_2\right) s_1\left(A\right) \varphi_2 I\right] M_2^{1/2}\right\}^{1/2} M_2^{-1/2}$$
$$\equiv P_{u3}\left(\varphi_2, X_2, A, Q, R\right), \tag{44}$$

where the positive definite matrix X_2 is chosen so that

$$M_2 \equiv R - A^T \left[\lambda_1\left(X_2^{-1}\right) I - X_2^{-1}\right] A > 0, \tag{45}$$

and φ_2 is defined by

$$\varphi_2 \equiv \left\{\frac{1}{\lambda_n\left(M_2\right)} \lambda_1\left[Q - X_2 + \lambda_1\left(X_2^{-1}\right) X_2^2\right]\right.$$
$$\left. + \frac{\kappa_2^2\left(X_2\right) s_1^2\left(A\right)}{\lambda_n^2\left(M_2\right)}\right\}^{1/2} + \frac{\kappa_2\left(X_2\right) s_1\left(A\right)}{\lambda_n\left(M_2\right)}. \tag{46}$$

Proof. By the use of the equality (39), from Lemmas 1 and 2, we can write

$$P\left(R + A^T X_2^{-1} A\right) P$$
$$= Q - X_2 + \left(PA^T + X_2\right) X_2^{-1} \left(PA^T + X_2\right)^T$$
$$\leq Q - X_2 + \lambda_1\left(X_2^{-1}\right) \left[PA^T AP + PA^T X_2 + X_2 AP + X_2^2\right]. \tag{47}$$

Having applied Lemmas 1, 6, and 5, respectively, to the part of $PA^T X_2 + X_2 AP$ in (47), since the following inequalities hold:

$$PA^T X_2 + X_2 AP \leq \lambda_1\left(PA^T X_2 + X_2 AP\right) I$$
$$\leq \left|\lambda_1\left(PA^T X_2 + X_2 AP\right)\right| I \leq 2s_1\left(PA^T X_2\right) \tag{48}$$
$$\leq 2\lambda_1\left(P\right) s_1\left(A\right) \lambda_1\left(X_2\right) I$$

via the definition of M_2 from (47), we arrive at

$$P\left[R - A^T\left(\lambda_1\left(X_2^{-1}\right)I - X_2^{-1}\right)A\right]P$$
$$= PM_2P \leq Q - X_2 + \lambda_1\left(X_2^{-1}\right)X_2^2 \qquad (49)$$
$$+ 2\kappa_2\left(X_2\right)s_1\left(A\right)\lambda_1\left(P\right)I.$$

Applying Lemmas 1 and 3 to (49), we have

$$P^2 \leq \frac{1}{\lambda_n\left(M_2\right)}\left[Q - X_2 + \lambda_1\left(X_2^{-1}\right)X_2^2\right]$$
$$+ 2\frac{\kappa_2\left(X_2\right)s_1\left(A\right)}{\lambda_n\left(M_2\right)}\lambda_1\left(P\right)I$$
$$\qquad\qquad (50)$$
$$\leq \frac{1}{\lambda_n\left(M_2\right)}\lambda_1\left[Q - X_2 + \lambda_1\left(X_2^{-1}\right)X_2^2\right]I$$
$$+ 2\frac{\kappa_2\left(X_2\right)s_1\left(A\right)}{\lambda_n\left(M_2\right)}\lambda_1\left(P\right)I.$$

Then,

$$\lambda_1\left(P^2\right) \leq \frac{1}{\lambda_n\left(M_2\right)}\lambda_1\left[Q - X_2 + \lambda_1\left(X_2^{-1}\right)X_2^2\right]$$
$$\qquad\qquad (51)$$
$$+ 2\frac{\kappa_2\left(X_2\right)s_1\left(A\right)}{\lambda_n\left(M_2\right)}\lambda_1\left(P\right).$$

Solving (51) with respect to $\lambda_1(P)$ gives

$$\lambda_1\left(P\right) \leq \left\{\frac{1}{\lambda_n\left(M_2\right)}\lambda_1\left[Q - X_2 + \lambda_1\left(X_2^{-1}\right)X_2^2\right]\right.$$
$$\left.+ \frac{\kappa_2^2\left(X_2\right)s_1^2\left(A\right)}{\lambda_n^2\left(M_2\right)}\right\}^{1/2} + \frac{\kappa_2\left(X_2\right)s_1\left(A\right)}{\lambda_n\left(M_2\right)} \equiv \varphi_2.$$
$$\qquad\qquad (52)$$

Substituting φ_2 into (49), we get

$$PM_2P \leq Q - X_2 + \lambda_1\left(X_2^{-1}\right)X_2^2 + 2\kappa_2\left(X_2\right)s_1\left(A\right)\varphi_2I. \quad (53)$$

Pre- and postmultiplying $M_2^{1/2}$ to (53) leads to

$$\left(M_2^{1/2}PM_2^{1/2}\right)^2$$
$$\leq M_2^{1/2}\left[Q - X_2 + \lambda_1\left(X_2^{-1}\right)X_2^2\right. \qquad (54)$$
$$\left.+ 2\kappa_2\left(X_2\right)s_1\left(A\right)\varphi_2I\right]M_2^{1/2}.$$

Therefore, by the nonsingularity of M_2, the upper matrix bound (44) is directly obtained by solving (54) with respect to P.

The proof is finished. $\qquad\qquad\square$

According to Theorem 17, we can propose the following corollaries.

Corollary 18. *The positive semidefinite solution P to the CARE (4) satisfies*

$$P \leq \left\{\frac{1}{\lambda_n\left(M_2\right)}\left[Q - X_2 + \lambda_1\left(X_2^{-1}\right)X_2^2\right.\right.$$
$$\qquad\qquad (55)$$
$$\left.\left.+ 2\kappa_2\left(X_2\right)s_1\left(A\right)\varphi_2I\right]\right\}^{1/2} \equiv P_{u3}^1,$$

where the positive definite matrices X_2, and M_2 and the positive constant φ_2 are defined by (45) and (46), respectively.

Proof. Substituting φ_2 into (50), having solved (50) regard to P, we obtain the upper bound (55). $\qquad\square$

Corollary 19. *The positive semidefinite solution P to the CARE (4) has the following eigenvalue upper bounds:*

$$\lambda_i\left(P\right) \leq \min \lambda_i\left(P_{u3}\left(\varphi_2, X_2, A, Q, R\right)\right) = P_{u3}^*$$
$$\leq \lambda_i\left(P_{u3}\left(\varphi_2, X_2, A, Q, R\right)\right)$$
$$\mathrm{tr}\left(P\right) \leq \sum_{i=1}^{n}P_{u3}^* \leq \min \mathrm{tr}\left(P_{u3}\left(\varphi_2, X_2, A, Q, R\right)\right)$$
$$\leq \mathrm{tr}\left(P_{u3}\left(\varphi_2, X_2, A, Q, R\right)\right)$$
$$\det\left(P\right) \leq \prod_{i=1}^{n}P_{u3}^* \leq \min \det\left(P_{u3}\left(\varphi_2, X_2, A, Q, R\right)\right)$$
$$\leq \det\left(P_{u3}\left(\varphi_2, X_2, A, Q, R\right)\right),$$
$$\qquad\qquad (56)$$

where the positive definite matrices X_2 and M_2 and the positive constant φ_2 are defined by (45) and (46), respectively.

As considered a diverse matrix identity, in the case that the matrix R is nonsingular, we can derive the following alternative upper bounds for the solution of the CARE (4).

Theorem 20. *If the positive definite matrix P is a unique solution matrix of the CARE (4), then*

$$P \leq \left\{\frac{1}{\lambda_n\left(M_3\right)}\left[Q - A^TX_3A + \lambda_1\left(X_3^{-1}\right)\right.\right.$$
$$\left.\left.\times\left[\eta + s_1\left(A^TX_3\right)\right]^2I\right]\right\}^{1/2} \qquad (57)$$
$$\equiv P_{u4}\left(\eta, X_3, A, Q, R\right),$$

where X_3 is a positive constant matrix such that $M_3 \equiv R + X_3^{-1} > 0$ and η is defined by (14).

Proof. When the term $PX_3^{-1}P + A^TX_3A$ is added and subtracted from the CARE (4), we can write

$$\left(P + A^TX_3\right)X_3^{-1}\left(P + A^TX_3\right)^T$$
$$\qquad\qquad (58)$$
$$- PRP + Q - A^TX_3A - PX_3^{-1}P = 0$$

which is equivalent to

$$0 < P\left(R + X_3^{-1}\right) P$$
$$= Q - A^T X_3 A + \left(P + A^T X_3\right) X_3^{-1} \left(P + A^T X_3\right)^T. \tag{59}$$

By the use of Lemmas 1, 2, 4, and 8 for the right side of the above equation, respectively, we obtain

$$0 < P\left(R + X_3^{-1}\right) P$$
$$\leq Q - A^T X_3 A + \lambda_1\left(X_3^{-1}\right)\left(P + A^T X_3\right)\left(P + A^T X_3\right)^T$$
$$\leq Q - A^T X_3 A + \lambda_1\left(X_3^{-1}\right) s_1^2\left(P + A^T X_3\right) I$$
$$\leq Q - A^T X_3 A + \lambda_1\left(X_3^{-1}\right)\left[\lambda_1(P) + s_1\left(A^T X_3\right)\right]^2 I$$
$$\leq Q - A^T X_3 A + \lambda_1\left(X_3^{-1}\right)\left[\eta + s_1\left(A^T X_3\right)\right]^2 I \tag{60}$$

and by the application of Lemma 1 to the term $P(R + X_3^{-1})P$ of (60), we can write

$$P^2 \leq \frac{1}{\lambda_n\left(R + X_3^{-1}\right)}$$
$$\times \left[Q - A^T X_3 A + \lambda_1\left(X_3^{-1}\right)\left[\eta + s_1\left(A^T X_3\right)\right]^2 I\right]. \tag{61}$$

Therefore, if the above inequality is solved with respect to P, we arrive at the upper bound P_{u4}.

Thus, the proof is established. □

Theorem 21. *Let P be the positive semidefinite solution of the CARE (4). Then*

$$P \leq \left\{\frac{1}{\lambda_n\left(M_4\right)}\left[Q - A^T X_4 A + \lambda_1\left(X_4^{-1}\right) A^T X_4^2 A\right.\right.$$
$$\left.\left. + 2s_1(A)\kappa_2\left(X_4\right)\varphi_3 I\right]\right\}^{1/2} \tag{62}$$

$$\equiv P_{u5}\left(\varphi_3, X_4, A, Q, R\right),$$

where the positive definite matrix X_4 is selected such that

$$M_4 \equiv R - \left[\lambda_1\left(X_4^{-1}\right) I - X_4^{-1}\right] > 0 \tag{63}$$

and the nonnegative constant φ_3 is defined by

$$\varphi_3 \equiv \left\{\frac{1}{\lambda_n\left(M_4\right)}\lambda_1\left[Q - A^T X_4 A + \lambda_1\left(X_4^{-1}\right) A^T X_4^2 A\right]\right.$$
$$\left. + \frac{\kappa_2^2\left(X_4\right) s_1^2(A)}{\lambda_n^2\left(M_4\right)}\right\}^{1/2} + \frac{\kappa_2\left(X_4\right) s_1(A)}{\lambda_n\left(M_4\right)}. \tag{64}$$

Proof. Consider (58). From Lemma 1, we can easily write

$$0 < P\left(R + X_4^{-1}\right) P$$
$$\leq Q - A^T X_4 A + \lambda_1\left(X_4^{-1}\right)\left(P + A^T X_4\right)\left(P + A^T X_4\right)^T \tag{65}$$

and then via the inequality obtained by using Lemmas 1, 6, and 5, respectively,

$$PX_4 A + A^T X_4 P \leq 2\lambda_1(P) s_1(A)\lambda_1\left(X_4\right) \tag{66}$$

and the definition (63) of M_4, from (65), we have

$$0 < P\left[R - \left(\lambda_1\left(X_4^{-1}\right) I - X_4^{-1}\right)\right] P = PM_4 P$$
$$\leq Q - A^T X_4 A + \lambda_1\left(X_4^{-1}\right) A^T X_4^2 A \tag{67}$$
$$+ 2s_1(A)\kappa_2\left(X_4\right)\lambda_1(P) I.$$

By the use of Lemmas 1 and 2, it is obtained that

$$P^2 \leq \frac{1}{\lambda_n\left(M_4\right)}$$
$$\times \left[Q - A^T X_4 A + \lambda_1\left(X_4^{-1}\right) A^T X_4^2 A\right. \tag{68}$$
$$\left. + 2s_1(A)\kappa_2\left(X_4\right)\lambda_1(P) I\right]$$

and thus applying Lemma 3 to (68) yields

$$\lambda_1\left(P^2\right) \leq \frac{1}{\lambda_n\left(M_4\right)}$$
$$\times \left\{\lambda_1\left[Q - A^T X_4 A + \lambda_1\left(X_4^{-1}\right) A^T X_4^2 A\right]\right. \tag{69}$$
$$\left. + 2s_1(A)\kappa_2\left(X_4\right)\lambda_1(P)\right\}.$$

As solving (69) according to $\lambda_1(P)$, one can reach the nonnegative constant φ_3 is defined by (64). If it is substituted φ_3 into (68), then

$$P^2 \leq \frac{1}{\lambda_n\left(M_4\right)}\left[Q - A^T X_4 A + \lambda_1\left(X_4^{-1}\right) A^T X_4^2 A\right.$$
$$\left. + 2s_1(A)\kappa_2\left(X_4\right)\varphi_3 I\right]. \tag{70}$$

Thus, solving the inequality (70) derives the upper bound (62) for the solution P of the CARE (4).

This concludes the proof of the theorem. □

Corollary 22. *The solution P to the CARE (4) has the following eigenvalue bounds for $j = 4, 5$:*

$$\lambda_i(P) \leq \min \lambda_i\left(P_{uj}\right) = P_{uj}^* \leq \lambda_i\left(P_{uj}\right),$$
$$\operatorname{tr}(P) \leq \sum_{i=1}^n P_{uj}^* \leq \min \operatorname{tr}\left(P_{uj}\right) \leq \operatorname{tr}\left(P_{uj}\right), \tag{71}$$
$$\det(P) \leq \prod_{i=1}^n P_{uj}^* \leq \min \det\left(P_{uj}\right) \leq \det\left(P_{uj}\right).$$

Remark 23. Chen and Lee in [16] indicated in it is hard or impossible to determine the best matrix bound among the parallel results. Since we find that it is difficult to compare the tightness of our results to the parallel result in [18], we will only make the comparisons on an example.

3. Numerical Example

In this section, we will give a numerical example to demonstrate the effectiveness of the proposed results of this paper.

Example 1. Consider the CARE (4) with

$$A = \begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix}, \qquad Q = \begin{pmatrix} 1 & 0 \\ 0 & 8 \end{pmatrix}, \qquad R = \begin{pmatrix} 9 & 4 \\ 4 & 16 \end{pmatrix}. \quad (72)$$

Choose $\varepsilon = 0.002$, then using (16) shows the following upper matrix bound:

$$P \le \begin{pmatrix} 0.8232 & -0.0001 \\ -0.0001 & 1.2853 \end{pmatrix} \equiv P_{u1},$$

$$\text{tr}(P_{u1}) = 2.1085, \qquad \det(P_{u1}) = 1.0580. \quad (73)$$

The upper bound (55) gives

$$P \le \begin{pmatrix} 0.8173 & 0.0068 \\ 0.0068 & 1.2815 \end{pmatrix} \equiv P_{u3}^1,$$

$$\text{tr}(P_{u3}^1) = 2.0988, \quad (74)$$

$$\det(P_{u3}^1) = 1.0473,$$

with $X_2 = \begin{pmatrix} 10 & 0.1 \\ 0.1 & 10 \end{pmatrix}$, and the upper bound (62) gives

$$P \le \begin{pmatrix} 0.8070 & 0 \\ 0 & 1.2750 \end{pmatrix} \equiv P_{u5},$$

$$\text{tr}(P_{u5}) = 2.082, \quad (75)$$

$$\det(P_{u5}) = 1.0289,$$

with $X_4 = \begin{pmatrix} 10 & 0.1 \\ 0.1 & 10 \end{pmatrix}$.

Using Theorem 3.1 of [19], we obtain the following upper matrix bound:

$$P \le \begin{pmatrix} 1.2328 & 0.0002 \\ 0.0002 & 1.2884 \end{pmatrix} \equiv P_{u[19]},$$

$$\text{tr}(P_{u[19]}) = 2.5212, \quad (76)$$

$$\det(P_{u[19]}) = 1.5883.$$

The bounds (18) and (23) proposed in [18] are

$$P \le \begin{pmatrix} 1.7135 & 0 \\ 0 & 1.0627 \end{pmatrix} \equiv P_{u1[18]},$$

$$\text{tr}(P_{u1[18]}) = 2.7762,$$

$$\det(P_{u1[18]}) = 1.8209,$$

$$P \le \begin{pmatrix} 1.2527 & 0.0050 \\ 0.0050 & 1.6025 \end{pmatrix} \equiv P_{u2[18]}, \quad (77)$$

$$\text{tr}(P_{u2[18]}) = 2.8552,$$

$$\det(P_{u2[18]}) = 2.0074,$$

for $\overline{R}_1 = X_4$.

By a simple computation, we have

$$P_{u5} \le P_{u3}^1 \le P_{u[19]} \le P_{u2[18]},$$

$$P_{u5} \le P_{u1} \le P_{[19]},$$

$$\text{tr}(P_{u5}) \le \text{tr}(P_{u3}^1) \le \text{tr}(P_{u1}) \le \text{tr}(P_{u[19]}),$$

$$\le \text{tr}(P_{u1[18]}) \le \text{tr}(P_{u2[18]}), \quad (78)$$

$$\det(P_{u5}) \le \det(P_{u3}^1) \le \det(P_{u1}) \le \det(P_{u[19]}),$$

$$\le \det(P_{u1[18]}) \le \det(P_{u2[18]})$$

which means that our upper bounds give more precise solution estimates than the results given by Theorem 3.1 in [19] and Theorems 2 and 3 in [18] for this case.

4. Conclusion

In this paper, new upper matrix bounds for the solution of the CARE are improved by using some linear algebraic techniques and matrix inequalities. A numerical example is given to show that the solution upper bounds presented in this paper are sharper than some results in the literature.

Acknowledgments

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