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Research Article

On the Global Stability Properties and Boundedness Results of Solutions of Third-Order Nonlinear Differential Equations

Muzaffer Ateş

Department of Electrical and Electronics Engineering, Yüzüncü Yıl University, 65080 Van, Turkey

Correspondence should be addressed to Muzaffer Ateş; ates.muzaffer65@gmail.com

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We studied the global stability and boundedness results of third-order nonlinear differential equations of the form $\dddot{x}+\psi(x,\dot{x},\ddot{x})\ddot{x}+f(x,\dot{x},\ddot{x})=P(t,x,\dot{x},\ddot{x})$. Particular cases of this equation have been studied by many authors over years. However, this particular form is a generalization of the earlier ones. A Lyapunov function was used for the proofs of the two main theorems: one with $P\equiv 0$ and the other with $P\neq 0$. The results in this paper generalize those of other authors who have studied particular cases of the differential equations. Finally, a concrete example is given to check our results.

1. Introduction

We will consider here the equations of the form

$$\dddot{x}+\psi(x,\dot{x},\ddot{x})\ddot{x}+f(x,\dot{x},\ddot{x})=P(t,x,\dot{x},\ddot{x}). \tag{1}$$

Now, (1) has an equivalent system

$$\begin{gathered}\dot{x}=y, \qquad \dot{y}=z,\\ \dot{z}=-\psi(x,y,z)z-f(x,y,z)+p(t,x,y,z),\end{gathered} \tag{2}$$

where $\psi\in C(R\times R\times R,R)$, $f\in C(R\times R\times R,R)$, and $P\in C([0,\infty)\times R\times R\times R,R)$. We also assume that the real functions ψ, f, and P depend only on the arguments displayed explicitly. The dots denote differentiation with respect to t.

Global stabilities of some special cases of (1) have been studied by a number of authors.

In 1953, Šimanov [1] investigated the global stability of the zero solution of the equation

$$\dddot{x}+\psi(x,\dot{x})\ddot{x}+b\dot{x}+cx=0, \tag{3a}$$

where b and c are constants.

Later, Ezeilo [2] and Ogurtsov [3] discussed the global stability of the zero solution of the equation of the form

$$\dddot{x}+\psi(x,\dot{x})\ddot{x}+\phi(\dot{x})+g(x)=0. \tag{3b}$$

Then, Goldwyn and Narendra [4] studied on the same subject for the following differential equation:

$$\dddot{x}+h(\dot{x})\ddot{x}+\mu(\dot{x})\dot{x}+k(x)x=0. \tag{3c}$$

Recently, Qian [5] and Omeike [6] have discussed the global stability, and in a recent paper, Tunç [7] has investigated the boundedness of solutions of the following differential equations:

$$\dddot{x}+\psi(x,\dot{x})\ddot{x}+f(x,\dot{x})=p(t,x,\dot{x},\ddot{x}). \tag{3d}$$

Plus, Tunç [8] and Omeike [9, 10] have studies on the global stability of solutions of the differential equation of the form

$$\dddot{x}+\psi(x,\dot{x},\ddot{x})\ddot{x}+f(x,\dot{x})=0. \tag{3e}$$

Moreover, Tunç and Omeike have studies on the asymptotic behavior of the following differential equations:

$$\dddot{x}+\psi(x,\dot{x},\ddot{x})\ddot{x}+f(x,\dot{x})=P(t,x,\dot{x},\ddot{x}), \tag{3f}$$

$$\dddot{x}+\psi(x,\dot{x})\ddot{x}+f(x,\dot{x})=P(t), \tag{3g}$$

respectively.

Motivation of this study has been based on recent studies of Qian [5], Tunç [7, 8, 11], and Omeike [6, 9, 10]. Equation (1) is a quite general third-order nonlinear differential equation.

Equations (3a), (3b), (3c), (3d), (3e), (3f), and (3g) are some special cases of (1), and our study is reducible to the studies in [1–11], but the inversions are not possible. Thus, the studies which have been done in [1–11] are some special cases of our study. Hence, our results extend and include those results obtained in [1–11].

2. Preliminaries

Before introducing our main results, we state some basic theorems and a Lyapunov function which will be required in future. Consider the autonomous system

$$\dot{x} = f(x), \tag{4}$$

where $f : \Omega \to R^n$ is continuous, with Ω being an open set in R^n containing the origin. Let $f(0) = 0$ and $f(x) \neq 0$ for $x \neq 0$.

Theorem 1. *Suppose that there exists a scalar function $V(x)$ such that*

(i) *$V(x)$ is positive definite on R^n, and $V(x) \to \infty$ as $\|x\| \to \infty$.*

(ii) *$\dot{V}(x) \leq 0$ on R^n.*

Then, all solutions of (4) are bounded as $t \to \infty$ (i.e., (4) is Lagrange stable).

Proof. See [12]. □

Theorem 2. *In addition to the assumptions of Theorem 1, if the origin is the only invariant subset of E, then the zero solution of (4) is globally asymptotically stable.*

Proof. E is the set where $\dot{V} = 0$, and assume that $(0, 0)$ is to be the only invariant subset of E; then $(0, 0)$ solution of (4) is globally asymptotically stable. □

Theorem 3. *In Theorem 1, if (ii) is replaced by the condition that $\dot{V}(x)$ is negative definite at all points $x \in R^n$, then the zero solution of (4) is globally asymptotically stable.*

Proof. See [12]. □

It is well known that the stability is a very important problem in the theory and applications of differential equations. So far, the most effective method to study the stability of nonlinear differential equations is still Lyapunov's second method. The major advantage of this method is that stability in large can be obtained without any prior knowledge of solutions. Today, this method is widely recognized as an excellent tool not only in the study of differential equations but also in the theory of control systems, dynamical systems, systems with time lag, power system analysis, time-varying nonlinear feedback systems, and so on. Its chief characteristic is the construction of a scalar function, namely, the *Lyapunov function*. Unfortunately, it is sometimes very difficult to find a proper Lyapunov function for a given system. Therefore, in this work, we construct a suitable Lyapunov function which is an excellent tool in the proof of the main theorems. Here, this function, $V = V(t) = V(x, y, z)$, is defined by

$$\begin{aligned} V(x,y,z) = {} & \int_0^x f(u,0,0)\,du + \frac{1}{a}\int_0^y f(x,v,0)\,dv \\ & + \int_0^y \psi(x,v,0)\,v\,dv + \frac{1}{2a}z^2 + yz. \end{aligned} \tag{5}$$

Rewrite the function $V(x, y, z)$ as follows:

$$\begin{aligned} V(x,y,z) = {} & \frac{1}{2a}(ay+z)^2 + \frac{1}{2ab}\left[f(x,0,0)+by\right]^2 \\ & + \int_0^y \left[\psi(x,v,0)-a\right]v\,dv \\ & + \frac{1}{a}\int_0^y \left[f_v(x,\theta_1 v,0)-b\right]v\,dv \\ & + \int_0^x \left[1-\frac{1}{ab}f_u(u,0,0)\right] f(u,0,0)\,du, \end{aligned} \tag{6}$$

where

$$f_v(x,\theta_1 v,0) = \frac{f(x,v,0)-f(x,0,0)}{v}, \quad (v \neq 0, 0 \leq \theta_1 \leq 1). \tag{7}$$

3. Main Result

In the case $P \equiv 0$, we have the following.

Theorem 4. *Let δ_0, a, b, and c be positive constants such that δ_0 is sufficiently small, and $ab > c$, and assume that the following conditions are satisfied:*

(i) *$f(x,0,0)/x \geq \delta_0$ $(x \neq 0)$,*

(ii) *$f_x(x,0,0) \leq c$,*

(iii) *$\psi(x,y,z) \geq a$ for all x, y, and z,*

(iv) *$f_y(x,y,0) \geq b$ for all x, y, and z,*

(v) *$y\psi_z(x,y,z) \geq 0$, $f_z(x,y,z) \geq 0$ for all x, y and z,*

(vi) *$a[f(x,y,z) - f(x,0,0) - \int_0^y \psi_x(x,v,0)v\,dv]y \geq y\int_0^y f_x(x,v,0)dv$.*

Then, the zero solution of (1) is globally asymptotically stable.

Proof. From conditions (i)–(iv) of Theorem 4, we obtain

$$\begin{aligned} V(x,y,z) = {} & \frac{1}{2a}(ay+z)^2 \\ & + \frac{1}{2ab}\left[f(x,0,0)+by\right]^2 + \frac{1}{2}\delta_1 x^2, \end{aligned} \tag{8}$$

where $\delta_1 = (1/ab)(ab-c)\delta_0 > 0$. It follows that there exists a constant $D_0 > 0$ small enough that

$$V(x,y,z) \geq D_0\left(x^2+y^2+z^2\right). \tag{9}$$

Hence, V is a positive definite function (see Global Asymptotic Stability on page 223, Theorem 5.2.12 that of Rao [12]).

Now, we show that the derivative of V with respect to t along the solution path of system (2) is negative semidefinite.

Let

$$\dot{V} = -U, \tag{10}$$

where

$$\begin{aligned} U = \Bigg[& \frac{\psi(x,y,z) - \psi(x,y,0)}{z} y \\ & + \frac{1}{a} \frac{f(x,y,z) - f(x,y,0)}{z} \Bigg] z^2 \\ & + \left[\frac{1}{a} \psi(x,y,z) - 1 \right] z^2 \\ & + y \Bigg[f(x,y,z) - f(x,0,0) \\ & \quad - \int_0^y \psi_x(x,v,0)\, v\, dv \\ & \quad - \frac{1}{a} \int_0^y f_x(x,v,0)\, dv \Bigg]. \end{aligned} \tag{11}$$

First, from condition (v) of Theorem 4, we obtain

$$\begin{gathered} \left[\frac{\psi(x,y,z) - \psi(x,y,0)}{z} \right] yz^2 = y\psi_z(x,y,\theta_2 z) z^2 \geq 0, \\ 0 \leq \theta_2 \leq 1, \\ \left[\frac{f(x,y,z) - f(x,y,0)}{z} \right] z^2 = f_z(x,y,\theta_3 z) z^2 \geq 0, \\ 0 \leq \theta_3 \leq 1. \end{gathered} \tag{12}$$

Next, from conditions (iii), (v), and (vi) of Theorem 4 we obtain that $U \geq 0$.

Hence

$$\dot{V}_{(2)}(x,y,z) \leq 0. \tag{13}$$

In addition, we can conclude that $V(x,y,z) \to \infty$ as $x^2 + y^2 + z^2 \to \infty$ (see Global Asymptotic Stability on page 223, Theorem 5.2.12 of Rao [12]).

The whole discussions (conditions of Theorems 1, 2, 3, and 4) show that the zero solution of system (2) is globally asymptotically stable (also see Theorem 1.5 that of Reissing et al. [13]). Then, the rest of the proof may now follow as in [2]. Thus, the proof of Theorem 4 is completed. □

In the case $P(t,x,y,z) \neq 0$, we have the following.

Theorem 5. *Suppose that the following conditions are satisfied:*

(i) *all the conditions of Theorem 4 hold;*

(ii) $|P(t,x,y,z)| \leq q(t)$, *where* $q \in L^1(0,\infty)$, $L^1(0,\infty)$ *is a space of integrable Lebesgue functions.*

Then, there is a finite positive constant K such that every solution $(x(t), y(t), z(t))$ of system (2) *satisfies*

$$|x(t)|^2 \leq K, \qquad |y(t)|^2 \leq K, \qquad |z(t)|^2 \leq K. \tag{14}$$

Proof. Consider the function $V(t) = V$ defined as above. Since $P \neq 0$, then the total derivative of $V(t)$ can be revised as

$$\dot{V}(t) \leq \dot{V}_{(2)} + \left[y + \frac{1}{a} z \right] \times P(t,x,y,z). \tag{15}$$

Let $D_1 = \max(1, a^{-1})$. Then, we have

$$\dot{V}(t) \leq D_1 \left[|y| + |z| \right] q(t). \tag{16}$$

Using the inequality

$$|y| \leq 1 + y^2, \tag{17}$$

we obtain

$$\dot{V}(t) \leq D_1 \left[2 + y^2 + z^2 \right] q(t). \tag{18}$$

From (9), we have

$$\dot{V}(t) \leq D_2 q(t) + D_3 V(t) q(t), \tag{19}$$

where

$$D_2 = 2D_1, \qquad D_3 = D_1 D_0^{-1}. \tag{20}$$

Integrating (19) from 0 to t, we obtain

$$V(t) \leq V(0) + D_2 \int_0^t q(s)\, ds + D_3 \int_0^t V(s) q(s)\, ds. \tag{21}$$

Setting

$$D_4 = V(0) + D_2 \int_0^t q(s)\, ds, \tag{22}$$

then, we have

$$V(t) \leq D_4 + D_3 \int_0^t V(s) q(s)\, ds. \tag{23}$$

Using Gronwall-Bellman inequality (see Rao [12]) yields

$$V(t) \leq D_4 \exp\left[D_3 \int_0^t q(s)\, ds \right]. \tag{24}$$

The proof of Theorem 5 is complete. □

4. Example

Consider the equation

$$\begin{aligned} & \dddot{x} + \left[(\sin x)\dot{x} + (\dot{x})^2 + e^{\dot{x}\ddot{x}} + 3 \right] \ddot{x} + (\dot{x})^3 + \dot{x} + \frac{x}{1+x^2} + \dot{x} e^{\dot{x}\ddot{x}} \\ & \quad = \frac{1}{1 + t^2 + x^2 + (\dot{x})^2 + (\ddot{x})^2}. \end{aligned} \tag{25}$$

Equation (25) is in the form of (1), where

$$\psi(x,y,z) = (\sin x)\,y + y^2 + e^{yz} + 3,$$
$$f(x,y,z) = y^3 + y + \frac{x}{1+x^2} + ye^{yz}, \tag{26}$$
$$P(t,x,y,z) = \frac{1}{1+t^2+x^2+y^2+z^2}.$$

Case 1 ($P \equiv 0$). With $a = 2$, $b = 1$, $c = 1$, from condition (vi) of Theorem 4, we have

$$2\left[\left(1 - \frac{1}{3}\cos x\right)y^2 + e^{yz} + 1\right]y^2 \geq \frac{1-x^2}{\left(1+x^2\right)^2}y^2. \tag{27}$$

Hence, condition (vi) of Theorem 4 is satisfied.

Then, it is easy to check that all the other conditions [(i)–(vi)] of Theorem 4 are satisfied. Hence, the trivial solution of (25) is globally asymptotically stable.

Case 2 ($P \neq 0$). Assume that ψ, f, a, b, c are the same as in Case 1 of the example, and

$$P(t,x,y,z) = \frac{1}{1+t^2+x^2+y^2+z^2} \leq \frac{1}{1+t^2} = q(t) \tag{28}$$

and that

$$\int_0^\infty q(s)\,ds = \int_0^\infty \frac{1}{1+s^2}ds = \frac{\pi}{2} < \infty. \tag{29}$$

Then, all the conditions of Theorem 5 are satisfied. Hence, the solutions of (25) are bounded.

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