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Research Article

A Discrete-Time $Geo/G/1$ Retrial Queue with J Vacations and Two Types of Breakdowns

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This paper is concerned with a discrete-time $Geo/G/1$ retrial queueing model with J vacations and two types of breakdowns. If the orbit is empty, the server takes at most J vacations repeatedly until at least one customer appears in the orbit upon returning from a vacation. It is assumed that the server is subject to two types of different breakdowns and is sent immediately for repair. We analyze the Markov chain underlying the considered queueing system and derive the system state distribution as well as the orbit size and the system size distributions in terms of their generating functions. Then, we obtain some performance measures through the generating functions. Moreover, the stochastic decomposition property and the corresponding continuous-time queueing system are investigated. Finally, some numerical examples are provided to illustrate the effect of vacations and breakdowns on several performance measures of the system.

1. Introduction

During the past few decades, retrial queueing systems have been widely studied due to their important applications in many practical systems such as inventory systems, computer systems, and telecommunication networks. Retrial queueing systems are characterized by the fact that arriving customers who find the server busy may join a retrial group to retry their requests after some random time. For excellent bibliographies on retrial queues, the readers are referred to [1–3].

Although many continuous-time retrial queueing models have been studied extensively in the past years, only few researchers studied the discrete-time retrial queues. In fact, the discrete-time retrial queueing systems are suitable for the description of the various phenomena in slotted time computer and communication systems such as broad-band integrated services digital network (B-ISDN) and time-division multiple access (TDMA) systems. Yang and Li [4] firstly studied a discrete-time $Geo/G/1$ retrial queue and obtained some performance measures of the systems. This work was generalized to discrete-time $Geo/G/1$ retrial queue with general retrial times by Atencia and Moreno [5]. In recent years, there has been a growing interest in the investigation of discrete-time retrial queue with server breakdowns. Such phenomena occur in day-to-day life, for example, the components of the computer and communication systems are subject to random breakdowns. Discrete-time retrial queues that take into account server breakdowns were studied by Atencia and Moreno [6, 7], Moreno [8], Wang and Zhao [9], Wang [10], Wang and Zhang [11], and Atencia et al. [12].

On the other hand, queueing systems with server vacations have been widely used to analyze the performance of some systems such as manufacturing systems and computer systems. In recent years, retrial queues with server vacations have drawn more and more attention. Li and Yang [13] studied a $M/G/1$ retrial queue with a finite number of input sources and Bernoulli vacations by the method of supplementary variables. Since the work of Li and Yang, the retrial queues with Bernoulli vacations have been studied by many authors, see Kumar and Arivudainambi [14], Kumar et al. [15], and Choudhury [16]. In addition, the retrial queues with exhaustive service vacations were also studied by several authors. For example, Artalejo [17] analyzed an $M/G/1$ retrial queue with multiple vacations and obtained two stochastic decomposition results for the system size. As a generalization of single and multiple vacation policy, Chang and Ke [18] and Ke and Chang [19] introduced the concept of J vacations into $M/G/1$ retrial queueing systems. Ke and Chang [20] and Wu

[21] studied the models concerning both server breakdowns and vacations, simultaneously.

The majority of articles studied the continuous-time retrial queues with server vacations. Most recently, Wang [22] extended the continuous-time $M/G/1$ retrial queue with Bernoulli vacations to discrete-time counterpart. Li et al. [23], Liu and Song [24] considered the discrete-time $Geo/Geo/1$ retrial queue with working vacation. Zhang et al. [25] studied a discrete-time $Geo/G/1$ retrial queue with single vacation and starting failure. To the best of our knowledge, no work appeared till now concerning the discrete-time retrial queue with exhaustive J vacations policy and server's different types of breakdowns. In fact, the discrete-time retrial queues with J vacations and server breakdowns are more appropriate to analyze some real situations, such as e-mail systems, IP access networks, and computer systems. This motivates us to analyze the discrete-time retrial queue with J vacations and and general retrial times. In this work, we consider a discrete-time $Geo/G/1$ retrial queue with J vacations where the server may be subjected to two different types of breakdowns.

The rest of the paper is organized as follows. In Section 2, the description of our model is given. In Section 3, We derive the generating functions of the number of customers in the orbit and in the system. The closed-form expressions of some performance measures of the system are also obtained. In Section 4, we give two stochastic decomposition results for the system size. In Section 5, the relationship between our model and the continuous-time counterpart is given. In Section 6, we present some numerical results to show the influence of the parameters on some performance measures of the system.

2. Description of the Mathematical Model

We consider a discrete-time $Geo/G/1$ retrial queue with J vacations and two types of server breakdowns where the time axis is segmented into slots of equal length and all queueing activities occur at the slot boundaries. Specifically, let the time axis be marked by $0, 1, \ldots, m, \ldots$. Consider the epoch m and assume that the departures, the end of the vacations and the end of repairs occur in the interval (m^-, m), while arrivals, retrials, the beginning of the vacations, and the beginning of repairs occur in the interval (m, m^+) in sequence. That is, we consider an early arrival system (EAS) policy in our model.

Customers arrive at the system according to a geometrical arrival process with parameter p, where p $(0 < p < 1)$ is the probability that an arrival occurs in a slot. If an arriving customer finds that the server is idle, he commences his service immediately. Otherwise, if the server is busy, under repair or on vacation, the arriving customer joins the retrial orbit. It is assumed that only the customer at the head of the orbit is permitted to access to the server. The time between retrials is assumed to follow a general probability distribution $\{a_i\}_{i=0}^{\infty}$ with the generating function $A(x) = \sum_{i=0}^{\infty} a_i x^i$.

The service times are independent and identically distributed with arbitrary distribution $\{s_i\}_{i=1}^{\infty}$, generating function $S(x) = \sum_{i=1}^{\infty} s_i x^i$ and nth factorial moment S_n. It is assumed that the server is subject to two different types of breakdowns. The first type of breakdown is starting failure. In this model, an arriving customer who finds the server idle must turn on the server. If the server is activated successfully (with a probability $\overline{\theta}_1 = 1 - \theta_1$), the customer begins his service immediately; otherwise, if the server is started unsuccessfully (with a complementary probability θ_1), the server is sent to repair immediately and the customer being served must join the orbit. The second type of breakdown is normal breakdown, that is, the server may break down during serving customers. The customer just being served before breakdown waits until the server is repaired to complete his remaining service time. The lifetime of the server is assumed to be geometrically distributed with parameter θ_2. The repair times for the server of two types of breakdowns are independent and identically distributed with arbitrary distribution $\{r_{l,i}\}_{i=1}^{\infty}$, generating functions $R_l(x) = \sum_{i=0}^{\infty} r_{l,i} x^i$, and nth factorial moment are $R_{l,n}$, $l = 1, 2$, respectively.

As soon as the system is empty, the server takes a vacation immediately. The vacation time is assumed to follow a general probability distribution $\{v_i\}_{i=1}^{\infty}$ with generating function $V(x) = \sum_{i=1}^{\infty} v_i x^i$ and nth factorial moment V_n. If there is at least one customer in the orbit at the end of a vacation, the server begins to provide service for customers immediately. Otherwise, if no customers appear in the orbit when the server returns from the vacation, it leaves for another vacation. This pattern continues until the server has taken a maximum number J of vacations. If the system is empty at the end of the Jth vacation, the server becomes idle and waits for new arriving customers.

Finally, we assume that the interarrival times, the service times, the retrial times, the repair times and the vacation times are mutually independent.

3. The Markov Chain and Some Performance Measures

At time m^+, the system can be described by the process $\{Y_m; m \geq 1\}$ with $Y_m = (C_m, \xi_m, \zeta_m, L_m, N_m)$, where C_m denotes the state of the server and N_m be the number of the customers in the orbit. It is assumed that $C_m = 0, 1, 2, 3$, or 4 according to whether the server is free, busy, under repair for first type of breakdown, under repair for second type of breakdown, or on vacations, respectively. If $C_m = 0$, ξ_m represents the remaining retrial time. If $C_m = 1$, ξ_m represents the remaining service time of the customer currently being served. If $C_m = 2$, ξ_m represents the remaining repair time for the first type of breakdown. If $C_m = 3$, ξ_m represents the remaining service time of a customer currently being served and ζ_m represents the remaining repair time of the second type of breakdowns. If $C_m = 4$, let $L_m = n$ represent that the server takes the nth vacation and let ξ_m represent the remaining vacation time, $n = 1, 2, \ldots, J$. It can be shown that $\{Y_m; m \geq 1\}$ is a Markov chain with the following state space:

$$\Omega = \{(0,0); (0,i,k) : i \geq 1, k \geq 1;$$
$$(1,i,k) : i \geq 1, k \geq 0; (2,i,k) : i \geq 1, k \geq 1;$$

$$(3,i,j,k): i \geq 1, k \geq 0\ (4,i,k): i \geq 1, k \geq 0, n = 1,2,\ldots,J\}. \tag{1}$$

Our object is to find the stationary distribution of the Markov chain which is defined as follows:

$$\begin{aligned}
\pi_{0,0} &= \lim_{m\to\infty} P\{C_m = 0, N_m = 0\},\\
\pi_{0,i,k} &= \lim_{m\to\infty} P\{C_m = 0, \xi_m = i, N_m = k\}; \quad i \geq 1,\ k \geq 1,\\
\pi_{1,i,k} &= \lim_{m\to\infty} P\{C_m = 1, \xi_m = i, N_m = k\}; \quad i \geq 1,\ k \geq 0,\\
\pi_{2,i,k} &= \lim_{m\to\infty} P\{C_m = 2, \xi_m = i, N_m = k\}; \quad i \geq 1,\ k \geq 1,\\
\pi_{3,i,j,k} &= \lim_{m\to\infty} P\{C_m = 3, \xi_m = i, \zeta_m = j, N_m = k\};\\
&\qquad i \geq 1,\ j \geq 1,\ k \geq 0,\\
\omega_{n,i,k} &= \lim_{m\to\infty} P\{C_m = 4, L_m = n, \xi_m = i, N_m = k\};\\
&\qquad i \geq 1,\ k \geq 0,\ n = 1,2,\ldots,J.
\end{aligned} \tag{2}$$

The Kolmogorov equations for the stationary distribution of the system are

$$\pi_{0,0} = \overline{p}\pi_{0,0} + \overline{p}\omega_{J,1,0}, \tag{3}$$

$$\begin{aligned}
\pi_{0,i,k} &= \overline{p}\pi_{0,i+1,k} + \overline{p}a_i\pi_{1,1,k} + \overline{p}a_i\pi_{2,1,k}\\
&\quad + \overline{p}a_i\sum_{n=1}^{J}\omega_{n,1,k}, \quad i \geq 1,\ k \geq 1,
\end{aligned} \tag{4}$$

$$\begin{aligned}
\pi_{1,i,k} &= \delta_{0,k}p\overline{\theta}_1\overline{\theta}_2 s_i\pi_{0,0} + \overline{\delta}_{0,k}p\overline{\theta}_1\overline{\theta}_2 s_i\sum_{j=1}^{\infty}\pi_{0,j,k}\\
&\quad + \overline{p}\overline{\theta}_1\overline{\theta}_2 s_i\pi_{0,1,k+1} + p\overline{\theta}_1\overline{\theta}_2 s_i\omega_{J,1,k}\\
&\quad + \overline{p}a_0\overline{\theta}_1\overline{\theta}_2 s_i\pi_{1,1,k+1} + \overline{\delta}_{0,k}p\overline{\theta}_1\overline{\theta}_2 s_i\pi_{1,1,k}\\
&\quad + \overline{p}\overline{\theta}_2\pi_{1,i+1,k} + \overline{\delta}_{0,k}p\overline{\theta}_2\pi_{1,i+1,k-1}\\
&\quad + \overline{\delta}_{0,k}p\overline{\theta}_1\overline{\theta}_2 s_i\pi_{2,1,k} + \overline{p}a_0\overline{\theta}_1\overline{\theta}_2 s_i\pi_{2,1,k+1}\\
&\quad + \overline{p}\overline{\theta}_2\pi_{3,i,1,k} + \overline{\delta}_{0,k}p\overline{\theta}_2\pi_{3,i,1,k-1}\\
&\quad + \overline{p}a_0\overline{\theta}_1\overline{\theta}_2 s_i\sum_{n=1}^{J}\omega_{n,1,k+1}\\
&\quad + \overline{\delta}_{0,k}p\overline{\theta}_1\overline{\theta}_2 s_i\sum_{n=1}^{J-1}\omega_{n,1,k}, \quad i \geq 1,\ k \geq 0,
\end{aligned} \tag{5}$$

$$\begin{aligned}
\pi_{2,i,k} &= \delta_{1,k}p\theta_1 r_{1,i}\pi_{0,0} + \overline{\delta}_{1,k}p\theta_1 r_{1,i}\sum_{j=1}^{\infty}\pi_{0,j,k-1}\\
&\quad + \overline{p}\theta_1 r_{1,i}\pi_{0,1,k} + \overline{p}a_0\theta_1 r_{1,i}\pi_{1,1,k}\\
&\quad + \overline{\delta}_{1,k}p\theta_1 r_{1,i}\pi_{1,1,k-1} + \overline{\delta}_{1,k}p\pi_{2,i+1,k-1}\\
&\quad + \overline{p}\pi_{2,i+1,k} + \overline{p}a_0\theta_1 r_{1,i}\pi_{2,1,k}\\
&\quad + \overline{\delta}_{1,k}p\theta_1 r_{1,i}\pi_{2,1,k-1} + \overline{p}a_0\theta_1 r_{1,i}\sum_{n=1}^{J}\omega_{n,1,k}\\
&\quad + \overline{\delta}_{1,k}p\theta_1 r_{1,i}\sum_{n=1}^{J-1}\omega_{n,1,k-1}\\
&\quad + p\theta_1 r_{1,i}\omega_{J,1,k-1}, \quad i \geq 1,\ k \geq 1,
\end{aligned} \tag{6}$$

$$\begin{aligned}
\pi_{3,i,j,k} &= \delta_{0,k}p\overline{\theta}_1\theta_2 s_i r_{2,j}\pi_{0,0} + \overline{\delta}_{0,k}p\overline{\theta}_1\theta_2 s_i r_{2,j}\sum_{l=1}^{\infty}\pi_{0,l,k}\\
&\quad + \overline{p}\overline{\theta}_1\theta_2 s_i r_{2,j}\pi_{0,1,k+1} + p\overline{\theta}_1\theta_2 s_i r_{2,j}\omega_{J,1,k}\\
&\quad + \overline{p}a_0\overline{\theta}_1\theta_2 s_i r_{2,j}\pi_{1,1,k+1} + \overline{\delta}_{0,k}p\overline{\theta}_1\theta_2 s_i r_{2,j}\pi_{1,1,k}\\
&\quad + \overline{p}\theta_2 r_{2,j}\pi_{1,i+1,k} + \overline{\delta}_{0,k}p\theta_2 r_{2,j}\pi_{1,i+1,k-1}\\
&\quad + \overline{\delta}_{0,k}p\overline{\theta}_1\theta_2 s_i r_{2,j}\pi_{2,1,k} + \overline{p}a_0\overline{\theta}_1\theta_2 s_i r_{2,j}\pi_{2,1,k+1}\\
&\quad + \overline{p}\pi_{3,i,j+1,k} + \overline{p}\theta_2 r_{2,j}\pi_{3,i,1,k}\\
&\quad + \overline{p}a_0\overline{\theta}_1\theta_2 s_i r_{2,j}\sum_{n=1}^{J}\omega_{n,1,k+1}\\
&\quad + \overline{\delta}_{0,k}p\overline{\theta}_1\theta_2 s_i r_{2,j}\sum_{n=1}^{J-1}\omega_{n,1,k}\\
&\quad + \overline{\delta}_{0,k}p\theta_2 r_{2,j}\pi_{3,i,1,k-1} + \overline{\delta}_{0,k}p\pi_{3,i,j+1,k-1},\\
&\qquad i \geq 1, j \geq 1, k \geq 0,
\end{aligned} \tag{7}$$

$$\begin{aligned}
\omega_{1,i,k} &= \overline{p}\omega_{1,i+1,k} + \overline{\delta}_{0,k}p\omega_{1,i+1,k-1} + \delta_{0,k}\overline{p}v_i\pi_{1,1,0}\\
&\quad + \delta_{1,k}pv_i\pi_{1,1,0}, \quad i \geq 1,\ k \geq 0,
\end{aligned} \tag{8}$$

$$\begin{aligned}
\omega_{n,i,k} &= \overline{p}\omega_{n,i+1,k} + \overline{\delta}_{0,k}p\omega_{n,i+1,k-1} + \delta_{0,k}\overline{p}v_i\omega_{n-1,1,0}\\
&\quad + \delta_{1,k}pv_i\omega_{n-1,1,0}, \quad i \geq 1,\ k \geq 0,\ n = 2,3,\ldots,J,
\end{aligned} \tag{9}$$

where $\delta_{i,j}$ is the Kronecker's symbol, $\overline{\delta}_{i,j} = 1 - \delta_{i,j}$, and the normalizing condition is

$$\begin{aligned}
\pi_{0,0} &+ \sum_{i=1}^{\infty}\sum_{k=1}^{\infty}(\pi_{0,i,k} + \pi_{2,i,k}) + \sum_{i=1}^{\infty}\sum_{k=0}^{\infty}\pi_{1,i,k}\\
&+ \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\sum_{k=0}^{\infty}\pi_{3,i,j,k} + \sum_{n=1}^{J}\sum_{i=1}^{\infty}\sum_{k=0}^{\infty}\omega_{n,i,k} = 1.
\end{aligned} \tag{10}$$

To solve (3)–(10), we introduce the following generating functions:

$$\varphi_0(x,z) = \sum_{i=1}^{\infty}\sum_{k=1}^{\infty}\pi_{0,i,k}x^i z^k,$$

$$\varphi_1(x,z) = \sum_{i=1}^{\infty}\sum_{k=0}^{\infty}\pi_{1,i,k}x^i z^k,$$

$$\varphi_2(x,z)=\sum_{i=1}^{\infty}\sum_{k=1}^{\infty}\pi_{2,i,k}x^i z^k,$$

$$\varphi_3(x,y,z)=\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\sum_{k=0}^{\infty}\pi_{3,i,j,k}x^i y^i z^k,$$

$$\psi_n(x,z)=\sum_{i=1}^{\infty}\sum_{k=0}^{\infty}\omega_{n,i,k}x^i z^k,\quad n=1,2,\ldots,J,$$

$$\varphi_4(x,z)=\sum_{n=1}^{J}\psi_n(x,z),\tag{11}$$

and the following auxiliary generating functions:

$$\varphi_{0,i}(z)=\sum_{k=1}^{\infty}\pi_{0,i,k}z^k,\qquad \varphi_{1,i}(z)=\sum_{k=0}^{\infty}\pi_{1,i,k}z^k,$$

$$\varphi_{2,i}(z)=\sum_{k=1}^{\infty}\pi_{2,i,k}z^k,\qquad \psi_{n,i}(z)=\sum_{k=1}^{\infty}\omega_{n,i,k}z^k,$$

$$\varphi_{3,i,j}(z)=\sum_{k=0}^{\infty}\pi_{3,i,j,k}z^k,\qquad \varphi_{3,i}(y,z)=\sum_{k=0}^{\infty}\pi_{3,i,j,k}y^j z^k,\tag{12}$$

$$\phi_1(x,z)=\sum_{i=1}^{\infty}\sum_{k=0}^{\infty}\pi_{3,i,1,k}x^i z^k.$$

Now, we can solve (3)–(10) by using the generating function technique. We first give some lemmas which will be used later on and their proof which can be readily obtained. Thus, they are omitted. The following inequalities hold.

Lemma 1. *If $\overline{\theta}_1\rho_1+\theta_1+\theta_1\rho_2<p+\overline{p}A(\overline{p})$, then the inequality*

$$\left[z+(1-z)\overline{p}A(\overline{p})\right]\Omega(z)-z\gamma(z)>0\tag{13}$$

holds for $0\le z<1$, where

$$\gamma(z)=\overline{p}+pz,\qquad \rho_1=pS_1\left(1+\frac{\theta_2}{\overline{\theta}_2}R_{2,1}\right),$$

$$\rho_2=pR_{1,1},\tag{14}$$

$$\Omega(z)=\overline{\theta}_1S(\tau(\gamma(z)))+z\theta_1R_1(\gamma(z)),$$

$$\tau(z)=\frac{\overline{\theta}_2z}{1-\theta_2R_2(z)}.$$

Lemma 2. *The following limits exist, if $\overline{\theta}_1\rho_1+\theta_1+\theta_1\rho_2<p+\overline{p}A(\overline{p})$*

$$\lim_{z\to1}\frac{\gamma(z)\Gamma(z)-\eta_3\Omega(z)}{\Lambda(z)}=\frac{pV_1\eta_2+p\eta_3\left[\overline{\theta}_1S_1\left(1+\left(\theta_2/\overline{\theta}_2\right)R_{2,1}\right)-\overline{\theta}_1+\theta_1R_{1,1}\right]}{p+\overline{p}A(\overline{p})-\overline{\theta}_1\rho_1-\theta_1-\theta_1\rho_2},$$

$$\lim_{z\to1}\frac{\Theta(z)}{\Lambda(z)}=\frac{pV_1\eta_2+\overline{p}A(\overline{p})\eta_3}{p+\overline{p}A(\overline{p})-\overline{\theta}_1\rho_1-\theta_1-\theta_1\rho_2},\tag{15}$$

where

$$\xi=V(\overline{p}),\qquad \eta_1=1-\xi^{J+1},$$

$$\eta_2=1-\xi^J,\qquad \eta_3=(1-\xi)\xi^J,$$

$$\Theta(z)=z\eta_2\left[1-V(\gamma(z))\right]+\overline{p}A(\overline{p})(1-z)\Gamma(z),\tag{16}$$

$$\Gamma(z)=\eta_1-\eta_2V(\gamma(z)),$$

$$\Lambda(z)=\left[z+\overline{p}A(\overline{p})(1-z)\right]\Omega(z)-z\gamma(z).$$

By using Lemmas 1 and 2, we can obtain the generating functions of the stationary distribution of the system which are given by the following theorem.

Theorem 3. *If $\overline{\theta}_1\rho_1+\theta_1+\theta_1\rho_2<p+\overline{p}A(\overline{p})$, the stationary distribution of the Markov chain $\{Y_m, m=0,1,2,\ldots\}$ has the following generating functions:*

$$\varphi_0(x,z)=\frac{A(x)-A(\overline{p})}{x-\overline{p}}\frac{pxz\left[\gamma(z)\Gamma(z)-\eta_3\Omega(z)\right]}{\eta_3\Lambda(z)}\pi_{0,0},$$

$$\varphi_1(x,z)=\frac{S(x)-S(\tau(\gamma(z)))}{x-\tau(\gamma(z))}\frac{\Theta(z)px\overline{\theta}_1\tau(\gamma(z))}{\overline{p}\eta_3\Lambda(z)}\pi_{0,0},$$

$$\varphi_2(x,z)=\frac{R_1(x)-R_1(\gamma(z))}{x-\gamma(z)}\frac{\Theta(z)px\theta_1z\gamma(z)}{\overline{p}\eta_3\Lambda(z)}\pi_{0,0},$$

$$\varphi_3(x,y,z)=\frac{S(x)-S(\tau(\gamma(z)))}{x-\tau(\gamma(z))}\times\frac{R_2(y)-R_2(\gamma(z))}{y-\gamma(z)}\frac{\Theta(z)pxy\overline{\theta}_1\tau(\gamma(z))}{\overline{p}\eta_3\Lambda(z)}\frac{\theta_2}{\overline{\theta}_2}\pi_{0,0},$$

$$\varphi_4(x,z)=\frac{p\eta_2x\gamma(z)\left[V(x)-V(\gamma(z))\right]}{\overline{p}\eta_3\left[x-\gamma(z)\right]}\pi_{0,0},\tag{17}$$

where

$$\pi_{0,0}=\frac{\overline{p}\eta_3\left[p+\overline{p}A(\overline{p})-\overline{\theta}_1\rho_1-\theta_1-\theta_1\rho_2\right]}{\overline{\theta}_1\left[pV_1\eta_2+\overline{p}A(\overline{p})\eta_3\right]}.\tag{18}$$

Proof. Multiplying (3)–(9) by z^k and summing over k, we get the following equations:

$$\begin{aligned}\varphi_{0,i}(z) &= \overline{p}\varphi_{0,i+1}(z) \\ &\quad + \overline{p}a_i\left[\varphi_{1,1}(z)+\varphi_{2,1}(z)+\sum_{n=1}^{J}\psi_{n,1}(z)\right] \\ &\quad - \overline{p}a_i\pi_{1,1,0} - \overline{p}a_i\sum_{n=1}^{J}\omega_{n,1,0}, \quad i\ge 1,\end{aligned} \tag{19}$$

$$\begin{aligned}\varphi_{1,i}(z) &= p\overline{\theta}_1\overline{\theta}_2 s_i\pi_{0,0} + p\overline{\theta}_1\overline{\theta}_2 s_i\varphi_0(1,z) \\ &\quad + \frac{\overline{p}}{z}\overline{\theta}_1\overline{\theta}_2 s_i\varphi_{0,1}(z) + \overline{\theta}_2\gamma(z)\varphi_{1,i+1}(z) \\ &\quad + \left(p+\frac{\overline{p}}{z}a_0\right)\overline{\theta}_1\overline{\theta}_2 s_i\left[\varphi_{1,1}(z) + \varphi_{2,1}(z)+\sum_{n=1}^{J}\psi_{n,1}(z)\right] \\ &\quad - \left(p+\frac{\overline{p}}{z}a_0\right)\overline{\theta}_1\overline{\theta}_2 s_i\pi_{1,1,0}, - p\overline{\theta}_1\overline{\theta}_2 s_i\sum_{n=1}^{J-1}\omega_{n,1,0} \\ &\quad - \frac{\overline{p}}{z}a_0\overline{\theta}_1\overline{\theta}_2 s_i\sum_{n=1}^{J}\omega_{n,1,0} + \overline{\theta}_2\gamma(z)\varphi_{3,i,1}(z), \quad i\ge 1,\end{aligned} \tag{20}$$

$$\begin{aligned}\varphi_{2,i}(z) &= p\theta_1 z r_{1,i}\pi_{0,0} + p\theta_1 z r_{1,i}\varphi_0(1,z) \\ &\quad + \overline{p}\theta_1 r_{1,i}\varphi_{0,1}(z) + \gamma(z)\varphi_{2,i+1}(z) \\ &\quad + (pz+\overline{p}a_0)\theta_1 r_{1,i}\left[\varphi_{1,1}(z)+\varphi_{2,1}(z) + \sum_{n=1}^{J}\psi_{n,1}(z)\right] \\ &\quad - (pz+\overline{p}a_0)\theta_1 r_{1,i}\pi_{1,1,0} \\ &\quad - \overline{p}a_0\theta_1 r_{1,i}\sum_{n=1}^{J}\omega_{n,1,0} \\ &\quad - p\theta_1 r_{1,i} z\sum_{n=1}^{J-1}\omega_{n,1,0}, \quad i\ge 1,\end{aligned} \tag{21}$$

$$\begin{aligned}\varphi_{3,i,j}(z) &= p\overline{\theta}_1\theta_2 s_i r_{2,j}\pi_{0,0} + p\overline{\theta}_1\theta_2 s_i r_{2,j}\varphi_0(1,z) \\ &\quad + \frac{\overline{p}}{z}\overline{\theta}_1\theta_2 s_i r_{2,j}\varphi_{0,1}(z) + \theta_2\gamma(z) r_{2,j}\varphi_{1,i+1}(z) \\ &\quad + \left(p+\frac{\overline{p}}{z}a_0\right)\overline{\theta}_1\theta_2 s_i r_{2,j}\left[\varphi_{1,1}(z)+\varphi_{2,1}(z) + \sum_{n=1}^{J}\psi_{n,1}(z)\right] \\ &\quad + \gamma(z)\varphi_{3,i,j+1}(z) - p\overline{\theta}_1\theta_2 s_i r_{2,j}\sum_{n=1}^{J-1}\omega_{n,1,0} \\ &\quad - \frac{\overline{p}}{z}a_0\overline{\theta}_1\theta_2 s_i r_{2,j}\sum_{n=1}^{J}\omega_{n,1,0} + \theta_2\gamma(z)\varphi_{3,i,1}(z) \\ &\quad - \left(p+\frac{\overline{p}}{z}a_0\right)\overline{\theta}_1\theta_2 s_i r_{2,j}\pi_{1,1,0}, \quad i\ge 1,\end{aligned} \tag{22}$$

$$\begin{aligned}\psi_{1,i}(z) &= \overline{p}\psi_{1,i+1}(z) + pz\psi_{1,i+1}(z) \\ &\quad + \gamma(z)v_i\pi_{1,1,0}, \quad i\ge 1,\ k\ge 0,\end{aligned} \tag{23}$$

$$\begin{aligned}\psi_{n,i}(z) &= \overline{p}\psi_{n,i+1}(z) + pz\psi_{n,i+1}(z) \\ &\quad + \gamma(z)v_i\omega_{n-1,1,0}, \quad i\ge 1,\ k\ge 0,\ 2\le n\le J.\end{aligned} \tag{24}$$

Multiplying (23) and (24) by x^i, and summing over i, we get

$$\frac{x-\gamma(z)}{x}\psi_1(x,z) = -\gamma(z)\psi_{1,1}(z) + \gamma(z)V(x)\pi_{1,1,0}, \tag{25}$$

$$\frac{x-\gamma(z)}{x}\psi_n(x,z) = -\gamma(z)\psi_{n,1}(z) + \gamma(z)V(x)\omega_{n-1,1,0}, \quad 2\le n\le J. \tag{26}$$

Setting $x=\gamma(z)$ in (25) and (26), we get

$$\psi_{1,1}(z) = V(\gamma(z))\pi_{1,1,0}, \tag{27}$$

$$\psi_{n,1}(z) = V(\gamma(z))\omega_{n-1,1,0}, \quad 2\le n\le J. \tag{28}$$

Substituting (27) and (28) into (25) and (26), respectively, we obtain

$$\begin{aligned}\psi_1(x,z) &= \frac{\overline{p}x(V(x)-V[\gamma(z)])}{x-\gamma(z)}\pi_{1,1,0}, \\ \psi_n(x,z) &= \frac{\overline{p}x(V(x)-V[\gamma(z)])}{x-\gamma(z)}\omega_{n-1,1,0}, \quad 2\le n\le J.\end{aligned} \tag{29}$$

Taking the derivative of (29) with respective to x, and letting $x=z=0$, we get

$$\omega_{1,1,0} = \xi\pi_{1,1,0}, \tag{30}$$

$$\omega_{n,1,0} = \xi\omega_{n-1,1,0} = \xi^n\pi_{1,1,0}, \quad 2\le n\le J. \tag{31}$$

Substituting (31) into (3), we get

$$\pi_{1,1,0} = \frac{p\pi_{0,0}}{\overline{p}\xi^J}. \tag{32}$$

Multiplying (19) by x^j, summing over j and taking into account (30)–(32), we obtain

$$\begin{aligned} &\frac{x-\overline{p}}{x}\varphi_0(x,z) \\ &\quad= \overline{p}\left(A(x)-a_0\right)\left(\varphi_{1,1}(z)+\varphi_{2,1}(z)\right) \\ &\qquad - \overline{p}\varphi_{0,1}(z) - p\left(A(x)-a_0\right)\frac{\eta_1-\eta_2 V(\gamma(z))}{\eta_3}\pi_{0,0}. \end{aligned} \tag{33}$$

Note that we can find $\varphi_0(1,z)$ by setting $x = 1$ in (33), then multiplying (20) and (21) by x^j, summing over j, and taking into account (30)–(32), we obtain

$$\begin{aligned} &\frac{x-\gamma(z)\overline{\theta}_2}{x}\varphi_1(x,z) \\ &\quad= \frac{z+\overline{p}a_0(1-z)}{z}\overline{\theta}_1\overline{\theta}_2 S(x)\left[\varphi_{1,1}(z)+\varphi_{2,1}(z)\right] \\ &\qquad - \gamma(z)\overline{\theta}_2\varphi_{1,1}(z) + \frac{\overline{p}(1-z)}{z}\overline{\theta}_1\overline{\theta}_2 S(x)\varphi_{0,1}(z) \\ &\qquad + \gamma(z)\overline{\theta}_2\phi_1(x,z) - \frac{K(z)}{z\overline{p}\eta_3}p\overline{\theta}_1\overline{\theta}_2 S(x)\pi_{0,0}, \end{aligned} \tag{34}$$

$$\begin{aligned} \frac{x-\gamma(z)}{x}\varphi_2(x,z) &= \left[z+\overline{p}a_0(1-z)\right]\theta_1 R_1(x) \\ &\quad\times\left(\varphi_{1,1}(z)+\varphi_{2,1}(z)\right) - \gamma(z)\varphi_{2,1}(z) \\ &\quad + \overline{p}\theta_1 R_1(x)(1-z)\varphi_{0,1}(z) \\ &\quad - \frac{K(z)}{\overline{p}\eta_3}p\theta_1 R_1(x)\pi_{0,0}, \end{aligned} \tag{35}$$

where $K(z) = z\eta_2[1-V(\gamma(z))] + \overline{p}a_0(1-z)\Gamma(z)$.

Multiplying (22) by y^j and x^i, summing over j and i, respectively, and taking into account (30)–(32), we obtain

$$\begin{aligned} \frac{y-\gamma(z)}{y}\varphi_3(x,y,z) &= \frac{z+\overline{p}a_0(1-z)}{z}\overline{\theta}_1\theta_2 S(x)R_2(y) \\ &\quad\times\left[\varphi_{1,1}(z)+\varphi_{2,1}(z)\right] \\ &\quad - \gamma(z)\theta_2 R_2(y)\varphi_{1,1}(z) \\ &\quad + \frac{\overline{p}(1-z)}{z}\overline{\theta}_1\theta_2 S(x)R_{2(y)}\varphi_{0,1}(z) \\ &\quad + \gamma(z)\left(1-\theta_2 R_2(y)\right)\phi_1(x,z) \\ &\quad - \frac{K(z)}{z\overline{p}\eta_3}p\overline{\theta}_1\theta_2 S(x)R_2(y)\pi_{0,0} \\ &\quad + \frac{\gamma(z)\theta_2 R_2(y)}{x}\varphi_1(x,z). \end{aligned} \tag{36}$$

Note that we can find $\phi_1(x,z)$ by setting $y = \gamma(z)$ in (36) and substituting the expression of $\phi_1(x,z)$ into (34), we get

$$\begin{aligned} &\frac{x-\tau_1(\gamma(z))}{x}\left[1-\theta_2 R_2(\gamma(z))\right]\varphi_1(x,z) \\ &\quad= \overline{\theta}_2\left\{\frac{z+\overline{p}a_0(1-z)}{z}\overline{\theta}_1 S(x)\left[\varphi_{1,1}(z)+\varphi_{2,1}(z)\right]\right. \\ &\qquad - \gamma(z)\varphi_{1,1}(z) + \frac{\overline{p}(1-z)}{z}\overline{\theta}_1 S(x)\varphi_{0,1}(z) \\ &\qquad \left. +\gamma(z)\phi_1(x,z) - \frac{K(z)}{z\overline{p}\eta_3}p\overline{\theta}_1 S(x)\pi_{0,0}\right\}. \end{aligned} \tag{37}$$

Setting $x = \overline{p}$ in (33), $x = \gamma(z)$ in (35), and $x = \tau(\gamma(z))$ in (37), respectively, we can get the equations for $\varphi_{0,1}(z)$, $\varphi_{1,1}(z)$ and $\varphi_{2,1}(z)$. By solving these equations, we get the the generating functions as follows:

$$\begin{aligned} \varphi_{0,1}(z) &= pz\left(A(\overline{p})-a_0\right)\frac{\gamma(z)\Gamma(z)-\eta_3\Omega(z)}{\overline{p}\eta_3\Lambda(z)}\pi_{0,0}, \\ \varphi_{1,1}(z) &= \frac{\Theta(z)\,p\overline{\theta}_1 S(\tau(\gamma(z)))}{\overline{p}\eta_3\Lambda(z)}\pi_{0,0}, \\ \varphi_{2,1}(z) &= \frac{\Theta(z)\,p\theta_1 zR(\gamma(z))}{\overline{p}\eta_3\Lambda(z)}\pi_{0,0}. \end{aligned} \tag{38}$$

Using Lemmas 1 and 2, it is easy to show that $\varphi_{0,1}(z)$, $\varphi_{1,1}(z)$, and $\varphi_{2,1}(z)$ are defined for $z \in [0,1)$ and can be extended by continuity in $z = 1$, if $\overline{\theta}_1\rho_1 + \theta_1 + \theta_1\rho_2 < p + \overline{p}A(\overline{p})$. Now substituting (30)–(32) into (29), we can get $\varphi_4(x,z)$. Similarly, substituting (38) into (33), (35)–(37) we derive $\varphi_0(x,z)$, $\varphi_1(x,z)$, $\varphi_2(x,z)$ and $\varphi_3(x,y,z)$. Using the normalizing condition, we can find the unknown constant $\pi_{0,0}$. This completes the proof. □

Based on Theorem 3, we can easily obtain the marginal generating functions of the number of customers when the server is in various states and some performance measures. They are summarized in the following Corollary. Their proofs are very easy, and thus are omitted. For convenience, we define variable N be the orbit size and L be the system size.

Corollary 4.

(1) The marginal generating function of the number of customers in the orbit when the server is idle or vacation is given by

$$\begin{aligned} &\pi_{0,0} + \varphi_0(1,z) \\ &\quad= \frac{z\left(1-A(\overline{p})\right)\left[\gamma(z)\Gamma(z)-\eta_3\Omega(z)\right]+\eta_3\Lambda(z)}{\eta_3\Lambda(z)}\pi_{0,0}. \end{aligned} \tag{39}$$

(2) The marginal generating function of the number of customers in the orbit when the server is busy is given by

$$\varphi_1(1,z) = \frac{1-S(\tau(\gamma(z)))}{1-\tau(\gamma(z))}\frac{\Theta(z)\overline{\theta}_1\tau(\gamma(z))}{\overline{p}\eta_3\Lambda(z)}\pi_{0,0}. \tag{40}$$

(3) The marginal generating function of the number of customers in the orbit when the server is under the first type of repair

$$\varphi_2(1,z)=\frac{1-R_1(\gamma(z))}{1-z}\frac{\Theta(z)\theta_1 z\gamma(z)}{\overline{p}\eta_3\Lambda(z)}\pi_{0,0}. \quad (41)$$

(4) The marginal generating function of the number of customers in the orbit when the server is under the second type of repair

$$\begin{aligned}\varphi_3(1,1,z)&=\frac{1-S(\tau(\gamma(z)))}{1-\tau(\gamma(z))}\frac{1-R_2(\gamma(z))}{1-z}\\&\quad\times\frac{\Theta(z)\overline{\theta}_1\tau(\gamma(z))}{\overline{p}\eta_3\Lambda(z)}\pi_{0,0}.\end{aligned} \quad (42)$$

(5) The marginal generating function of the number of customers in the orbit when the server is on vacation is given by

$$\varphi_4(1,z)=\frac{\eta_2\gamma(z)[1-V(\gamma(z))]}{\overline{p}\eta_3(1-z)}\pi_{0,0}. \quad (43)$$

(6) The generating function of the number of customers in the orbit is given by

$$\begin{aligned}\Psi(z)&=\pi_{0,0}+\varphi_0(1,z)+\varphi_1(1,z)\\&\quad+\varphi_2(1,z)+\varphi_3(1,1,z)+\varphi_4(1,z)\\&=\frac{\Theta(z)\overline{\theta}_1\gamma(z)}{\overline{p}\eta_3\Lambda(z)}\pi_{0,0}.\end{aligned} \quad (44)$$

(7) The probability generating function of the number of customers in the system is given by

$$\begin{aligned}\Phi(z)&=\pi_{0,0}+\varphi_0(1,z)+z\varphi_1(1,z)\\&\quad+\varphi_2(1,z)+z\varphi_3(1,1,z)+\varphi_4(1,z)\\&=\frac{\Theta(z)\overline{\theta}_1\gamma(z)S(\tau(\gamma(z)))}{\overline{p}\eta_3\Lambda(z)}\pi_{0,0}.\end{aligned} \quad (45)$$

Corollary 5.

(1) The system is idle with probability

$$\pi_{0,0}=\frac{\overline{p}\eta_3\left[p+\overline{p}A(\overline{p})-\overline{\theta}_1\rho_1-\theta_1-\theta_1\rho_2\right]}{\overline{\theta}_1\left[pV_1\eta_2+\overline{p}A(\overline{p})\eta_3\right]}. \quad (46)$$

(2) The probability that the server is free is

$$\pi_{0,0}+\varphi_0(1,1)+\varphi_4(1,1)=1-\rho_1-\frac{\theta_1}{\overline{\theta}_1}\rho_2. \quad (47)$$

(3) The probability that the server is busy is

$$\varphi_1(1,1)=pS_1. \quad (48)$$

(4) The probability that the server is under the first type of repair is

$$\varphi_2(1,1)=pR_{1,1}\frac{\theta_1}{\overline{\theta}_1}. \quad (49)$$

(5) The probability that the server is under the second type of repair is

$$\varphi_3(1,1,1)=pS_1R_{2,1}\frac{\theta_2}{\overline{\theta}_2}. \quad (50)$$

(6) The probability that the server is on vacation is

$$\varphi_4(1,1)=\frac{p\eta_2V_1\left[p+\overline{p}A(\overline{p})-\overline{\theta}_1\rho_1-\theta_1-\theta_1\rho_2\right]}{\overline{\theta}_1\left[pV_1\eta_2+\overline{p}A(\overline{p})\eta_3\right]}. \quad (51)$$

(7) The mean number of customers in the orbit is

$$\begin{aligned}E(N)&=\Psi'(z)\big|_{z=1}\\&=\frac{\Omega''(1)+2\overline{p}(1-A(\overline{p}))\left[\Omega'(1)-p\right]}{2\left[p+\overline{p}A(\overline{p})-\Omega'(1)\right]}\\&\quad+\frac{\eta_2\left[2pV_1(1-\overline{p}A(\overline{p}))+p^2V_2\right]}{2\left[pV_1\eta_2+\overline{p}A(\overline{p})\eta_3\right]},\end{aligned} \quad (52)$$

where

$$\Omega'(1)=\overline{\theta}_1\rho_1+\theta_1+\theta_1\rho_2, \quad (53)$$

$$\begin{aligned}\Omega''(1)&=\overline{\theta}_1\left\{p^2S_2\left(1+\frac{\theta_2}{\overline{\theta}_2}\right)^2+p^2S_1\right.\\&\quad\left.\times\left[\frac{\theta_2}{\overline{\theta}_2}R_{2,2}+2\frac{\theta_2}{\overline{\theta}_2}R_{2,1}\left(1+\frac{\theta_2}{\overline{\theta}_2}\right)\right]\right\}\\&\quad+2\theta_1pR_{1,1}+\theta_1p^2R_{1,2}.\end{aligned} \quad (54)$$

(8) The mean number of customers in the system is

$$E(L)=\Phi'(z)\big|_{z=1}=E(N)+\rho_1. \quad (55)$$

Remark 6. Consider some special cases as follows.

(i) When $J=0$, $\theta_1=\theta_2=0$, our model becomes a discrete-time $Geo/G/1$ queue with general retrial times. In this case, $\Phi(z)$ reduces to

$$\Phi(z)=\frac{(1-z)\gamma(z)\left[p+\overline{p}A(\overline{p})-pS_1\right]S[\gamma(z)]}{\left[z+\overline{p}A(\overline{p})(1-z)\right]S(\gamma(z))-z\gamma(z)}, \quad (56)$$

which coincides with results of the model studied by Atencia and Moreno [5].

(ii) When $J = 0$, $\theta_2 = 0$, the present model becomes a discrete-time *Geo*/*G*/1 queue with general retrial times and starting failure. In this case $\Phi(z)$ reduces to

$$\Phi(z) = \Big((1-z)\gamma(z) S(\gamma(z)) \times \left[p + \overline{p}A(\overline{p}) - \overline{\theta}_1 p S_1 - \theta_1 - \theta_1 p R_{2,1}\right] \Big) \times \Big([z + \overline{p}A(\overline{p})(1-z)] \times \left[\overline{\theta}_1 S(\gamma(z)) + z\theta_1 R_1(\gamma(z))\right] - z\gamma(z)\Big)^{-1}. \tag{57}$$

(iii) When $\theta_1 = \theta_2 = 0$, our model becomes a discrete-time *Geo*/*G*/1 queue with J vacations. In this case, $\Phi(z)$ reduces to

$$\Phi(z) = \frac{z\eta_2(1 - V(\gamma(z))) + \overline{p}A(\overline{p})(1-z)\Gamma(z)}{[z + \overline{p}A(\overline{p})(1-z)] S(\gamma(z)) - z\gamma(z)} S[\gamma(z)]\pi_{0,0}, \tag{58}$$

where

$$\pi_{0,0} = \frac{\overline{p}\eta_3[p + \overline{p}A(\overline{p}) - pS_1]}{pV_1\eta_2 + \overline{p}A(\overline{p})\eta_3}. \tag{59}$$

4. Stochastic Decomposition

The property of stochastic decomposition was proposed firstly in queueing system with vacations. This property has also been studied for retrial queues with vacations by Artalejo [17]. The property of stochastic decomposition shows that the steady-state system size at an arbitrary point can be represented as the sum of two independent random variable, one of which is the system size of the corresponding queueing system without server vacations and the other is the orbit size given that the server is on vacations. In this section, we present two different stochastic decompositions of the system size distribution in our model.

Theorem 7. *The total number of customers* (L) *in the system can be represented as the sum of two independent random variables* $L = L_1 + M_1$. L_1 *is the number of customers in the Geo/G/1/ queue with server breakdowns and* M_1 *is the number of repeated customers given that the server is idle, under the first type of repair or on vacation.*

Proof. After some algebra operation, $\Phi(z)$ can be expressed by

$$\Phi(z) = \Phi_1(z)\Phi_2(z), \tag{60}$$

where

$$\Phi_1(z) = \frac{(1-\rho_1) S(\tau(\gamma(z)))(1-z)}{S(\tau(\gamma(z))) - z} \tag{61}$$

is the generating function of the *Geo*/*G*/1 queue server with breakdowns, and

$$\begin{aligned}\Phi_2(z) &= \frac{S(\tau(\gamma(z))) - z}{1-z} \times \frac{z\eta_2[1 - V(\gamma(z))] + \overline{p}A(\overline{p})(1-z)\Gamma(z)}{(1-\rho_1)\Lambda(z)} \times \frac{p + \overline{p}A(\overline{p}) - \overline{\theta}_1\rho_1 - \theta_1 - \theta_1\rho_2}{pV_1\eta_2 + \overline{p}A(\overline{p})\eta_3} \\ &= \frac{\pi_{0,0} + \varphi_0(1,z) + \varphi_2(1,z) + \varphi_4(1,z)}{\pi_{0,0} + \varphi_0(1,1) + \varphi_2(1,1) + \varphi_4(1,1)}\end{aligned} \tag{62}$$

is the generating function given that the server is idle, under first type of repair or on vacations. This completes the proof. □

Theorem 8. *The total number of customers* (L) *in the system can be expressed as the sum of two independent random variables* $L = L_2 + M_2$. L_2 *is the number of customers in the Geo/G/1/ queue with two types of breakdowns and* M_2 *is the number of repeated customers given that the server is idle or on vacation.*

Proof. After some algebra operation, $\Phi(z)$ can be expressed by

$$\Phi(z) = \Phi_1(z)\Phi_2(z), \tag{63}$$

where

$$\Phi_1(z) = \frac{\left(1 - \overline{\theta}_1\rho_1 - \theta_1 - \theta_1\rho_2\right) S(\tau(\gamma(z)))(1-z)}{\overline{\theta}_1 S(\tau(\gamma(z))) + z\theta_1 R_1(\gamma(z)) - z} \tag{64}$$

is the generating function of the unreliable *Geo*/*G*/1 queue, and

$$\begin{aligned}\Phi_2(z) &= \frac{\overline{\theta}_1 S(\tau(\gamma(z))) + z\theta_1 R_1(\gamma(z)) - z}{1-z} \times \frac{z\eta_2[1 - V(\gamma(z))] + \overline{p}A(\overline{p})(1-z)\Gamma(z)}{\left(1 - \overline{\theta}_1\rho_1 - \theta_1 - \theta_1\rho_2\right)\Lambda(z)} \times \frac{p + \overline{p}A(\overline{p}) - \overline{\theta}_1\rho_1 - \theta_1 - \theta_1\rho_2}{pV_1\eta_2 + \overline{p}A(\overline{p})\eta_3} \\ &= \frac{\pi_{0,0} + \varphi_0(1,z) + \varphi_4(1,z)}{\pi_{0,0} + \varphi_0(1,1) + \varphi_4(1,1)}\end{aligned} \tag{65}$$

is the generating function given that the server is idle or on vacations. This completes the proof. □

5. Relationship to the Corresponding Continuous-Time Model

In this section, we investigate the relationship between our discrete-time model and the corresponding continuous-time counterpart. We consider a continuous-time $M/G/1$ general retrial queue with J vacations and two types of breakdowns. The customers arrive according to a Poisson process with rate λ. If an arriving customer finds that the server is busy, under repair or on vacations, the customer joins the orbit and only the first customer in the orbit is permitted to retry for service. The server is subject to two types of different breakdowns. As soon as the server fails, it is repaired immediately. The first type of breakdown is starting failure, that is, an arriving customer must start the server, which takes negligible time and the server is started successfully with a probability $\overline{\theta}_1$. In addition to starting failure, the server may break down during serving an customer and the lifetime of the server follows exponential distribution with rate θ_2.

Once the system becomes empty, the server takes vacations immediately according to J vacation policy. The retail times, the service times, the repair times for the two types of breakdowns and the vacation times are all assumed to follow general continuous-time distributions denoted by $A(x), S(x)$, $R_1(x)$, $R_2(x)$, and $V(x)$, respectively, and their Laplace-Stieltjes transforms denoted by $\widetilde{A}(s)$, $\widetilde{S}(s)$, $\widetilde{R_1}(s)$, $\widetilde{R_2}(s)$, and $\widetilde{V}(s)$, respectively, and their finite means are denoted by A_1, S_1 and $R_{1,1}$, $R_{2,1}$, and V_1 respectively.

We suppose that the time is divided into sufficiently small intervals of equal length Δ. Then, the continuous-time $M/G/1$ general retrial queue with J vacations and two types of breakdowns can be approximated by the following:

$$p = \lambda\Delta, \qquad a_i = \int_{i\Delta}^{(i+1)\Delta} dA(x), \qquad s_i = \int_{(i-1)\Delta}^{i\Delta} dS(x),$$
$$R_{l,i} = \int_{(i-1)\Delta}^{i\Delta} dR_l(x), \quad l = 1, 2, \qquad v_i = \int_{(i-1)\Delta}^{i\Delta} dV(x). \tag{66}$$

By using the definition of Lebesgue integration, we can get the following equalities:

$$\lim_{\Delta\to 0}\rho_2 = \lambda R_{2,1} = \widetilde{\rho}_2, \qquad \lim_{\Delta\to 0} pV_i = \lambda V_1,$$
$$\lim_{\Delta\to 0}\rho_1 = \lambda S_1\left(1 + \frac{\theta_2}{\overline{\theta}_2}R_{2,1}\right) = \widetilde{\rho}_1$$
$$\lim_{\Delta\to 0} S(\tau(\gamma(z))) = \widetilde{S}\left[\lambda(1-z) + \theta_2\widetilde{\overline{R}}_2(\lambda(1-z))\right]$$
$$= \widetilde{S}(\tau(\gamma(z))),$$
$$\lim_{\Delta\to 0} V_2(\overline{p}) = \widetilde{V_2}(\lambda), \qquad \lim_{\Delta\to 0} R_1(\gamma(z)) = \widetilde{R}_1[\lambda(1-z)],$$
$$\lim_{\Delta\to 0} A(\overline{p}) = A(\lambda), \qquad \lim_{\Delta\to 0} V(\gamma(z)) = \widetilde{V}[\lambda(1-z)],$$
$$\lim_{\Delta\to 0}\eta_i = \widetilde{\eta}_i, \quad i = 1, 2, 3. \tag{67}$$

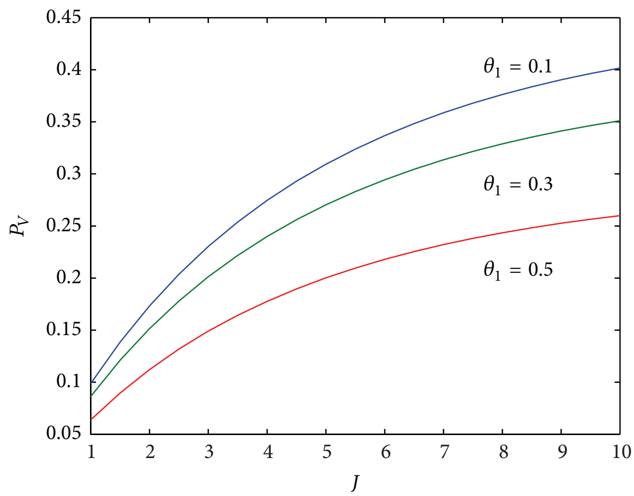

Figure 1: The curve of P_V with J.

From the above relations, we obtain the generating function of the system size for the corresponding continuous-time model by

$$\begin{aligned}\lim_{\Delta\to 0}\Phi(z) = {} & \Big(z\widetilde{\eta}_2\left(1 - \widetilde{V}[\lambda(1-z)]\right) \\ & + \widetilde{A}(\lambda)(1-z)\left(\widetilde{\eta}_1 - \widetilde{\eta}_2\widetilde{V}[\lambda(1-z)]\right)\Big) \\ & \times\left(\left[\overline{\theta}_1\widetilde{S}(\tau(\gamma(z))) + z\theta_1\widetilde{R_1}[\lambda(1-z)] - z\right]\right. \\ & \left.\times\left[z + \widetilde{A}(\lambda)(1-z)\right] - z\right)^{-1} \\ & \times\frac{\widetilde{A}(\lambda) - \overline{\theta}_1\widetilde{\rho}_1 - \theta_1 - \theta_1\widetilde{\rho}_2}{\lambda V_1\widetilde{\eta}_2 + \widetilde{A}(\lambda)\widetilde{\eta}_3}\widetilde{S}(\tau(\gamma(z))).\end{aligned} \tag{68}$$

6. Numerical Examples

In this section, we present some numerical examples to illustrate the effect of the parameters on the main performance measures of the system. We consider two important performance measures: the probability that server is on vacation P_V and the mean orbit size $E(N)$. It is assumed that the retrial time is geometric distribution with generating function $A(x) = (1-r)/(1-rx)$. The service time, the repair times of the server for the two types of breakdowns, and the vacation time are also assumed to follow geometric distribution with generating functions $S(x) = (1-\mu)x/(1-\mu x)$, $R_l(x) = (1-r_l)x/(1-r_l x)$, $l = 1, 2$, and $V(x) = (1-v)x/(1-vx)$, respectively. In general, we have chosen the arrival rate $p = 0.1$, the retrial rate $r = 0.25$, the service rate $\mu = 0.5$, and the vacation rate $v = 0.5$, and we have presented three curves which correspond to $\theta_1 = 0.1, 0.3$ and 0.5, respectively, in Figures 1–4, where θ_1 is the probability that the server experience the first type of breakdown.

Figure 1 describes P_V with varying values of the maximum number of the server's vacations J. As illustrated in

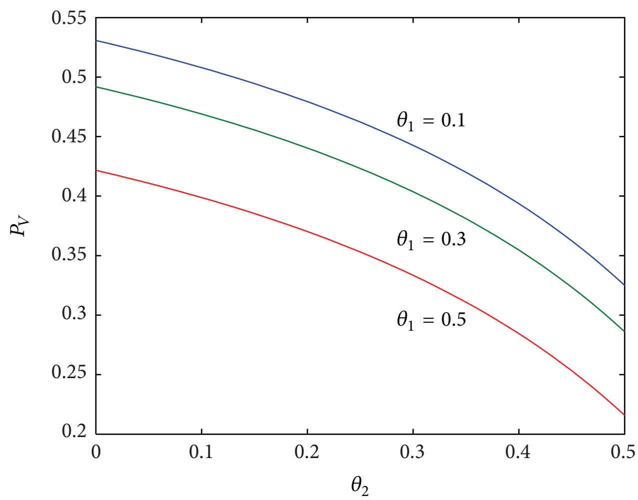

FIGURE 2: The curve of P_V with θ_2.

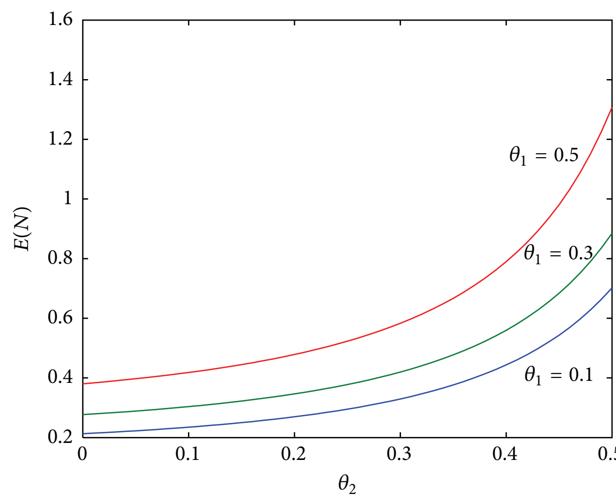

FIGURE 4: The curve of $E(N)$ with θ_2.

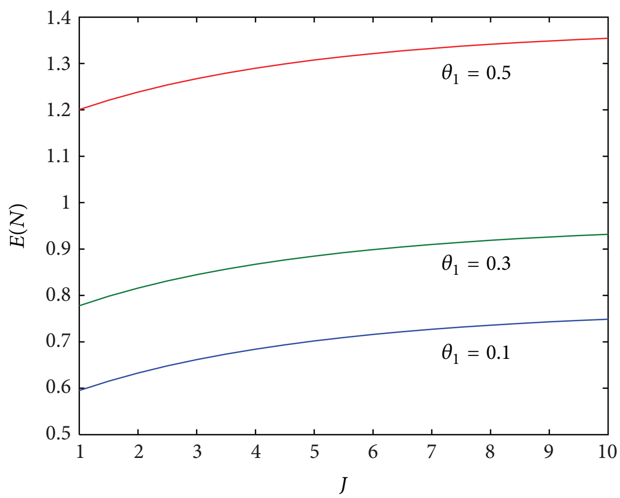

FIGURE 3: The curve of $E(N)$ with J.

Figure 1, P_V is increasing as a function of J. For different values of θ_1, we observe that P_V decreases with increasing values of θ_1. That is because the mean service time of a customer is getting longer with increasing values of θ_1 and P_V decreases accordingly. Figure 2 describes the effect of θ_2 on P_V. The curve in Figure 2 shows that P_V is decreasing as a function of θ_2 which agrees with intuitive expectations.

Figures 3 and 4 describe the influence of the parameter J and θ_2 on the mean orbit size $E(N)$, respectively. As to be expected, $E(N)$ increases with increasing value of J and θ_2, that is, there are more customers in the orbit as J and θ_2 increase.

As we can observe in Figures 3 and 4, the mean orbit size $E(N)$ increases with increasing values of θ_1. As the the value of θ_1 increases, the mean sojourn time time for an customer increases, and therefore the number of customers in the system increases. Moreover, the parameter θ_1 affects $E(N)$ more apparently, when the value of the parameter θ_1 is getting greater.

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