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Research Article

The Explicit Expression of the Drazin Inverse and Its Application

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We give explicit expressions of $(P \pm Q)_d$ of two matrices P and Q, in terms of P, Q, P_d, and Q_d, $(P \pm I)_d$, under the condition that $PQ = P$, and apply the result to finding an explicit representation for the Drazin inverse of some 2×2 block matrix.

1. Introduction

In recent years, the representations and perturbation analysis of the Drazin inverse for matrices or operators have been investigated (see [1–6]). In [7], the author presented the presentations of the Drazin inverse of sum and product of two operators over Banach spaces on the condition of the commutativity up to a factor. And, in [8], the same author discussed explicit representations of Drazin inverses of sums and differences of two idempotents over Hilbert spaces.

These investigations motivate us to deal with an explicit expression of the Drazin inverse of differences and sums of two matrices. The paper is organized as follows. In this section, we will introduce some notions and lemmas. In Section 2, we will present these explicit expressions of differences and sums of two matrices P and Q under the condition $PQ = P$. In Section 3, we will deduce an explicit representation for the Drazin inverse of the 2×2 block matrix $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ with $A = BC$ and $B = BD$ in terms of its subblocks and their Drazin inverses and $(A + I)_d$. In Section 4, we will present a numerical example to demonstrate the main result in Section 2.

Throughout this paper the symbol $\mathbb{C}^{m\times n}$ stands for the set of $m \times n$ complex matrices, and $I \in \mathbb{C}^{n\times n}$ stands for the unit matrix. Let $A \in \mathbb{C}^{n\times n}$; the Drazin inverse, denoted by A_d, of matrix A is defined as the unique matrix satisfying

$$A^{k+1}A_d = A^k, \qquad A_dAA_d = A_d, \qquad AA_d = A_dA, \quad (1)$$

where $k = \mathrm{Ind}(A)$ is the index of A. In particular, if $\mathrm{Ind}(A) = 1$, then A_d is called the group inverse, denoted by A_g, of A. Apparently, if A is nonsingular, then $\mathrm{Ind}(A) = 0$; otherwise, $\mathrm{Ind}(A) \geq 1$, especially $\mathrm{Ind}(0) = 1$. If A is nilpotent, then $A_d = 0$. If X is nonsingular and $B = XAX^{-1}$, then $B_d = XA_dX^{-1}$ (see [6, 9, 10]). For convenience, we write $A^\pi = I - AA_d$ and use the convention $\sum_{i=n}^{m} = 0$, if $m < n$.

Before we start the discussion, we need some preparations.

Lemma 1 (see [11, Theorem 3.2]). *Let* $M = \left[\begin{smallmatrix} A & B \\ 0 & D \end{smallmatrix}\right]$ *with* $t = \mathrm{Ind}(A)$ *and* $l = \mathrm{Ind}(D)$. *Then,*

$$M_d = \begin{bmatrix} A_d & S \\ 0 & D_d \end{bmatrix}, \quad (2)$$

where

$$S = A^\pi \sum_{n=0}^{t-1} A^n B D_d^{n+2} + \sum_{n=0}^{l-1} A_d^{n+2} B D^n D^\pi - A_d B D_d. \quad (3)$$

Lemma 2. *Let* $P, Q \in \mathbb{C}^{m\times m}$ *with* $\mathrm{Ind}(Q) = s$. *If* $PQ = P$, *then*

$$PQ_d = P,$$
$$(Q_dQP)^k = Q_dQP^k, \qquad (QP^\pi)^k = Q^kP^\pi, \quad (4)$$
$$(Q_dP^\pi)^k = Q_d^kP^\pi, \quad k \geq 1.$$

Proof. Since $PQ = P$ and $\mathrm{Ind}(Q) = s$,

$$P = PQ^s = PQ^{s+1}Q_d = PQ_d, \tag{5}$$

and then $PQQ_d = P$. So from this, by induction, it follows that $(Q_dQP)^k = Q_dQP^k$ for $k \geq 1$.

Now, we will show inductively that $(QP^\pi)^k = Q^kP^\pi$. It holds for $k = 1$. Assume it holds for $k = n$; that is, $(QP^\pi)^n = Q^nP^\pi$. Then,

$$\left(QP^\pi\right)^{n+1} = \left(QP^\pi\right)\left(QP^\pi\right)^n = Q\left(I - P_dP\right)Q^nP^\pi = Q^{n+1}P^\pi. \tag{6}$$

So it holds for any $k \geq 1$.

From $PQ_d = P$, we can similarly show that $(Q_dP^\pi)^k = Q_d^kP^\pi$. □

Lemma 3. *Let P be nilpotent of index $t > 1$ and $S = \sum_{i=0}^{t-1} a_i^{[1]}P^i$. If $a_i^{[1]} = 1$, then*

$$S^n = \sum_{i=0}^{t-1} a_i^{[n]}P^i, \quad n \geq 2, \tag{7}$$

where $a_i^{[n]} = \sum_{u=0}^{i} a_u^{[n-1]}$, $i = 0, \ldots, t-1$.

Proof. Since $P^t = 0$, we can easily write S^n as

$$S^n = \sum_{i=0}^{t-1} a_i^{[n]}P^i, \quad n \geq 2. \tag{8}$$

We will prove the relationship

$$a_i^{[n]} = \sum_{h=0}^{i} a_h^{[n-1]}, \quad i = 0, \ldots, t-1 \tag{9}$$

by induction on n. Obviously, (9) holds for $n = 2$. Assume inductively that (9) holds for $n = k$.

Since $a_i^{[1]} = 1$ and $P^t = 0$,

$$\begin{aligned} S^{k+1} = S^kS &= \sum_{i=0}^{t-1} a_i^{[k]}P^i \sum_{j=0}^{t-1} P^j \\ &= \sum_{h=0}^{t-1} \sum_{i+j=h} a_i^{[k]}P^{i+j} = \sum_{h=0}^{t-1} \sum_{i=0}^{h} a_i^{[k]}P^h. \end{aligned} \tag{10}$$

On the other hand,

$$S^{k+1} = \sum_{i=0}^{t-1} a_i^{[k+1]}P^i. \tag{11}$$

So, (9) holds for $n = k + 1$. Hence, (9) holds for $n \geq 2$. □

Lemma 4 (see [9, Lemma 7.7.2]). *Let*

$$N = \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} \quad \text{or } N = \begin{bmatrix} 0 & 0 \\ B & A \end{bmatrix}. \tag{12}$$

Then, $\mathrm{Ind}(A) \leq \mathrm{Ind}(N) \leq \mathrm{Ind}(A) + 1$.

2. The Drazin Inverse of Differences and Sums of Two Matrices

In this section, we will investigate how to express $P \pm Q$ as a function of P, Q, P_d, and Q_d, $(P \pm I)_d$, under the condition $PQ = P$. We begin with the following theorem, in which P is assumed to be nilpotent.

Theorem 5. *Let $P, Q \in \mathbb{C}^{n\times n}$ with $\mathrm{Ind}(Q) = s$. If $PQ = P$ and P is nilpotent of index t, then*

$$\begin{aligned} (P - Q)_d = &\sum_{n=0}^{s-1}\sum_{i=0}^{t-2}\left(a_i^{[n+2]}I - a_i^{[n+1]}Q_d\right)Q^nP^{i+1} \\ &- Q_dQ^s\sum_{i=0}^{t-2} a_i^{[s+1]}P^{i+1} - Q_d, \end{aligned} \tag{13}$$

where $a_i^{[n]} = \sum_{k=0}^{i} a_k^{[n-1]}$ and $a_i^{[1]} = 1$, $i = 0, \ldots, t-1$.

Proof. If $t = 1$, then, from the convention in Section 1, (13) clearly holds. Now assume $t > 1$. From $s = \mathrm{Ind}(Q)$, there exists a nonsingular matrix W_1 such that

$$\begin{aligned} Q &= W_1\begin{bmatrix} Q_1 & 0 \\ 0 & Q_2 \end{bmatrix}W_1^{-1}, \\ Q_d &= W_1\begin{bmatrix} Q_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}W_1^{-1}, \end{aligned} \tag{14}$$

where Q_1 is nonsingular and Q_2 is nilpotent of index s. Partitioning $W_1^{-1}PW_1$ conformably with $W_1^{-1}QW_1$, we have

$$P = W_1\begin{bmatrix} P_1 & P_4 \\ P_3 & P_2 \end{bmatrix}W_1^{-1}. \tag{15}$$

Since $PQ = P$, $PQ_d = P$ by Lemma 2, and then $P_2 = 0$, $P_4 = 0$, and $P_iQ_1 = P_i$, $i = 1, 3$. Let t be the index of nilpotent matrix P. Then,

$$P^t = W_1\begin{bmatrix} P_1^t & 0 \\ P_3P_1^{t-1} & 0 \end{bmatrix}W_1^{-1} = 0, \tag{16}$$

and therefore P_1 is also nilpotent, and $(P_1 - I)^{-1} = -\sum_{i=0}^{t-1} P_1^i$. Thus,

$$\begin{aligned} P - Q &= W_1\begin{bmatrix} P_1 - Q_1 & 0 \\ P_3 & -Q_2 \end{bmatrix}W_1^{-1} \\ &= W_1\begin{bmatrix} \left(P_1 - I\right)Q_1 & 0 \\ P_3 & -Q_2 \end{bmatrix}W_1^{-1}. \end{aligned} \tag{17}$$

By Lemma 1, we have

$$\begin{aligned} (P - Q)_d &= W_1\begin{bmatrix} Q_1^{-1}\left(P_1 - I\right)^{-1} & 0 \\ X & 0 \end{bmatrix}W_1^{-1} \\ &= W_1\begin{bmatrix} -Q_1^{-1}\sum_{i=0}^{t-1} P_1^i & 0 \\ X & 0 \end{bmatrix}W_1^{-1}, \end{aligned} \tag{18}$$

where

$$X = \sum_{n=0}^{s-1}(-1)^n Q_2^n P_3 \left[\left(P_1 - I\right) Q_1\right]^{-(n+2)}. \tag{19}$$

By Lemma 3,

$$\begin{aligned}(-1)^n P_3\left[\left(P_1 - I\right) Q_1\right]^{-(n+2)} &= P_3\left(Q_1^{-1}\sum_{i=0}^{t-1}P_1^i\right)^{n+2} \\ &= P_3\left(\sum_{i=0}^{t-1}P_1^i\right)^{n+2} \\ &= P_3\sum_{i=0}^{t-1}a_i^{[n+2]}P_1^i \\ &= \sum_{i=0}^{t-2}a_i^{[n+2]}P_3P_1^i.\end{aligned} \tag{20}$$

Thus,

$$\begin{aligned}W_1\begin{bmatrix}0 & 0\\ X & 0\end{bmatrix}W_1^{-1} &= \sum_{n=0}^{s-1}\sum_{i=0}^{t-2}W_1\begin{bmatrix}0 & 0\\ a_i^{[n+2]}Q_2^nP_3P_1^i & 0\end{bmatrix}W_1^{-1} \\ &= \sum_{n=0}^{s-1}\sum_{i=0}^{t-2}a_i^{[n+2]}Q^\pi Q^n P^{i+1}.\end{aligned} \tag{21}$$

So, by (18),

$$\begin{aligned}(P-Q)_d &= -\sum_{i=0}^{t-1}Q_dP^i + Q^\pi\sum_{n=0}^{s-1}\sum_{i=0}^{t-2}a_i^{[n+2]}Q^nP^{i+1} \\ &= -\sum_{i=0}^{t-2}Q_dP^{i+1} - Q_d \\ &\quad + \sum_{n=0}^{s-1}\sum_{i=0}^{t-2}a_i^{[n+2]}Q^nP^{i+1} \\ &\quad - Q_d\sum_{n=1}^{s}\sum_{i=0}^{t-2}a_i^{[n+1]}Q^nP^{i+1} \\ &= -Q_d + \sum_{n=0}^{s-1}\sum_{i=0}^{t-2}a_i^{[n+2]}Q^nP^{i+1} \\ &\quad - Q_d\sum_{n=0}^{s}\sum_{i=0}^{t-2}a_i^{[n+1]}Q^nP^{i+1} \\ &= \sum_{n=0}^{s-1}\sum_{i=0}^{t-2}\left(a_i^{[n+2]}I - a_i^{[n+1]}Q_d\right)Q^nP^{i+1} \\ &\quad - Q_dQ^s\sum_{i=0}^{t-2}a_i^{[s+1]}P^{i+1} - Q_d.\end{aligned} \tag{22}$$

□

If $PQ = P$, then $(-P)Q = -P$. So, it immediately follows from the above theorem.

Corollary 6. *Let $P, Q \in \mathbb{C}^{n\times n}$ with $\mathrm{Ind}(Q) = s$. If $PQ = P$ and P is nilpotent of index k, then*

$$\begin{aligned}(P+Q)_d &= \sum_{n=0}^{s-1}\sum_{i=0}^{t-2}(-1)^i\left(a_i^{[n+2]}I - a_i^{[n+1]}Q_d\right)Q^nP^{i+1} \\ &\quad - Q_dQ^s\sum_{i=0}^{t-2}(-1)^i a_i^{[s+1]}P^{i+1} + Q_d,\end{aligned} \tag{23}$$

where $a_i^{[n]} = \sum_{k=0}^{i}a_k^{[n-1]}$, $a_i^{[1]} = 1$, $i = 0, \ldots, t-1$.

If the nilpotency of P is taken out in Theorem 5, then we can obtain our main result, a more general result.

Theorem 7. *Let $P, Q \in \mathbb{C}^{n\times n}$ with $\mathrm{Ind}(QP^\pi) = s$, $\mathrm{Ind}(P) = t$, $\mathrm{Ind}[(P-I)PP_d] = l$, and $\mathrm{Ind}[(P-Q)P^\pi] = k$. If $PQ = P$, then*

$$\begin{aligned}(P-Q)_d &= \sum_{n=0}^{s-1}\sum_{i=0}^{t-2}\left(a_i^{[n+2]}I - a_i^{[n+1]}Q_d\right)Q^nP^{i+1}P^\pi - Q_dP^\pi \\ &\quad - \sum_{n=0}^{k-1}(-1)^nQ^nQ^\pi\left(Q-I\right)PP_d(P-I)_d^{n+2} \\ &\quad + \left(Q^\pi + Q_d\right)PP_d(P-I)_d \\ &\quad + \sum_{n=0}^{l-1}(-1)^{n+1}Q_d^n\left(Q_d - Q_d^2\right)PP_d(P-I)^n(P-I)^\pi \\ &\quad - Q_dQ^s\sum_{i=0}^{t-2}a_i^{[s+1]}P^\pi P^{i+1},\end{aligned} \tag{24}$$

where $a_i^{[n]} = \sum_{k=0}^{i}a_k^{[n-1]}$, $a_i^{[1]} = 1$, $i = 0, \ldots, t-1$.

Proof. There exists a nonsingular matrix W_1 such that

$$P = W_1\begin{bmatrix}P_1 & 0\\ 0 & P_2\end{bmatrix}W_1^{-1}, \qquad Q = W_1\begin{bmatrix}Q_1 & Q_3\\ Q_4 & Q_2\end{bmatrix}W_1^{-1}, \tag{25}$$

where P_1 is nonsingular and P_2 is nilpotent of index t. From $PQ = P$, we get

$$Q = W_1\begin{bmatrix}I & 0\\ Q_4 & Q_2\end{bmatrix}W_1^{-1}, \tag{26}$$

where $P_2Q_4 = 0$ and $P_2Q_2 = P_2$. So

$$P - Q = W_1\begin{bmatrix}P_1 - I & 0\\ -Q_4 & P_2 - Q_2\end{bmatrix}W_1^{-1}. \tag{27}$$

By Lemma 1, we have

$$(P-Q)_d = W_1\begin{bmatrix}\left(P_1 - I\right)_d & 0\\ X & \left(P_2 - Q_2\right)_d\end{bmatrix}W_1^{-1}, \tag{28}$$

where

$$\begin{aligned} X = &\sum_{n=0}^{l-1}(P_2-Q_2)_d^{n+2}(-Q_4)(P_1-I)^n(P_1-I)^\pi \\ &+(P_2-Q_2)^\pi\sum_{n=0}^{k-1}(P_2-Q_2)^n(-Q_4)(P_1-I)_d^{n+2} \\ &-(P_2-Q_2)_d(-Q_4)(P_1-I)_d, \end{aligned} \tag{29}$$

and $k=\operatorname{Ind}(P_2-Q_2)=\operatorname{Ind}[(P-Q)P^\pi]$ and $l=\operatorname{Ind}(P_1-I)=\operatorname{Ind}[(P-I)PP_d]$.

Since $P_2Q_4=0$ and $P_2Q_2=P_2$, by Theorem 5 and Lemma 2,

$$\begin{aligned} (P_2-Q_2)_d(Q_2)_d^jQ_4 &= -\sum_{n=0}^{t-1}(Q_2)_dP_2^n(Q_2)_d^jQ_4 \\ &\quad+\sum_{n=0}^{s-1}Q_2^\pi Q_2^nP_2\left(\sum_{i=0}^{t-2}P_2^i\right)^{n+2}(Q_2)_d^jQ_4 \\ &= -(Q_2)_d^{j+1}Q_4, \quad j\ge 0, \\ (P_2-Q_2)^\pi Q_4 &= Q_4+(P_2-Q_2)(Q_2)_dQ_4 \\ &= Q_4-Q_2(Q_2)_dQ_4 \\ &= Q_2^\pi Q_4, \\ P_2Q_2^\pi &= 0, \end{aligned} \tag{30}$$

and then

$$\begin{aligned} &(P_2-Q_2)^\pi\sum_{n=0}^{k-1}(P_2-Q_2)^n(-Q_4) \\ &\quad= -\sum_{n=0}^{k-1}(P_2-Q_2)^nQ_2^\pi Q_4 \\ &\quad= -\sum_{n=0}^{k-1}(-Q_2)^nQ_2^\pi Q_4. \end{aligned} \tag{31}$$

Thus,

$$\begin{aligned} X = &\sum_{n=0}^{l-1}(-1)^{n+1}(Q_2)_d^{n+2}Q_4(P_1-I)^n(P_1-I)^\pi \\ &-\sum_{n=0}^{k-1}(-1)^nQ_2^nQ_2^\pi Q_4(P_1-I)_d^{n+2} \\ &-(Q_2)_dQ_4(P_1-I)_d. \end{aligned} \tag{32}$$

Since

$$P-I=W_1\begin{bmatrix}P_1-I & 0\\ 0 & P_2-I\end{bmatrix}W_1^{-1}, \tag{33}$$

we have that, for $n\ge 0$,

$$\begin{aligned} PP_d(P-I)_d^n &= W_1\begin{bmatrix}(P_1-I)_d^n & 0\\ 0 & 0\end{bmatrix}W_1^{-1}, \\ (P-I)^nPP_d(P-I)^\pi &= W_1\begin{bmatrix}(P_1-I)^n(P_1-I)^\pi & 0\\ 0 & 0\end{bmatrix}W_1^{-1}. \end{aligned} \tag{34}$$

Obviously,

$$\begin{aligned} (QP^\pi)_d &= W_1\begin{bmatrix}0 & 0\\ 0 & (Q_2)_d\end{bmatrix}W_1^{-1}=Q_dP^\pi, \\ (Q-I)PP_d &= W_1\begin{bmatrix}0 & 0\\ Q_4 & 0\end{bmatrix}W_1^{-1}, \\ Y^nP^\pi &= W_1\begin{bmatrix}0 & 0\\ 0 & Y_2^n\end{bmatrix}W_1^{-1}, \quad n\ge 0, \end{aligned} \tag{35}$$

where the symbol Y denotes Q or P. Also,

$$P^\pi(QP^\pi)^\pi = P^\pi-P^\pi QP^\pi Q_dP^\pi = P^\pi-QQ_dP^\pi = Q^\pi P^\pi. \tag{36}$$

Note that $P^\pi(Q-I)=(Q-I)$. Then, by (32),

$$\begin{aligned} &W_1\begin{bmatrix}0 & 0\\ X & 0\end{bmatrix}W_1^{-1} \\ &\quad=\sum_{n=0}^{l-1}(-1)^{n+1}Q_d^{n+2}P^\pi(Q-I)PP_d(P-I)^nPP_d(P-I)^\pi \\ &\qquad-\sum_{n=0}^{k-1}(-1)^nQ^nP^\pi(QP^\pi)^\pi(Q-I)PP_d(P-I)_d^{n+2} \\ &\qquad-Q_dP^\pi(Q-I)PP_d(P-I)_d \\ &\quad=\sum_{n=0}^{l-1}(-1)^{n+1}Q_d^{n+2}(Q-I)PP_d(P-I)^n(P-I)^\pi \\ &\qquad-\sum_{n=0}^{k-1}(-1)^nQ^nQ^\pi(Q-I)PP_d(P-I)_d^{n+2} \\ &\qquad-Q_d(Q-I)PP_d(P-I)_d. \end{aligned} \tag{37}$$

Note that $P^\pi Q^nP^\pi=Q^nP^\pi$. So, by Theorem 5,

$$\begin{aligned} &W_1\begin{bmatrix}0 & 0\\ 0 & (P_2-Q_2)_d\end{bmatrix}W_1^{-1} \\ &\quad=\sum_{n=0}^{s-1}\sum_{i=0}^{t-2}\left(a_i^{[n+2]}I-a_i^{[n+1]}Q_dP^\pi\right)Q^nP^\pi P^{i+1}P^\pi \\ &\qquad-Q_dP^\pi-Q_dP^\pi Q^sP^\pi\sum_{i=0}^{t-2}a_i^{[s+1]}P^{i+1}P^\pi \\ &\quad=\sum_{n=0}^{s-1}\sum_{i=0}^{t-2}\left(a_i^{[n+2]}I-a_i^{[n+1]}Q_d\right)Q^nP^{i+1}P^\pi \\ &\qquad-Q_dP^\pi-Q_dQ^s\sum_{i=0}^{t-2}a_i^{[s+1]}P^\pi P^{i+1}. \end{aligned} \tag{38}$$

Hence, putting (34), (37), and (38) into (28) yields (24). □

Corollary 8. *Let $P, Q \in \mathbb{C}^{n\times n}$ with $\operatorname{Ind}(QP^{\pi}) = s$, $\operatorname{Ind}(P) = t$, $\operatorname{Ind}[(P+I)PP_d] = l$, and $\operatorname{Ind}[(P+Q)P^{\pi}] = k$. If $PQ = P$, then*

$$\begin{aligned}
&(P+Q)_d \\
&= \sum_{n=0}^{s-1}\sum_{i=0}^{t-2}(-1)^i\left(a_i^{[n+2]}I - a_i^{[n+1]}Q_d\right)Q^nP^{i+1}P^{\pi} + Q_dP^{\pi} \\
&\quad + \sum_{n=0}^{k-1}Q^nQ^{\pi}(Q-I)PP_d(P+I)_d^{n+2} \\
&\quad + \left(Q^{\pi}+Q_d\right)PP_d(P+I)_d \\
&\quad + \sum_{n=0}^{l-1}Q_d^n\left(Q_d - Q_d^2\right)PP_d(P+I)^n(P+I)^{\pi} \\
&\quad - Q_dQ^s\sum_{i=0}^{t-2}(-1)^i a_i^{[s+1]}P^{\pi}P^{i+1},
\end{aligned} \tag{39}$$

where $a_i^{[n]} = \sum_{k=0}^{i} a_k^{[n-1]}$, $a_i^{[1]} = 1$, $i = 0, \ldots, t-1$.

3. The Drazin Inverse of Some 2×2 Block Matrix

In this section, we will apply the results in Section 2 to studying the representation for the Drazin inverse of a 2×2 block matrix, in terms of its subblocks.

Theorem 9. *Let $M = \left[\begin{smallmatrix} A & B \\ C & D\end{smallmatrix}\right] \in \mathbb{C}^{n\times n}$ with $\operatorname{Ind}[(CB)^{\pi}D] = s-1$, $\operatorname{Ind}(A) = t-1$, $\operatorname{Ind}[(A+I)AA_d] = l-1$, and $\operatorname{Ind}\left[\left[\begin{smallmatrix} A^{\pi} & 0 \\ 0 & (CB)^{\pi}\end{smallmatrix}\right]M\right] = k$. If $A = BC$ and $B = BD$, then*

$$\begin{aligned}
&M_d \\
&= \begin{bmatrix} (I-(A+I)_d)(A+I)_dA_d & 0 \\ (D_d+D^{\pi})C(A+I)_d^2A_d + \left(D_d^2 - D_d\right)C(A+I)_dA_d & D_d^2(CB)^{\pi}\end{bmatrix}M \\
&\quad + \sum_{n=1}^{s-1}\sum_{i=0}^{t-2}(-1)^i\begin{bmatrix} 0 & 0 \\ \left(a_i^{[n+2]}I - a_i^{[n+1]}D_d\right)D^{n-1}CA^iA^{\pi} & 0\end{bmatrix}M \\
&\quad + \sum_{n=1}^{k-1}\begin{bmatrix} 0 & 0 \\ D^{\pi}\left(D^n - D^{n-1}\right)C(A+I)_d^{n+2}A_d & 0\end{bmatrix}M \\
&\quad + \sum_{n=0}^{l-1}\begin{bmatrix} 0 & 0 \\ \left(D_d^{n+2} - D_d^{n+3}\right)C(A+I)^n(A+I)^{\pi}A_d & 0\end{bmatrix}M \\
&\quad + \sum_{i=0}^{t-2}(-1)^i\begin{bmatrix} a_i^{[2]}A^iA^{\pi} & 0 \\ -\left(a_i^{[1]}D_d^2 + a_i^{[s+1]}D_dD^{s-1}\right)CA^iA^{\pi} & 0\end{bmatrix}M,
\end{aligned} \tag{40}$$

where $a_i^{[n]} = \sum_{k=0}^{i} a_k^{[n-1]}$, $a_i^{[1]} = 1$, $i = 0, \ldots, t-1$.

Proof. Let

$$P = \begin{bmatrix} A & B \\ 0 & 0\end{bmatrix}, \qquad Q = \begin{bmatrix} 0 & 0 \\ C & D\end{bmatrix}. \tag{41}$$

Then, $M = P + Q$, and, for $n \geq 1$,

$$\begin{gathered}
Q^n = \begin{bmatrix} 0 & 0 \\ D^{n-1}C & D^n\end{bmatrix}, \qquad Q_d^n = \begin{bmatrix} 0 & 0 \\ D_d^{n+1}C & D_d^n\end{bmatrix}, \\
Q^{\pi} = \begin{bmatrix} I & 0 \\ -D_dC & D^{\pi}\end{bmatrix}, \qquad P^n = \begin{bmatrix} A^n & A^{n-1}B \\ 0 & 0\end{bmatrix}, \\
P_d = \begin{bmatrix} A_d & A_d^2B \\ 0 & 0\end{bmatrix}, \qquad PP_d = \begin{bmatrix} AA_d & A_dB \\ 0 & 0\end{bmatrix} = \begin{bmatrix} A_d & 0 \\ 0 & 0\end{bmatrix}M, \\
P^{\pi} = \begin{bmatrix} A^{\pi} & -A_dB \\ 0 & I\end{bmatrix}, \qquad (P+I)_{\alpha}^n = \begin{bmatrix} (A+I)_{\alpha}^n & * \\ 0 & I\end{bmatrix},
\end{gathered} \tag{42}$$

where the subscript α stands for d or its absence.

Since $WW_d(W+I) = (W+I)WW_d$, $WW_d(W+I)_d^n = (W+I)_d^nWW_d$ where W denotes P or A. Thus,

$$PP_d(P+I)_{\alpha}^n = (P+I)_{\alpha}^nPP_d = \begin{bmatrix} (A+I)_{\alpha}^nA_d & 0 \\ 0 & 0\end{bmatrix}M, \tag{43}$$

$$\begin{aligned}
PP_d(P+I)^{\pi} &= (P+I)^{\pi}PP_d = \begin{bmatrix} (A+I)^{\pi} & * \\ 0 & I\end{bmatrix}PP_d \\
&= \begin{bmatrix} (A+I)^{\pi}A_d & 0 \\ 0 & 0\end{bmatrix}M, \\
a_i^{[n+2]}I - a_i^{[n+1]}Q_d &= \begin{bmatrix} a_i^{[n+2]}I & 0 \\ 0 & a_i^{[n+2]}I\end{bmatrix} - \begin{bmatrix} 0 & 0 \\ a_i^{[n+1]}D_d^2C & a_i^{[n+1]}D_d\end{bmatrix} \\
&= \begin{bmatrix} a_i^{[n+2]}I & 0 \\ -a_i^{[n+1]}D_d^2C & a_i^{[n+2]}I - a_i^{[n+1]}D_d\end{bmatrix}, \\
P^{i+1}P^{\pi} &= \begin{bmatrix} A^{i+1} & A^iB \\ 0 & 0\end{bmatrix}\begin{bmatrix} A^{\pi} & -A_dB \\ 0 & I\end{bmatrix} \\
&= \begin{bmatrix} A^{i+1}A^{\pi} & A^iA^{\pi}B \\ 0 & 0\end{bmatrix} \\
&= \begin{bmatrix} A^iA^{\pi} & 0 \\ 0 & 0\end{bmatrix}M, \\
Q^nP^{i+1}P^{\pi} &= \begin{bmatrix} 0 & 0 \\ D^{n-1}C & D^n\end{bmatrix}P^{i+1}P^{\pi} \\
&= \begin{bmatrix} 0 & 0 \\ D^{n-1}CA^iA^{\pi} & 0\end{bmatrix}M.
\end{aligned} \tag{44}$$

So, for $n \geq 1$,

$$\begin{aligned}
&\left(a_i^{[n+2]} I - a_i^{[n+1]} Q_d\right) Q^n P^{i+1} P^{\pi} \\
&\quad = \begin{bmatrix} 0 & 0 \\ \left(a_i^{[n+2]} I - a_i^{[n+1]} D_d\right) D^{n-1} C A^i A^{\pi} & 0 \end{bmatrix} M, \\
&\left(a_i^{[2]} I - a_i^{[1]} Q_d\right) P^{i+1} P^{\pi} = \begin{bmatrix} a_i^{[2]} A^i A^{\pi} & 0 \\ -a_i^{[1]} D_d^2 C A^i A^{\pi} & 0 \end{bmatrix} M, \\
&a_i^{[s+1]} Q_d Q^s P^{\pi} P^{i+1} \\
&\quad = a_i^{[s+1]} \begin{bmatrix} 0 & 0 \\ D_d^2 C & D_d \end{bmatrix} \begin{bmatrix} 0 & 0 \\ D^{s-1} C A^i A^{\pi} & 0 \end{bmatrix} M \\
&\quad = \begin{bmatrix} 0 & 0 \\ a_i^{[s+1]} D_d D^{s-1} C A^{\pi} A^i & 0 \end{bmatrix} M.
\end{aligned} \tag{45}$$

Also, for $n \geq 1$,

$$\begin{aligned}
&Q^{\pi} \left(Q^{n+1} - Q^n\right) P P_d (P+I)_d^{n+2} \\
&\quad = \begin{bmatrix} I & 0 \\ -D_d C & D^{\pi} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ \left(D^n - D^{n-1}\right) C & D^{n+1} - D^n \end{bmatrix} \\
&\qquad \times \begin{bmatrix} (A+I)_d^{n+2} A_d & 0 \\ 0 & 0 \end{bmatrix} M \\
&\quad = \begin{bmatrix} 0 & 0 \\ D^{\pi} \left(D^n - D^{n-1}\right) C (A+I)_d^{n+2} A_d & 0 \end{bmatrix} M, \\
&Q^{\pi} (Q - I) P P_d (P+I)_d^2 \\
&\quad = \begin{bmatrix} I & 0 \\ -D_d C & D^{\pi} \end{bmatrix} \begin{bmatrix} -I & 0 \\ C & D - I \end{bmatrix} \begin{bmatrix} (A+I)_d^2 A_d & 0 \\ 0 & 0 \end{bmatrix} M \\
&\quad = \begin{bmatrix} -(A+I)_d^2 A_d & 0 \\ \left(D_d + D^{\pi}\right) C (A+I)_d^2 A_d & 0 \end{bmatrix} M, \\
&\left(Q^{\pi} + Q_d\right) P P_d (P+I)_d \\
&\quad = \begin{bmatrix} I & 0 \\ D_d^2 C - D_d C & D^{\pi} + D_d \end{bmatrix} \begin{bmatrix} (A+I)_d A_d & 0 \\ 0 & 0 \end{bmatrix} M \\
&\quad = \begin{bmatrix} (A+I)_d A_d & 0 \\ \left(D_d^2 - D_d\right) C (A+I)_d A_d & 0 \end{bmatrix} M,
\end{aligned} \tag{46}$$

and, for $n \geq 0$,

$$\begin{aligned}
&\left(Q_d^{n+1} - Q_d^{n+2}\right) P P_d (P+I)^n (P+I)^{\pi} \\
&\quad = \begin{bmatrix} 0 & 0 \\ \left(D_d^{n+2} - D_d^{n+3}\right) C & D^{n+1} - D^{n+2} \end{bmatrix} \\
&\qquad \times \begin{bmatrix} (A+I)^n A_d & 0 \\ 0 & 0 \end{bmatrix} M \begin{bmatrix} (A+I)^{\pi} A_d & 0 \\ 0 & 0 \end{bmatrix} M \\
&\quad = \begin{bmatrix} 0 & 0 \\ \left(D_d^{n+2} - D_d^{n+3}\right) C (A+I)^n (A+I)^{\pi} A_d & 0 \end{bmatrix} M.
\end{aligned} \tag{47}$$

Since

$$\begin{aligned}
D - C A_d B &= D - C B (C B)_d^2 C B D = (C B)^{\pi} D, \\
C A^{\pi} &= C - C B C B (C B)_d^2 C = (C B)^{\pi} C,
\end{aligned} \tag{48}$$

we have

$$\begin{aligned}
Q_d P^{\pi} &= \begin{bmatrix} 0 & 0 \\ D_d^2 C & D_d \end{bmatrix} \begin{bmatrix} A^{\pi} & -A_d B \\ 0 & I \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 \\ D_d^2 (C B)^{\pi} C & D_d^2 (C B)^{\pi} D \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 \\ 0 & D_d^2 (C B)^{\pi} \end{bmatrix} M,
\end{aligned} \tag{49}$$

$$\begin{aligned}
Q P^{\pi} &= \begin{bmatrix} 0 & 0 \\ C A^{\pi} & D - C A_d B \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ (C B)^{\pi} C & (C B)^{\pi} D \end{bmatrix}, \\
M P^{\pi} &= \begin{bmatrix} A A^{\pi} & B - A A_d B \\ C A^{\pi} & D - C A_d B \end{bmatrix} = \begin{bmatrix} A A^{\pi} & B A^{\pi} \\ (C B)^{\pi} C & (C B)^{\pi} D \end{bmatrix} \\
&= \begin{bmatrix} A^{\pi} & 0 \\ 0 & (C B)^{\pi} \end{bmatrix} M,
\end{aligned} \tag{50}$$

and then, by Lemma 4, $\operatorname{Ind}(Q P^{\pi}) \leq \operatorname{Ind}[(C B)^{\pi} D] + 1 = s$, and $\operatorname{Ind}[(P+Q) P^{\pi}] \leq \operatorname{Ind}\left[\begin{bmatrix} A^{\pi} & 0 \\ 0 & (C B)^{\pi} \end{bmatrix} M\right] = k$.

By Lemma 4, $\operatorname{Ind}(P) \leq \operatorname{Ind}(A) + 1 = t$. Further, by (43), $\operatorname{Ind}[(P+I) P P_d] \leq \operatorname{Ind}[(A+I) A A_d] + 1 = l$.

Hence, putting (45)~(49) into (39) yields (40). □

4. Example

In this section, we present a numerical example to demonstrate Theorem 7.

Example 1. Taking P, Q as follows:

$$\begin{aligned}
P &= \begin{bmatrix} 0.1321 & 0.8459 & 0.2893 & -0.0597 \\ -0.2830 & 0.0802 & 0.0943 & 0.3066 \\ -0.3396 & 0.1462 & 1.1132 & 0.6179 \\ -0.6226 & 0.9764 & 0.2075 & 0.6745 \end{bmatrix}, \\
Q &= \begin{bmatrix} 0.5472 & 1.5283 & 0.1509 & -0.5094 \\ 0.0000 & 1.0000 & -0.0000 & 0 \\ 0.1132 & -0.3821 & 0.9623 & 0.1274 \\ -0.4528 & 1.5283 & 0.1509 & 0.4906 \end{bmatrix},
\end{aligned} \tag{51}$$

we can get

$$P_d = \begin{bmatrix} 0.2830 & 0.0031 & 0.2390 & -0.2233 \\ -0.2830 & 0.0802 & 0.0943 & 0.3066 \\ 0.8491 & 0.0094 & 0.7170 & -0.6698 \\ -0.8491 & 0.2406 & 0.2830 & 0.9198 \end{bmatrix},$$

$$P^{\pi} = \begin{bmatrix} 0.9057 & -0.0566 & -0.3019 & 0.0189 \\ 0.2830 & 0.9198 & -0.0943 & -0.3066 \\ -0.2830 & -0.1698 & 0.0943 & 0.0566 \\ 0.8491 & -0.2406 & -0.2830 & 0.0802 \end{bmatrix},$$

$$Q_d = \begin{bmatrix} 0.5472 & 1.5283 & 0.1509 & -0.5094 \\ 0.0000 & 1.0000 & -0.0000 & 0 \\ 0.1132 & -0.3821 & 0.9623 & 0.1274 \\ -0.4528 & 1.5283 & 0.1509 & 0.4906 \end{bmatrix},$$

$$Q^{\pi} = \begin{bmatrix} 0.4528 & -1.5283 & -0.1509 & 0.5094 \\ 0 & 0 & 0 & 0 \\ -0.1132 & 0.3821 & 0.0377 & -0.1274 \\ 0.4528 & -1.5283 & -0.1509 & 0.5094 \end{bmatrix}, \tag{52}$$

and $\mathrm{Ind}(QP^{\pi}) = 1$, $\mathrm{Ind}(P) = 2$, $\mathrm{Ind}[(P - I)PP_d] = 2$, and $\mathrm{Ind}[(P - Q)P^{\pi}] = 1$. By Theorem 7, we have

$$(P - Q)_d = \begin{bmatrix} -0.2264 & -0.7358 & 0.0755 & 0.2453 \\ -0.2830 & -0.9198 & 0.0943 & 0.3066 \\ 0.1132 & 0.3679 & -0.0377 & -0.1226 \\ -0.1698 & -0.5519 & 0.0566 & 0.1840 \end{bmatrix}. \tag{53}$$

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