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Research Article

Almost Periodic Solution of a Modified Leslie-Gower Predator-Prey Model with Beddington-DeAngelis Functional Response

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We consider a predator-prey model with modified Leslie-Gower and Beddington-DeAngelis functional response. By applying the comparison theorem of the differential equation and constructing a suitable Lyapunov function, sufficient conditions which guarantee the permanence and existence of a unique globally attractive positive almost periodic solution of the system are obtained. Our results not only supplement but also improve some existing ones. One example is presented to verify our main results.

1. Introduction

Let $f(t)$ be a continuous bounded function on R, and we set

$$f^l = \inf_{t\in R} f(t), \qquad f^u = \sup_{t\in R} f(t). \tag{1}$$

Leslie [1] introduced the famous Leslie predator-prey system

$$\begin{aligned} \dot{x} &= x(a - bx) - p(x)y, \\ \dot{y} &= y\left(e - f\frac{y}{x}\right), \end{aligned} \tag{2}$$

where x, y stand for the population (the density) of the prey and the predator at time t, respectively, and $p(x)$ is the so-called predator functional response to prey. The term y/x is the Leslie-Gower term which measures the loss in the predator population due to rarity (per capita y/x) of its favorite food. Leslie and Gower [2], Pielou [3] obtain some excellent results on the system (2) with the functional response $p(x) = cx$ which is called Holling-type I. By applying the Dulac's criterion and constructing Lyapunov functions, Hsu and Huang [4] establish the global stability of system (2).

Recently, Aziz-Alaoui and Daher Okiye [5] pointed out that in the case of severe scarcity, y can switch over to other populations, but its growth will be limited by the fact that its most favorite food x is not available in abundance. To solve such problem, they suggested to add a positive constant d to the denominator and proposed the following predator-prey model with modified Leslie-Gower and Holling-type II schemes:

$$\begin{aligned} \dot{x} &= \left(r_1 - bx - \frac{a_1 y}{x + k_1}\right)x, \\ \dot{y} &= \left(r_2 - \frac{a_2 y}{x + k_2}\right)y, \end{aligned} \tag{3}$$

with initial conditions $x(0) > 0$ and $y(0) > 0$, where x and y represent the population densities at time t. r_1, a_1, b, k_1, r_2, a_2, and k_2 are model parameters assuming only positive values. r_1 is the growth rate of prey x, b measures the strength of competition among individuals of species x, a_1 is the maximum value of the per capita reduction rate of x due to y, k_1 (resp., k_2) measures the extent to which the environment provides protection to prey x (resp., to the predator y), r_2 describes the growth rate of y, and a_2 has a similar meaning to a_1. The authors studied the boundedness and global stability of positive equilibrium of the system (3). Since then, system (2) and its nonautonomous versions have been studied by incorporating delay, impulses, harvesting, and so on (see, e.g., [6–13]). In particular, Zhu and Wang [13] consider the following nonautonomous model:

$$\begin{aligned}\dot{x}(t) &= x(t)\left(r_1(t) - b(t)x(t) - \frac{a_1(t)y(t)}{x(t)+k_1}\right),\\ \dot{y}(t) &= y(t)\left(r_2(t) - \frac{a_2(t)y(t)}{x(t)+k_2}\right).\end{aligned} \tag{4}$$

Under the assumption that the coefficients of the system (4) are all continuous T-periodic functions, by utilizing the coincidence degree theorem and constructing a suitable Lyapunov function, they obtained sufficient conditions for the existence and global attractivity of positive periodic solutions of the system (4). More precisely, Zhu and Wang [13] obtained the following theorem (see [13, Theorems 3.1 and 3.2]).

Theorem 1. *Suppose that*

(C_1)

$$\left[k_1 r_1 - a_1\left[\left(k_2 + \left[\frac{r_1}{b}\right]^u\right)\left[\frac{r_2}{a_2}\right]^u\right]\right]^l > 0 \tag{5}$$

holds, and further suppose that one of the following conditions:

(C_2)

$$k_1\overline{r}_1 - \overline{a}_1\left[\frac{\overline{r}_2\left(k_2+e^{A_5}\right)}{\overline{a}_2}\right] > 0, \quad \text{where } A_5 = \ln\frac{\overline{r}_1}{\overline{b}} \tag{6}$$

(C_3)

$$k_1 > k_2 \tag{7}$$

holds; then system (4) *has at least one positive T-periodic solution.*

Corollary 6 given in Section 2 of this paper shows that when (C_2) or (C_3) does not satisfy, the conclusion of Theorem 1 also holds. Moreover, as we know, permanence is one of the most important topics in the study of population dynamics; however, Zhu and Wang [13] did not investigate this property of the system (4). One aim of this work is to obtain a set of sufficient conditions which guarantee the permanence of the system (4).

On the other hand, as Fan and Kuang [14] have mentioned, the Holling II type functional responses fail to model the interference among predators and have been facing challenges from the biology and physiology communities (see [15–17]). Some biologists have argued that in many situations, especially when predators have to search for food (and therefore have to share or compete for food), the functional response in a prey-predator model should be predator-dependent.

Stimulated by the previous reasons, in this paper we will incorporate the Beddington-DeAngelis functional response into model (4) and consider the following model which is the generalization of the model (4):

$$\begin{aligned}\dot{x}(t) &= x(t)\left(r_1(t) - b(t)x(t) - \frac{c(t)y(t)}{\alpha(t)+\beta(t)x(t)+\gamma(t)y(t)}\right),\\ \dot{y}(t) &= y(t)\left(r_2(t) - \frac{d(t)y(t)}{x(t)+k(t)}\right),\end{aligned} \tag{8}$$

where $x(t)$ is the size of prey population and $y(t)$ is the size of predator population.

Li and Zhang [18] pointed out that in real world phenomenon, the environment varies due to the factors such as seasonal effects of weather, food supplies, mating habits, and harvesting. So it is usual to assume the periodicity of parameters in the system (8). However, if the various constituent components of the temporally nonuniform environment are with incommensurable (nonintegral multiples) periods, then one has to consider the environment to be almost periodic since there is no a priori reason to expect the existence of periodic solutions. For this reason, the assumption of almost periodicity is more realistic, more important, and more general when we consider the effects of the environmental factors. So it is assumed that the coefficients $b(t)$, $c(t)$, $d(t)$, $k(t)$, $\alpha(t)$, $\beta(t)$, $\gamma(t)$, $r_i(t)$ $(i = 1, 2)$ are all continuous, almost periodic functions and satisfy

$$\begin{aligned}&\min_{i=1,2}\left\{b^l, c^l, d^l, k^l, \alpha^l, \beta^l, \gamma^l, r_i^l\right\} > 0,\\ &\max_{i=1,2}\left\{b^u, c^u, d^u, k^u, \alpha^u, \beta^u, \gamma^u, r_i^u\right\} < +\infty.\end{aligned} \tag{9}$$

We consider system (8) with the following initial conditions:

$$x(0) > 0, \qquad y(0) > 0. \tag{10}$$

One can easily show that the solution of (8) with the initial condition (10) is defined and remains positive for all $t \geq 0$.

The aim of this paper is to obtain sufficient conditions for the existence of a unique globally attractive almost periodic solution of systems (8) and (10), by utilizing the comparison theorem of the differential equation and applying the analysis technique of papers [19–21].

The organization of this paper is as follows. In Section 2, by applying the theory of differential inequality, we present the permanence results for system (8). In Section 3, by constructing a suitable Lyapunov function, a set of sufficient conditions which ensure the existence and uniqueness of almost periodic solutions of system (8) are obtained. Then, in Section 4, a suitable example together with its numeric simulations is given to illustrate the feasibility of the main results. We end this paper by a brief discussion.

2. Permanence

Now let us state several definitions and lemmas which will be useful in the proving of the main result of this section.

Lemma 2 (see [22]). *If $a > 0$, $b > 0$ and $\dot{x} \geq x(b - ax)$, when $t \geq 0$ and $x(0) > 0$, we have*

$$\liminf_{t\to+\infty} x(t) \geq \frac{b}{a}. \tag{11}$$

If $a > 0$, $b > 0$ and $\dot{x} \leq x(b - ax)$, when $t \geq 0$ and $x(0) > 0$, we have

$$\limsup_{t\to+\infty} x(t) \leq \frac{b}{a}. \tag{12}$$

Theorem 3. *Suppose that system* (8) *with initial condition* (10) *satisfies the following conditions:*

(H_1)

$$r_1^l\alpha^l - \frac{r_2^u c^u (M_1 + k^u)}{d^l} > 0, \quad \text{where } M_1 = \frac{r_1^u}{b^l}. \tag{13}$$

Then system (8) *is permanent, that is, any positive solution $(x(t), y(t))^T$ of the system* (8) *satisfies*

$$\begin{aligned} 0 < m_1 \leq \liminf_{t\to+\infty} x(t) \leq \limsup_{t\to+\infty} x(t) \leq M_1, \\ 0 < m_2 \leq \liminf_{t\to+\infty} y(t) \leq \limsup_{t\to+\infty} y(t) \leq M_2, \end{aligned} \tag{14}$$

where $m_1 = (r_1^l\alpha^l - c^u M_2)/b^u\alpha^l$, $M_2 = r_2^u(M_1 + k^u)/d^l$, $m_2 = r_2^l(m_1 + k^l)/d^u$.

Proof. From condition (H_1), we can choose a small enough ε such that

$$r_1^l\alpha^l > \frac{r_2^u c^u (M_1 + \varepsilon + k^u)}{d^l}. \tag{15}$$

The first equation of (8) yields

$$\dot{x}(t) \leq x(t)\left[r_1^u - b^l x(t)\right]. \tag{16}$$

Applying Lemma 2 to (16) leads to

$$\limsup_{t\to+\infty} x(t) \leq \frac{r_1^u}{b^l} \triangleq M_1. \tag{17}$$

Equation (17) shows that there exists a large enough $T_1 > 0$ such that for all $t \geq T_1$,

$$x(t) \leq M_1 + \varepsilon \triangleq M_{1\varepsilon}. \tag{18}$$

It follows from (18) and the second equation of system (8) that, for $t \geq T_1$,

$$\dot{y}(t) \leq y(t)\left[r_2^u - \frac{d^l y(t)}{M_{1\varepsilon} + k^u}\right]. \tag{19}$$

According to Lemma 2 and (19), one has

$$\limsup_{t\to+\infty} y(t) \leq \frac{r_2^u (M_{1\varepsilon} + k^u)}{d^l}. \tag{20}$$

Thus, for previous ε, there exists a $T_2 \geq T_1$, such that

$$y(t) \leq \frac{r_2^u (M_{1\varepsilon} + k^u)}{d^l} + \varepsilon \triangleq M_{2\varepsilon} \quad \forall t \geq T_2. \tag{21}$$

Equation (21) together with the first equation of (8) leads to

$$\dot{x}(t) \geq x(t)\left[r_1^l - b^u x(t) - \frac{c^u M_{2\varepsilon}}{\alpha^l}\right] \quad \forall t \geq T_2. \tag{22}$$

From (22), according to (15) and Lemma 2, we can obtain

$$\liminf_{t\to+\infty} x(t) \geq \frac{r_1^l\alpha^l - c^u M_{2\varepsilon}}{b^u\alpha^l}. \tag{23}$$

Hence, for previous ε, there exists a $T_3 \geq T_2$, such that

$$x(t) \geq \frac{r_1^l\alpha^l - c^u M_{2\varepsilon}}{b^u\alpha^l} - \varepsilon \triangleq m_{1\varepsilon} \quad \forall t \geq T_3. \tag{24}$$

From (24) and the second equation of system (8), we know that for $t \geq T_3$,

$$\dot{y}(t) \geq \left[r_2^l - \frac{d^u y(t)}{m_{1\varepsilon} + k^l}\right]. \tag{25}$$

Applying Lemma 2 to (25) leads to

$$\liminf_{t\to+\infty} y(t) \geq \frac{r_2^l\left(m_{1\varepsilon} + k^l\right)}{d^u}. \tag{26}$$

Setting $\varepsilon \to 0$, it follows from the previous discussion that

$$\begin{aligned} \limsup_{t\to+\infty} y(t) &\leq \frac{r_2^u (M_1 + k^u)}{d^l} \triangleq M_2, \\ \liminf_{t\to+\infty} x(t) &\geq \frac{r_1^l\alpha^l - c^u M_2}{b^u\alpha^l} \triangleq m_1, \\ \liminf_{t\to+\infty} y(t) &\geq \frac{r_2^l\left(m_1 + k^l\right)}{d^u} \triangleq m_2. \end{aligned} \tag{27}$$

Obviously, m_i and M_i $(i = 1, 2)$ are independent of the solution of system (8); (17) and (27) show that the conclusion of Theorem 3 holds. The proof is completed. □

Theorem 4. *Suppose that system* (8) *with initial condition* (10) *satisfies the following conditions:*

(H_2)

$$r_1^l\gamma^l - c^u > 0. \tag{28}$$

Then system (8) *is permanent, that is, any positive solution $(x(t), y(t))^T$ of the system* (8) *satisfies*

$$\begin{aligned} 0 < m_1 \leq \liminf_{t\to+\infty} x(t) \leq \limsup_{t\to+\infty} x(t) \leq M_1, \\ 0 < m_2 \leq \liminf_{t\to+\infty} y(t) \leq \limsup_{t\to+\infty} y(t) \leq M_2, \end{aligned} \tag{29}$$

where M_1, M_2 are defined in Theorem 3 and $m_1 = (r_1^l\gamma^l - c^u)/b^u\gamma^l$, $m_2 = r_2^l(m_1 + k^l)/d^u$.

Proof. From (17) and (20), one can derive

$$\limsup_{t\to+\infty} x(t) \le M_1, \qquad \limsup_{t\to+\infty} y(t) \le M_2, \tag{30}$$

where M_1 and M_2 are defined in Theorem 3.

It follows from the first equation of system (8) that

$$x(t) \ge x(t)\left(r_1^l - b^u x(t) - \frac{c^u}{\gamma^l}\right). \tag{31}$$

According to (H_2), by applying Lemma 2 to (31), we obtain

$$\liminf_{t\to+\infty} x(t) \ge \frac{r_1^l\gamma^l - c^u}{b^u\gamma^l} \triangleq m_1. \tag{32}$$

Hence, for a small enough ε, there exists a $T > 0$, such that

$$x(t) \ge \frac{r_1^l\gamma^l - c^u}{b^u\gamma^l} - \varepsilon \triangleq m_{1\varepsilon} \quad \forall t \ge T. \tag{33}$$

From (33) and the second equation of system (8), we know that for $t \ge T_3$,

$$\dot{y}(t) \ge \left[r_2^l - \frac{d^u y(t)}{m_{1\varepsilon} + k^l}\right]. \tag{34}$$

Applying Lemma 2 to (34) leads to

$$\liminf_{t\to+\infty} y(t) \ge \frac{r_2^l\left(m_{1\varepsilon} + k^l\right)}{d^u}. \tag{35}$$

Setting $\varepsilon \to 0$, it follows from the previous discussion that

$$\liminf_{t\to+\infty} y(t) \ge \frac{r_2^l\left(m_1 + k^l\right)}{d^u} \triangleq m_2. \tag{36}$$

Obviously, m_i and M_i $(i = 1, 2)$ are independent of the solution of system (8); (30), (32) and (36) show that the conclusion of Theorem 4 holds. The proof is completed. □

As a direct corollary of Theorem 2 in [23], from Theorems 3 or 4, we have the following.

Corollary 5. *Suppose that (H_1) or (H_2) holds, then system (8) admits at least one positive T-periodic solution if $b(t)$, $c(t)$, $d(t)$, $k(t)$, $\alpha(t)$, $\beta(t)$, $\gamma(t)$, $r_i(t)$ $(i = 1, 2)$ are all continuous positive T-periodic functions.*

When $\alpha(t) = k_1, \beta(t) = 1, \gamma(t) = 0, k(t) = k_2, c(t) = a_1(t)$, $d(t) = a_2(t)$, where k_1, k_2 are positive constants, (8) becomes (4) which was discussed in [13]. According to Theorem 3 and Corollary 5, we obtain the following.

Corollary 6. *Suppose that*

(H_{11})

$$r_1^l k_1 - \frac{a_1^u r_2^u\left(M_1 + k_2\right)}{a_2^l} > 0, \quad \text{where } M_1 = \frac{r_1^u}{b^l} \tag{37}$$

holds; then system (4) is permanent and admits at least one positive T-periodic solution if $a_i(t)$, $b(t)$, $r_i(t)$ $(i = 1, 2)$ are all continuous positive T-periodic functions.

Remark 7. Comparing with Theorem 1, it follows from Corollary 6 that (C_2) or (C_3) is superfluous, so our results improve the main results in [13].

3. Existence of a Unique Almost Periodic Solution

Now let us state several definitions and lemmas which will be useful in the proving of the main result of this section.

Definition 8 (see [24, 25]). A function $f(t, x)$, where f is an m-vector, t is a real scalar, and x is an n-vector, is said to be almost periodic in t uniformly with respect to $x \in X \subset R^n$, if $f(t, x)$ is continuous in $t \in R$ and $x \in X$, and if for any $\varepsilon > 0$, there is a constant $l(\varepsilon) > 0$, such that in any interval of length $l(\varepsilon)$ there exists τ such that the inequality

$$\|f(t+\tau) - f(t)\| = \sum_{i=1}^{m} \left|f_i(t+\tau, x) - f_i(t, x)\right| < \varepsilon \tag{38}$$

is satisfied for all $t \in (-\infty, +\infty)$, $x \in X$. The number τ is called an ε-translation number of $f(t, x)$.

Definition 9 (see [24, 25]). A function $f : R \to R$ is said to be asymptotically almost periodic function if there exists an almost periodic function $q(t)$ and a continuous function $r(t)$ such that

$$f(t) = q(t) + r(t), \quad t \in R, \ r(t) \longrightarrow 0 \text{ as } t \longrightarrow \infty. \tag{39}$$

We denote by $S(E)$ the set of all solutions $z(t) = (x(t), y(t))^T$ of system (8) satisfying $m_1 \le x(t) \le M_1$, $m_2 \le y(t) \le M_2$ for all $t \in R$.

Lemma 10. $S(E) \neq \emptyset$.

Proof. Since $b(t), c(t), d(t), k(t), \alpha(t), \beta(t), \gamma(t), r_i(t)$ $(i = 1, 2)$ are almost periodic functions, there exists a sequence $\{t_n\}$, $t_n \to \infty$ as $n \to \infty$ such that

$$\begin{aligned}
&b(t+t_n) \longrightarrow b(t), \qquad c(t+t_n) \longrightarrow c(t),\\
&d(t+t_n) \longrightarrow d(t), \qquad k(t+t_n) \longrightarrow k(t),\\
&\alpha(t+t_n) \longrightarrow \alpha(t), \qquad \beta(t+t_n) \longrightarrow \beta(t),\\
&\gamma(t+t_n) \longrightarrow \gamma(t), \qquad r_i(t+t_n) \longrightarrow r_i(t) \quad (i = 1, 2),
\end{aligned} \tag{40}$$

as $n \to \infty$ uniformly on R. Let $z(t) = (x(t), y(t))^T$ be solution of systems (8) and (10) satisfying $m_1 \le x(t) \le M_1$, $m_2 \le y(t) \le M_2$ for $t > T$. Obviously, the sequence $z(t + t_n)$ is uniformly bounded and equicontinuous on each bounded subset of R. Therefore, by Ascoli-Arzela theorem, there exists a subsequence of $\{t_n\}$, and we still denote it as $\{t_n\}$, such that $x(t+t_n) \to p_1(t)$, $y(t+t_n) \to p_2(t)$, as $n \to \infty$ uniformly on each bounded subset of R. For any $T_1 \in R$, we may assume that $t_n + T_1 \ge T$ for all n. For $t \ge 0$, we have

$$\begin{aligned}
&x(t+t_n+T_1)-x(t_n+T_1)\\
&\quad=\int_{T_1}^{t+T_1}x(s+t_n)\Bigg(r_1(s+t_n)-b(s+t_n)x(s+t_n)\\
&\qquad-\frac{c(s+t_n)y(s+t_n)}{\alpha(s+t_n)+\beta(s+t_n)x(s+t_n)+\gamma(s+t_n)y(s+t_n)}\Bigg)ds,\\
&y(t+t_n+T_1)-y(t_n+T_1)=\int_{T_1}^{t+T_1}y(s+t_n)\left(r_2(s+t_n)-\frac{d(s+t_n)y(s+t_n)}{x(s+t_n)+k(s+t_n)}\right)ds.
\end{aligned}\tag{41}$$

Applying Lebesgue's dominated convergence theorem and letting $n\to\infty$ in the previous equations, we obtain

$$\begin{aligned}
&p_1(t+T_1)-p_1(T_1)\\
&\quad=\int_{T_1}^{t+T_1}p_1(s)\Bigg(r_1(s)-b(s)p_1(s)\\
&\qquad-\frac{c(s)p_2(s)}{\alpha(s)+\beta(s)p_1(s)+\gamma(s)p_2(s)}\Bigg)ds,\\
&p_2(t+T_1)-p_2(T_1)\\
&\quad=\int_{T_1}^{t+T_1}p_2(s)\left(r_2(s)-\frac{d(s)p_2(s)}{p_1(s)+k(s)}\right)ds,
\end{aligned}\tag{42}$$

for all $t\geq 0$. Since $T_1\in R$ is arbitrarily given, $(p_1(t),p_2(t))^T$ is a solution of system (8) on R. It is clear that $m_1\leq p_1(t)\leq M_1$, $m_2\leq p_2(t)\leq M_2$ for $t\in R$. That is to say $(p_1(t),p_2(t))^T\in S(E)$. This completes the proof. □

Lemma 11 (see [26]). *Let f be a nonnegative function defined on $[0,+\infty)$ such that f is integrable on $[0,+\infty)$ and is uniformly continuous on $[0,+\infty)$. Then $\lim_{t\to+\infty}f(t)=0$.*

Theorem 12. *In addition to (H_1) or (H_2), further suppose that*

(H_3)

$$\left[b(t)-\frac{c(t)\beta(t)M_2}{\Delta^2(t,m_1,m_2)}-\frac{d(t)M_2}{(m_1+k(t))^2}\right]^l>0,\tag{43}$$

(H_4)

$$\left[\frac{d(t)}{M_1+k(t)}-\frac{c(t)}{\Delta(t,m_1,m_2)}-\frac{c(t)\gamma(t)M_2}{\Delta^2(t,m_1,m_2)}\right]^l>0,\tag{44}$$

where m_i and M_i $(i=1,2)$ are defined in the proof of Theorem 3 (or Theorem 4) and

$$\Delta(t,x(t),y(t))=\alpha(t)+\beta(t)x(t)+\gamma(t)y(t).\tag{45}$$

Then system (8) *with initial conditions* (10) *is globally attractive.*

Proof. It follows from conditions (H_3) and (H_4) that there exists a small enough $\varepsilon>0$ such that

$$\begin{aligned}
&A_1(\varepsilon)\\
&\quad=\Bigg[b(t)-\frac{c(t)\beta(t)(M_2+\varepsilon)}{\Delta^2(t,m_1-\varepsilon,m_2-\varepsilon)}\\
&\qquad-\frac{d(t)(M_2+\varepsilon)}{(m_1-\varepsilon+k(t))^2}\Bigg]^l>\varepsilon,\\
&A_2(\varepsilon)\\
&\quad=\Bigg[\frac{d(t)}{M_1+\varepsilon+k(t)}-\frac{c(t)}{\Delta(t,m_1-\varepsilon,m_2-\varepsilon)}\\
&\qquad-\frac{c(t)\gamma(t)(M_2+\varepsilon)}{\Delta^2(t,m_1-\varepsilon,m_2-\varepsilon)}\Bigg]^l>\varepsilon.
\end{aligned}\tag{46}$$

Let $z_1(t)=(x(t),y(t))^T$, $z_2(t)=(x_*(t),y_*(t))^T$ be any two positive solutions of system (8) with initial conditions (10). For previous ε, according to Lemma 10 and Theorem 3 (or Theorem 4), there exists a $T>0$, when $t\geq T$,

$$\begin{aligned}
&m_1-\varepsilon\leq x(t)\leq M_1+\varepsilon, &&m_2-\varepsilon\leq y(t)\leq M_2+\varepsilon,\\
&m_1-\varepsilon\leq x_*(t)\leq M_1+\varepsilon, &&m_2-\varepsilon\leq y_*(t)\leq M_2+\varepsilon.
\end{aligned}\tag{47}$$

Let $V(t)=V_1(t)+V_2(t)$, where

$$\begin{aligned}
V_1(t)&=\left|\ln x(t)-\ln x_*(t)\right|,\\
V_2(t)&=\left|\ln y(t)-\ln y_*(t)\right|.
\end{aligned}\tag{48}$$

Calculating the upper right derivatives of $V_1(t)$ along the solution of (8) leads to

$$\begin{aligned}
D^+V_1(t) &= \operatorname{sgn}(x(t)-x_*(t))\left[-b(t)(x(t)-x_*(t)) - c(t)\left(\frac{y(t)}{\Delta(t,x(t),y(t))} - \frac{y_*(t)}{\Delta(t,x_*(t),y_*(t))}\right)\right] \\
&= -b(t)\left|x(t)-x_*(t)\right| - \operatorname{sgn}(x(t)-x_*(t))\,c(t)\left(\frac{y(t)}{\Delta(t,x(t),y(t))} - \frac{y_*(t)}{\Delta(t,x(t),y(t))}\right. \\
&\qquad \left. + \frac{y_*(t)}{\Delta(t,x(t),y(t))} - \frac{y_*(t)}{\Delta(t,x_*(t),y_*(t))}\right) \\
&= -b(t)\left|x(t)-x_*(t)\right| - \operatorname{sgn}(x(t)-x_*(t)) \\
&\quad \times c(t)\left(\frac{y(t)-y_*(t)}{\Delta(t,x(t),y(t))} + \frac{y_*(t)\left[\beta(t)(x_*(t)-x(t)) + \gamma(t)(y_*(t)-y(t))\right]}{\Delta(t,x_*(t),y_*(t))\cdot\Delta(t,x(t),y(t))}\right), \\
D^+V_1(t) &\le -b(t)\left|x(t)-x_*(t)\right| \\
&\quad + c(t)\left(\frac{\left|y(t)-y_*(t)\right|}{\Delta(t,x(t),y(t))} + \frac{y_*(t)\left[\beta(t)\left|x(t)-x_*(t)\right| + \gamma(t)\left|y(t)-y_*(t)\right|\right]}{\Delta(t,x_*(t),y_*(t))\cdot\Delta(t,x(t),y(t))}\right).
\end{aligned} \tag{49}$$

Calculating the upper right derivatives of $V_2(t)$ along the solution of (8), one has

$$\begin{aligned}
&D^+V_2(t) \\
&\quad = \operatorname{sgn}\left[y(t)-y_*(t)\right] d(t)\left(\frac{y_*(t)}{x_*(t)+k(t)} - \frac{y(t)}{x(t)+k(t)}\right) \\
&\quad = \operatorname{sgn}\left[y(t)-y_*(t)\right] d(t) \\
&\qquad \times\left(\frac{y_*(t)}{x_*(t)+k(t)} - \frac{y_*(t)}{x(t)+k(t)} + \frac{y_*(t)}{x(t)+k(t)} - \frac{y(t)}{x(t)+k(t)}\right) \\
&\quad = \operatorname{sgn}\left[y(t)-y_*(t)\right] d(t) \\
&\qquad \times\left(\frac{y_*(t)(x(t)-x_*(t))}{(x_*(t)+k(t))(x(t)+k(t))} + \frac{y_*(t)-y(t)}{x(t)+k(t)}\right) \\
&\quad \le \frac{d(t)y_*(t)\left|x(t)-x_*(t)\right|}{(x_*(t)+k(t))(x(t)+k(t))} - \frac{d(t)\left|y(t)-y_*(t)\right|}{x(t)+k(t)}.
\end{aligned} \tag{50}$$

It follows from (49)-(50) that for $t \ge T$,

$$\begin{aligned}
&D^+V(t) \\
&\quad \le \left[-b(t) + \frac{c(t)y_*(t)\beta(t)}{\Delta(t,x_*(t),y_*(t))\cdot\Delta(t,x(t),y(t))} + \frac{d(t)y_*(t)}{(x_*(t)+k(t))(x(t)+k(t))}\right]\left|x(t)-x_*(t)\right| \\
&\qquad + \left[-\frac{d(t)}{x(t)+k(t)} + \frac{c(t)}{\Delta(t,x(t),y(t))} + \frac{c(t)y_*(t)\gamma(t)}{\Delta(t,x_*(t),y_*(t))\cdot\Delta(t,x(t),y(t))}\right] \\
&\qquad \times\left|y(t)-y_*(t)\right| \\
&\quad \le -\left[b(t) - \frac{c(t)\beta(t)(M_2+\varepsilon)}{\Delta^2(t,m_1-\varepsilon,m_2-\varepsilon)} - \frac{d(t)(M_2+\varepsilon)}{(m_1-\varepsilon+k(t))^2}\right] \\
&\qquad \times\left|x(t)-x_*(t)\right| \\
&\qquad - \left[\frac{d(t)}{M_1+\varepsilon+k(t)} - \frac{c(t)}{\Delta(t,m_1-\varepsilon,m_2-\varepsilon)} - \frac{c(t)\gamma(t)(M_2+\varepsilon)}{\Delta^2(t,m_1-\varepsilon,m_2-\varepsilon)}\right]\left|y(t)-y_*(t)\right|.
\end{aligned} \tag{51}$$

It follows from (46) and (51) that for $t \ge T$,

$$D^+V(t) \le -\varepsilon\left|x(t)-x_*(t)\right| - \varepsilon\left|y(t)-y_*(t)\right|, \tag{52}$$

which implies $V(t)$ is nonincreasing on $[T,+\infty)$. Integrating the previous inequality from T to t leads to

$$\begin{aligned}
&V(t) + \varepsilon\int_T^t \left|x(s)-x_*(s)\right| ds + \varepsilon \\
&\quad \times \int_T^t \left|y(s)-y_*(s)\right| ds < V(T) < +\infty, \quad t \ge T.
\end{aligned} \tag{53}$$

Then for $t \ge T$, we obtain that

$$\begin{aligned}
&\int_T^t \left|x(s)-x_*(s)\right| ds < \frac{V(T)}{\varepsilon} < +\infty, \\
&\int_T^t \left|y(s)-y_*(s)\right| ds < \frac{V(T)}{\varepsilon} < +\infty.
\end{aligned} \tag{54}$$

Hence, $|x(t)-x_*(t)|, |y(t)-y_*(t)| \in L^1([T,+\infty))$. By system (8) and Theorem 3 (or Theorem 4), we get $x(t)$, $x_*(t)$, $y(t)$, $y_*(t)$, and their derivatives are bounded on $[T,+\infty)$, which implies that both $|x(t)-x_*(t)|$ and $|y(t)-y(t)|$ are uniformly continuous on $[T,+\infty)$. By Lemma 11, we obtain

$$\lim_{t\to+\infty}|x(t)-x_*(t)|=0, \qquad \lim_{t\to+\infty}|y(t)-y_*(t)|=0. \tag{55}$$

Then the solution of systems (8) and (10) is globally attractive. □

Theorem 13. *Suppose all conditions of Theorem 12 hold; then there exists a unique almost periodic solution of systems* (8) *and* (10).

Proof. According to Lemma 10, there exists a bounded positive solution $u(t)=(u_1(t),u_2(t))^T$ of (8) with initial condition (10). Then there exists a sequence $\{t'_k\}$, $\{t'_k\}\to\infty$ as $k\to\infty$, such that $(u_1(t+t'_k),u_2(t+t'_k))^T$ is a solution of the following system:

$$\begin{aligned}\dot{x}(t)&=x(t)\left(r_1(t+t'_k)-b(t+t'_k)x(t)\right.\\&\qquad\left.-\frac{c(t+t'_k)y(t)}{\alpha(t+t'_k)+\beta(t+t'_k)x(t)+\gamma(t+t'_k)y(t)}\right),\\\dot{y}(t)&=y(t)\left(r_2(t+t'_k)-\frac{d(t+t'_k)y(t)}{x(t)+k(t+t'_k)}\right).\end{aligned}\tag{56}$$

According to Theorem 3 (or Theorem 4) and the fact that $b(t)$, $c(t)$, $d(t)$, $k(t)$, $\alpha(t)$, $\beta(t)$, $\gamma(t)$, $r_i(t)$ $(i=1,2)$ are all continuous, positive almost periodic functions, we know that both $\{u_i(t+t'_k)\}$ $(i=1,2)$ and their derivative function $\{\dot{u}(t+t'_k)\}$ $(i=1,2)$ are uniformly bounded; thus $\{u_i(t+t'_k)\}$ $(i=1,2)$ are uniformly bounded and equicontinuous. By Ascoli's theorem, there exists a uniformly convergent subsequence $\{u_i(t+t_k)\}\subseteq\{u_i(t+t'_k)\}$ such that for any $\varepsilon>0$, there exists a $K(\varepsilon)>0$ with the property that if $m,k\geq K(\varepsilon)$, then

$$|u_i(t+t_m)-u_i(t+t_k)|<\varepsilon, \quad i=1,2. \tag{57}$$

That is to say $u_i(t)$ $(i=1,2)$ are asymptotically almost periodic functions; hence there exists two almost periodic functions $p_i(t+t_k)(i=1,2)$ and two continuous functions $q_i(t+t_k)$ $(i=1,2)$ such that

$$u_i(t+t_k)=p_i(t+t_k)+q_i(t+t_k), \quad i=1,2, \tag{58}$$

where

$$\lim_{k\to+\infty}p_i(t+t_k)=p_i(t), \qquad \lim_{k\to+\infty}q_i(t+t_k)=0, \quad i=1,2, \tag{59}$$

$p_i(t)(i=1,2)$ are also almost periodic functions.

Therefore,

$$\lim_{k\to+\infty}u_i(t+t_k)=p_i(t), \quad i=1,2. \tag{60}$$

On the other hand,

$$\begin{aligned}\lim_{k\to+\infty}\dot{u}_i(t+t_k)&=\lim_{k\to+\infty}\lim_{h\to0}\frac{u_i(t+t_k+h)-u_i(t+t_k)}{h}\\&=\lim_{h\to0}\lim_{k\to+\infty}\frac{u_i(t+t_k+h)-u_i(t+t_k)}{h}\\&=\lim_{h\to0}\frac{p_i(t+h)-p_i(t)}{h}, \quad i=1,2.\end{aligned}\tag{61}$$

So $\dot{p}_i(t)$ $(i=1,2)$ exist. Now we will prove that $(p_1(t),p_2(t))^T$ is an almost solution of system (8).

From properties of almost periodic function, there exists an sequence $\{t_n\}$, $\{t_n\}\to\infty$ as $n\to\infty$, such that

$$\begin{aligned}&b(t+t_n)\longrightarrow b(t), && c(t+t_n)\longrightarrow c(t),\\&d(t+t_n)\longrightarrow d(t), && k(t+t_n)\longrightarrow k(t),\\&\alpha(t+t_n)\longrightarrow\alpha(t), && \beta(t+t_n)\longrightarrow\beta(t),\\&\gamma(t+t_n)\longrightarrow\gamma(t), && r_i(t+t_n)\longrightarrow r_i(t), \quad (i=1,2)\end{aligned}\tag{62}$$

as $n\to\infty$ uniformly on R.

It is easy to know that $u_i(t+t_n)\to p_i(t)(i=1,2)$ as $n\to\infty$; then we have

$$\begin{aligned}\dot{p}_1(t)&=\lim_{n\to+\infty}\dot{u}_1(t+t_n)\\&=\lim_{n\to+\infty}u_1(t+t_n)\left(r_1(t+t_n)-b(t+t_n)u_1(t+t_n)\right.\\&\qquad\left.-\frac{c(t+t_n)u_2(t+t_n)}{\alpha(t+t_n)+\beta(t+t_n)u_1(t+t_n)+\gamma(t+t_n)u_2(t+t_n)}\right)\\&=p_1(t)(r_1(t)-b(t)p_1(t))-\frac{c(t)p_2(t)}{\alpha(t)+\beta(t)p_1(t)+\gamma(t)p_2(t)},\end{aligned}$$

$$\begin{aligned}\dot{p}_2(t) &= \lim_{n\to+\infty}\dot{u}_2(t+t_n)\\ &= \lim_{n\to+\infty}u_2(t+t_n)\left(r_2(t+t_n)-\frac{d(t+t_n)u_2(t+t_n)}{u_1(t+t_n)+k(t+t_n)}\right)\\ &= p_2(t)\left(r_2(t)-\frac{d(t)p_2(t)}{p_1(t)+k(t)}\right).\end{aligned} \tag{63}$$

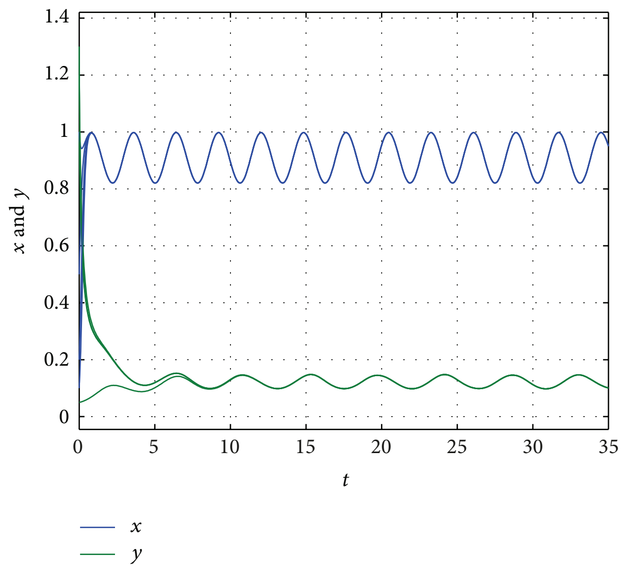

Figure 1: Dynamic behavior of the system (64) with the initial condition $(x(0), y(0)) = (0.5, 1)^T$, $(1, 1.2)^T$, $(0, 2, 0.05)^T$, and $(0.1, 1.3)^T$, respectively.

These show that $(p_1(t), p_2(t))^T$ satisfies system (8). Hence, $(p_1(t), p_2(t))^T$ is a positive almost periodic solution of (8). Then, it follows from Theorem 12 that system (8) has a unique positive almost periodic solution. The proof is completed. □

4. Examples and Numeric Simulations

Consider the following example:

$$\begin{aligned}\dot{x}(t) &= x(t)\left[10+\sin\sqrt{5}t-11x(t)\right.\\ &\quad\left.-\frac{\left(0.3+0.2\sin\sqrt{3}t\right)y(t)}{8+\cos\sqrt{11}t+\left(10+\sin\sqrt{3}t\right)x(t)+5y(t)}\right],\\ \dot{y}(t) &= y(t)\\ &\quad\times\left[0.5+0.3\sin\sqrt{2}t-\frac{\left(12+0.2\sin\sqrt{13}t\right)y(t)}{x(t)+2}\right].\end{aligned} \tag{64}$$

In this case, we have $r_1^u = 11$, $r_1^l = 9$, $b^u = b^l = 11$, $c^u = 0.5$, $c^l = 0.1$, $\alpha^u = 9$, $\alpha^l = 7$, $\beta^u = 11$, $\beta^l = 9$, $\gamma^u = \gamma^l = 5$, $r_2^u = 0.8$, $r_2^l = 0.2$, $d^u = 12.2$, $d^l = 11.8$, $k^u = k^l = 2$. According to Theorem 3 (or Theorem 4), we have

$$\begin{aligned}&m_1 \approx 0.81686 \ (\text{or } 0.80909),\\ &m_2 \approx 0.04618 \ (\text{or } 0.04605),\\ &M_1 = 1, \qquad M_2 \approx 0.20339.\end{aligned} \tag{65}$$

Considering (H_3) and (H_4), we choose $m_1 = 0.80909$, $m_2 = 0.04605$. Hence,

$$\begin{aligned}&r_1^l\alpha^l - \frac{r_2^u c^u (M_1+k^u)}{d^l} \approx 62.8983 > 0,\\ &r_1^l\gamma^l - c^u \approx 44.5 > 0,\\ &\left[b(t)-\frac{c(t)\beta(t)M_2}{\Delta^2(t,m_1,m_2)}-\frac{d(t)M_2}{(m_1+k(t))^2}\right]^l\\ &\quad\geq b^l - \frac{c^u\beta^u M_2}{(\alpha^l+\beta^l m_1+\gamma^l m_2)^2}\\ &\qquad-\frac{d^u M_2}{(m_1+k^l)^2} \approx 10.6802 > 0,\\ &\left[\frac{d(t)}{M_1+K(t)}-\frac{c(t)}{\Delta(t,m_1,m_2)}-\frac{c(t)\gamma(t)M_2}{\Delta^2(t,m_1,m_2)}\right]^l\\ &\quad\geq \frac{d^l}{M_1+k^u}-\frac{c^u}{\alpha^l+\beta^l m_1+\gamma^l m_2}\\ &\qquad-\frac{c^u\gamma^u M_2}{(\alpha^l+\beta^l m_1+\gamma^l m_2)^2}\\ &\quad\approx 3.89646 > 0.\end{aligned} \tag{66}$$

Equation (66) means that all conditions of Theorem 13 are satisfied in system (64). Thus, it admits a unique, globally attractive, positive, almost periodic solution. Figure 1 shows the dynamic behaviors of the solution $(x(t), y(t))^T$ with the four group initial values $(x(0), y(0)) = (0.5, 1)^T$, $(1, 1.2)^T$, $(0, 2, 0.05)^T$, and $(0.1, 1.3)^T$. From the figure, we could easily see that the solution $(x(t), y(t))^T$ is asymptotic to the unique, almost periodic solution of the system (64).

5. Conclusion

In this paper, we consider a predator-prey with modified Leslie-Gower model and Beddington-DeAngelis functional response. When $\alpha(t) = k_1$, $\beta(t) = 1$, $\gamma(t) = 0$, $k(t) = k_2$, $c(t) = a_1(t)$, $d(t) = a_2(t)$, (8) we discussed reduces to (4) which was studied by Zhu and Wang [13]. By utilizing the coincidence degree theorem and constructing a suitable Lyapunov function, the authors in [13] investigated the existence and global attractivity of positive periodic solutions of (4) and obtained Theorem 1. More precisely, comparing Theorem 1 with Corollary 6, we find that conditions (C_2) or (C_3) of Theorem 1 are redundant, which implies that our results improve those of [13]. Example together with numeric simulation shows the feasibility of our main results.

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