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Research Article

Lower Bounds of Periods of Periodic Solutions for a Class of Differential Equations with Variable Delays

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Based on generalized Wirtinger's inequality, periods of periodic solutions of the nonautonomous differential equations with variable delays are investigated. Based on Hölder inequality, lower bounds of periods of periodic solutions for a class of functional differential equations with variable delays are obtained by a simple method.

1. Introduction

The existence and multiplicity of periodic solutions, bifurcations of periodic solutions, and stability of solutions of functional differential equations have attracted the attention of many mathematicians [1–5]. A lot of remarkable results have been achieved [6–10]. However, only a few works on periods of periodic solutions have been done (see, e.g., [11–13]). Suppose f is Lipschitz continuous in a Banach space with constant L and $x(t)$ is a T-periodic nonconstant solution of $x'(t) = f(x(t))$. Lasota and Yorke [12] have showed that $TL \geq 4$. Busenberg et al. [14] refined the earlier estimate of TL in [12]; they [14] showed that $TL \geq 6$. At the same time, they [14] also gave a simple proof of the better lower bound $TL \geq 2\pi$ in spaces with the norm defined via an inner product. Mawhin and Walter [15] showed how some lower bounds on the period of the possible periodic solutions of autonomous ordinary differential equations due to Yorke [11] are easy consequences of the general principle. Zevin and Pinsky [16] investigated a class of Lipschitzian differential equations of even order; they obtained the minimal periods of periodic solutions. In 2012, Domoshnitsky et al. [17] investigated componentwise positivity of solutions to periodic boundary problem for linear functional differential system. Recently, Cheng and Zhang [18] proved a generalized Wirtingers inequality. Based on this inequality, they [18] studied estimates for lower bounds of periods of periodic solutions for the following autonomous delay differential equation:

$$\dot{x}(t) = -\sum_{k=1}^{n} f(x(t-kr)), \tag{1}$$

where $x \in R^p$, $f \in C(R^p, R^p)$, and $r > 0$ is a given constant. In their paper [18], delays are required to be constants with the form of kr. In this paper, we will replace the constant delay kr with the generalized delay function $r_k(t)$ with $|r_k'(t)| < 1$, $k = 1, 2, \ldots, n$. Furthermore, the method used in our paper is simpler than that in [18]. Lower bounds of periods of periodic solutions for a class of functional differential equations with variable delays are obtained.

Consider the lower bounds of periods of periodic solutions for the following delay differential equations:

$$\dot{x}(t) = -\sum_{k=1}^{n} f(x(t-r_k(t))), \tag{2}$$

where $x \in R^p$, $f \in C(R^p, R^p)$, $r_k(t) = r_k(t+T)$, $r_k(t) > 0$, and $|r_k'(t)| < 1$ for $t \in R$.

In order to estimate the lower bounds of periods of periodic solution of (2), we need the following definitions and lemmas.

Definition 1. For a positive constant L, $f(x) \in C(R^p, R^p)$ is called L-Lipschitz continuous if, for all $x, y \in R^p$,

$$|f(x) - f(y)| \leq L|x - y|, \quad (3)$$

where $|\cdot|$ denotes the Euclidean norm in R^p.

Let $H_T^1(R, R^p)$ be the Hilbert space consisting of the T-periodic functions x on R which together with weak derivatives belong to $L^2(0, T; R^p)$. For all $x, y \in L^2(0, T; R^p)$, let $\langle x, y\rangle = \int_0^T (x, y)dt$ and $\|x\| = \sqrt{\langle x, x\rangle}$ denote the inner product and the norm in $L^2(0, T; R^p)$, respectively, where $(\cdot, \cdot)$ is the inner product in R^p. Let $S = \{x \mid x(t) = (x_1(t), x_2(t), \ldots, x_p(t))\}$, where $x_i(t)$ has the second derivative, $i = 1, 2, \ldots, p$.

Lemma 2 (see [8]). *Suppose $\tau \in C_\omega^1$ and $\tau'(t) < 1$, for all $t \in [0, \omega]$. Then the function $t - \tau(t)$ has an inverse $\mu(t)$ satisfying $\mu \in C(R, R)$ with $\mu(a + \omega) = \mu(a) + \omega$.*

Lemma 3 (see [18]). *If $x \in H_T^1$ and $\int_0^T x(t)dt = 0$, then*

$$\int_0^T |x(t)|^2 dt \leq \frac{T^2}{4\pi^2} \int_0^T |\dot{x}(t)|^2 dt. \quad (4)$$

2. Main Results

Since $r_k'(t) < 1$, by Lemma 2, the inverse of $t - r_k(t)$ exists. Let $\mu_k(s)$ be the inverse of $t - r_k(t)$.

Theorem 4. *Let x be a nonconstant T-periodic solution of the nonautonomous delay differential equation (2) and $x \in S$. Suppose that the function $f : R^p \to R^p$ is L-Lipschitz continuous and $r_k(t) = r_k(t + T)$, $r_k(t) > 0$, $|r_k'(t)| < 1$. Then $T > \sqrt{2}\pi/nL$.*

Proof. Since x is a nonconstant T-periodic solution of the nonautonomous delay differential equation (2), for all $t, u \in R$, we have

$$\begin{aligned} &|\dot{x}(t+u) - \dot{x}(t)| \\ &\quad = \left|\sum_{k=1}^n f(x(t+u-r_k(t+u))) - \sum_{k=1}^n f(x(t-r_k(t)))\right| \\ &\quad \leq \sum_{k=1}^n |f(x(t+u-r_k(t+u))) - f(x(t-r_k(t)))| \\ &\quad \leq L\sum_{k=1}^n |x(t+u-r_k(t+u)) - x(t-r_k(t))|. \end{aligned} \quad (5)$$

We claim that if $u \neq 0$, then, for $t \in R$, there exists at least one k such that

$$u - r_k(t+u) + r_k(t) \neq 0. \quad (6)$$

Otherwise, if $u - r_k(t+u) + r_k(t) = 0$ for $k = 1, 2, \ldots, n$, then

$$x(t+u-r_k(t+u)) - x(t-r_k(t)) = 0. \quad (7)$$

From (5), one has

$$|\dot{x}(t+u) - \dot{x}(t)| = 0. \quad (8)$$

Noting that $x \in S$, we obtain

$$|\ddot{x}(t)| = 0. \quad (9)$$

Noting that $x(t) = x(t+T)$, then $\int_0^T \dot{x}(t)dt = 0$. From Lemma 3, we have

$$2\pi\|\dot{x}\| \leq T\|\ddot{x}\|. \quad (10)$$

Then $|\dot{x}| = 0$. x is a constant T-periodic solution. This contradicts the assumption that x is a nonconstant T-periodic solution.

For simplicity of proof, we suppose that $u - r_k(t+u) + r_k(t) \neq 0$ for $k = 1, 2, \ldots, n$, (5) can be rewritten as

$$\begin{aligned} &\left|\frac{\dot{x}(t+u) - \dot{x}(t)}{u}\right| \\ &\quad \leq L\sum_{k=1}^n \left|\frac{x(t+u-r_k(t+u)) - x(t-r_k(t))}{u - r_k(t+u) + r_k(t)}\right| \\ &\qquad \times \left|\frac{u - r_k(t+u) + r_k(t)}{u}\right|. \end{aligned} \quad (11)$$

Let $u \to 0$; one has

$$\begin{aligned} |\ddot{x}(t)| &\leq L\sum_{k=1}^n |\dot{x}(t-r_k(t))|[1 - r_k'(t)] \\ &= L\sum_{k=1}^n |\dot{x}(t-r_k(t))|[1 - r_k'(t)]^{1/2}[1 - r_k'(t)]^{1/2}. \end{aligned} \quad (12)$$

Applying Hölder inequality gives

$$\begin{aligned} &|\ddot{x}(t)| \\ &\quad \leq L\sum_{k=1}^n |\dot{x}(t-r_k(t))|[1 - r_k'(t)]^{1/2}[1 - r_k'(t)]^{1/2} \\ &\quad \leq L\left\{\sum_{k=1}^n |\dot{x}(t-r_k(t))|^2[1 - r_k'(t)]\right\}^{1/2} \\ &\qquad \times \left\{\sum_{k=1}^n [1 - r_k'(t)]\right\}^{1/2} \\ &\quad < (2n)^{1/2} L\left\{\sum_{k=1}^n |\dot{x}(t-r_k(t))|^2[1 - r_k'(t)]\right\}^{1/2}. \end{aligned} \quad (13)$$

Raising both sides of inequality (13) to power 2 and integrating both sides from 0 to T, we have

$$\begin{aligned} &\int_0^T |\ddot{x}(t)|^2 dt \\ &\quad < 2nL^2 \int_0^T \sum_{k=1}^n |\dot{x}(t-r_k(t))|^2[1 - r_k'(t)]dt. \end{aligned} \quad (14)$$

Since $\mu_k(s)$ is the inverse of $t - r_k(t)$, by using Lemma 2, we have

$$\begin{aligned}
&\int_0^T |\ddot{x}(t)|^2 dt \\
&\quad < 2nL^2 \sum_{k=1}^{n} \int_0^T \left|\dot{x}\left(t - r_k(t)\right)\right|^2 \left[1 - r_k'(t)\right] dt \\
&\quad = 2nL^2 \sum_{k=1}^{n} \int_{-r_k(0)}^{T-r_k(T)} |\dot{x}(s)|^2 \\
&\qquad\qquad \times \left[1 - r_k'\left(\mu_k(s)\right)\right] \frac{1}{1 - r_k'\left(\mu_k(s)\right)} ds \\
&\quad = 2nL^2 \sum_{k=1}^{n} \int_{-r_k(0)}^{T-r_k(T)} |\dot{x}(s)|^2 ds \\
&\quad = 2nL^2 \sum_{k=1}^{n} \int_0^T |\dot{x}(s)|^2 ds \\
&\quad = 2n^2L^2 \int_0^T |\dot{x}(s)|^2 ds.
\end{aligned} \tag{15}$$

That is,

$$\|\ddot{x}\| < \sqrt{2}nL \|\dot{x}\|. \tag{16}$$

Since $x(t) = x(t+T)$, obviously, $\int_0^T \dot{x}(t)dt = 0$. By Lemma 3, we have $2\pi\|\dot{x}\| \le T\|\ddot{x}\|$. So

$$T > \frac{\sqrt{2}\pi}{nL}. \tag{17}$$

□

Remark 5. When delay $r_k(t) = kr, k = 1, 2, \ldots, n$, from the second inequality of (13), we can easily obtain Theorem 1 in [18].

We can easily obtain the following result.

Corollary 6. *Let x be a nonconstant T-periodic solution of the nonautonomous delay differential equation* (2) *and $x \in S$. Suppose that the function $f : R^p \to R^p$ is L-Lipschitz continuous and $r_k(t) = r_k(t+T)$, $r_k(t) > 0$, $-1/n \le r_k'(t) < 1$. Then $T \ge 2\pi/\sqrt{n(n+1)}L$.*

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