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Research Article

The Traveling Wave Solutions and Their Bifurcations for the BBM-Like $B(m,n)$ Equations

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We investigate the traveling wave solutions and their bifurcations for the BBM-like $B(m,n)$ equations $u_t+\alpha u_x+\beta(u^m)_x-\gamma(u^n)_{xxt}=0$ by using bifurcation method and numerical simulation approach of dynamical systems. Firstly, for BBM-like $B(3,2)$ equation, we obtain some precise expressions of traveling wave solutions, which include periodic blow-up and periodic wave solution, peakon and periodic peakon wave solution, and solitary wave and blow-up solution. Furthermore, we reveal the relationships among these solutions theoretically. Secondly, for BBM-like $B(4,2)$ equation, we construct two periodic wave solutions and two blow-up solutions. In order to confirm the correctness of these solutions, we also check them by software Mathematica.

1. Introduction

In recent years, the nonlinear phenomena exist in all fields including either the scientific work or engineering fields, such as fluid mechanics, plasma physics, optical fibers, biology, solid-state physics, chemical kinematics, and chemical physics. Many nonlinear evolution equations are playing important roles in the analysis of the phenomena.

In order to find the traveling wave solutions of these nonlinear evolution equations, there have been many methods, such as Jacobi elliptic function method [1, 2], F-expansion and extended F-expansion method [3, 4], (G'/G)-expansion method [5, 6], and the bifurcation method of dynamical systems [7–11].

BBM equation or regularized long-wave equation (RLW equation)

$$u_t + uu_x - u_{xxt} = 0 \tag{1}$$

was derived by Peregine [12, 13] and Benjamin et al. [14] as an alternative model to Korteweg-de Vries equation for small-amplitude, long wavelength surface water waves.

There are various generalized form related to (1). Shang [15] introduced a family of BBM-like equations with nonlinear dispersion

$$u_t + (u^m)_x - (u^n)_{xxt} = 0, \quad m,n > 1, \tag{2}$$

which were called BBM-like $B(m,n)$ equations as alternative model to the nonlinear dispersive $K(m,n)$ equations [16–18]. He presented a method called the extend sine-cosine method to seek exact solitary-wave solutions with compact support and exact special solutions with solitary patterns of (2).

When $m = n = 2$, (2) reduces to the BBM-like $B(2,2)$ equation

$$u_t + (u^2)_x - (u^2)_{xxt} = 0. \tag{3}$$

Jiang et al. [19] employed the bifurcation method of dynamical systems to investigate (3). Under different parametric conditions, they gave various sufficient conditions to guarantee the existence of smooth and nonsmooth traveling wave solutions. Furthermore, through some special phase orbits, they obtained some solitary wave solutions expressed by implicit functions, periodic cusp wave solution, compacton solution, and peakon solution.

Wazwaz [20] introduced a system of nonlinear variant RLW equations

$$u_t + au_x - k(u^n)_x + b(u^n)_{xxt} = 0 \tag{4}$$

and derived some compact and noncompact exact solutions by using the sine-cosine method and tanh method.

Feng et al. [21] studied the following generalized variant RLW equations:

$$u_t + au_x - k(u^m)_x + b(u^n)_{xxt} = 0. \tag{5}$$

By using four different ansatzs, they obtained some exact solutions such as compactons, solitary pattern solutions, solitons, and periodic solutions.

Kuru [22–24] considered the following BBM-like equations with a fully nonlinear dispersive term:

$$u_t + u_x + a(u^m)_x - (u^n)_{xxt} = 0, \quad m, n > 1. \tag{6}$$

By means of the factorization technique, he obtained the traveling wave solutions of (6) in terms of the Weierstrass functions.

In the present paper, we use the bifurcation method and numerical simulation approach of dynamical systems to study the following BBM-like $B(m,n)$ equations:

$$u_t + \alpha u_x + \beta(u^m)_x - \gamma(u^n)_{xxt} = 0 \quad m, n > 1. \tag{7}$$

For BBM-like $B(3,2)$ equation, we obtain some precise expressions of traveling wave solutions, which include periodic blow-up and periodic wave solution, peakon and periodic peakon wave solution, and solitary wave and blow-up solution. We also reveal the relationships among these solutions theoretically. For BBM-like $B(4,2)$ equation, we construct two elliptic periodic wave solutions and two hyperbolic blow-up solutions.

This paper is organized as follows. In Section 2, we state our main results which are included in two propositions. In Sections 3 and 4, we give the derivations for the two propositions, respectively. A brief conclusion is given in Section 5.

2. Main Results and Remarks

In this section, we list our main results and give some remarks. Firstly, let us recall some symbols. The symbols sn u and cn u denote the Jacobian elliptic functions sine amplitude u and cosine amplitude u. cosh u, sinh u, sech u, and csch u are the hyperbolic functions. Secondly, for the sake of simplification, we only consider the case $\alpha > 0$, $\beta > 0$, and $\gamma > 0$ (the other cases can be considered similarly). To relate conveniently, for given constant wave speed c, let

$$\xi = x - ct, \qquad g_0 = \frac{4}{27}\sqrt{\frac{5|c-\alpha|^3}{\beta}}. \tag{8}$$

Proposition 1. *Consider BBM-like $B(3,2)$ equation*

$$u_t + \alpha u_x + \beta(u^3)_x - \gamma(u^2)_{xxt} = 0 \tag{9}$$

and its traveling wave equation

$$(\alpha - c)\varphi + \beta\varphi^3 + 2\gamma c(\varphi')^2 + 2\gamma c\varphi\varphi'' = g. \tag{10}$$

For given constants c and g, there are the following results.

(1) When $0 < c < \alpha$ or $c > \alpha$, $g < -g_0$, (9) has two elliptic periodic blow-up solutions

$$u_1(\xi) = \varphi_1 + A_1 - \frac{2A_1}{1 - \mathrm{cn}(\eta_1\xi, k_1)}, \qquad u_2(\xi) = \varphi_1 + A_1 - \frac{2A_1}{1 + \mathrm{cn}(\eta_1\xi, k_1)}, \tag{11}$$

where

$$A_1 = \sqrt{(\varphi_1 - \varphi_2)(\varphi_1 - \varphi_3)}, \tag{12}$$

$$\eta_1 = \sqrt{\frac{\beta A_1}{5\gamma c}}, \tag{13}$$

$$k_1 = \sqrt{\frac{2A_1 + 2\varphi_1 - \varphi_2 - \varphi_3}{4A_1}}, \tag{14}$$

$$\varphi_1 = \frac{2\sqrt[3]{100}\beta(\alpha - c) - \sqrt[3]{10}\Omega^{2/3}}{6\beta\Omega^{1/3}}, \tag{15}$$

$$\varphi_2 = \frac{\sqrt[3]{5}\left(2\beta\sqrt[3]{10}\left(1 - \sqrt{3}i\right)(c-\alpha) + \left(1 + \sqrt{3}i\right)\Omega^{2/3}\right)}{6\sqrt[3]{4}\beta\Omega^{1/3}}, \tag{16}$$

$$\varphi_3 = \frac{\sqrt[3]{5}\left(2\beta\sqrt[3]{10}\left(1 + \sqrt{3}i\right)(c-\alpha) + \left(1 - \sqrt{3}i\right)\Omega^{2/3}\right)}{6\sqrt[3]{4}\beta\Omega^{1/3}}, \tag{17}$$

$$\Omega = \sqrt{729g^2\beta^4 - 80(c-\alpha)^3\beta^3} - 27g\beta^2. \tag{18}$$

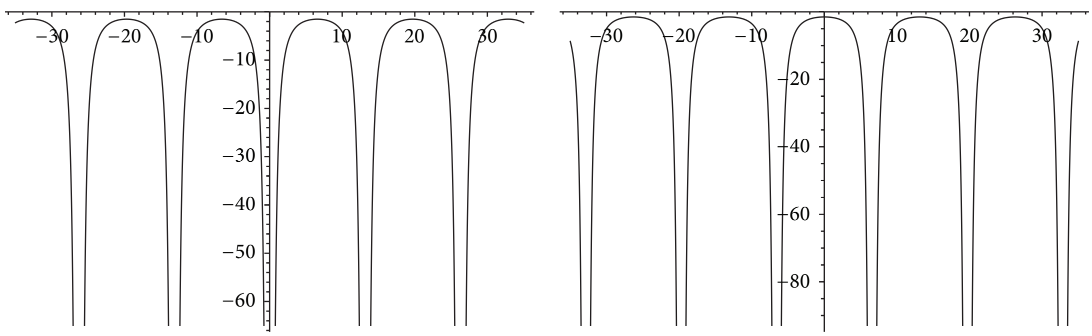
For the graphs of $u_1(\xi)$ and $u_2(\xi)$, see Figures 1(a) and 1(b).

(2) When $c > \alpha$ and $g = -g_0$, (9) has two trigonometric periodic blow-up solutions

$$u_3(\xi) = \sqrt{\frac{5(c-\alpha)}{9\beta}}\left(1 - 3\csc^2\left(\sqrt{\frac{\beta}{\gamma c}}\sqrt[4]{\frac{c-\alpha}{80\beta}}\xi\right)\right), \tag{19}$$

$$u_4(\xi) = \sqrt{\frac{5(c-\alpha)}{9\beta}}\left(1 - 3\sec^2\left(\sqrt{\frac{\beta}{\gamma c}}\sqrt[4]{\frac{c-\alpha}{80\beta}}\xi\right)\right). \tag{20}$$

The graphs of $u_3(\xi)$ and $u_4(\xi)$ are similar to Figure 1.



(a) $u = u_1(\xi)$

(b) $u = u_2(\xi)$.

Figure 1: The graphs of $u_1(\xi)$ and $u_2(\xi)$ when $\alpha = \beta = \gamma = 1$, $c = 2$, and $g = -2\sqrt{3}/9$.

(3) When $c > \alpha$ and $-g_0 < g < g_0$, (9) has two elliptic periodic blow-up solutions $u_5(\xi)$, $u_6(\xi)$ and two symmetric elliptic periodic wave solutions $u_7(\xi)$, $u_8(\xi)$

$$\begin{gathered} u_5(\xi) = \varphi_3 - (\varphi_3 - \varphi_1)\,\mathrm{sn}^{-2}(\eta_2\xi, k_2), \\ u_6(\xi) = \frac{\varphi_1 - \varphi_2 \mathrm{sn}^2(\eta_2\xi, k_2)}{1 - \mathrm{sn}^2(\eta_2\xi, k_2)}, \\ u_7(\xi) = \varphi_3 - (\varphi_3 - \varphi_2)\,\mathrm{sn}^2(\eta_2\xi, k_2), \\ u_8(\xi) = \frac{\varphi_2 - \varphi_1 k_2^2 \mathrm{sn}^2(\eta_2\xi, k_2)}{1 - k_2^2 \mathrm{sn}^2(\eta_2\xi, k_2)}, \end{gathered} \tag{21}$$

where

$$\eta_2 = \frac{1}{2}\sqrt{\frac{\beta(\varphi_3 - \varphi_1)}{5\gamma c}}, \qquad k_2 = \sqrt{\frac{\varphi_3 - \varphi_2}{\varphi_3 - \varphi_1}}. \tag{22}$$

For the graphs of $u_i(\xi)$ ($i = 5$–8), see Figures 2–4.

(4) When $c > \alpha$ and $g = g_0$, (9) has a hyperbolic smooth solitary wave solution

$$u_9(\xi) = -\sqrt{\frac{5(c-\alpha)}{9\beta}}\left(1 - 3\,\mathrm{sech}^2\left(\sqrt{\frac{\beta}{\gamma c}}\sqrt[4]{\frac{c-\alpha}{80\beta}}\,\xi\right)\right), \tag{23}$$

a hyperbolic blow-up solution

$$u_{10}(\xi) = -\sqrt{\frac{5(c-\alpha)}{9\beta}}\left(1 + 3\,\mathrm{csch}^2\left(\sqrt{\frac{\beta}{\gamma c}}\sqrt[4]{\frac{c-\alpha}{80\beta}}\,\xi\right)\right), \tag{24}$$

a hyperbolic peakon wave solution

$$u_{11}(\xi) = -\sqrt{\frac{5(c-\alpha)}{9\beta}} \times \left(1 - 3\,\mathrm{sech}^2\left(\delta_1 - \sqrt{\frac{\beta}{\gamma c}}\sqrt[4]{\frac{c-\alpha}{80\beta}}\,|\xi|\right)\right), \tag{25}$$

and a hyperbolic periodic peakon wave solution

$$\begin{gathered} u_{12}(\xi) = -\sqrt{\frac{5(c-\alpha)}{9\beta}} \times \left(1 - 3\,\mathrm{sech}^2\left(\delta_1 + \sqrt{\frac{\beta}{\gamma c}}\sqrt[4]{\frac{c-\alpha}{80\beta}}\,|\xi|\right)\right), \\ \xi \in [(2n-1)T, (2n+1)T), \end{gathered} \tag{26}$$

where

$$\begin{gathered} \delta_1 = \frac{1}{2}\ln\left(5 - 2\sqrt{6}\right), \\ T = \sqrt{\frac{5\gamma c}{\beta}}\sqrt[4]{\frac{\beta}{5(c-\alpha)}}\left|\ln\left(5 - 2\sqrt{6}\right)\right|, \\ n = 0, \pm 1, \pm 2, \ldots. \end{gathered} \tag{27}$$

For the graphs of $u_9(\xi)$ and $u_{10}(\xi)$, see Figures 4(d) and 3(d). For the graphs of $u_{11}(\xi)$ and $u_{12}(\xi)$, see Figures 7(a) and 7(b).

(5) When $c > \alpha$ and $g > g_0$, (9) has two elliptic periodic blow-up solutions

$$\begin{gathered} u_{13}(\xi) = \varphi_3 + A_2 - \frac{2A_2}{1 - \mathrm{cn}(\eta_3\xi, k_3)}, \\ u_{14}(\xi) = \varphi_3 + A_2 - \frac{2A_2}{1 + \mathrm{cn}(\eta_3\xi, k_3)}, \end{gathered} \tag{28}$$

where

$$\begin{gathered} A_2 = \sqrt{(\varphi_3 - \varphi_2)(\varphi_3 - \varphi_1)}, \\ \eta_3 = \sqrt{\frac{\beta A_2}{5\gamma c}}, \\ k_3 = \sqrt{\frac{2A_2 + 2\varphi_3 - \varphi_2 - \varphi_1}{4A_2}}. \end{gathered} \tag{29}$$

For the graphs of $u_{13}(\xi)$ and $u_{14}(\xi)$, see Figures 5(a) and 6(a).

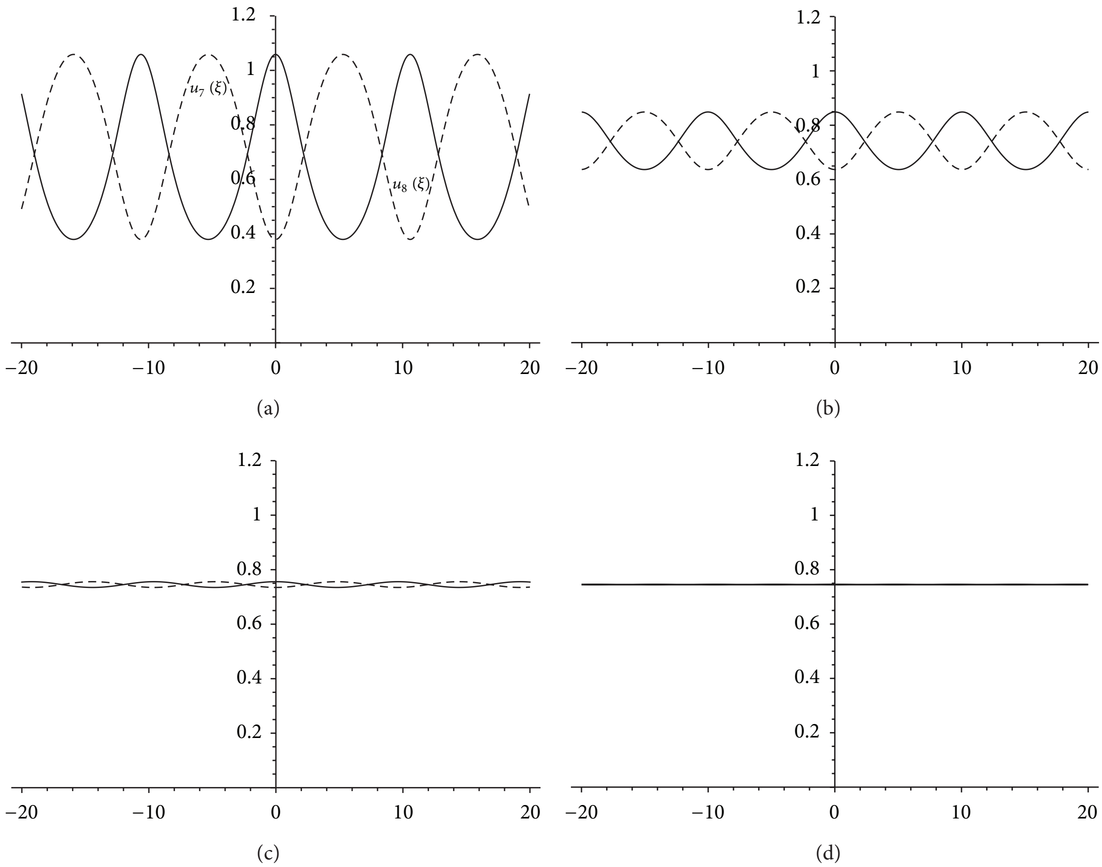


FIGURE 2: The varying process for graphs of $u_7(\xi)$ and $u_8(\xi)$ when $c > \alpha$ and $g \to -g_0 + 0$ where $\alpha = \beta = \gamma = 1$, $c = 2$, and (a) $g = -g_0 + 10^{-1}$. (b) $g = -g_0 + 10^{-2}$. (c) $g = -g_0 + 10^{-4}$. (d) $g = -g_0 + 10^{-6}$.

(6) When $c < 0$, (9) has two elliptic periodic blow-up solutions

$$u_{15}(\xi) = \varphi_1 - A_1 + \frac{2A_1}{1 - \mathrm{cn}(\eta_4 \xi, k_4)},$$
$$u_{16}(\xi) = \varphi_1 - A_1 + \frac{2A_1}{1 + \mathrm{cn}(\eta_4 \xi, k_4)}, \tag{30}$$

$$\eta_4 = \sqrt{\frac{\beta A_1}{-5\gamma c}},$$
$$k_4 = \sqrt{\frac{2A_1 - 2\varphi_1 + \varphi_2 + \varphi_3}{4A_1}}. \tag{31}$$

For the graphs of $u_{15}(\xi)$ and $u_{16}(\xi)$, see Figures 7(c) and 7(d).

Remark 2. When $c > \alpha$ and $g \to -g_0 - 0$, the elliptic periodic blow-up solutions $u_1(\xi)$ and $u_2(\xi)$ become the trigonometric periodic blow-up solutions $u_3(\xi)$ and $u_4(\xi)$, respectively.

Remark 3. When $c > \alpha$ and $g \to -g_0 + 0$, the elliptic periodic blow-up solutions $u_5(\xi)$ and $u_6(\xi)$ become the trigonometric periodic blow-up solutions $u_3(\xi)$ and $u_4(\xi)$, respectively. The symmetric elliptic periodic wave solutions $u_7(\xi)$ and $u_8(\xi)$ become a trivial solution $u(\xi) = \sqrt{5(c-\alpha)/9\beta}$, and for the varying process, see Figure 2.

Remark 4. When $c > \alpha$ and $g \to g_0 - 0$, the elliptic periodic blow-up solution $u_5(\xi)$ becomes the hyperbolic blow-up solution $u_{10}(\xi)$, for the varying process, see Figure 3. The elliptic periodic wave solutions $u_7(\xi)$ become the hyperbolic smooth solitary wave solution $u_9(\xi)$, and for the varying process, see Figure 4. The elliptic solutions $u_6(\xi)$ and $u_8(\xi)$ become a trivial solution $u(\xi) = -\sqrt{5(c-\alpha)/9\beta}$.

Remark 5. When $c > \alpha$ and $g \to g_0 + 0$, the elliptic periodic blow-up solution $u_{13}(\xi)$ becomes the hyperbolic blow-up solution $u_{10}(\xi)$, and for the varying process, see Figure 5. The elliptic periodic blow-up solution $u_{14}(\xi)$ becomes the hyperbolic smooth solitary wave solution $u_9(\xi)$, and for the varying process, see Figure 6.

Proposition 6. *Consider BBM-like* $B(4,2)$ *equation*

$$u_t + \alpha u_x + \beta\left(u^4\right)_x - \gamma\left(u^2\right)_{xxt} = 0 \tag{32}$$

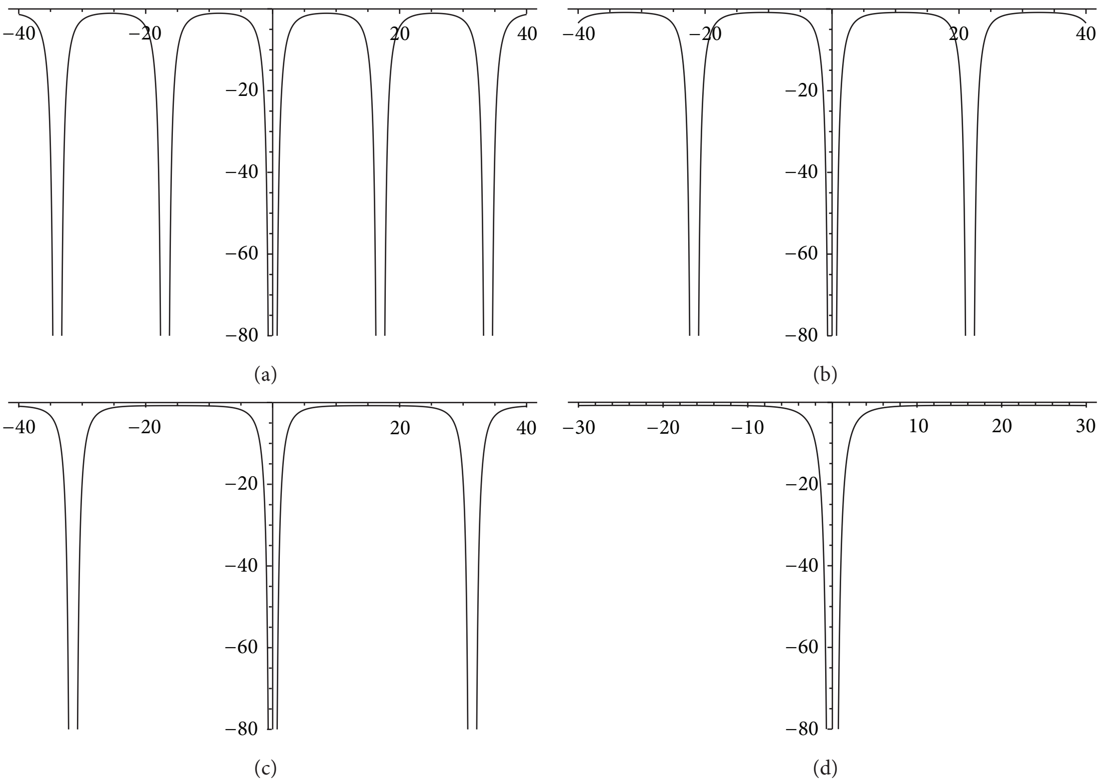


Figure 3: The varying process for graphs of $u_5(\xi)$ when $c > \alpha$ and $g \rightarrow g_0 - 0$ where $\alpha = \beta = \gamma = 1$, $c = 2$, and (a) $g = g_0 - 10^{-1}$. (b) $g = g_0 - 10^{-2}$. (c) $g = g_0 - 10^{-4}$. (d) $g = g_0 - 10^{-6}$.

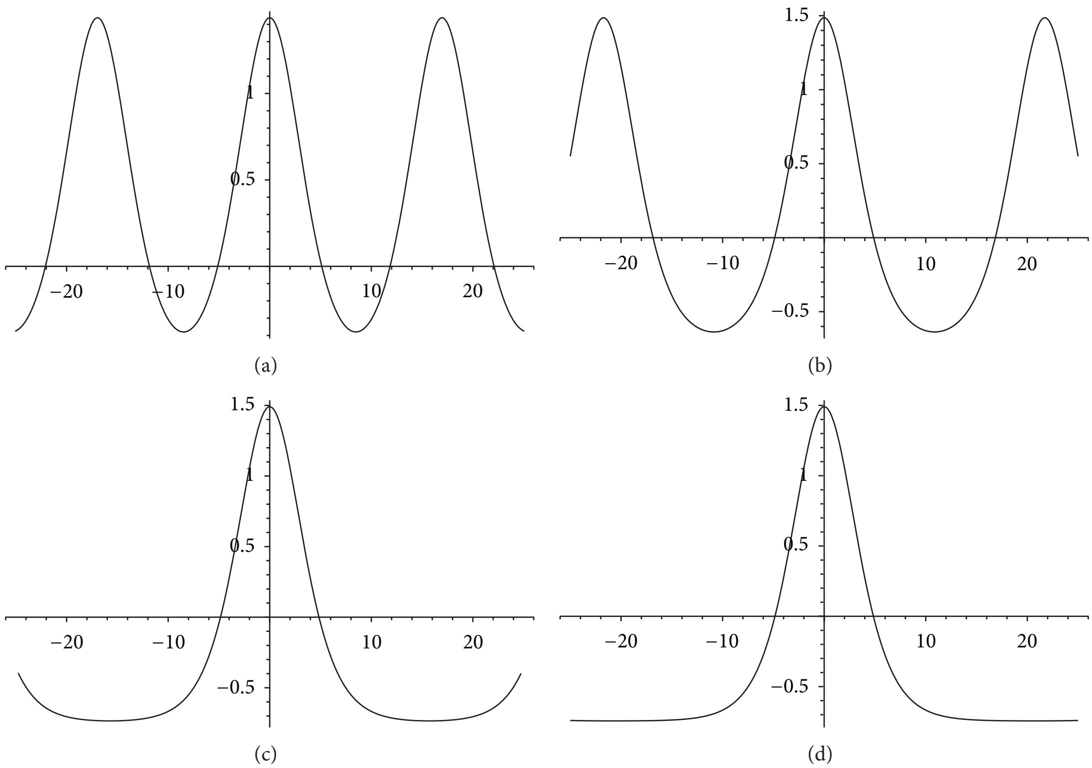


Figure 4: The varying process for graphs of $u_7(\xi)$ when $c > \alpha$ and $g \rightarrow g_0 - 0$ where $\alpha = \beta = \gamma = 1$, $c = 2$, and (a) $g = g_0 - 10^{-1}$. (b) $g = g_0 - 10^{-2}$. (c) $g = g_0 - 10^{-4}$. (d) $g = g_0 - 10^{-6}$.

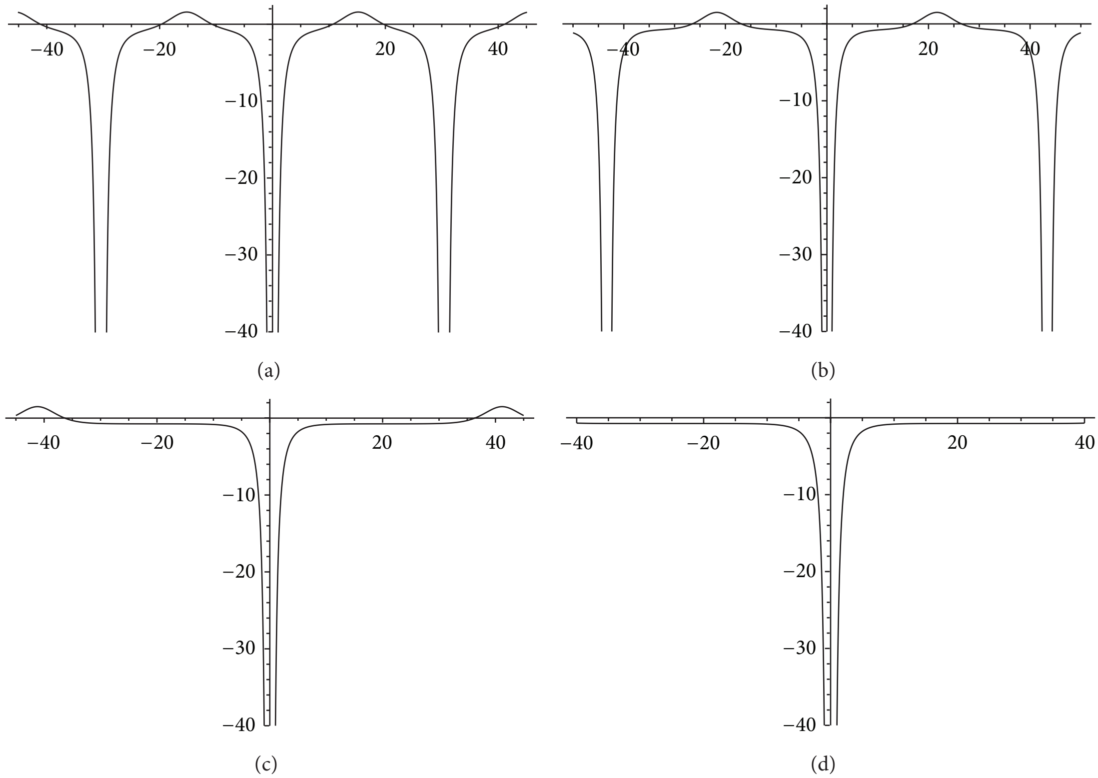


FIGURE 5: The varying process for graphs of $u_{13}(\xi)$ when $c > \alpha$ and $g \rightarrow g_0 + 0$ where $\alpha = \beta = \gamma = 1$, $c = 2$, and (a) $g = g_0 + 1/5$. (b) $g = g_0 + 10^{-2}$. (c) $g = g_0 + 10^{-6}$. (d) $g = g_0 + 10^{-8}$.

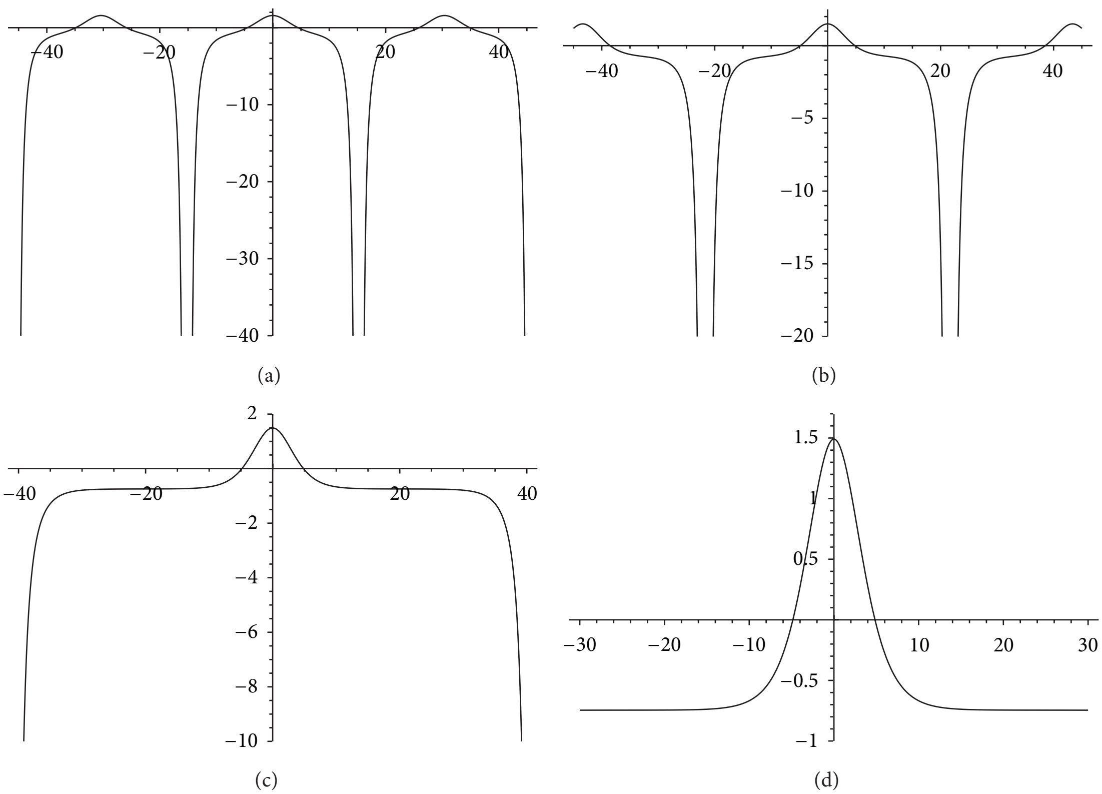
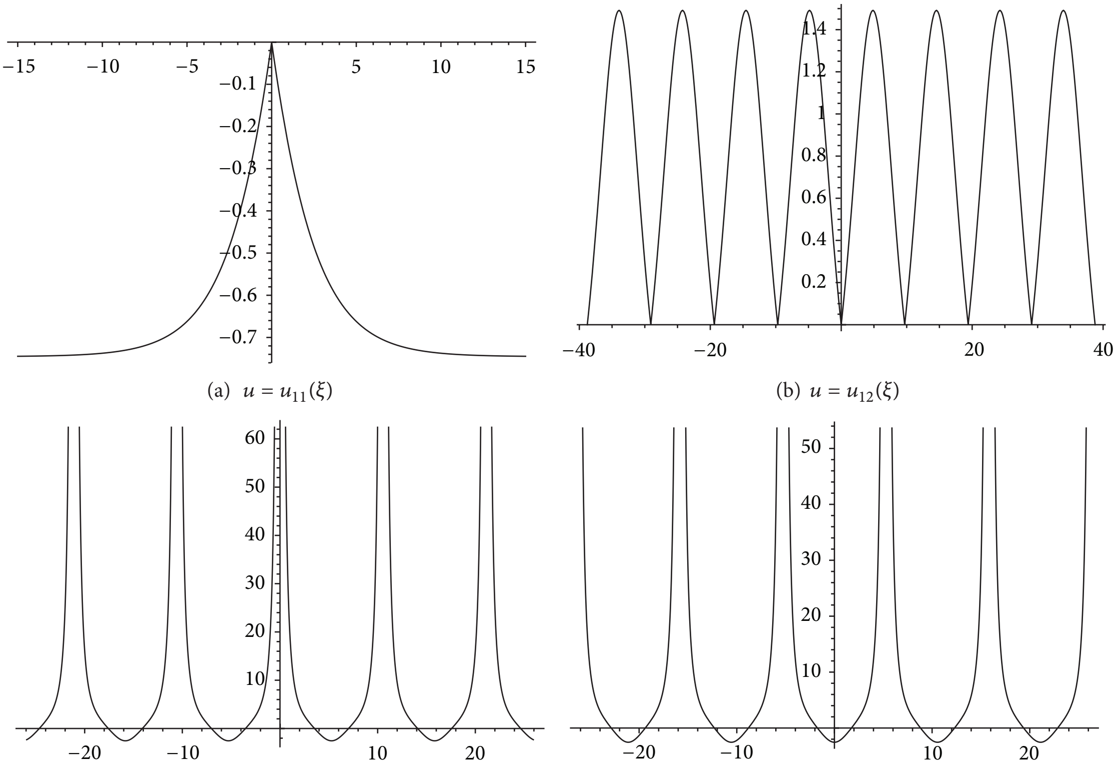


FIGURE 6: The varying process for graphs of $u_{14}(\xi)$ when $c > \alpha$ and $g \rightarrow g_0 + 0$ where $\alpha = \beta = \gamma = 1$, $c = 2$, and (a) $g = g_0 + 1/5$. (b) $g = g_0 + 10^{-2}$. (c) $g = g_0 + 10^{-6}$. (d) $g = g_0 + 10^{-8}$.



(a) $u = u_{11}(\xi)$ (b) $u = u_{12}(\xi)$ (c) $u = u_{15}(\xi)$ (d) $u = u_{16}(\xi)$

Figure 7: The graphs of $u_{11}(\xi)$, $u_{12}(\xi)$ when $c = 2$, $g = 4\sqrt{5}/27$ and $u_{15}(\xi)$, $u_{16}(\xi)$ when $c = -1$, $g = -10$.

and its traveling wave equation

$$(\alpha - c)\,\psi + \beta\psi^4 + 2\gamma c\left(\psi'\right)^2 + 2\gamma c\psi\psi'' = g. \tag{33}$$

For given constants c and g, there are the following results.

(1°) When $c > \alpha$ and $g = 0$, (32) has two elliptic periodic wave solutions

$$u_{17}(\xi) = \frac{1 - \operatorname{cn}(\eta_5\xi, k_5)}{\sqrt[3]{\beta/2(c-\alpha)}\left(1 + \sqrt{3} + \left(\sqrt{3} - 1\right)\operatorname{cn}(\eta_5\xi, k_5)\right)},$$
$$u_{18}(\xi) = \frac{1 + \operatorname{cn}(\eta_5\xi, k_5)}{\sqrt[3]{\beta/2(c-\alpha)}\left(1 + \sqrt{3} - \left(\sqrt{3} - 1\right)\operatorname{cn}(\eta_5\xi, k_5)\right)}, \tag{34}$$

where

$$\eta_5 = \sqrt[4]{3}\sqrt{\frac{c-\alpha}{3\gamma c}}\sqrt[6]{\frac{\beta}{2(c-\alpha)}},$$
$$k_5 = \frac{\sqrt{3} - 1}{2\sqrt{2}}. \tag{35}$$

(2°) When $c < 0$ and $g = -\sqrt[3]{(c-\alpha)^4/16\beta}$, (32) has two hyperbolic blow-up solutions

$$u_{19}(\xi) = \frac{\sqrt[3]{c-\alpha}\left(-4 - 2\cosh(\eta_6\xi) + \sqrt{6}\sinh(\eta_6|\xi|)\right)}{\sqrt[3]{2\beta}\left(2 - 2\cosh(\eta_6\xi) + \sqrt{6}\sinh(\eta_6|\xi|)\right)},$$
$$u_{20}(\xi) = \frac{\sqrt[3]{c-\alpha}\left(4 + 2\cosh(\eta_6\xi) + \sqrt{6}\sinh(\eta_6|\xi|)\right)}{\sqrt[3]{2\beta}\left(-2 + 2\cosh(\eta_6\xi) + \sqrt{6}\sinh(\eta_6|\xi|)\right)}, \tag{36}$$

where

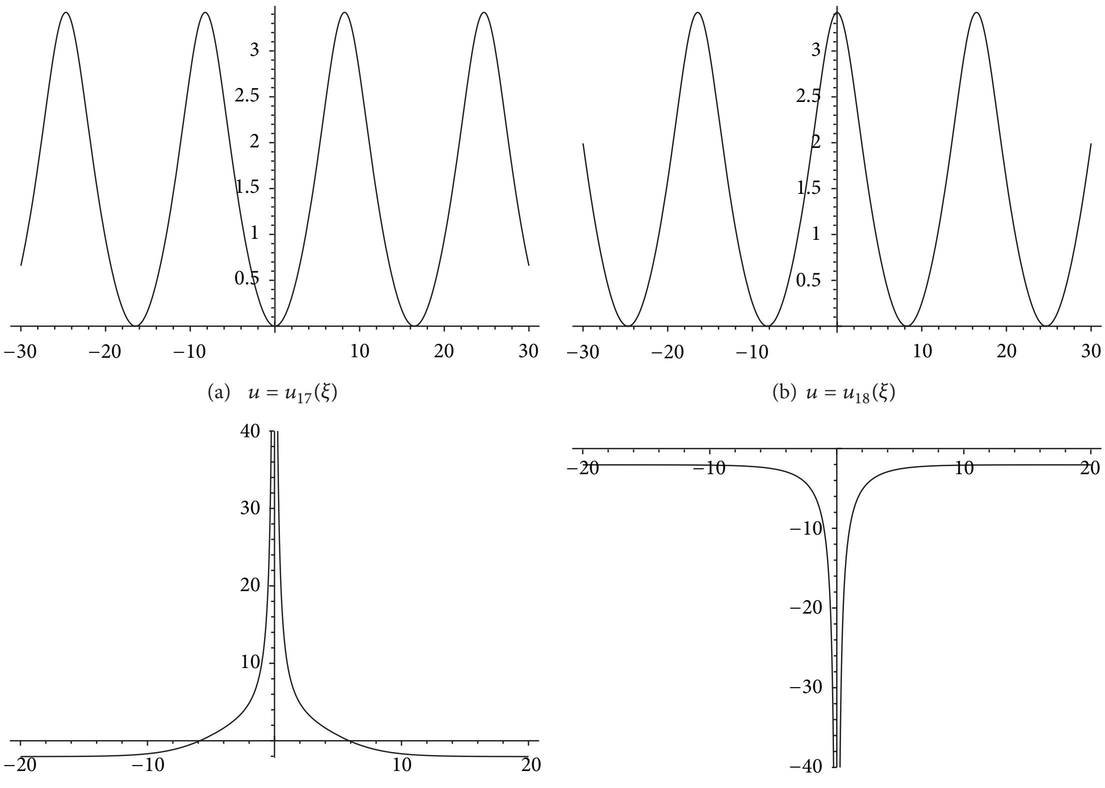
$$\eta_6 = -\sqrt{\frac{\beta}{-\gamma c}\sqrt[3]{\frac{c-\alpha}{2\beta}}}. \tag{37}$$

For the graphs of $u_i(\xi)$ ($i = 17$–20), see Figure 8.

Remark 7. In order to confirm the correctness of these solutions, we have verified them by using the software Mathematica; for instance, about $u_{20}(\xi)$ the commands are as follows:

$$\xi = x - ct$$

$$\eta = -\sqrt{\frac{\beta}{-\gamma c}\frac{\sqrt[3]{c-\alpha}}{\sqrt[3]{2\beta}}}$$



(a) $u = u_{17}(\xi)$ (b) $u = u_{18}(\xi)$ (c) $u = u_{19}(\xi)$ (d) $u = u_{20}(\xi)$

FIGURE 8: The graphs of $u_{17}(\xi)$, $u_{18}(\xi)$ when $c = 20$, $g = 0$ and $u_{19}(\xi)$, $u_{20}(\xi)$ when $c = -16$.

$$u = \frac{\sqrt[3]{c-\alpha}\left(4 + 2\cosh[\eta\xi] + \sqrt{6}\sinh[\eta\xi]\right)}{\sqrt[3]{2\beta}\left(-2 + 2\cosh[\eta\xi] + \sqrt{6}\sinh[\eta\xi]\right)}.$$

$$\begin{aligned}&\text{Simplify}\left[D[u,t] + \alpha D[u,x]\right.\\ &\qquad \left. + \beta D\left[u^4, x\right] - \gamma D\left[u^2, \{x,2\}, t\right]\right]\\ &\qquad 0.\end{aligned} \tag{38}$$

3. The Derivations for Proposition 1

In this section, firstly, we derive the precise expressions of the traveling wave solutions for BBM-like $B(3,2)$ equation. Secondly we show the relationships among these solutions theoretically. Substituting $u(x,t) = \varphi(\xi)$ with $\xi = x - ct$ into (9), it follows that

$$(\alpha - c)\varphi' + \beta\left(\varphi^3\right)' + \gamma c\left(\varphi^2\right)''' = 0. \tag{39}$$

Integrating (39) once, we have

$$(\alpha - c)\varphi + \beta\varphi^3 + 2\gamma c\left(\varphi'\right)^2 + 2\gamma c\varphi\varphi'' = g, \tag{40}$$

where g is an integral constant.

Letting $y = \varphi'$, we obtain the following planar system

$$\begin{aligned}\frac{d\varphi}{d\xi} &= y,\\ \frac{dy}{d\xi} &= \frac{g + (c-\alpha)\varphi - \beta\varphi^3 - 2\gamma c y^2}{2\gamma c\varphi}.\end{aligned} \tag{41}$$

Under the transformation

$$d\xi = 2\gamma c\varphi\, d\tau, \tag{42}$$

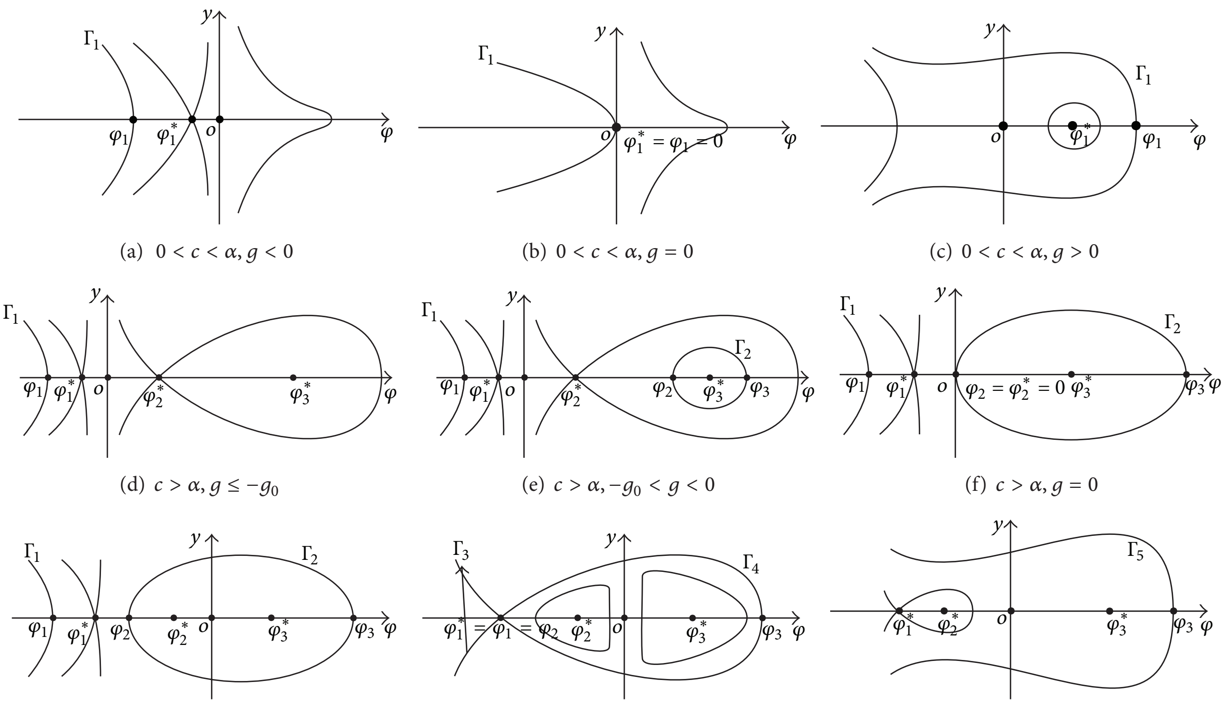
system (41) becomes

$$\begin{aligned}\frac{d\varphi}{d\tau} &= 2\gamma c\varphi y,\\ \frac{dy}{d\tau} &= g + (c-\alpha)\varphi - \beta\varphi^3 - 2\gamma c y^2.\end{aligned} \tag{43}$$

Clearly, system (41) and system (43) have the same first integral

$$\gamma c\varphi^2 y^2 - \frac{g}{2}\varphi^2 - \frac{c-\alpha}{3}\varphi^3 + \frac{\beta}{5}\varphi^5 = h, \tag{44}$$

where h is an integral constant. Consequently, these two systems have the same topological phase portraits except for

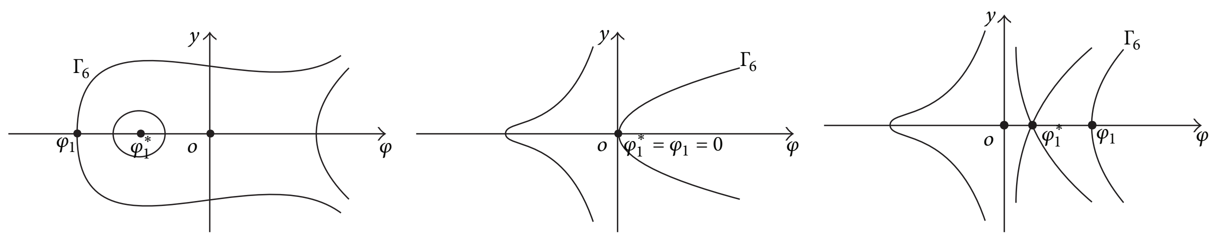


(a) $0 < c < \alpha, g < 0$ (b) $0 < c < \alpha, g = 0$ (c) $0 < c < \alpha, g > 0$

(d) $c > \alpha, g \le -g_0$ (e) $c > \alpha, -g_0 < g < 0$ (f) $c > \alpha, g = 0$

(g) $c > \alpha, 0 < g < g_0$ (h) $c > \alpha, g = g_0$ (i) $c > \alpha, g > g_0$

FIGURE 9: The phase portraits of system (43) when $c > 0$ and $c \neq \alpha$.



(a) $g < 0$ (b) $g = 0$ (c) $g > 0$

FIGURE 10: The phase portraits of system (43) when $c < 0$.

the straight line $\varphi = 0$. Thus, we can understand the phase portraits of system (41) from that of system (43).

When the integral constant $h = 0$, (44) becomes

$$\gamma c \varphi^2 y^2 - \varphi^2 \left(\frac{g}{2} + \frac{c-\alpha}{3}\varphi - \frac{\beta}{5}\varphi^3 \right) = 0. \tag{45}$$

Solving equation $g/2 + ((c-\alpha)/3)\varphi - (\beta/5)\varphi^3 = 0$, we get three roots φ_1, φ_2, and φ_3 as (15), (16), and (17).

On the other hand, solving equation

$$\begin{gathered} y = 0, \\ g + (c-\alpha)\varphi - \beta\varphi^3 - 2\gamma c y^2 = 0, \end{gathered} \tag{46}$$

we get three three singular points $(\varphi_i^*, 0)$ $(i = 1, 2, 3)$, where

$$\begin{gathered} \varphi_1^* = \frac{2\beta\sqrt[3]{18}(\alpha - c) - \sqrt[3]{12}\Delta^{2/3}}{6\beta\Delta^{1/3}}, \\ \varphi_2^* = \frac{2\beta\sqrt[3]{2}\sqrt[6]{3}\left(\sqrt{3} - 3i\right)(c-\alpha) + \sqrt[3]{4}\sqrt[6]{9}\left(1 + \sqrt{3}i\right)\Delta^{2/3}}{12\beta\Delta^{1/3}}, \\ \varphi_3^* = \frac{2\beta\sqrt[3]{2}\sqrt[6]{3}\left(\sqrt{3} + 3i\right)(c-\alpha) + \sqrt[3]{4}\sqrt[6]{9}\left(1 - \sqrt{3}i\right)\Delta^{2/3}}{12\beta\Delta^{1/3}}, \\ \Delta = \sqrt{81g^2\beta^4 - 12(c-\alpha)^3\beta^3} - 9g\beta^2. \end{gathered} \tag{47}$$

According to the qualitative theory, we obtain the phase portraits of system (43) as Figures 9 and 10.

From Figures 9 and 10, one can see that there are six kinds of orbits Γ_i $(i = 1\text{–}6)$, where $\Gamma_1(\Gamma_6)$ passes $(\varphi_1, 0)$, Γ_2 passes $(\varphi_2, 0)$ and $(\varphi_3, 0)$, Γ_3 and Γ_4 pass $(\varphi_1, 0)$ and $(\varphi_3, 0)$, and Γ_5 passes $(\varphi_3, 0)$. Now, we will derive the explicit expressions of solutions for the BBM-like $B(3,2)$ equation, respectively.

(1) When $0 < c < \alpha$ or $c > \alpha$, $g < -g_0$, Γ_1 has the expression

$$\Gamma_1 : y = \pm\sqrt{\frac{\beta}{5\gamma c}}\sqrt{(\varphi_1-\varphi)(\varphi-\varphi_2)(\varphi-\varphi_3)}, \quad \varphi \le \varphi_1, \tag{48}$$

where φ_2 and φ_3 are complex numbers.

Substituting (48) into $d\varphi/d\xi = y$ and integrating it, we have

$$\begin{gathered}\int_{-\infty}^{\varphi}\frac{ds}{\sqrt{(\varphi_1-s)(s-\varphi_2)(s-\varphi_3)}} = \sqrt{\frac{\beta}{5\gamma c}}|\xi|, \\ \int_{\varphi}^{\varphi_1}\frac{ds}{\sqrt{(\varphi_1-s)(s-\varphi_2)(s-\varphi_3)}} = \sqrt{\frac{\beta}{5\gamma c}}|\xi|.\end{gathered} \tag{49}$$

Completing the integrals in the above two equations and noting that $u = \varphi(\xi)$, we obtain $u_1(\xi)$ and $u_2(\xi)$ as (11).

(2) When $c > \alpha$ and $g = -g_0$, Γ_1 has the expression

$$\Gamma_1 : y = \pm\sqrt{\frac{\beta}{5\gamma c}}(\varphi_2-\varphi)\sqrt{\varphi_1-\varphi}, \quad \varphi \le \varphi_1, \tag{50}$$

where $\varphi_1 = -2\sqrt{5(c-\alpha)/9\beta}$, $\varphi_2 = \sqrt{5(c-\alpha)/9\beta}$.

Substituting (50) into $d\varphi/d\xi = y$ and integrating it, we have

$$\begin{gathered}\int_{-\infty}^{\varphi}\frac{ds}{(\varphi_2-s)\sqrt{\varphi_1-s}} = \sqrt{\frac{\beta}{5\gamma c}}|\xi|, \\ \int_{\varphi}^{\varphi_1}\frac{ds}{(\varphi_2-s)\sqrt{\varphi_1-s}} = \sqrt{\frac{\beta}{5\gamma c}}|\xi|.\end{gathered} \tag{51}$$

Completing the integrals in the two above equations and noting that $u = \varphi(\xi)$, we obtain $u_3(\xi)$ and $u_4(\xi)$ as (19) and (20).

(3) When $c > \alpha$ and $-g_0 < g < g_0$, Γ_1 and Γ_2 have the expressions

$$\begin{gathered}\Gamma_1 : y = \pm\sqrt{\frac{\beta}{5\gamma c}}\sqrt{(\varphi_1-\varphi)(\varphi_2-\varphi)(\varphi_3-\varphi)}, \quad \varphi \le \varphi_1, \\ \Gamma_2 : y = \pm\sqrt{\frac{\beta}{5\gamma c}}\sqrt{(\varphi-\varphi_1)(\varphi-\varphi_2)(\varphi_3-\varphi)}, \quad \varphi_2 \le \varphi \le \varphi_3.\end{gathered} \tag{52}$$

Substituting (52) into $d\varphi/d\xi = y$ and integrating them, we have

$$\begin{gathered}\int_{-\infty}^{\varphi}\frac{ds}{\sqrt{(\varphi_1-s)(\varphi_2-s)(\varphi_3-s)}} = \sqrt{\frac{\beta}{5\gamma c}}|\xi|, \\ \int_{\varphi}^{\varphi_1}\frac{ds}{\sqrt{(\varphi_1-s)(\varphi_2-s)(\varphi_3-s)}} = \sqrt{\frac{\beta}{5\gamma c}}|\xi|, \\ \int_{\varphi_2}^{\varphi}\frac{ds}{\sqrt{(s-\varphi_1)(s-\varphi_2)(\varphi_3-s)}} = \sqrt{\frac{\beta}{5\gamma c}}|\xi|, \\ \int_{\varphi}^{\varphi_3}\frac{ds}{\sqrt{(s-\varphi_1)(s-\varphi_2)(\varphi_3-s)}} = \sqrt{\frac{\beta}{5\gamma c}}|\xi|.\end{gathered} \tag{53}$$

Completing the integrals in the above four equations and noting that $u = \varphi(\xi)$, we obtain $u_i(\xi)$ $(i = 5\text{–}8)$ as (21).

(4) When $c > \alpha$ and $g = g_0$, Γ_3 and Γ_4 have the expressions

$$\begin{gathered}\Gamma_3 : y = \pm\sqrt{\frac{\beta}{5\gamma c}}(\varphi_1-\varphi)\sqrt{\varphi_3-\varphi}, \quad \varphi < \varphi_1, \\ \Gamma_4 : y = \pm\sqrt{\frac{\beta}{5\gamma c}}(\varphi-\varphi_1)\sqrt{\varphi_3-\varphi}, \quad \varphi_1 < \varphi \le \varphi_3,\end{gathered} \tag{54}$$

where $\varphi_1 = -\sqrt{5(c-\alpha)/9\beta}$, $\varphi_3 = 2\sqrt{5(c-\alpha)/9\beta}$.

Substituting (54) into $d\varphi/d\xi = y$ and integrating them, we have

$$\begin{gathered}\int_{\varphi}^{\varphi_3}\frac{ds}{(s-\varphi_1)\sqrt{\varphi_3-s}} = \sqrt{\frac{\beta}{5\gamma c}}|\xi|, \\ \int_{-\infty}^{\varphi_1}\frac{ds}{(\varphi_1-s)\sqrt{\varphi_3-s}} = \sqrt{\frac{\beta}{5\gamma c}}|\xi|.\end{gathered} \tag{55}$$

Completing the integrals in the two equations above and noting that $u = \varphi(\xi)$, we obtain $u_9(\xi)$ and $u_{10}(\xi)$ as (23) and (24).

(5) When $c > \alpha$ and $g_0 < g$, Γ_5 has the expression

$$\Gamma_5 : y = \pm\sqrt{\frac{\beta}{5\gamma c}}\sqrt{(\varphi_3-\varphi)(\varphi-\varphi_2)(\varphi-\varphi_1)}, \quad \varphi \le \varphi_3, \tag{56}$$

where φ_1 and φ_2 are complex numbers.

Substituting (56) into $d\varphi/d\xi = y$ and integrating it, we have

$$\begin{gathered}\int_{-\infty}^{\varphi}\frac{ds}{\sqrt{(\varphi_3-s)(s-\varphi_2)(s-\varphi_1)}} = \sqrt{\frac{\beta}{5\gamma c}}|\xi|, \\ \int_{\varphi}^{\varphi_3}\frac{ds}{\sqrt{(\varphi_3-s)(s-\varphi_2)(s-\varphi_1)}} = \sqrt{\frac{\beta}{5\gamma c}}|\xi|.\end{gathered} \tag{57}$$

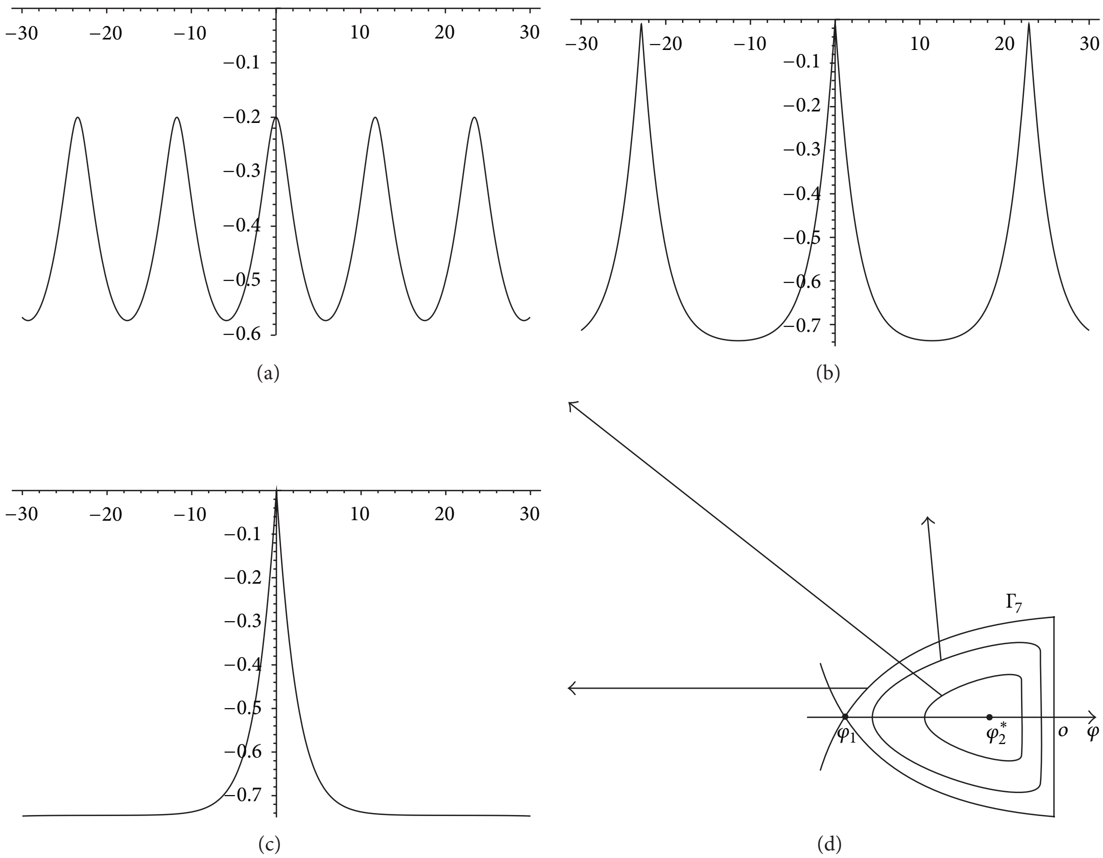


Figure 11: The simulation of integral curves of (40) when $\alpha = \beta = \gamma = 1$, $c = 2$, $g = 4\sqrt{5}/27$, (a) initial value $\varphi(0) = -0.2$, (b) initial value $\varphi(0) = -0.01$, and (c) initial value $\varphi(0) = -0.00001$.

Completing the integrals in the two above equations and noting that $u = \varphi(\xi)$, we obtain $u_{13}(\xi)$ and $u_{14}(\xi)$ as (28).

(6) When $c < 0$, Γ_6 has the expression

$$\Gamma_6 : y = \pm\sqrt{\frac{\beta}{-5\gamma c}}\sqrt{(\varphi - \varphi_1)(\varphi - \varphi_2)(\varphi - \varphi_3)}, \quad \varphi_1 \leq \varphi, \tag{58}$$

where φ_2 and φ_3 are complex numbers.

Substituting (58) into $d\varphi/d\xi = y$ and integrating it, we have

$$\int_{\varphi}^{+\infty} \frac{ds}{\sqrt{(s-\varphi_1)(s-\varphi_2)(s-\varphi_3)}} = \sqrt{\frac{\beta}{-5\gamma c}}|\xi|,$$
$$\int_{\varphi_1}^{\varphi} \frac{ds}{\sqrt{(s-\varphi_1)(s-\varphi_2)(s-\varphi_3)}} = \sqrt{\frac{\beta}{-5\gamma c}}|\xi|. \tag{59}$$

Completing the integrals in the two above equations and noting that $u = \varphi(\xi)$, we obtain $u_{15}(\xi)$ and $u_{16}(\xi)$ as (30).

(7) When $c > \alpha$ and $g = g_0$, there are two special kinds of orbits Γ_7 surrounding the center point $(\varphi_2^*, 0)$ (see Figure 11(d)) and Γ_8 surrounding the center point $(\varphi_3^*, 0)$ (see Figure 12(a)), which are the boundaries of two families of closed orbits. Note that the periodic waves of (9) correspond to the periodic integral curves of (40), and the periodic integral curves correspond to the closed orbits of system (41). For given constants c, g and the corresponding initial value $\varphi(0)$, we simulate the integral curves of (40) as shown in Figures 11 and 12.

From Figure 11, we see that when the initial value $\varphi(0)$ tends to $0 - 0$, the periodic integral curve tends to peakon. This implies that the orbit Γ_7 corresponds to peakon. On $\varphi - y$ plane, Γ_7 has the expression

$$\Gamma_7 : y = \pm\sqrt{\frac{\beta}{5\gamma c}}(\varphi - \varphi_1)\sqrt{\varphi_3 - \varphi}, \quad \varphi_1 < \varphi < 0, \tag{60}$$

where $\varphi_1 = -\sqrt{5(c-\alpha)/9\beta}$, $\varphi_3 = 2\sqrt{5(c-\alpha)/9\beta}$.

Substituting (60) into $d\varphi/d\xi = y$ and integrating it, we have

$$\int_{\varphi}^{0} \frac{ds}{(s-\varphi_1)\sqrt{\varphi_2 - s}} = \sqrt{\frac{\beta}{5\gamma c}}|\xi|. \tag{61}$$

Completing the integral in the above equation and noting that $u = \varphi(\xi)$, we obtain $u_{11}(\xi)$ as (25).

Similarly, from Figure 12, we see that when the initial value $\varphi(0)$ tends to $0 + 0$, the periodic integral curve tends to

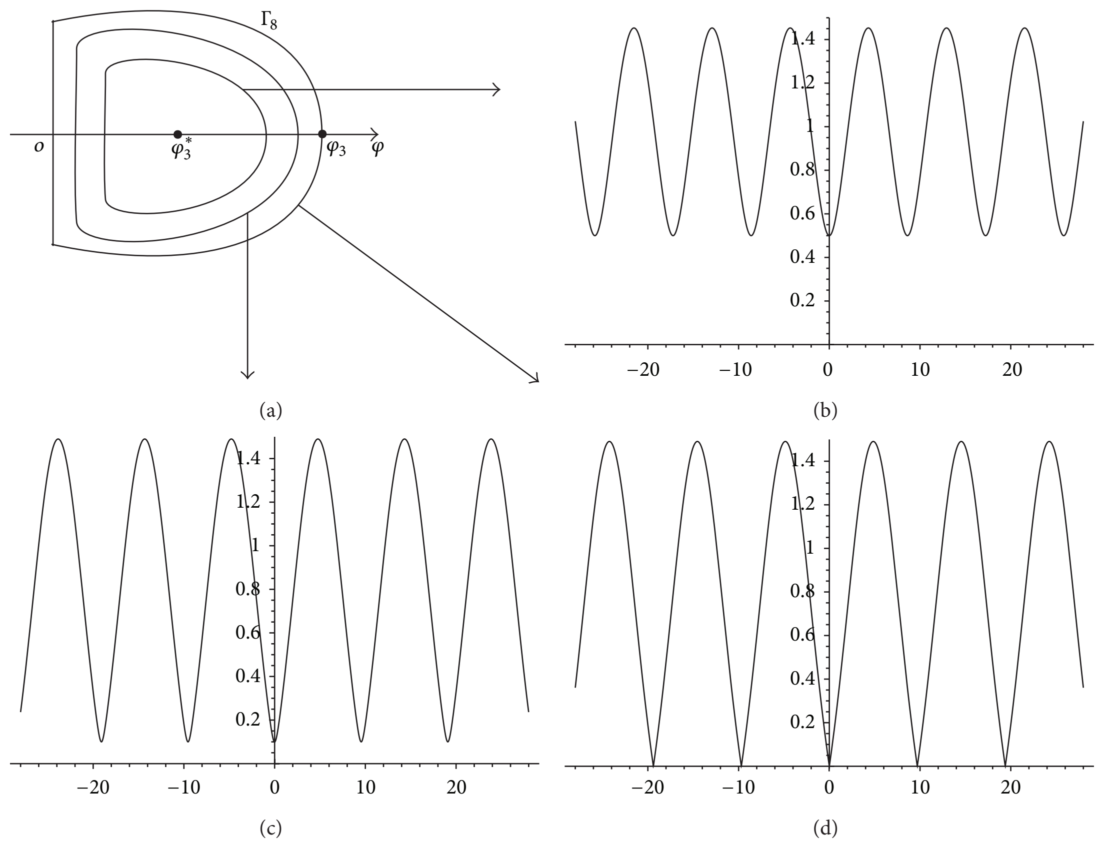


Figure 12: The simulation of integral curves of (40) when $\alpha = \beta = \gamma = 1$, $c = 2$, $g = 4\sqrt{5}/27$, (b) initial value $\varphi(0) = 0.5$, (c) initial value $\varphi(0) = 0.1$, and (d) initial value $\varphi(0) = 0.0001$.

a periodic peakon. This implies that the orbit Γ_8 corresponds to a periodic peakon. On $\varphi - y$ plane, Γ_8 has the expression

$$\Gamma_8 : y = \pm\sqrt{\frac{\beta}{5\gamma c}}\left(\varphi - \varphi_1\right)\sqrt{\varphi_3 - \varphi}, \quad 0 < \varphi \le \varphi_3. \tag{62}$$

Substituting (62) into $d\varphi/d\xi = y$ and integrating it, we have

$$\int_0^{\varphi} \frac{ds}{\left(s - \varphi_1\right)\sqrt{\varphi_3 - s}} = \sqrt{\frac{\beta}{5\gamma c}}\left|\xi\right|. \tag{63}$$

Completing the integral in the above equation and noting that $u = \varphi(\xi)$, we obtain $u_{12}(\xi)$ as (26), where

$$\begin{aligned} T &= \sqrt{\frac{5\gamma c}{\beta}}\left|\int_0^{\varphi_3} \frac{ds}{\left(s - \varphi_1\right)\sqrt{\varphi_3 - s}}\right| \\ &= \sqrt{\frac{5\gamma c}{\beta}}\sqrt[4]{\frac{\beta}{5\left(c - \alpha\right)}}\left|\ln\left(5 - 2\sqrt{6}\right)\right|. \end{aligned} \tag{64}$$

Hereto, we have finished the derivations for the solutions $u_i(\xi)$ $(i = 1\text{–}16)$. In what follows, we shall derive the relationships among these solutions.

(1°) When $c > \alpha$ and $g \longrightarrow -g_0$, it follows that

$$\varphi_1 \longrightarrow -2\sqrt{\frac{5\left(c - \alpha\right)}{9\beta}},$$

$$\varphi_2 \longrightarrow \sqrt{\frac{5\left(c - \alpha\right)}{9\beta}},$$

$$\varphi_3 \longrightarrow \sqrt{\frac{5\left(c - \alpha\right)}{9\beta}},$$

$$A_1 \longrightarrow \sqrt{\frac{5\left(c - \alpha\right)}{\beta}},$$

$$\eta_1 \longrightarrow \sqrt{\frac{\beta}{\gamma c}}\sqrt[4]{\frac{c - \alpha}{5\beta}},$$

$$k_1 \longrightarrow 0,$$

$$\begin{aligned} \operatorname{cn}\left(\eta_1\xi, k_1\right) &\longrightarrow \operatorname{cn}\left(\sqrt{\frac{\beta}{\gamma c}}\sqrt[4]{\frac{c - \alpha}{5\beta}}\xi, 0\right) \\ &= \cos\left(\sqrt{\frac{\beta}{\gamma c}}\sqrt[4]{\frac{c - \alpha}{5\beta}}\xi\right), \end{aligned}$$

$$\eta_2 \longrightarrow \sqrt{\frac{\beta}{\gamma c}}\sqrt[4]{\frac{c-\alpha}{80\beta}},$$
$$k_2 \longrightarrow 0,$$
$$\operatorname{sn}(\eta_2\xi, k_2) \longrightarrow \operatorname{sn}\left(\sqrt{\frac{\beta}{\gamma c}}\sqrt[4]{\frac{c-\alpha}{80\beta}}\xi, 0\right) = \sin\left(\sqrt{\frac{\beta}{\gamma c}}\sqrt[4]{\frac{c-\alpha}{80\beta}}\xi\right). \quad (65)$$

Thus, we have

$$\begin{aligned}
u_1(\xi) &\longrightarrow -2\sqrt{\frac{5(c-\alpha)}{9\beta}} + \sqrt{\frac{5(c-\alpha)}{\beta}} - \frac{2\sqrt{5(c-\alpha)/\beta}}{1-\operatorname{cn}\left(\sqrt{\beta/\gamma c}\sqrt[4]{(c-\alpha)/5\beta}\,\xi, 0\right)} \\
&= \sqrt{\frac{5(c-\alpha)}{9\beta}} - \frac{2\sqrt{5(c-\alpha)/\beta}}{1-\cos\left(\sqrt{\beta/\gamma c}\sqrt[4]{(c-\alpha)/5\beta}\,\xi\right)} \\
&= \sqrt{\frac{5(c-\alpha)}{9\beta}} - \sqrt{\frac{5(c-\alpha)}{\beta}}\csc^2\left(\sqrt{\frac{\beta}{\gamma c}}\sqrt[4]{\frac{c-\alpha}{80\beta}}\xi\right) \\
&= u_3(\xi)\ (\text{see } (19)), \\
u_2(\xi) &\longrightarrow -2\sqrt{\frac{5(c-\alpha)}{9\beta}} + \sqrt{\frac{5(c-\alpha)}{\beta}} - \frac{2\sqrt{5(c-\alpha)/\beta}}{1+\operatorname{cn}\left(\sqrt{\beta/\gamma c}\sqrt[4]{(c-\alpha)/5\beta}\,\xi, 0\right)} \\
&= \sqrt{\frac{5(c-\alpha)}{9\beta}} - \frac{2\sqrt{5(c-\alpha)/\beta}}{1+\cos\left(\sqrt{\beta/\gamma c}\sqrt[4]{(c-\alpha)/5\beta}\,\xi\right)} \\
&= \sqrt{\frac{5(c-\alpha)}{9\beta}} - \sqrt{\frac{5(c-\alpha)}{\beta}}\sec^2\left(\sqrt{\frac{\beta}{\gamma c}}\sqrt[4]{\frac{c-\alpha}{80\beta}}\xi\right) \\
&= u_4(\xi)\ (\text{see } (20)), \\
u_5(\xi) &\longrightarrow \sqrt{\frac{5(c-\alpha)}{9\beta}} - \sqrt{\frac{5(c-\alpha)}{\beta}}\operatorname{sn}^{-2}\left(\sqrt{\frac{\beta}{\gamma c}}\sqrt[4]{\frac{c-\alpha}{80\beta}}\xi, 0\right) \\
&= \sqrt{\frac{5(c-\alpha)}{9\beta}} - \sqrt{\frac{5(c-\alpha)}{\beta}}\sin^{-2}\left(\sqrt{\frac{\beta}{\gamma c}}\sqrt[4]{\frac{c-\alpha}{80\beta}}\xi\right) \\
&= \sqrt{\frac{5(c-\alpha)}{9\beta}} - \sqrt{\frac{5(c-\alpha)}{\beta}}\csc^2\left(\sqrt{\frac{\beta}{\gamma c}}\sqrt[4]{\frac{c-\alpha}{80\beta}}\xi\right) \\
&= u_3(\xi)\ (\text{see } (19)), \\
u_6(\xi) &\longrightarrow \left(-2\sqrt{\frac{5(c-\alpha)}{9\beta}} - \sqrt{\frac{5(c-\alpha)}{9\beta}} \times \operatorname{sn}^2\left(\sqrt{\frac{\beta}{\gamma c}}\sqrt[4]{\frac{c-\alpha}{80\beta}}\xi, 0\right)\right) \times \left(1-\operatorname{sn}^2\left(\sqrt{\frac{\beta}{\gamma c}}\sqrt[4]{\frac{c-\alpha}{80\beta}}\xi, 0\right)\right)^{-1} \\
&= \left(-2\sqrt{\frac{5(c-\alpha)}{9\beta}} - \sqrt{\frac{5(c-\alpha)}{9\beta}} \times \sin^2\left(\sqrt{\frac{\beta}{\gamma c}}\sqrt[4]{\frac{c-\alpha}{80\beta}}\xi\right)\right) \times \left(1-\sin^2\left(\sqrt{\frac{\beta}{\gamma c}}\sqrt[4]{\frac{c-\alpha}{80\beta}}\xi\right)\right)^{-1} \\
&= \left(-\sqrt{\frac{5(c-\alpha)}{\beta}} + \sqrt{\frac{5(c-\alpha)}{9\beta}} \times \cos^2\left(\sqrt{\frac{\beta}{\gamma c}}\sqrt[4]{\frac{c-\alpha}{80\beta}}\xi\right)\right) \times \left(\cos^2\left(\sqrt{\frac{\beta}{\gamma c}}\sqrt[4]{\frac{c-\alpha}{80\beta}}\xi\right)\right)^{-1} \\
&= \sqrt{\frac{5(c-\alpha)}{9\beta}} - \sqrt{\frac{5(c-\alpha)}{\beta}} \times \sec^2\left(\sqrt{\frac{\beta}{\gamma c}}\sqrt[4]{\frac{c-\alpha}{80\beta}}\xi\right) \\
&= u_4(\xi)\ (\text{see } (20)), \\
u_7(\xi) &\longrightarrow \sqrt{\frac{5(c-\alpha)}{9\beta}} - \left(\sqrt{\frac{5(c-\alpha)}{9\beta}} - \sqrt{\frac{5(c-\alpha)}{9\beta}}\right) \times \operatorname{sn}^2\left(\sqrt{\frac{\beta}{\gamma c}}\sqrt[4]{\frac{c-\alpha}{80\beta}}\xi, 0\right) \\
&= \sqrt{\frac{5(c-\alpha)}{9\beta}}, \\
u_8(\xi) &\longrightarrow \frac{\sqrt{5(c-\alpha)/9\beta} - 0}{1-0} \\
&= \sqrt{\frac{5(c-\alpha)}{9\beta}}.
\end{aligned} \quad (66)$$

(2°) When $c > \alpha$ and $g \longrightarrow g_0$, it follows that

$$\begin{aligned}
\varphi_1 &\longrightarrow -\sqrt{\frac{5(c-\alpha)}{9\beta}},\\
\varphi_2 &\longrightarrow -\sqrt{\frac{5(c-\alpha)}{9\beta}},\\
\varphi_3 &\longrightarrow 2\sqrt{\frac{5(c-\alpha)}{9\beta}},\\
\eta_2 &\longrightarrow \sqrt{\frac{\beta}{\gamma c}}\sqrt[4]{\frac{c-\alpha}{80\beta}},\\
k_2 &\longrightarrow 1,\\
\operatorname{sn}(\eta_2\xi, k_2) &\longrightarrow \operatorname{sn}\left(\sqrt{\frac{\beta}{\gamma c}}\sqrt[4]{\frac{c-\alpha}{80\beta}}\xi, 1\right)\\
&= \tanh\left(\sqrt{\frac{\beta}{\gamma c}}\sqrt[4]{\frac{c-\alpha}{80\beta}}\xi\right).\\
A_2 &\longrightarrow \sqrt{\frac{5(c-\alpha)}{\beta}},\\
\eta_3 &\longrightarrow \sqrt{\frac{\beta}{\gamma c}}\sqrt[4]{\frac{c-\alpha}{5\beta}},\\
k_3 &\longrightarrow 1,\\
\operatorname{cn}(\eta_3\xi, k_3) &\longrightarrow \operatorname{cn}\left(\sqrt{\frac{\beta}{\gamma c}}\sqrt[4]{\frac{c-\alpha}{5\beta}}\xi, 1\right)\\
&= \operatorname{sech}\left(\sqrt{\frac{\beta}{\gamma c}}\sqrt[4]{\frac{c-\alpha}{5\beta}}\xi\right).
\end{aligned} \tag{67}$$

Thus, we have

$$\begin{aligned}
u_5(\xi) &\longrightarrow 2\sqrt{\frac{5(c-\alpha)}{9\beta}}\\
&\quad - \left(2\sqrt{\frac{5(c-\alpha)}{9\beta}} + \sqrt{\frac{5(c-\alpha)}{9\beta}}\right)\\
&\quad \times \operatorname{sn}^{-2}\left(\sqrt{\frac{\beta}{\gamma c}}\sqrt[4]{\frac{c-\alpha}{80\beta}}\xi, 1\right)\\
&= 2\sqrt{\frac{5(c-\alpha)}{9\beta}} - \sqrt{\frac{5(c-\alpha)}{\beta}}\\
&\quad \times \tanh^{-2}\left(\sqrt{\frac{\beta}{\gamma c}}\sqrt[4]{\frac{c-\alpha}{80\beta}}\xi\right)\\
&= 2\sqrt{\frac{5(c-\alpha)}{9\beta}} - \sqrt{\frac{5(c-\alpha)}{\beta}}\\
&\quad \times \left(1 + \operatorname{csch}^2\left(\sqrt{\frac{\beta}{\gamma c}}\sqrt[4]{\frac{c-\alpha}{80\beta}}\xi\right)\right)\\
&= u_{10}(\xi)\ (\text{see } (24)),\\
u_6(\xi)\ (\text{or } u_8(\xi)) &\longrightarrow \left(-\sqrt{\frac{5(c-\alpha)}{9\beta}} + \sqrt{\frac{5(c-\alpha)}{9\beta}}\right.\\
&\quad \left.\times \operatorname{sn}^2\left(\sqrt{\frac{\beta}{\gamma c}}\sqrt[4]{\frac{c-\alpha}{80\beta}}\xi, 1\right)\right)\\
&\quad \times \left(1 - \operatorname{sn}^2\left(\sqrt{\frac{\beta}{\gamma c}}\sqrt[4]{\frac{c-\alpha}{80\beta}}\xi, 1\right)\right)^{-1}\\
&= -\sqrt{\frac{5(c-\alpha)}{9\beta}},\\
u_7(\xi) &\longrightarrow 2\sqrt{\frac{5(c-\alpha)}{9\beta}}\\
&\quad - \left(2\sqrt{\frac{5(c-\alpha)}{9\beta}} + \sqrt{\frac{5(c-\alpha)}{9\beta}}\right)\\
&\quad \times \operatorname{sn}^2\left(\sqrt{\frac{\beta}{\gamma c}}\sqrt[4]{\frac{c-\alpha}{80\beta}}\xi, 1\right)\\
&= 2\sqrt{\frac{5(c-\alpha)}{9\beta}} - \sqrt{\frac{5(c-\alpha)}{\beta}}\\
&\quad \times \tanh^2\left(\sqrt{\frac{\beta}{\gamma c}}\sqrt[4]{\frac{c-\alpha}{80\beta}}\xi\right)\\
&= 2\sqrt{\frac{5(c-\alpha)}{9\beta}} - \sqrt{\frac{5(c-\alpha)}{\beta}}\\
&\quad \times \left(1 - \operatorname{sech}^2\left(\sqrt{\frac{\beta}{\gamma c}}\sqrt[4]{\frac{c-\alpha}{80\beta}}\xi\right)\right)\\
&= u_9(\xi)\ (\text{see } (23)),\\
u_{13}(\xi) &\longrightarrow 2\sqrt{\frac{5(c-\alpha)}{9\beta}} + \sqrt{\frac{5(c-\alpha)}{\beta}}\\
&\quad - \frac{2\sqrt{5(c-\alpha)/\beta}}{1 - \operatorname{cn}\left(\sqrt{\beta/\gamma c}\sqrt[4]{(c-\alpha)/5\beta}\,\xi, 1\right)}
\end{aligned}$$

$$
\begin{aligned}
&= 5\sqrt{\frac{5(c-\alpha)}{9\beta}} \\
&\quad - \frac{2\sqrt{5(c-\alpha)/\beta}}{1-\operatorname{sech}\left(\sqrt{\beta/\gamma c}\sqrt[4]{(c-\alpha)/5\beta}\,\xi\right)} \\
&= 5\sqrt{\frac{5(c-\alpha)}{9\beta}} \\
&\quad - \left(2\sqrt{\frac{5(c-\alpha)}{\beta}}\right. \\
&\qquad \left.\times \cosh\left(\sqrt{\frac{\beta}{\gamma c}}\sqrt[4]{\frac{c-\alpha}{5\beta}}\xi\right)\right) \\
&\quad \times\left(\cosh\left(\sqrt{\frac{\beta}{\gamma c}}\sqrt[4]{\frac{c-\alpha}{5\beta}}\xi\right)-1\right)^{-1} \\
&= 5\sqrt{\frac{5(c-\alpha)}{9\beta}} \\
&\quad - \left(2\sqrt{\frac{5(c-\alpha)}{\beta}}+4\sqrt{\frac{5(c-\alpha)}{\beta}}\right. \\
&\qquad \left.\times \sinh^2\left(\sqrt{\frac{\beta}{\gamma c}}\sqrt[4]{\frac{c-\alpha}{80\beta}}\xi\right)\right) \\
&\quad \times\left(2\sinh^2\left(\sqrt{\frac{\beta}{\gamma c}}\sqrt[4]{\frac{c-\alpha}{80\beta}}\xi\right)\right)^{-1} \\
&= -\sqrt{\frac{5(c-\alpha)}{9\beta}} - \sqrt{\frac{5(c-\alpha)}{\beta}} \\
&\quad \times \operatorname{csch}^2\left(\sqrt{\frac{\beta}{\gamma c}}\sqrt[4]{\frac{(c-\alpha)}{80\beta}}\xi\right) \\
&= u_{10}(\xi)\ (\text{see } (24)), \\
u_{14}(\xi) &\longrightarrow 2\sqrt{\frac{5(c-\alpha)}{9\beta}} + \sqrt{\frac{5(c-\alpha)}{\beta}} \\
&\quad - \frac{2\sqrt{5(c-\alpha)/\beta}}{1+\operatorname{cn}\left(\sqrt{\beta/\gamma c}\sqrt[4]{(c-\alpha)/5\beta}\,\xi,1\right)} \\
&= 5\sqrt{\frac{5(c-\alpha)}{9\beta}} \\
&\quad - \frac{2\sqrt{5(c-\alpha)/\beta}}{1+\operatorname{sech}\left(\sqrt{\beta/\gamma c}\sqrt[4]{(c-\alpha)/5\beta}\,\xi\right)} \\
&= 5\sqrt{\frac{5(c-\alpha)}{9\beta}} \\
&\quad - \left(2\sqrt{\frac{5(c-\alpha)}{\beta}}\right. \\
&\qquad \left.\times \cosh\left(\sqrt{\frac{\beta}{\gamma c}}\sqrt[4]{\frac{c-\alpha}{5\beta}}\xi\right)\right) \\
&\quad \times\left(\cosh\left(\sqrt{\frac{\beta}{\gamma c}}\sqrt[4]{\frac{c-\alpha}{5\beta}}\xi\right)+1\right)^{-1} \\
&= 5\sqrt{\frac{5(c-\alpha)}{9\beta}} \\
&\quad - \left(-2\sqrt{\frac{5(c-\alpha)}{\beta}}+4\sqrt{\frac{5(c-\alpha)}{\beta}}\right. \\
&\qquad \left.\times \cosh^2\left(\sqrt{\frac{\beta}{\gamma c}}\sqrt[4]{\frac{c-\alpha}{80\beta}}\xi\right)\right) \\
&\quad \times\left(2\cosh^2\left(\sqrt{\frac{\beta}{\gamma c}}\sqrt[4]{\frac{c-\alpha}{80\beta}}\xi\right)\right)^{-1} \\
&= -\sqrt{\frac{5(c-\alpha)}{9\beta}} + \sqrt{\frac{5(c-\alpha)}{\beta}} \\
&\quad \times \operatorname{sech}^2\left(\sqrt{\frac{\beta}{\gamma c}}\sqrt[4]{\frac{c-\alpha}{80\beta}}\xi\right) \\
&= u_9(\xi)\ (\text{see } (23)).
\end{aligned} \tag{68}
$$

Hereto, we have completed the derivations for Proposition 1.

4. The Derivations for Proposition 6

In this section, we derive the precise expressions of the traveling wave solutions for BBM-like $B(4,2)$ equation. Similar to the derivations in Section 3, substituting $u(x,t) = \psi(\xi)$ with $\xi = x - ct$ into (32) and integrating it, we have the following planar system:

$$
\begin{aligned}
\frac{d\psi}{d\xi} &= y, \\
\frac{dy}{d\xi} &= \frac{g+(c-\alpha)\psi-\beta\psi^4-2\gamma c y^2}{2\gamma c\psi},
\end{aligned} \tag{69}
$$

with the first integral

$$
\gamma c\psi^2 y^2 - \frac{g}{2}\psi^2 - \frac{c-\alpha}{3}\psi^3 + \frac{\beta}{6}\psi^6 = h, \tag{70}
$$

where g and h are the integral constants.

When $c > \alpha$, $h = 0$, and $g = 0$, (70) becomes

$$y = \pm\sqrt{\frac{c-\alpha}{3\gamma c}}\sqrt{\psi\left(1-\frac{\beta}{2(c-\alpha)}\psi^3\right)}, \quad 0 \le \psi \le \sqrt[3]{\frac{2(c-\alpha)}{\beta}}. \tag{71}$$

Substituting (71) into the first equation of (69) and integrating it from 0 to ψ or ψ to $\sqrt[3]{2(c-\alpha)/\beta}$, respectively, it follows that

$$\begin{aligned} \int_0^{\psi}\frac{ds}{\sqrt{s\left(1-\left(\beta/2(c-\alpha)\right)s^3\right)}} &= \sqrt{\frac{c-\alpha}{3\gamma c}}\,|\xi|, \\ \int_{\psi}^{\sqrt[3]{2(c-\alpha)/\beta}}\frac{ds}{\sqrt{s\left(1-\left(\beta/2(c-\alpha)\right)s^3\right)}} &= \sqrt{\frac{c-\alpha}{3\gamma c}}\,|\xi|. \end{aligned} \tag{72}$$

Completing the integrals in the two above equations and noting that $u = \psi(\xi)$, we obtain the two elliptic periodic wave solutions $u_{17}(\xi)$ and $u_{18}(\xi)$ as (34).

When $c < 0$, $h = 0$, and $g = -\sqrt[3]{(c-\alpha)^4/16\beta}$, (70) becomes

$$y = \pm\sqrt{\frac{\beta}{-6\gamma c}}\,|\psi - p|\,\sqrt{\psi^2 + q\psi + r}, \tag{73}$$

where

$$p = \sqrt[3]{\frac{c-\alpha}{2\beta}}, \qquad q = \sqrt[3]{\frac{4(c-\alpha)}{\beta}}, \qquad r = \sqrt[3]{\frac{27(c-\alpha)^2}{4\beta^2}}. \tag{74}$$

Substituting (73) into the first equation of (69) and integrating it from $-\infty$ to ψ or ψ to $+\infty$, respectively, it follows that

$$\begin{aligned} \int_{-\infty}^{\psi}\frac{ds}{(p-s)\sqrt{s^2+qs+r}} &= \frac{\beta}{\sqrt{-6\gamma c}}\,|\xi|, \quad (\psi < p), \\ \int_{\psi}^{+\infty}\frac{ds}{(s-p)\sqrt{s^2+qs+r}} &= \frac{\beta}{\sqrt{-6\gamma c}}\,|\xi|, \quad (p < \psi). \end{aligned} \tag{75}$$

Completing the integrals in the two above equations and noting that $u = \psi(\xi)$, we obtain the two hyperbolic blow-up solutions $u_{19}(\xi)$ and $u_{20}(\xi)$ as (36).

Hereto, we have completed the derivations for Proposition 6.

5. Conclusion

In this paper, we have investigated the nonlinear wave solutions and their bifurcations for BBM-like $B(m,2)$ ($m = 3,4$) equations. For BBM-like $B(3,2)$ equation, we obtain some precise expressions of traveling wave solutions (see $u_i(\xi)$ (i = 1–16)), which include periodic blow-up and periodic wave solution, peakon and periodic peakon wave solution, and solitary wave and blow-up solution. We also reveal the relationships among these solutions theoretically (see Remarks 2–5 and the corresponding derivations). For BBM-like $B(4,2)$ equation, we construct two elliptic periodic wave solutions and two hyperbolic blow-up solutions (see $u_i(\xi)$ (i = 17–20)). We would like to study the BBM-like $B(m,n)$ equations further.

Conflict of Interests

The authors declare that they do not have any commercial or associative interest that represents a conflict of interests in connection with the work submitted.

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