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Research Article

Application of Optimal Homotopy Asymptotic Method to Burger Equations

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We apply optimal homotopy asymptotic method (OHAM) for finding approximate solutions of the Burger's-Huxley and Burger's-Fisher equations. The results obtained by proposed method are compared to those of Adomian decomposition method (ADM) (Ismail et al., (2004)). As a result it is concluded that the method is explicit, effective, and simple to use.

1. Introduction

Nonlinear phenomena play a vital role in applied mathematics, physics, and engineering sciences. The Burger's equation models efficiently certain problems of a fluid flow nature, in which either shocks or viscous dissipation is a significant factor. It can be used as a model for any nonlinear wave propagation problem subject to dissipation [1]. The first steady-state solutions of Burger equation were given by Young et al. [2] However, the equation gets its name from the extensive research of Burger's [3]. The generalized Burger's-Huxley introduced by Satsuma shows a prototype model for describing the communication among reaction mechanisms, convection effects, and diffusion transports [4]. Burger-Fisher equation has significant applications in various fields of applied mathematics and has physical applications such as gas dynamic, traffic flow, convection effect, and diffusion transport [5–12]. Marinca and Herişanu et al. introduced a new semianalytic method OHAM for approximate solution of nonlinear problems of thin film flow of a fourth-grade fluid down a vertical cylinder. In progression of papers Marinca and Herişanu et al. have applied this method for the solution of nonlinear equations arising in the steady state flow of a fourth-grade fluid past a porous plate and for the solution of nonlinear equations arising in heat transfer [13–15]. The method has been applied by a number of researchers for solution of ordinary and partial differential equations [16–21]. The motivation of this paper is to show the effectiveness of OHAM for the solution of Burger's-Huxley and Burger's-Fisher equations. We consider Burger's-Huxley equation of the form

$$\begin{aligned}\frac{\partial u(x,t)}{\partial t} &+ \alpha u^{\delta}(x,t)\frac{\partial u(x,t)}{\partial x} - \frac{\partial^2 u(x,t)}{\partial x^2}\\ &- \beta u(x,t)\left(1-u^{\delta}(x,t)\right)\left(u^{\delta}(x,t)-\gamma\right)=0,\\ &\forall 0\le x\le 1,\ t\ge 0\end{aligned} \tag{1}$$

and Burger's-Fisher equation of the form

$$\begin{aligned}\frac{\partial u(x,t)}{\partial t} &+ \alpha u^{\delta}(x,t)\frac{\partial u(x,t)}{\partial x} - \frac{\partial^2 u(x,t)}{\partial x^2}\\ &- \beta u(x,t)\left(1-u^{\delta}(x,t)\right)=0,\\ &\forall 0\le x\le 1,\ t\ge 0,\end{aligned} \tag{2}$$

where α, β, γ, and δ are parameters and $\beta \ge 0$, $\delta \ge 0$, $\gamma \in (0,1)$.

The present paper is divided into three sections. In Section 2 fundamental mathematical theory of OHAM is presented. In Section 3 comparisons are made between the results of the proposed method and HAM for Burger's-Huxley. In Section 4 solution of Burger's-Fisher equation is presented, and absolute error of approximate solution of proposed method is compared with approximate solution of HAM. In all cases the proposed method yields better results than those of ADM.

2. Fundamental Theory of OHAM

Here we start by describing the basic idea of OHAM. Consider the partial differential equation of the form:

$$\mathcal{L}(u(x,t)) + \mathcal{N}(u(x,t)) + g(x,t) = 0, \quad x \in \Omega, \tag{3}$$

$$\mathcal{B}\left(u, \frac{\partial u}{\partial t}\right) = 0, \tag{4}$$

where $\mathcal{L}$ is a linear operator and $\mathcal{N}$ is nonlinear operator. $\mathcal{B}$ is boundary operator, $u(x,t)$ is an unknown function, and x and t denote spatial and time variables, respectively; Ω is the problem domain and $g(x,t)$ is a known function.

According to the basic idea of OHAM, one can construct the optimal homotopy $\psi(x,t;q) : \Omega \times [0,1] \to R$ which satisfies

$$\begin{aligned}(1-q)&\{\mathcal{L}(\psi(x,t;q)) + g(x,t)\}\\ &= \mathcal{H}(q)\{\mathcal{L}(\psi(x,t;q)) + \mathcal{N}(\psi(x,t;q)) + g(x,t)\},\end{aligned} \tag{5}$$

where $q \in [0,1]$ is an embedding parameter, $H(q)$ is a nonzero auxiliary function for $q \neq 0$, $H(0) = 0$. Equation (3) is called optimal homotopy equation. Clearly, we have

$$q = 0 \Longrightarrow \mathcal{H}(\psi(x,t;0),0) = \mathcal{L}(\psi(x,t;0)) + g(x,t) = 0, \tag{6}$$

$$\begin{aligned}q = 1 &\Longrightarrow \mathcal{H}(\psi(x,t;1),1)\\ &= \mathcal{H}(1)\{\mathcal{L}(\psi(x,t;q)) + \mathcal{N}(\psi(x,t;q)) + g(x,t)\} = 0.\end{aligned} \tag{7}$$

Clearly, when $q = 0$ and $q = 1$, it holds that $\psi(x,t;0) = u_0(x,t)$ and $\psi(x,t;1) = u(x,t)$, respectively. Thus, as q varies from 0 to 1, the solution $\psi(x,t;q)$ approaches from $u_0(x,t)$ to $u(x,t)$, where $u_0(x,t)$ is obtained from (3) for $q = 0$:

$$\mathcal{L}(u_0(x,t)) + g(x,t) = 0, \qquad \mathcal{B}\left(u_0, \frac{\partial u_0}{\partial t}\right) = 0. \tag{8}$$

Next, we choose auxiliary function $H(q)$ in the form

$$\mathcal{H}(q) = qC_1 + q^2C_2 + \cdots. \tag{9}$$

Here $C_1, C_2, \ldots$ are constants to be determined later.

To get an approximate solution, we expand $\psi(x,t;q,C_i)$ in Taylor's series about q in the following manner:

$$\psi(x,t;q,C_i) = u_0(x,t) + \sum_{k=1}^{\infty} u_k(x,t;C_i)\, q^k, \quad i = 1,2,\ldots. \tag{10}$$

Substituting (10) into (4) and equating the coefficient of like powers of q, we obtain Zeroth-order problem, given by (6), the first- and second-order problems are given by (11)-(12), respectively, and the general governing equations for $u_k(x,t)$ are given by (13):

$$\mathcal{L}(u_1(x,t)) = C_1\mathcal{N}_0(u_0(x,t)), \qquad \mathcal{B}\left(u_1, \frac{\partial u_1}{\partial t}\right) = 0, \tag{11}$$

$$\begin{aligned}\mathcal{L}&(u_2(x,t)) - \mathcal{L}(u_1(x,t))\\ &= C_2\mathcal{N}_0(u_0(x,t))\\ &\quad + C_1[\mathcal{L}(u_1(x,t)) + \mathcal{N}_1(u_0(x,t), u_1(x,t))],\\ &\qquad \mathcal{B}\left(u_2, \frac{\partial u_2}{\partial t}\right) = 0,\end{aligned} \tag{12}$$

$$\begin{aligned}\mathcal{L}&(u_k(x,t)) - \mathcal{L}(u_{k-1}(x,t))\\ &= C_k\mathcal{N}_0(u_0(x,t))\\ &\quad + \sum_{i=1}^{k-1} C_i[\mathcal{L}(u_{k-i}(x,t)) + \mathcal{N}_{k-i}\\ &\qquad \times (u_0(x,t), u_1(x,t), \ldots, u_{k-i}(x,t))],\\ &\qquad \mathcal{B}\left(u_k, \frac{\partial u_k}{\partial t}\right) = 0, \quad k = 2,3,\ldots,\end{aligned} \tag{13}$$

where $\mathcal{N}_{k-i}(u_0(x,t), u_1(x,t), \ldots, u_{k-i}(x,t))$ is the coefficient of q^{k-i} in the expansion of $\mathcal{N}(\psi(x,t;q))$ about the embedding parameter q:

$$\begin{aligned}\mathcal{N}(\psi(x,t;q,C_i)) &= \mathcal{N}_0(u_0(x,t))\\ &\quad + \sum_{k\geq 1}\mathcal{N}_k(u_0,u_1,u_2,\ldots,u_k)\, q^k.\end{aligned} \tag{14}$$

Here u_k for $k \geq 0$ are set of linear equations with the linear boundary conditions, which can be easily solved.

The convergence of the series in (10) depends upon the auxiliary constants $C_1, C_2, \ldots$. If it is convergent at $q = 1$, one has:

$$\tilde{u}(x,t;C_i) = u_0(x,t) + \sum_{k\geq 1} u_k(x,t;C_i). \tag{15}$$

Substituting (15) into (1) results in the following expression for residual:

$$\mathcal{R}(x,t;C_i) = \mathcal{L}(\tilde{u}(x,t;C_i)) + g(x,t) + \mathcal{N}(\tilde{u}(x,t;C_i)). \tag{16}$$

If $R(x,t;C_i) = 0$, then $\tilde{u}(x,t;C_i)$ will be the exact solution.

For computing the auxiliary constants, C_i, $i = 1,2,\ldots,m$, there are many methods like Galerkin's Method, Ritz Method, Least Squares Method, and Collocation Method to find the optimal values of C_i, $i = 1,2,3,\ldots$, One can apply the Method of Least Squares as

$$\mathcal{J}(C_i) = \int_0^t \int_\Omega \mathcal{R}^2(x,t,C_i)\, dx\, dt, \tag{17}$$

where R is the residual, $R(x,t;C_i) = L(\tilde{u}(x,t;C_i)) + g(x,t) + N(\tilde{u}(x,t;C_i))$, and

$$\frac{\partial \mathcal{J}}{\partial C_1} = \frac{\partial \mathcal{J}}{\partial C_2} = \cdots = \frac{\partial \mathcal{J}}{\partial C_m} = 0. \tag{18}$$

The constants C_i can also be determined by another method as

$$\mathcal{R}(h_1;C_i) = \mathcal{R}(h_2;C_i) = \cdots = \mathcal{R}(h_m;C_i) = 0, \quad i = 1,2,\ldots,m, \tag{19}$$

at any time t, where $h_i \in \Omega$. The convergence depends upon constants $C_1, C_2, \ldots$, can be optimally identified and minimized by (18).

3. Application of OHAM

In this section we apply OHAM for the two problems: the first is the Burger's-Huxley equation (1) and the second is the Burger's-Fisher equation (2).

3.1. Application of OHAM for Burger's-Huxley Equation. Let us consider Burger's-Huxley equation of form (1):

$$\begin{aligned}\frac{\partial u(x,t)}{\partial t} &+ \alpha u^{\delta}(x,t)\frac{\partial u(x,t)}{\partial x} - \frac{\partial^2 u(x,t)}{\partial x^2} \\ &- \beta u(x,t)\left(1 - u^{\delta}(x,t)\right)\left(u^{\delta}(x,t) - \gamma\right) = 0, \\ &\forall 0 \le x \le 1,\ t \ge 0,\end{aligned} \tag{20}$$

Subject to constant initial condition

$$u(x,0) = \left(0.5\gamma + 0.5\gamma \tanh\left(A_1 x\right)\right)^{1/\delta}. \tag{21}$$

The exact solution of (26) with given condition is given by

$$u(x,0) = \left(0.5\gamma + 0.5\gamma \tanh\left(A_1\left(x - A_2 t\right)\right)\right)^{1/\delta}, \tag{22}$$

where

$$\begin{aligned} A_1 &= \frac{-\alpha\delta + \delta\sqrt{\alpha^2 + 4\beta(1+\delta)}}{4(1+\delta)}\gamma, \\ A_2 &= \frac{\alpha\gamma}{(1+\delta)} - \frac{(1+\delta-\gamma)\left(-\alpha + \sqrt{\alpha^2 + 4\beta(1+\delta)}\right)}{2(1+\delta)}. \end{aligned} \tag{23}$$

For computational work, we have taken $\alpha = 1$, $\beta = 1$, $\delta = 1$, and $\gamma = 0.001$ for various values of x and t.

Following the basic idea of OHAM presented in preceding section we start with

Zeroth-Order Problem

$$\begin{aligned}\frac{\partial u_0(x,t)}{\partial t} &= 0, \\ u_0(x,0) &= (0.0005 + 0.0005\tanh(0.00025x)).\end{aligned} \tag{24}$$

Its solution is

$$u_0(x,t) = (0.0005 + 0.0005\tanh(0.00025x)). \tag{25}$$

First-Order Problem

$$\begin{aligned}\frac{\partial u_1(x,t)}{\partial t} &- \left(1 + C_1\right)\frac{\partial u_0(x,t)}{\partial t} - C_1 u_0(x,t)\frac{\partial u_0(x,t)}{\partial x} \\ &+ C_1\frac{\partial^2 u_0(x,t)}{\partial x^2} - 0.001C_1 u_0(x,t) + 0.001C_1 u_0^2(x,t) \\ &- C_1 u_0^3(x,t) = 0, \\ &u_1(x,0) = 0.\end{aligned} \tag{26}$$

Its solution is

$$\begin{aligned} u_1&(x,t) \\ &= -t\left(-2.4987500000000003 \times 10^{-7}C_1\right. \\ &\quad - 6.25 \times 10^{-11}C_1\operatorname{sech}^2(0.00025x) \\ &\quad + 1.249999999999317 \\ &\quad \times 10^{-10}C_1\tanh(0.00025x) \\ &\quad - 1.25 \times 10^{-10}C_1\operatorname{sech}^2(0.00025x) \\ &\quad \times \tanh(0.00025x) \\ &\quad + 2.49875 \times 10^{-7}C_1\tanh^2(0.00025x) \\ &\quad \left.-1.25 \times 10^{-10}C_1\tanh^3(0.00025x)\right). \end{aligned} \tag{27}$$

Second-Order Problem

$$\begin{aligned}\frac{\partial u_2(x,t)}{\partial t} &- \left(1 + C_1\right)\frac{\partial u_1(x,t)}{\partial t} - C_2 u_0(x,t)\frac{\partial u_0(x,t)}{\partial x} \\ &+ C_2\frac{\partial^2 u_0(x,t)}{\partial x^2} - C_2\frac{\partial u_0(x,t)}{\partial t} - C_1 u_1(x,t)\frac{\partial u_0(x,t)}{\partial x} \\ &- C_1 u_0(x,t)\frac{\partial u_1(x,t)}{\partial x} + C_1\frac{\partial^2 u_1(x,t)}{\partial x^2} - 0.001C_2 u_0(x,t) \\ &+ 1.001C_2 u_0^2(x,t) - C_2 u_0^3(x,t) - 0.001C_1 u_1(x,t) \\ &+ 2.002C_1 u_0(x,t)u_1(x,t) - 3C_1 u_0^2(x,t)u_1(x,t) = 0, \\ &u_2(x,0) = 0.\end{aligned} \tag{28}$$

Its solution is

$$\begin{aligned} u_2&(x,t,C_1) \\ &= -t - 2.4987500000000003 \times 10^{-7}C_1 - 6.25 \\ &\quad \times 10^{-11}C_1\operatorname{sech}^2(0.00025x) \\ &\quad + 1.249999999999317 \times 10^{-10}C_1 \end{aligned}$$

$$\times \tanh(0.00025x) - 1.25\times10^{-10}C_1 \times \operatorname{sech}^2(0.00025x)\tanh(0.00025x) + 2.49875\times10^{-7}C_1\tanh^2(0.00025x) - 1.25\times10^{-10}C_1\tanh^3(0.00025x). \tag{29}$$

Third-Order Problem

$$\begin{aligned}
&\frac{\partial u_3(x,t)}{\partial t} - (1+C_1)\frac{\partial u_2(x,t)}{\partial t} - C_3u_0(x,t)\frac{\partial u_0(x,t)}{\partial x} \\
&\quad + C_3\frac{\partial^2 u_0(x,t)}{\partial x^2} - C_3\frac{\partial u_0(x,t)}{\partial t} - C_2u_1(x,t)\frac{\partial u_0(x,t)}{\partial x} \\
&\quad - C_2u_0(x,t)\frac{\partial u_1(x,t)}{\partial x} + C_2\frac{\partial^2 u_1(x,t)}{\partial x^2} - C_2\frac{\partial u_1(x,t)}{\partial t} \\
&\quad - C_1u_2(x,t)\frac{\partial u_0(x,t)}{\partial x} - C_1u_1(x,t)\frac{\partial u_1(x,t)}{\partial x} \\
&\quad + C_1\frac{\partial^2 u_2(x,t)}{\partial x^2} - C_1u_0(x,t)\frac{\partial u_2(x,t)}{\partial x} \\
&\quad - 0.001C_3u_0(x,t) + 1.001C_3u_0^2(x,t) \\
&\quad - C_3u_0^3(x,t) - 0.001C_2u_1(x,t) \\
&\quad - 3C_2u_0^2(x,t)u_1(x,t) + 1.001C_1u_1^2(x,t) \\
&\quad - 3C_1u_0(x,t)u_1^2(x,t) - 0.001C_1u_2(x,t) \\
&\quad + 2.002C_1u_0(x,t)u_2(x,t) - 3C_1u_0^2(x,t)u_2(x,t) \\
&\quad + 2.002C_2u_0(x,t)u_1(x,t) = 0, \\
&\qquad u_3(x,0) = 0.
\end{aligned} \tag{30}$$

Its solution is

$$\begin{aligned}
&u_3(x,t,C_1,C_2,C_3) \\
&= \Bigg(\frac{1}{(1+e^{0.5x})^2}t\operatorname{sech}^2(0.25x) \\
&\quad \times (-0.0625C_2 - 0.0625C_3 \\
&\qquad + C_1(-0.0625 + C_1(-0.125 - 0.015625t) \\
&\qquad\quad + C_2(-0.125 - 0.015625t) \\
&\qquad\quad - 0.000651042C_1^2(5.0718+t) \\
&\qquad\quad \times (18.9282+t)) \\
&\qquad + \big((288C_2 + 288C_3 + C_1 \\
&\qquad\quad \times (288 + 576C_2 + C_1(576 - 24t) \\
&\qquad\quad - 24C_2t - 3C_1^2(-6.58301+t) \\
&\qquad\quad \times (14.583+t)\cosh(0.25x) \\
&\qquad\quad + (-96C_2 - 96C_3 + C_1 \\
&\qquad\qquad \times (-96 - 192C_2 + C_1(-192 - 24t) \\
&\qquad\qquad - 24C_2t + 5C_1^2(-7.396+t) \\
&\qquad\qquad \times (2.596+t)))\sinh(0.25x)) \\
&\qquad -24C_2t + 5C_1^2(-7.396+t)(2.596+t)\big)\big) \\
&\qquad \times \sinh(0.25x) \\
&\qquad \times (-0.000651042\cosh(0.75x) \\
&\qquad\quad -0.000651042\sinh(0.75x)))\Bigg).
\end{aligned} \tag{31}$$

Adding (25), (27), (29), and (31) we obtain

$$\tilde{u}(x,t,C_1,C_2) = u_0(x,t) + u_1(x,t,C_1) + u_2(x,t,C_1,C_2) + u_3(x,t,C_1,C_2,C_3). \tag{32}$$

For the calculations of the constants C_1, C_2, and C_3 using the collocation method, we have computed that

$$C_1 = -1.0000010231545267,$$
$$C_2 = -9.98159444155818\times10^{-7},$$
$$C_3 = -2.041939789528322\times10^{-12}.$$

Putting the values of these constants into (32) the third order approximate solution using OHAM is

$$\begin{aligned}
&u_3(x,t) \\
&= 0.5 - 5.2607845247854\times10^{-6}t\operatorname{sech}^2(0.25x) \\
&\quad + \frac{1}{(1+e^{0.5x})^2} \\
&\quad \times \big(6.360090415451543\times10^{-9} \\
&\qquad + \big(1.3153067008227217\times10^{-6} \\
&\qquad\quad + 0.0006512058813062292t\big)t \\
&\qquad + \big(1.2720180853076355\times10^{-8} \\
&\qquad\quad - 0.002604823525224917t^2\big)\cosh(0.5x)
\end{aligned}$$

TABLE 1: Comparison of absolute errors of OHAM and ADM [5] for $\alpha = 1$, $\beta = 1$, $\delta = 1$, and $\gamma = 0.001$.

t	ADM $x = 0.1$	OHAM $x = 0.1$	ADM $x = 0.5$	OHAM $x = 0.5$	ADM $x = 0.9$	OHAM $x = 0.9$
0.05	1.93715×10^{-7}	1.87406×10^{-8}	1.9373×10^{-7}	1.87406×10^{-8}	1.93745×10^{-7}	1.87406×10^{-8}
0.1	3.87434×10^{-7}	3.74812×10^{-8}	3.87464×10^{-7}	3.74812×10^{-8}	3.87494×10^{-7}	3.74812×10^{-8}
1	3.87501×10^{-6}	3.74812×10^{-7}	3.87531×10^{-6}	3.74812×10^{-7}	3.87561×10^{-6}	3.74812×10^{-7}

TABLE 2: Comparison of absolute errors obtained by OHAM and ADM [5] for $\alpha = 0$, $\beta = 1$, $\delta = 1$, and $\gamma = 0.001$.

t	ADM $x = 0.1$	OHAM $x = 0.1$	ADM $x = 0.5$	OHAM for $x = 0.5$	ADM $x = 0.9$	OHAM $x = 0.9$
0.05	1.93715×10^{-7}	2.49875×10^{-8}	1.9373×10^{-7}	2.49875×10^{-8}	1.93745×10^{-7}	2.49875×10^{-8}
0.1	3.87434×10^{-7}	4.9975×10^{-8}	3.87464×10^{-7}	4.9975×10^{-8}	3.87494×10^{-7}	4.9975×10^{-8}
1	3.87501×10^{-6}	4.9975×10^{-7}	3.87531×10^{-6}	4.9975×10^{-7}	3.87561×10^{-6}	4.9975×10^{-7}

TABLE 3: Comparison of absolute errors obtained by OHAM and ADM [5] for $\alpha = 0$, $\beta = 1$, $\delta = 2$, and $\gamma = 0.001$.

t	Error ADM	Error OHAM
0.05	5.58836×10^{-7}	2.7938×10^{-7}
0.1	1.11766×10^{-6}	5.58771×10^{-7}
1	1.00741×10^{-5}	5.5896×10^{-6}

$$\begin{aligned} &+ \Big(6.36009042653818 \times 10^{-9} \\ &\quad + \big(-1.3153067008227217 \times 10^{-6} \\ &\qquad + 0.0006512058813062292t\big)\, t\Big) \cosh(x) \\ &+ \Big(1.2720180853076355 \times 10^{-8} \\ &\quad - 0.002604823525224917t^2\Big) \sinh(0.5x) \\ &+ \Big(6.36009042653818 \times 10^{-9} \\ &\quad + \big(-1.3153067008227217 \times 10^{-9} \\ &\qquad + 0.0006512058813062292t\big)\, t\Big) \sinh(x) \\ &- 0.5 \tanh(0.25x) + 0.007813813661895406t^2 \\ &\times \operatorname{sech}^2(0.25x) \tanh(0.25x) + \operatorname{sech}^2(0.25x) \\ &\times \big(0.06250525442670962t\big)\big). \end{aligned} \tag{33}$$

Table 1 shows a comparison between OHAM solution and ADM solution for $\alpha = 1$ and $\gamma = 0.001$. For $\alpha = 0$ (1) is reduced to the generalized Huxley equation which describes nerve pulse propagation in nerve fibers and wall motion in liquid crystals [22]. Tables 2 and 3 show a comparison between ADM solution and OHAM solution for $\alpha = 0$ and $\beta = 1$ respectively. Table 4 shows absolute errors of OHAM solution for larger domain for $\alpha = 0, 1$, $\beta = 1$, and $\delta = 1, 2$ respectively.

3.2. Application of OHAM for Burger's-Fisher Equation. Consider the Burger's-Fisher equation of form (2):

$$\begin{gathered} \frac{\partial u(x,t)}{\partial t} + \alpha u^{\delta}(x,t) \frac{\partial u(x,t)}{\partial x} - \frac{\partial^2 u(x,t)}{\partial x^2} \\ - \beta u(x,t)\left(1 - u^{\delta}(x,t)\right) = 0, \\ \forall 0 \le x \le 1,\ t \ge 0, \end{gathered} \tag{34}$$

subject to constant initial condition

$$u(x,0) = \left(0.5 + 0.5 \tanh\left(\frac{-\alpha\delta}{2(\delta+1)} x\right)\right)^{1/\delta}, \tag{35}$$

with exact solution given by

$$\begin{aligned} u(x,t) &= 0.5 + 0.5 \tanh \\ &\times \left(\frac{-\alpha\delta}{2(\delta+1)}\right. \\ &\left.\times \left(x - \left(\frac{\alpha}{(\delta+1)} + \frac{\beta(\delta+1)}{\alpha}\right)t\right)\right)^{1/\delta}. \end{aligned} \tag{36}$$

For computational work, we have taken $\alpha = 0.001$, $\beta = 0.001$, and $\delta = 1$ for various values of x and t.

Zeroth-Order Problem

$$\begin{gathered} \frac{\partial u_0(x,t)}{\partial t} = 0, \\ u_0(x,0) = \left(0.5 + 0.5 \tanh\left(\frac{-0.001}{4} x\right)\right). \end{gathered} \tag{37}$$

Its solution is

$$u_0(x,t) = \left(0.5 - 0.5 \tanh\left(\frac{0.001}{4} x\right)\right). \tag{38}$$

Table 4: Absolute errors of OHAM for $\alpha = 0, 1$, $\beta = 1$, $\delta = 1, 2$, and $x = 2$.

t	$\alpha = 1, \beta = 1, \delta = 1$, and $\gamma = 0.001$	$\alpha = 0, \beta = 1, \delta = 1$, and $\gamma = 0.001$	$\alpha = 0, \beta = 1, \delta = 2$, and $\gamma = 0.001$
0.1	3.74812×10^{-8}	2.49875×10^{-8}	2.23403×10^{-6}
0.2	7.49625×10^{-8}	4.9975×10^{-8}	4.46806×10^{-6}
0.3	1.12444×10^{-7}	7.49625×10^{-8}	6.70209×10^{-6}
0.4	1.49925×10^{-7}	9.995×10^{-8}	8.93612×10^{-6}
0.5	1.87406×10^{-7}	1.24937×10^{-7}	1.11702×10^{-5}
0.6	2.24887×10^{-7}	1.49925×10^{-7}	1.34042×10^{-5}
0.7	2.62369×10^{-7}	1.74912×10^{-7}	1.56382×10^{-5}
0.8	2.9985×10^{-7}	1.999×10^{-7}	1.78722×10^{-5}
0.9	3.37331×10^{-7}	2.24887×10^{-7}	2.01063×10^{-5}
1.0	3.74812×10^{-7}	2.49875×10^{-7}	2.23403×10^{-5}

Table 5: Comparison of absolute errors obtained by OHAM and ADM [5] for $\alpha = 0.001$, $\beta = 0.001$, and $\delta = 1$.

t	ADM for $x = 0.1$	OHAM $x = 0.1$	ADM $x = 0.5$	OHAM $x = 0.5$	ADM $x = 0.9$	OHAM $x = 0.9$
0.005	9.68763×10^{-6}	1.12257×10^{-7}	9.68691×10^{-6}	2.28888×10^{-7}	9.68619×10^{-6}	2.28888×10^{-7}
0.001	1.93753×10^{-6}	2.24513×10^{-8}	1.93738×10^{-6}	4.57775×10^{-8}	1.93724×10^{-6}	4.57775×10^{-8}
0.01	1.93752×10^{-5}	2.24514×10^{-7}	1.93738×10^{-5}	4.57777×10^{-7}	1.93724×10^{-5}	4.57777×10^{-7}

First-Order Problem

$$\begin{gathered}\frac{\partial u_1(x,t)}{\partial t} - (1 + C_1)\frac{\partial u_0(x,t)}{\partial t} - 0.001C_1u_0(x,t)\frac{\partial u_0(x,t)}{\partial x} \\ - 0.001C_1u_0(x,t)(1 - u_0(x,t)) + C_1\frac{\partial^2 u_0(x,t)}{\partial x^2} = 0, \\ u_1(x,0) = 0.\end{gathered} \tag{39}$$

Its solution is

$$\begin{aligned}u_1(x,t,C_1) &= -t\Big(0.00025C_1 + 6.25 \times 10^{-8}C_1 \\ &\quad \times \operatorname{sech}^2(0.00025x) \\ &\quad - 0.00025C_1\tanh^2(0.00025x)\Big).\end{aligned} \tag{40}$$

Second-Order Problem

$$\begin{gathered}\frac{\partial u_2(x,t)}{\partial t} - (1 + C_1)\frac{\partial u_1(x,t)}{\partial t} + 0.001C_2u_0(x,t) \\ \times (1 - u_0(x,t)) + C_2\frac{\partial^2 u_0(x,t)}{\partial x^2} \\ - C_2\frac{\partial u_0(x,t)}{\partial t} - 0.001C_2u_0(x,t)\frac{\partial u_0(x,t)}{\partial x} \\ - 0.001C_1u_1(x,t)\frac{\partial u_0(x,t)}{\partial x} - 0.001C_1u_0(x,t) \\ \times \frac{\partial u_1(x,t)}{\partial x} + C_1\frac{\partial^2 u_1(x,t)}{\partial x^2} - 0.001C_1u_1(x,t) \\ - 0.002C_1u_0(x,t)u_1(x,t) = 0, \\ u_2(x,0) = 0.\end{gathered} \tag{41}$$

Its solution is

$$\begin{aligned}&u_2(x,t,C_1,C_2) \\ &= \operatorname{sech}^5(0.00025x) \\ &\quad \times \Big(- 0.000187547C_1t - 0.000187547C_2t \\ &\quad + C_1^2(-0.000187547t)\Big)\cosh(0.00025x) \\ &\quad + -0.0000625156C_1t - 0.0000625156C_2t \\ &\quad + C_1^2(-0.0000625156t) + 2.71051 \\ &\quad \times 10^{-20}C_1t\sinh(0.00025x) + 2.71051 \\ &\quad \times 10^{-20}C_1^2t\sinh(0.00025x) + 2.71051 \\ &\quad \times 10^{-20}C_2t\sinh(0.00025x) + 3.12656 \\ &\quad \times 10^{-8}C_1^2t^2\sinh(0.00025x) + 3.12656 \\ &\quad \times 10^{-8}C_1^2t^2\sinh(0.00075x).\end{aligned} \tag{42}$$

Table 6: Comparison of absolute errors obtained by OHAM and ADM [5] for $\alpha = 1$, $\beta = 1$, and $\delta = 2$.

t	ADM for $x = 0.1$	OHAM $x = 0.1$	ADM $x = 0.5$	OHAM $x = 0.5$	ADM $x = 0.9$	OHAM $x = 0.9$
0.0005	1.40177×10^{-3}	5.87633×10^{-5}	1.34526×10^{-3}	1.06736×10^{-5}	1.27699×10^{-3}	4.64718×10^{-5}
0.0001	2.80396×10^{-4}	1.17539×10^{-5}	2.69094×10^{-4}	5.33686×10^{-5}	2.55438×10^{-4}	9.29303×10^{-6}
0.001	2.80301×10^{-3}	1.17512×10^{-4}	2.69000×10^{-3}	1.06739×10^{-4}	2.55346×10^{-3}	9.296×10^{-4}

Table 7: Absolute errors of OHAM for $x = 2$ and $t \in [0.1, 1]$.

t	$\alpha = 0.001$, $\beta = 0.001$, and $\delta = 1$	$\alpha = 0.001$, $\beta = 0.001$, and $\delta = 2$
0.1	1.98526×10^{-9}	1.09926×10^{-5}
0.2	3.20807×10^{-8}	2.19856×10^{-5}
0.3	1.63084×10^{-7}	2.9789×10^{-5}
0.4	5.16881×10^{-7}	4.39726×10^{-5}
0.5	1.26475×10^{-6}	5.49666×10^{-5}
0.6	2.62763×10^{-6}	6.5961×10^{-5}
0.7	4.87621×10^{-6}	7.69557×10^{-5}
0.8	8.33106×10^{-6}	8.79507×10^{-5}
0.9	1.33626×10^{-5}	9.89461×10^{-5}

The third order approximate solution using OHAM is given by

$$\tilde{u}(x,t,C_1,C_2) = u_0(x,t) + u_1(x,t,C_1) + u_2(x,t,C_1,C_2) + u_3(x,t,C_1,C_2,C_3), \tag{43}$$

where $u_3(x,t,C_1,C_2,C_3)$ is obtained in same lines as for first problem.

For the calculations of the constants C_1, C_2, and C_3 using the collocation method we have computed that

$$\begin{aligned} C_1 &= -5.928318703338053 \times 10^{-7}, \\ C_2 &= -465.9630543691778, \\ C_3 &= 1.8651679832921486. \end{aligned} \tag{44}$$

The third order OHAM solution yields very encouraging results after being compared with Fourth order approximate solution by ADM [5].

Table 5 shows a comparison between OHAM solution and ADM solution for $\alpha = 0.001$, $\beta = 0.001$, and $\delta = 1$. Table 6 compares between OHAM solution and ADM solution for $\alpha = 1$, $\beta = 1$, and $\delta = 2$. Table 7 shows the reliability of OHAM for larger domain.

4. Conclusion

We successfully applied OHAM for solution of Burger's-Huxley and Burger's-Fisher equations. The method is simple in applicability and is fast converging to the exact solution. The results obtained by OHAM are very consistent in comparison with ADM.

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