Research Article

Expanded Mixed Finite Element Method for the Two-Dimensional Sobolev Equation

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Expanded mixed finite element method is introduced to approximate the two-dimensional Sobolev equation. This formulation expands the standard mixed formulation in the sense that three unknown variables are explicitly treated. Existence and uniqueness of the numerical solution are demonstrated. Optimal order error estimates for both the scalar and two vector functions are established.

1. Introduction

In this paper, we consider the following Sobolev equation:

$$u_t(\mathbf{x},t) - \nabla \cdot \{a(\mathbf{x},t)\nabla u_t + b_1(\mathbf{x},t)\nabla u(\mathbf{x},t)\}$$
$$= f(\mathbf{x},t), \quad (\mathbf{x},t) \in \Omega \times (0,T],$$
$$u(\mathbf{x},t) = 0, \quad (\mathbf{x},t) \in \partial\Omega \times [0,T],$$
$$u(\mathbf{x},0) = u_0(\mathbf{x}), \quad \mathbf{x} \in \Omega,$$

in a bounded domain $\Omega \subset R^2$ with Lipschitz continuous boundary $\partial\Omega$. Equation (1) has a wide range of applications in many mathematical and physical problems [1, 2], for example, the percolation theory when the fluid flows through the cracks, the transfer problem of the moisture in the soil, and the heat conduction problem in different materials. So there exists great and actual significance to research Sobolev equation. Up till now, there are some different schemes studied to solve this kind of equation (see [1–4] for instance).

The mixed finite element method, which is a finite element method [5] with constrained conditions, plays an important role in the research of the numerical solution for partial differential equations. Its general theory was proposed by Babuska [6] and Brezzi [7]. Falk and Osborn [8] improved their theory and expanded the adaptability of the mixed finite element method. The mixed finite element method

(see [9, 10] for instance) is wildly used for the modeling of fluid flow and transport, as it provides accurate and locally mass conservative velocities.

The main motivation of the expanded mixed finite element method [11–15] is to introduce three (or more) auxiliary variables for practical problems, while the traditional finite element method and mixed finite element method can only approximate one and two variables, respectively. The expanded mixed finite element method also has some other advantages except introducing more variables. It can treat individual boundary conditions. Also, this method is suitable for differential equation with small coefficient (close to zero) which does not need to be inverted. Consequently, this method works for the problems with small diffusion or low permeability terms in fluid problems. Using this method, we can get optimal order error estimates for certain nonlinear problems, while standard mixed formulation sometimes gives only suboptimal error estimates.

The object of this paper is to present the expanded mixed finite element method for the Sobolev equation. We conduct theoretical analysis to study the existence and uniqueness and obtain optimal order error estimates. The rest of this paper is organized as follows. In Section 2, the mixed weak formulation and its mixed element approximation are considered. In Section 3, we prove the existence and uniqueness of approximation form. In Section 4, some lemmas are given.

In Section 5, optimal order semidiscrete error estimates are established.

Throughout the paper, we will use C, with or without subscript, to denote a generic positive constant which does not depend on the discretization parameter h. Vectors will be expressed in boldface. At the same time, we show a useful ε-Cauchy inequality

$$ab \le \varepsilon a^2 + \frac{1}{4\varepsilon}b^2, \quad \varepsilon > 0. \tag{2}$$

2. Mixed Weak Form and Mixed Element Approximation

For $1 \le q \le +\infty$ and m any nonnegative integer, let

$$W^{m,q}(\Omega) = \left\{ f \in L^q(\Omega) \mid D^\alpha f \in L^q(\Omega), \ |\alpha| \le m \right\} \tag{3}$$

denote the Sobolev spaces [16] endowed with the norm

$$\|f\|_{m,q,\Omega} = \left(\sum_{|\alpha| \le m} \|D^\alpha f\|_{L^q(\Omega)}^q \right)^{1/q} \tag{4}$$

(the subscript Ω will always be omitted).

Let $H^m(\Omega) = W^{m,2}(\Omega)$ with the norm $\|\cdot\|_m = \|\cdot\|_{m,2}$. The notation $\|\cdot\|$ will mean $\|\cdot\|_{L^2(\Omega)}$ or $\|\cdot\|_{L^2(\Omega)^2}$. We denote by $(\ ,\ )$ the inner product in either $L^2(\Omega)$ or $L^2(\Omega)^2$; that is,

$$(\xi, \eta) = \int_\Omega \xi \eta d\mathbf{x}, \qquad (\boldsymbol{\xi}, \boldsymbol{\eta}) = \int_\Omega \boldsymbol{\xi} \cdot \boldsymbol{\eta} d\mathbf{x}. \tag{5}$$

The notation $\langle\ ,\ \rangle$ denotes the L^2-inner product on the boundary of Ω

$$\langle \xi, \eta \rangle = \int_{\partial\Omega} \xi \eta ds. \tag{6}$$

To formulate the weak form, let $b = b_1/a$; we introduce two vector variables

$$\boldsymbol{\lambda} = -\nabla u_t - b\nabla u, \qquad \boldsymbol{\sigma} = a\boldsymbol{\lambda} = -a\nabla u_t - b_1 \nabla u. \tag{7}$$

Letting $\mathbf{c} = -\nabla b$, we rewrite (1) as the following system:

$$u_t + \nabla \cdot \boldsymbol{\sigma} = f, \quad (\mathbf{x}, t) \in \Omega \times (0, T],$$

$$\boldsymbol{\lambda} + \nabla u_t - \nabla (bu) + \mathbf{c}u = 0, \quad (\mathbf{x}, t) \in \Omega \times (0, T],$$

$$\boldsymbol{\sigma} - a\boldsymbol{\lambda} = 0, \quad (\mathbf{x}, t) \in \Omega \times (0, T], \tag{8}$$

$$u(\mathbf{x}, t) = 0, \quad (\mathbf{x}, t) \in \partial\Omega \times (0, T],$$

$$u(\mathbf{x}, 0) = u_0(\mathbf{x}), \quad \mathbf{x} \in \Omega.$$

Let $\mathbf{n}$ be the unit exterior normal vector to the boundary of Ω. Then (8) is formulated in the following expanded mixed weak form: find $(u, \boldsymbol{\lambda}, \boldsymbol{\sigma}) \in W \times \boldsymbol{\Lambda} \times \mathbf{V}$, such that

$$(u_t, w) + (\operatorname{div} \boldsymbol{\sigma}, w) = (f, w),$$

$$\forall w \in W, \ 0 < t \le T,$$

$$(\boldsymbol{\lambda}, \mathbf{v}) - (u_t + bu, \operatorname{div} \mathbf{v}) + (\mathbf{c}u, \mathbf{v}) = 0,$$

$$\forall \mathbf{v} \in \mathbf{V}, \ 0 < t \le T, \tag{9}$$

$$(\boldsymbol{\sigma}, \boldsymbol{\tau}) - (a\boldsymbol{\lambda}, \boldsymbol{\tau}) = 0, \quad \forall \boldsymbol{\tau} \in \boldsymbol{\Lambda}, \ 0 < t \le T,$$

$$(u(0), w) = (u_0, w), \quad \forall w \in W,$$

where

$$W = L^2(\Omega), \qquad \boldsymbol{\Lambda} = L^2(\Omega)^2,$$

$$\mathbf{V} = H(\operatorname{div}, \Omega) = \left\{ \mathbf{v} \in L^2(\Omega)^2 \mid \nabla \cdot \mathbf{v} \in L^2(\Omega) \right\}. \tag{10}$$

Let T_h be a quasiregular polygonalization of Ω (by triangles or rectangles), with h being the maximum diameter of the elements of the polygonalization. Let $W_h \times \boldsymbol{\Lambda}_h \times \mathbf{V}_h$ be a conforming mixed element space with index k and discretization parameter h. $W_h \times \boldsymbol{\Lambda}_h \times \mathbf{V}_h$ is an approximation to $W \times \boldsymbol{\Lambda} \times \mathbf{V}$. There are many conforming (or compatible) mixed element function spaces such as Raviart-Thomas elements [17], BDFM elements [18, 19]. Some RT type mixed elements are listed in Table 1. Here, $P_k(T)$ is the polynomial up to k order in two-dimensional domain used in triangle, while $Q_{k,l}(T)$ is the polynomial up to k, l in each dimension used in rectangle.

Replacing the three variables by their approximation, we get the expanded mixed finite element approximation problem: find $(u_h, \boldsymbol{\lambda}_h, \boldsymbol{\sigma}_h) \in W_h \times \boldsymbol{\Lambda}_h \times \mathbf{V}_h$, such that

$$(u_{h,t}, w) + (\operatorname{div} \boldsymbol{\sigma}_h, w) = (f, w),$$

$$\forall w \in W_h, \ 0 < t \le T,$$

$$(\boldsymbol{\lambda}_h, \mathbf{v}) - (u_{h,t} + bu_h, \operatorname{div} \mathbf{v}) + (\mathbf{c}u_h, \mathbf{v}) = 0,$$

$$\forall \mathbf{v} \in \mathbf{V}_h, \ 0 < t \le T, \tag{11}$$

$$(\boldsymbol{\sigma}_h, \boldsymbol{\tau}) - (a\boldsymbol{\lambda}_h, \boldsymbol{\tau}) = 0, \quad \forall \boldsymbol{\tau} \in \boldsymbol{\Lambda}_h, \ 0 < t \le T,$$

$$(u(0), w) = (\tilde{u}_0, w), \quad \forall w \in W_h,$$

where

$$W_h(E) = \left\{ \omega \in W : \omega|_E \in W_h(E), \forall E \in T_h \right\},$$

$$\boldsymbol{\Lambda}_h(E) = \left\{ \boldsymbol{\tau} \in \boldsymbol{\Lambda} : \boldsymbol{\tau}|_E \in \mathbf{V}_h(E), \forall E \in T_h \right\}, \tag{12}$$

$$\mathbf{V}_h(E) = \left\{ \mathbf{v} \in \mathbf{V} : \mathbf{v}|_E \in \mathbf{V}_h(E), \forall E \in T_h \right\}.$$

The error analysis next makes use of three projection operators. The first operator is the Raviart-Thomas projection (or Brezzi-Douglas-Marini projection) $\Pi_h : \mathbf{V} \to \mathbf{V}_h$; Π_h satisfies

$$(\nabla \cdot (\mathbf{v} - \Pi_h \mathbf{v}), \omega_h) = 0, \quad \forall \omega_h \in W_h. \tag{13}$$

Table 1: Some RT type mixed elements.

Dim	Element	$\mathbf{V}_h(E)$	$W_h(E)$
2D	Triangle	$\mathrm{RT}_k(T) = P_k(E)^2 \oplus \mathbf{x} P_k(E)$	$P_k(E)$
2D	Rectangle	$\mathrm{RT}_{[k]}(T) = Q_{k+1,k}(E) \oplus Q_{k,k+1}(E)$	$Q_{k,k}(E)$

The following approximation properties are well known.

$$\|\mathbf{v} - \Pi_h \mathbf{v}\| \le C h^r \|\mathbf{v}\|_r, \quad \frac{1}{2} \le r \le k+1, \tag{14}$$

$$\left\|\nabla \cdot (\mathbf{v} - \Pi_h \mathbf{v})\right\|_r \le C h^r \|\nabla \cdot \mathbf{v}\|_r, \quad 0 \le r \le k+1. \tag{15}$$

The other two operators are the standard L^2-projection [5] P_h and R_h onto W_h and Λ_h, respectively,

$$(\omega - P_h \omega, \nabla \cdot \mathbf{v}_h) = 0, \quad \forall \omega \in W, \ \mathbf{v}_h \in \mathbf{V}_h,$$
$$(\boldsymbol{\tau} - R_h \boldsymbol{\tau}, \boldsymbol{\tau}_h) = 0, \quad \forall \boldsymbol{\tau} \in \Lambda, \ \boldsymbol{\tau}_h \in \Lambda_h. \tag{16}$$

They have the approximation properties

$$\|\omega - P_h \omega\| \le C h^r \|\omega\|_r, \quad 0 \le r \le k+1,$$
$$\|\boldsymbol{\tau} - R_h \boldsymbol{\tau}\| \le C h^r \|\boldsymbol{\mu}\|_r, \quad 0 \le r \le k+1. \tag{17}$$

The two projections Π_h and P_h preserve the commuting property

$$\mathrm{div} \circ \Pi_h = P_h \circ \mathrm{div} : H^1(\Omega)^2 \longrightarrow W_h. \tag{18}$$

3. Existence and Uniqueness of Approximation Form

In this section, we consider the existence and uniqueness of the solution of (11).

Lemma 1. *Equation* (11) *has a unique solution.*

Proof. In fact, this equation is linear; it suffices to show that the associated homogeneous system

$$(u_{h,t}, w) + (\mathrm{div}\, \boldsymbol{\sigma}_h, w) = 0,$$
$$\forall w \in W_h, \ 0 < t \le T,$$
$$(\boldsymbol{\lambda}_h, \mathbf{v}) - (u_{h,t} + b u_h, \mathrm{div}\, \mathbf{v}) + (c u_h, \mathbf{v}) = 0,$$
$$\forall \mathbf{v} \in \mathbf{V}_h, \ 0 < t \le T, \tag{19}$$
$$(\boldsymbol{\sigma}_h, \boldsymbol{\tau}) - (a \boldsymbol{\lambda}_h, \boldsymbol{\tau}) = 0,$$
$$\forall \boldsymbol{\tau} \in \Lambda_h, \ 0 < t \le T,$$
$$(u(0), w) = 0, \quad \forall w \in W_h,$$

has only the trivial solution. In the first equation of (19), if we take $\omega = u_{h,t}$ and $\omega = \mathrm{div}\, \boldsymbol{\sigma}_h$, respectively, then we have that

$$\|u_{h,t}\| \le \|\mathrm{div}\, \boldsymbol{\sigma}_h\|,$$
$$\|\mathrm{div}\, \boldsymbol{\sigma}_h\| \le \|u_{h,t}\|. \tag{20}$$

By (20), it is easy to see that

$$\|\mathrm{div}\, \boldsymbol{\sigma}_h\| = \|u_{h,t}\|. \tag{21}$$

Choosing $\omega = \mathrm{div}\, \boldsymbol{\sigma}_h$, $\mathbf{v} = \boldsymbol{\sigma}_h$, and $\boldsymbol{\tau} = \boldsymbol{\lambda}_h$ in (19), we get

$$(a \boldsymbol{\lambda}_h, \boldsymbol{\lambda}_h) + (\mathrm{div}\, \boldsymbol{\sigma}_h, \mathrm{div}\, \boldsymbol{\sigma}_h) = (b u_h, \mathrm{div}\, \boldsymbol{\sigma}_h) - (c u_h, \boldsymbol{\sigma}_h). \tag{22}$$

In the third equation of (19), letting $\boldsymbol{\tau} = \boldsymbol{\sigma}_h$ and $\boldsymbol{\tau} = \boldsymbol{\lambda}_h$, respectively, then

$$\|\boldsymbol{\sigma}_h\| \le C \|\boldsymbol{\lambda}_h\|, \tag{23}$$
$$\|\boldsymbol{\lambda}_h\| \le C \|\boldsymbol{\sigma}_h\|. \tag{24}$$

Using the ε-Cauchy inequality to (22), we have

$$a_0 \|\boldsymbol{\lambda}_h\|^2 + \|\mathrm{div}\, \boldsymbol{\sigma}_h\|^2 \le C \|u_h\|^2 + \frac{1}{2} \|\mathrm{div}\, \boldsymbol{\sigma}_h\|^2 + \frac{a_0}{2} \|\boldsymbol{\lambda}_h\|^2. \tag{25}$$

Note that (21), we deserve

$$\|\boldsymbol{\lambda}_h\| \le C \|u_h\|, \tag{26}$$
$$\|u_{h,t}\| \le C \|u_h\|. \tag{27}$$

From (27), we know

$$\|u_h\| = \left\| \int_0^t u_{h,t} \, dt \right\| \le C \int_0^t \|u_{h,t}\| \, dt \le C \int_0^t \|u_h\| \, dt. \tag{28}$$

Using Gronwall's inequality to (28), we can prove $u_h = 0$. Further, from (26) and (23), we know that $\boldsymbol{\lambda}_h = \mathbf{0}$, $\boldsymbol{\sigma}_h = \mathbf{0}$. The proof is completed. $\square$

4. Some Lemmas

In the study of parabolic equations, we usually introduce a mixed elliptic projection associated with our equations. Define a map: find $(\widetilde{u}_h, \widetilde{\boldsymbol{\lambda}}_h, \widetilde{\boldsymbol{\sigma}}_h) \in W_h \times \Lambda_h \times \mathbf{V}_h$, such that

$$(\mathrm{div}\, (\boldsymbol{\sigma} - \widetilde{\boldsymbol{\sigma}}_h), w) = 0, \quad \forall w \in W_h, \ 0 < t \le T,$$
$$(\boldsymbol{\lambda} - \widetilde{\boldsymbol{\lambda}}_h, \mathbf{v}) - (u_t - \widetilde{u}_{h,t} + b(u - \widetilde{u}_h), \mathrm{div}\, \mathbf{v}) + (\mathbf{c}(u - \widetilde{u}_h), \mathbf{v})$$
$$= 0, \quad \forall \mathbf{v} \in \mathbf{V}_h, \ 0 < t \le T,$$
$$(\boldsymbol{\sigma} - \widetilde{\boldsymbol{\sigma}}_h, \boldsymbol{\tau}) - (a(\boldsymbol{\lambda} - \widetilde{\boldsymbol{\lambda}}_h), \boldsymbol{\tau}) = 0, \quad \forall \boldsymbol{\tau} \in \Lambda_h, \ 0 < t \le T,$$
$$(u(0) - \widetilde{u}_h(0), w) = 0, \quad \forall w \in W_h. \tag{29}$$

Similarly to Lemma 1, we can prove that system (29) has a unique solution.

Now we give some error estimates of $(\widetilde{u}_h, \widetilde{\boldsymbol{\lambda}}_h, \widetilde{\boldsymbol{\sigma}}_h)$. Define

$$u - \widetilde{u}_h = (p_h u - \widetilde{u}_h) + (u - p_h u) = u_1 + u_2,$$
$$\lambda_1 = \boldsymbol{\lambda} - \widetilde{\boldsymbol{\lambda}}_h, \qquad \sigma_1 = \boldsymbol{\sigma} - \widetilde{\boldsymbol{\sigma}}_h. \tag{30}$$

System (29) can be rewritten as follows:

$$(\operatorname{div}\boldsymbol{\sigma}_1, w) = 0, \quad \forall w \in W_h,$$

$$(\boldsymbol{\lambda}_1, \mathbf{v}) - (u_{1,t} + b(u_1 + u_2), \operatorname{div}\mathbf{v}) + (\mathbf{c}(u_1 + u_2), \mathbf{v}) = 0,$$
$$\forall \mathbf{v} \in \mathbf{V}_h,$$

$$(\boldsymbol{\sigma}_1, \boldsymbol{\tau}) - (a\boldsymbol{\lambda}_1, \boldsymbol{\tau}) = 0, \quad \forall \boldsymbol{\tau} \in \Lambda_h.$$

$$(u(0) - \tilde{u}_h(0), w) = 0, \quad \forall w \in W_h. \tag{31}$$

Now we consider the estimates of u_1 and $u_{1,t}$.

Lemma 2. *Let $(u, \boldsymbol{\lambda}_1, \boldsymbol{\sigma}_1)$ be the solution of system (31). If u, $\boldsymbol{\lambda}$, and $\boldsymbol{\sigma}$ are sufficiently smooth, then there exist positive constants C such that*

$$\|u_1\| \leq C \int_0^t \left(h\|\boldsymbol{\lambda}_1\| + h^{1-\delta_{k0}}\|\boldsymbol{\lambda}_1\| + h\|u_2\| + \|u_2\|_{-1} \right) dt,$$

$$\|u_{1,t}\| \leq C \left(h\|\boldsymbol{\lambda}_1\| + h^{1-\delta_{k0}}\|\boldsymbol{\lambda}_1\| + h\|u_2\| + \|u_2\|_{-1} + \|u_1\| \right), \tag{32}$$

where $\delta_{k0} = 0$ for $k = 0$, and $\delta_{k0} = 0$ for $k \geq 1$.

Proof. Let $\phi \in H^2(\Omega) \bigcap H_0^1(\Omega)$ be the solution of the following problem:

$$\nabla \cdot (a\nabla\phi) = \psi, \quad \forall x \in \Omega,$$
$$\phi|_{\partial\Omega} = 0. \tag{33}$$

Then we know

$$\|\phi\|_2 \leq C\|\psi\|. \tag{34}$$

For $0 < t \leq T$, from the second equation of (31), we have that

$$(u_{1,t}, \psi) = (u_{1,t}, \operatorname{div}(a\nabla\phi))$$
$$= (u_{1,t}, \operatorname{div}\Pi_h(a\nabla\phi))$$
$$= (\boldsymbol{\lambda}_1, \Pi_h(a\nabla\phi)) + (\mathbf{c}(u_1 + u_2), \Pi_h(a\nabla\phi)) \tag{35}$$
$$\quad - (b(u_1 + u_2), \operatorname{div}\Pi_h(a\nabla\phi))$$
$$= I_1 + I_2 + I_3.$$

It is easy to see that

$$I_1 = (\boldsymbol{\lambda}_1, \Pi_h(a\nabla\phi))$$
$$= (\boldsymbol{\lambda}_1, \Pi_h(a\nabla\phi) - a\nabla\phi) + (\boldsymbol{\lambda}_1, a\nabla\phi)$$
$$= (\boldsymbol{\lambda}_1, \Pi_h(a\nabla\phi) - a\nabla\phi) + (a\boldsymbol{\lambda}_1, \nabla\phi)$$
$$= E_1 + E_2,$$

$$E_1 \leq Ch\|\boldsymbol{\lambda}_1\|\|\phi\|_2 \leq Ch\|\boldsymbol{\lambda}_1\|\|\psi\|,$$

$$E_2 = (a\boldsymbol{\lambda}_1, \nabla(\phi - p_h\phi)) + (a\boldsymbol{\lambda}_1, \nabla(p_h\phi))$$
$$= (a\boldsymbol{\lambda}_1, \nabla(\phi - p_h\phi)) + (\boldsymbol{\sigma}_1, \nabla(p_h\phi)) \tag{36}$$
$$= (a\boldsymbol{\lambda}_1, \nabla(\phi - p_h\phi)) - (\operatorname{div}\boldsymbol{\sigma}_1, (p_h\phi))$$
$$= (a\boldsymbol{\lambda}_1, \nabla(\phi - p_h\phi))$$
$$\leq C\|\boldsymbol{\lambda}_1\|\|\nabla(\phi - p_h\phi)\|$$
$$\leq Ch^{1-\delta_{ko}}\|\phi\|_2\|\boldsymbol{\lambda}_1\|$$
$$\leq Ch^{1-\delta_{ko}}\|\psi\|\|\boldsymbol{\lambda}_1\|.$$

Now, we estimate I_2,

$$(\mathbf{c}u_1, \Pi_h(a\nabla\phi))$$
$$\leq C\|u_1\|\|\Pi_h(a\nabla\phi) - a\nabla\phi + a\nabla\phi\|$$
$$\leq C\|u_1\|(h\|\nabla\phi\| + \|\nabla\phi\|)$$
$$\leq C\|u_1\|\|\psi\|, \tag{37}$$
$$(\mathbf{c}u_2, \Pi_h(a\nabla\phi))$$
$$= (\mathbf{c}u_2, \Pi_h(a\nabla\phi) - a\nabla\phi) + (\mathbf{c}u_2, a\nabla\phi)$$
$$\leq C(h\|u_2\| + \|u_2\|_{-1})\|\psi\|.$$

We turn to consider I_3,

$$(-bu_1, \operatorname{div}\Pi_h(a\nabla\phi))$$
$$\leq C\|u_1\|\|\nabla\phi\| \leq C\|u_1\|\|\psi\|,$$
$$(-bu_2, \operatorname{div}\Pi_h(a\nabla\phi))$$
$$\leq -\left((b - p_h^0 b)u_2, \operatorname{div}\Pi_h(a\nabla\phi)\right) \tag{38}$$
$$\quad - \sum_{e \in T_h} \left((p_h^0 b)u_2, \operatorname{div}\Pi_h(a\nabla\phi)\right)_e$$
$$= -\left((b - p_h^0 b)u_2, \operatorname{div}\Pi_h(a\nabla\phi)\right)$$
$$\leq Ch\|u_2\|\|\psi\|,$$

where $p_h^0 b$ is the piecewise constant interpolation of function b. Using the previous estimates, we obtain

$$\|u_{1,t}\|$$
$$\leq C\left(h\|\boldsymbol{\lambda}_1\| + h^{1-\delta_{k0}}\|\boldsymbol{\lambda}_1\| + h\|u_2\| + \|u_2\|_{-1} + \|u_1\|\right). \tag{39}$$

So we deserve

$$\|u_1\| = \left\| \int_0^t u_{1,t} \, dt \right\|$$

$$\leq C \int_0^t \|u_{1,t}\| \, dt \tag{40}$$

$$\leq C \int_0^t \left(h \|\lambda_1\| + h^{1-\delta_{k0}} \|\lambda_1\| + h \|u_2\| + \|u_2\|_{-1} + \|u_1\| \right) dt.$$

Applying Gronwall's inequality to (40), we have

$$\|u_1\|$$

$$\leq C \int_0^t \left(h \|\lambda_1\| + h^{1-\delta_{k0}} \|\lambda_1\| + h \|u_2\| + \|u_2\|_{-1} \right) dt. \tag{41}$$

Noting the estimates of I_1, I_2, and I_3, the proof is completed. □

Define

$$\|u, \lambda, \sigma\|_r = \|u\|_r + \|\lambda\|_r + \|\sigma\|_r,$$

$$\||u, \lambda, \sigma|\|_r = \int_0^t \left(\|u\|_r + \|\lambda\|_r + \|\sigma\|_r \right) dt. \tag{42}$$

Lemma 3. *Let (u, λ, σ) and $(\tilde{u}_h, \tilde{\lambda}_h, \tilde{\sigma}_h)$ be the solution of (19) and (29), respectively. If u, λ, and σ are sufficiently smooth, then there exist positive constants C such that*

(I) *if $k \geq 0$, then*

$$\|\mathrm{div}\,(\sigma - \tilde{\sigma}_h)\| \leq Ch^r \|\sigma\|_{r+1}, \quad 0 \leq r \leq k+1, \tag{43}$$

$$\|p_h u - \tilde{u}_h\| \leq Ch^{r+1-\delta_{k0}} \||u, \lambda, \sigma|\|_r, \quad 1 \leq r \leq k+1, \tag{44}$$

$$\|p_h u_t - \tilde{u}_{h,t}\| \leq Ch^{r+1-\delta_{k0}} \|u, \lambda, \sigma\|_r, \quad 1 \leq r \leq k+1, \tag{45}$$

(II) *if $k = 0$, then*

$$\|u - \tilde{u}_h\| \leq Ch \left(\|u\|_1 + \||u, \lambda, \sigma|\|_1 \right), \tag{46}$$

$$\|u_t - \tilde{u}_{h,t}\| \leq Ch \left(\|u_t\|_1 + \|u, \lambda, \sigma\|_1 + \||u, \lambda, \sigma|\|_1 \right), \tag{47}$$

$$\|\lambda - \tilde{\lambda}_h\| \leq Ch \left(\|u, \lambda, \sigma\|_1 + \||u, \lambda, \sigma|\|_1 \right), \tag{48}$$

$$\|\sigma - \tilde{\sigma}_h\| \leq Ch \left(\|u, \lambda, \sigma\|_1 + \||u, \lambda, \sigma|\|_1 \right), \tag{49}$$

(III) *if $k \geq 1, 2 \leq r \leq k+1$, then*

$$\|u - \tilde{u}_h\| \leq Ch^r \left(\|u\|_r + \||u, \lambda, \sigma|\|_{r-1} \right), \tag{50}$$

$$\|u_t - \tilde{u}_{h,t}\| \leq Ch^r \left(\|u_t\|_r + \|u, \lambda, \sigma\|_r + \||u, \lambda, \sigma|\|_{r-1} \right), \tag{51}$$

$$\|\lambda - \tilde{\lambda}_h\| \leq Ch^r \left(\|u, \lambda, \sigma\|_r + \||u, \lambda, \sigma|\|_{r-1} \right), \tag{52}$$

$$\|\sigma - \tilde{\sigma}_h\| \leq Ch^r \left(\|u, \lambda, \sigma\|_r + \||u, \lambda, \sigma|\|_{r-1} \right). \tag{53}$$

Proof. For $0 < t \leq T$, the proof proceeds in three steps as follows.

(I) In the first equation of (29), taking $\omega = \mathrm{div}\,\sigma_1$, we know

$$(\mathrm{div}\,\sigma_1, \mathrm{div}\,\sigma_1)$$

$$= (\mathrm{div}\,\sigma_1, \mathrm{div}\,(\sigma - \Pi_h \sigma)) \tag{54}$$

$$\leq \|\mathrm{div}\,\sigma_1\| \|\mathrm{div}\,(\sigma - \Pi_h \sigma)\|,$$

which implies that

$$\|\mathrm{div}\,\sigma_1\| \leq \|\mathrm{div}\,(\sigma - \Pi_h \sigma)\|$$

$$\leq Ch^r \|\sigma\|_{r+1}, \quad 0 \leq r \leq k+1. \tag{55}$$

Choosing $\omega \in W_h$, note that $(\mathrm{div}(\sigma - \tilde{\sigma}_h), \omega) = 0$ and $(\mathrm{div}(\sigma - \Pi_h \sigma), \omega) = 0$, we get

$$\mathrm{div}\,(\Pi_h \sigma - \tilde{\sigma}_h) = 0. \tag{56}$$

Combing (15) with (56), we can prove (43).

(II) In (29), letting $\mathbf{v} = \Pi_h \sigma - \tilde{\sigma}_h$, together with (43), we obtain

$$\left(\lambda - \tilde{\lambda}_h, \Pi_h \sigma - \tilde{\sigma}_h \right) - \left(\mathbf{c}\,(u - \tilde{u}_h), \Pi_h \sigma - \tilde{\sigma}_h \right) = 0. \tag{57}$$

Hence

$$\left(R_h \lambda - \tilde{\lambda}_h, \Pi_h \sigma - \tilde{\sigma}_h \right) - \left(\mathbf{c}\,(u - \tilde{u}_h), \Pi_h \sigma - \tilde{\sigma}_h \right) = 0. \tag{58}$$

In the third equation of (29), choosing $\tau = R_h \lambda - \tilde{\lambda}_h$, we get

$$\left(\sigma - \tilde{\sigma}_h, R_h \lambda - \tilde{\lambda}_h \right) = \left(a\,(\lambda - \tilde{\lambda}_h), R_h \lambda - \tilde{\lambda}_h \right). \tag{59}$$

So we know

$$\left(\sigma - \Pi_h \sigma, R_h \lambda - \tilde{\lambda}_h \right) + \left(\Pi_h \sigma - \tilde{\sigma}_h, R_h \lambda - \tilde{\lambda}_h \right)$$

$$= \left(a\,(\lambda - \tilde{\lambda}_h), R_h \lambda - \tilde{\lambda}_h \right). \tag{60}$$

Note that (44) and (45), we obtain

$$\left(\Pi_h \sigma - \tilde{\sigma}_h, R_h \lambda - \tilde{\lambda}_h \right)$$

$$= \left(\mathbf{c}\,(u_h - \tilde{u}_h), \Pi_h \sigma - \tilde{\sigma}_h \right)$$

$$= \left(a\,(\lambda - \tilde{\lambda}_h), R_h \lambda - \tilde{\lambda}_h \right) - \left(\sigma - \Pi_h \sigma, R_h \lambda - \tilde{\lambda}_h \right), \tag{61}$$

which implies that

$$\left(\Pi_h \sigma - \tilde{\sigma}_h, R_h \lambda - \tilde{\lambda}_h \right)$$

$$\leq C \|u - \tilde{u}_h\| \|\Pi_h \sigma - \tilde{\sigma}_h\| \tag{62}$$

$$\leq C \left(\|u_1\| + \|u_2\| \right) \|\Pi_h \sigma - \tilde{\sigma}_h\|.$$

Now we estimate $\|\Pi_h \sigma - \tilde{\sigma}_h\|$,

$$\left(\sigma - \tilde{\sigma}_h, \Pi_h \sigma - \tilde{\sigma}_h \right) = \left(a\,(\lambda - \tilde{\lambda}_h), \Pi_h \sigma - \tilde{\sigma}_h \right). \tag{63}$$

It is easy to see that

$$
\begin{aligned}
&\left\|\Pi_h \boldsymbol{\sigma} - \widetilde{\boldsymbol{\sigma}}_h\right\|^2 \\
&\quad = \left(\Pi_h \boldsymbol{\sigma} - \boldsymbol{\sigma}, \Pi_h \boldsymbol{\sigma} - \widetilde{\boldsymbol{\sigma}}_h\right) + \left(a\left(\boldsymbol{\lambda} - \widetilde{\boldsymbol{\lambda}}_h\right), \Pi_h \boldsymbol{\sigma} - \widetilde{\boldsymbol{\sigma}}_h\right) \\
&\quad = \left(\Pi_h \boldsymbol{\sigma} - \boldsymbol{\sigma}, \Pi_h \boldsymbol{\sigma} - \widetilde{\boldsymbol{\sigma}}_h\right) + \left(a\left(\boldsymbol{\lambda} - R_h \boldsymbol{\lambda}\right), \Pi_h \boldsymbol{\sigma} - \widetilde{\boldsymbol{\sigma}}_h\right) \\
&\qquad + \left(a\left(R_h \boldsymbol{\lambda} - \widetilde{\boldsymbol{\lambda}}_h\right), \Pi_h \boldsymbol{\sigma} - \widetilde{\boldsymbol{\sigma}}_h\right) \\
&\quad \leq C\left(\left\|\Pi_h \boldsymbol{\sigma} - \boldsymbol{\sigma}\right\| + \left\|\boldsymbol{\lambda} - R_h \boldsymbol{\lambda}\right\| + \left\|u_1\right\| + \left\|u_2\right\|\right) \\
&\qquad \times \left\|\Pi_h \boldsymbol{\sigma} - \widetilde{\boldsymbol{\sigma}}_h\right\|.
\end{aligned}
\tag{64}
$$

By the properties of projection Π_h, we get

$$
\left\|\Pi_h \boldsymbol{\sigma} - \widetilde{\boldsymbol{\sigma}}_h\right\| \leq C h^r \left(\|u\|_r + \|\boldsymbol{\lambda}\|_r + \|\boldsymbol{\sigma}\|_r + \|u_1\|\right),
\tag{65}
$$
$$
0 \leq r \leq k + 1,
$$

which proves (49) and (53).

Now we estimate $\|R_h \boldsymbol{\lambda} - \widetilde{\boldsymbol{\lambda}}_h\|$,

$$
\begin{aligned}
&\left(a\left(R_h \boldsymbol{\lambda} - \widetilde{\boldsymbol{\lambda}}_h\right), R_h \boldsymbol{\lambda} - \widetilde{\boldsymbol{\lambda}}_h\right) \\
&\quad = \left(a\left(\boldsymbol{\lambda} - \widetilde{\boldsymbol{\lambda}}_h, R_h \boldsymbol{\lambda} - \widetilde{\boldsymbol{\lambda}}_h\right)\right. \\
&\qquad \left. + \left(a\left(R_h \boldsymbol{\lambda} - \boldsymbol{\lambda}\right), R_h \boldsymbol{\lambda} - \widetilde{\boldsymbol{\lambda}}_h\right)\right. \\
&\quad = \left(\boldsymbol{\sigma} - \Pi_h \boldsymbol{\sigma}, R_h \boldsymbol{\lambda} - \widetilde{\boldsymbol{\lambda}}_h\right) + \left(\Pi_h \boldsymbol{\sigma} - \widetilde{\boldsymbol{\sigma}}_h, R_h \boldsymbol{\lambda} - \widetilde{\boldsymbol{\lambda}}_h\right) \\
&\qquad + \left(a\left(R_h \boldsymbol{\lambda} - \boldsymbol{\lambda}\right), R_h \boldsymbol{\lambda} - \widetilde{\boldsymbol{\lambda}}_h\right) \\
&\quad = \left(\boldsymbol{\sigma} - \Pi_h \boldsymbol{\sigma}, R_h \boldsymbol{\lambda} - \widetilde{\boldsymbol{\lambda}}_h\right) + \left(\mathbf{c}\left(u - \widetilde{u}_h\right), \Pi_h \boldsymbol{\sigma} - \widetilde{\boldsymbol{\sigma}}_h\right) \\
&\qquad + \left(a\left(R_h \boldsymbol{\lambda} - \boldsymbol{\lambda}\right), R_h \boldsymbol{\lambda} - \widetilde{\boldsymbol{\lambda}}_h\right) \\
&\quad \leq \left(\left\|\boldsymbol{\sigma} - \Pi_h \boldsymbol{\sigma}\right\| + C\left\|\boldsymbol{\lambda} - R_h \boldsymbol{\lambda}\right\|\right) \left\|R_h \boldsymbol{\lambda} - \widetilde{\boldsymbol{\lambda}}_h\right\| \\
&\qquad + C\left\|u_1 + u_2\right\| \left\|\Pi_h \boldsymbol{\sigma} - \widetilde{\boldsymbol{\sigma}}_h\right\| \\
&\quad \leq C\left(\left\|\boldsymbol{\sigma} - \Pi_h \boldsymbol{\sigma}\right\|^2 + \left\|u_1\right\|^2 + \left\|u_2\right\|^2 + \left\|R_h \boldsymbol{\lambda} - \boldsymbol{\lambda}\right\|^2\right) \\
&\qquad + \frac{c_0}{2} \left\|R_h \boldsymbol{\lambda} - \widetilde{\boldsymbol{\lambda}}_h\right\|^2,
\end{aligned}
\tag{66}
$$

where

$$
\left(a\left(R_h \boldsymbol{\lambda} - \widetilde{\boldsymbol{\lambda}}_h\right), R_h \boldsymbol{\lambda} - \widetilde{\boldsymbol{\lambda}}_h\right) \geq c_0 \left\|R_h \boldsymbol{\lambda} - \widetilde{\boldsymbol{\lambda}}_h\right\|^2.
\tag{67}
$$

From (66) and (67), we know

$$
\left\|R_h \boldsymbol{\lambda} - \widetilde{\boldsymbol{\lambda}}_h\right\| \leq C\left(h^r\left(\|u\|_r + \|\boldsymbol{\lambda}\|_r + \|\boldsymbol{\sigma}\|_r\right) + \|u_1\|\right),
$$
$$
\left\|\boldsymbol{\lambda} - \widetilde{\boldsymbol{\lambda}}_h\right\| \leq C\left(h^r\left(\|u\|_r + \|\boldsymbol{\lambda}\|_r + \|\boldsymbol{\sigma}\|_r\right) + \|u_1\|\right),
\tag{68}
$$
$$
0 \leq r \leq k + 1.
$$

By Lemma 2, if $k = 0$, we have that

$$
\begin{aligned}
\left\|\boldsymbol{\lambda} - \widetilde{\boldsymbol{\lambda}}_h\right\| &= \|\boldsymbol{\lambda}_1\| \\
&\leq Ch\left(\|u\|_1 + \|\boldsymbol{\lambda}\|_1 + \|\boldsymbol{\sigma}\|_1\right. \\
&\qquad \left. + \int_0^t \left(\|u\|_1 + \|\boldsymbol{\lambda}\|_1 + \|\boldsymbol{\sigma}\|_1\right) d\tau\right).
\end{aligned}
\tag{69}
$$

If $k \geq 1$, $2 \leq r \leq k + 1$, we have that

$$
\begin{aligned}
\left\|\boldsymbol{\lambda} - \widetilde{\boldsymbol{\lambda}}_h\right\| &\leq Ch^r\left(\|u\|_r + \|\boldsymbol{\lambda}\|_r + \|\boldsymbol{\sigma}\|_r\right. \\
&\qquad \left. + \int_0^t \left(\|u\|_{r-1} + \|\boldsymbol{\lambda}\|_{r-1}\|\boldsymbol{\sigma}\|_{r-1}\right) dt\right).
\end{aligned}
\tag{70}
$$

Using the estimate of $\|\Pi_h \boldsymbol{\sigma} - \widetilde{\boldsymbol{\sigma}}_h\|$, we obtain

$$
\left\|\boldsymbol{\sigma} - \widetilde{\boldsymbol{\sigma}}_h\right\| \leq C\left(h^r\left(\|u\|_r + \|\boldsymbol{\lambda}\|_r + \|\boldsymbol{\sigma}\|_r\right) + \|u_1\|\right),
\tag{71}
$$
$$
0 \leq r \leq k + 1.
$$

If $k = 0$, we get

$$
\begin{aligned}
&\left\|\boldsymbol{\sigma} - \widetilde{\boldsymbol{\sigma}}_h\right\| \\
&\quad \leq Ch\left(\|u\|_1 + \|\boldsymbol{\lambda}\|_1 + \|\boldsymbol{\sigma}\|_1\right. \\
&\qquad \left. + \int_0^t \left(\|u\|_1 + \|\boldsymbol{\lambda}\|_1 + \|\boldsymbol{\sigma}\|_1\right) dt\right).
\end{aligned}
\tag{72}
$$

If $k \geq 1$, $2 \leq r \leq k + 1$, we get

$$
\begin{aligned}
&\left\|\boldsymbol{\sigma} - \widetilde{\boldsymbol{\sigma}}_h\right\| \\
&\quad \leq Ch^r\left(\|u\|_r + \|\lambda\|_r + \|\boldsymbol{\sigma}\|_r\right. \\
&\qquad \left. + \int_0^t \left(\|u\|_{r-1} + \|\boldsymbol{\lambda}\|_{r-1} + \|\boldsymbol{\sigma}\|_{r-1}\right) dt\right).
\end{aligned}
\tag{73}
$$

(III) By Lemma 2, we have that

$$
\begin{aligned}
\left\|p_h u - \widetilde{u}_h\right\| &= \|u_1\| \leq Ch^{r+1-\delta_{k0}} \\
&\quad \times \int_0^t \left(\|u\|_r + \|\boldsymbol{\lambda}\|_r + \|\boldsymbol{\sigma}\|_r\right) d\tau, \quad 0 \leq r \leq k + 1, \\
\left\|p_h u_t - \widetilde{u}_{h,t}\right\| & \\
&\leq Ch^{r+1-\delta_{k0}}\left(\|u\|_r + \|\boldsymbol{\lambda}\|_r + \|\boldsymbol{\sigma}\|_r\right. \\
&\qquad \left. + \int_0^t \left(\|u\|_r + \|\boldsymbol{\lambda}\|_r + \|\boldsymbol{\sigma}\|_r\right) dt\right), \\
&\qquad\qquad 0 \leq r \leq k + 1.
\end{aligned}
\tag{74}
$$

If $k = 0$, we know

$$\|u - \tilde{u}_h\| \le \|u - p_h u\| + \|u_1\|$$
$$\le Ch\left(\|u\|_1 + \int_0^t (\|u\|_1 + \|\lambda\|_1 + \|\sigma\|_1)\,dt\right),$$
$$\|u_t - \tilde{u}_{h,t}\| \tag{75}$$
$$\le Ch\left(\|u_t\|_1 + \|u\|_1 + \|\lambda\|_1 + \|\sigma\|_1 \right.$$
$$\left. + \int_0^t (\|u\|_1 + \|\lambda\|_1 + \|\sigma\|_1)\,dt\right).$$

If $k \ge 1, 2 \le r \le k + 1$, we know

$$\|u - \tilde{u}_h\|$$
$$\le Ch^r\left(\|u\|_r + \int_0^t (\|u\|_{r-1} + \|\lambda\|_{r-1} + \|\sigma\|_{r-1})\,dt\right),$$
$$\|u_t - \tilde{u}_{h,t}\|$$
$$\le Ch^r\left(\|u_t\|_r + \|u\|_{r-1} + \|\lambda\|_{r-1} \right.$$
$$\left. + \int_0^t (\|u\|_{r-1} + \|\lambda\|_{r-1} + \|\sigma\|_{r-1})\,dt\right). \tag{76}$$

The proof is completed. □

Remark 4. The estimate results of (46)–(53) are optimal. If $k \ge 1$, the estimate results of (43)–(45) are superconvergent.

5. Main Result

In this section, we consider error estimates for the continuous-in-time mixed finite element approximation. Define

$$u - u_h = (u - \tilde{u}_h) + (\tilde{u}_h - u_h) = u_3 + u_4,$$
$$\lambda - \lambda_h = (\lambda - \tilde{\lambda}_h) + (\tilde{\lambda}_h - \lambda_h) = \lambda_1 + \lambda_2,$$
$$\sigma - \sigma_h = (\sigma - \tilde{\sigma}_h) + (\tilde{\sigma}_h - \sigma_h) = \sigma_1 + \sigma_2, \tag{77}$$
$$\|u, u_t, \lambda, \sigma\|_r = \|u\|_r + \|u_t\|_r + \|\lambda\|_r + \|\sigma\|_r,$$
$$\||u, u_t, \lambda, \sigma\||_r = \int_0^t (\|u\|_r + \|u_t\|_r + \|\lambda\|_r + \|\sigma\|_r)\,dt.$$

Theorem 5. *Let (u, λ, σ) and $(u_h, \lambda_h, \sigma_h)$ be the solution of (9) and (11), respectively. If u, λ, and σ are sufficiently smooth, then there exist positive constants C such that*

(I) *if $k = 0$, then*

$$\|u - u_h\| \le Ch\left(\|u\|_1 + \||u, u_t, \lambda, \sigma\||_1\right), \tag{78}$$
$$\|u_t - u_{h,t}\| \le Ch\left(\|u, u_t, \lambda, \sigma\|_1 + \||u, \lambda, \sigma\||_1\right), \tag{79}$$
$$\|\lambda - \lambda_h\| \le Ch\left(\|u, u_t, \lambda, \sigma\|_1 + \||u, u_t, \lambda, \sigma\||_1\right), \tag{80}$$
$$\|\sigma - \sigma_h\| \le Ch\left(\|u, u_t, \lambda, \sigma\|_1 + \||u, u_t, \lambda, \sigma\||_1\right), \tag{81}$$
$$\|\mathrm{div}\,(\sigma - \sigma_h)\|$$
$$\le Ch\left(\|u\|_1 + \|u_t\|_1 + \|\sigma\|_2 + \||u, u_t, \lambda, \sigma\||_1\right), \tag{82}$$

(II) *if $k \ge 1$ and $2 \le r \le k + 1$, then*

$$\|u - u_h\| \le Ch^r\left(\|u\|_r + \||u, u_t, \lambda, \sigma\||_{r-1}\right),$$
$$\|u_t - u_{h,t}\|$$
$$\le Ch^r\left(\|u_t\|_r + \|u\|_{r-1} + \|\lambda\|_{r-1} + \|\sigma\|_{r-1} \right.$$
$$\left. + \int_0^t (\|u\|_{r-1} + \|u_t\|_r + \|\lambda\|_{r-1} + \|\sigma\|_{r-1})\,dt\right),$$
$$\|\lambda - \lambda_h\|$$
$$\le Ch^r\left(\|u, \lambda, \sigma\|_r \right.$$
$$\left. + \int_0^t (\|u_t\|_r + \|u\|_{r-1} + \|\lambda\|_{r-1} + \|\sigma\|_{r-1})\,dt\right),$$
$$\|\sigma - \sigma_h\|$$
$$\le Ch^r\left(\|u, \lambda, \sigma\|_r \right.$$
$$\left. + \int_0^t (\|u_t\|_r + \|u\|_{r-1} + \|\lambda\|_{r-1} + \|\sigma\|_{r-1})\,dt\right), \tag{83}$$
$$\|\mathrm{div}\,(\sigma - \sigma_h)\|$$
$$\le Ch^r\left(\|u_t\|_r + \|\lambda\|_r + \|\sigma\|_{r+1} + \|u\|_{r-1} \right.$$
$$\left. + \int_0^t (\|u_t\|_r + \|u\|_{r-1} + \|\lambda\|_{r-1} + \|\sigma\|_{r-1})\,dt\right). \tag{84}$$

Proof. From (9) and (19), we have the following error equation:

$$(u_{4,t} + u_{3,t}, w) + (\mathrm{div}\,\sigma_2, w) = 0,$$
$$\forall w \in W_h, \ 0 < t \le T,$$

$$(\boldsymbol{\sigma}_2, \boldsymbol{\tau}) - (a\boldsymbol{\lambda}_2, \boldsymbol{\tau}) = 0,$$

$$\forall \boldsymbol{\tau} \in \Lambda_h, \ 0 < t \le T,$$

$$(\boldsymbol{\lambda}_2, \mathbf{v}) - (u_{4,t} + bu_4, \operatorname{div} \mathbf{v}) + (cu_4, \mathbf{v}) = 0,$$

$$\forall \mathbf{v} \in \mathbf{V}_h, \ 0 < t \le T,$$

$$u_4(0) = 0.$$

$$(85)$$

In (85), choosing $\omega = u_{4,t} + p_h(bu_4)$, $\boldsymbol{\tau} = \boldsymbol{\lambda}_2$, and $\mathbf{v} = \boldsymbol{\sigma}_2$, we have

$$
\begin{aligned}
(\boldsymbol{\lambda}_2, \boldsymbol{\sigma}_2) &+ (u_{4,t}, u_{4,t}) \\
&= -(u_{3,t}, u_{4,t} + p_h(bu_4)) - (u_{4,t}, p_h(bu_4)) \\
&\quad - (cu_4, \boldsymbol{\sigma}_2).
\end{aligned}
\tag{86}
$$

In the second equation of (85), letting $\boldsymbol{\tau} = \boldsymbol{\sigma}_2$, we know

$$\boldsymbol{\sigma}_2 \le C\boldsymbol{\lambda}_2. \tag{87}$$

Further, using ε-inequality to (86), we get

$$\|u_{4,t}\|^2 + \|\boldsymbol{\lambda}_2\|^2 \le C\left(\|u_4\|^2 + \|u_{3,t}\|^2\right). \tag{88}$$

Hence,

$$
\begin{aligned}
\|u_4\| &= \left\|\int_0^t u_{4,t} d\tau\right\| \\
&\le C \int_0^t \|u_{4,t}\| d\tau \le C \int_0^t (\|u_4\| + \|u_{3,t}\|) dt.
\end{aligned}
\tag{89}
$$

Using Gronwall's inequality to (89), we obtain

$$\|u_4\| \le C \int_0^t \|u_{3,t}\| dt. \tag{90}$$

Further, we know

$$\|\boldsymbol{\sigma}_2\| + \|u_{4,t}\| + \|\boldsymbol{\lambda}_2\| \le C\left(\|u_{3,t}\| + \int_0^t \|u_{3,t}\| dt\right). \tag{91}$$

Noting the first equation of (78), we get

$$\|\operatorname{div} \boldsymbol{\sigma}_2\| \le \|u_{3,t}\| + \|u_{4,t}\| \le C\left(\|u_{3,t}\| + \int_0^t \|u_{3,t}\| dt\right). \tag{92}$$

So we have that

$$
\begin{aligned}
\|u - u_h\| &\le \|u_3\| + C \int_0^t \|u_{3,t}\| dt, \\
\|u_t - u_{h,t}\| &\le C\left(\|u_{3,t}\| + \int_0^t \|u_{3,t}\| dt\right), \\
\|\boldsymbol{\lambda} - \boldsymbol{\lambda}_h\| &\le \|\boldsymbol{\lambda}_1\| + C\left(\|u_{3,t}\| + \int_0^t \|u_{3,t}\| dt\right), \\
\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\| &\le \|\boldsymbol{\sigma}_1\| + C\left(\|u_{3,t}\| + \int_0^t \|u_{3,t}\| dt\right), \\
\|\operatorname{div}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)\| &\le \|\operatorname{div} \boldsymbol{\sigma}_1\| + \|\operatorname{div} \boldsymbol{\sigma}_2\|.
\end{aligned}
\tag{93}
$$

Together with the results of Lemmas 2 and 3, the proof is completed. $\qquad\square$

Remark 6. The estimate results of (78)–(84) are optimal.

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