Research Article

On the Nature of Bifurcation in a Ratio-Dependent Predator-Prey Model with Delays

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A ratio-dependent predator-prey model with two delays is investigated. The conditions which ensure the local stability and the existence of Hopf bifurcation at the positive equilibrium of the system are obtained. It shows that the two different time delays have different effects on the dynamical behavior of the system. An example together with its numerical simulations shows the feasibility of the main results. Finally, main conclusions are included.

1. Introduction

After the seminal models of Volterra and Lotka in the mid-1920s, understanding the dynamics of predator-prey models has been the focus of intense research in recent years. A great deal of excellent and interesting results have been reported. For example, Bhattacharyya and Mukhopadhyay [1] made a detailed discussion on the local and global dynamical behavior of an ecoepidemiological model, Kar and Ghorai [2] analyzed the local stability, global stability, influence of harvesting, and bifurcation of a delayed predator-prey model with harvesting, and Chakraborty et al. [3] studied the bifurcation and control of a bioeconomic model of a delayed prey-predator model. Bhattacharyya and Mukhopadhyay [4] focused on the spatial dynamics of nonlinear prey-predator models with prey migration and predator switching, and Chang and Wei [5] considered the bifurcation nature and optimal control of a diffusive predator-prey system with time delay and prey harvesting. For more related research, one can see [6–26].

In 2011, Wang and Pei [27] investigated the stability and Hopf bifurcation of the following delayed ratio-dependent predator-prey system:

$$\dot{x}(t) = ax(t - \tau_1) - bx^2 - \frac{cxy}{my + x},$$

$$\dot{y}(t) = \frac{x(t - \tau_2)\, y(t - \tau_2)}{my(t - \tau_2) + x(t - \tau_2)} - dy, \qquad (1)$$

where $x(t)$ and $y(t)$ represent the densities of the prey and the predator population at time t, respectively. The parameters a, k, c, m, f, and d are positive constants that stand for prey intrinsic growth rate, carrying capacity, capturing rate, half capturing saturation constant, conversion rate, predator and death rate, respectively. The constants $\tau_1 \geq 0$ and $\tau_2 \geq 0$ denote the time delays due to gestation of the prey and predator, respectively. For more detailed biological meaning of the coefficients of system (1), one can see [27]. Applying the Nyquist criteria and the theory of Hopf bifurcation, Wang and Pei [27] considered the stability of the positive equilibrium and the existence of the local Hopf bifurcation of system (1). By means of the center manifold and normal form theories, they obtained explicit formulae which determine the stability, direction, and other properties of bifurcating periodic solutions.

We would like to point out that Wang and Pei [27] investigated the local stability of system (1) under the assumption $\tau_1 + \tau_2$ in a certain range and focused on the local Hopf bifurcation by choosing the delay τ_2 as bifurcation parameter. A natural problem arising from this is what effect the different delays τ_1 and τ_2 have on the dynamical behavior of system (1). In [27], Wang and Pei did not analyze this aspect. Thus,
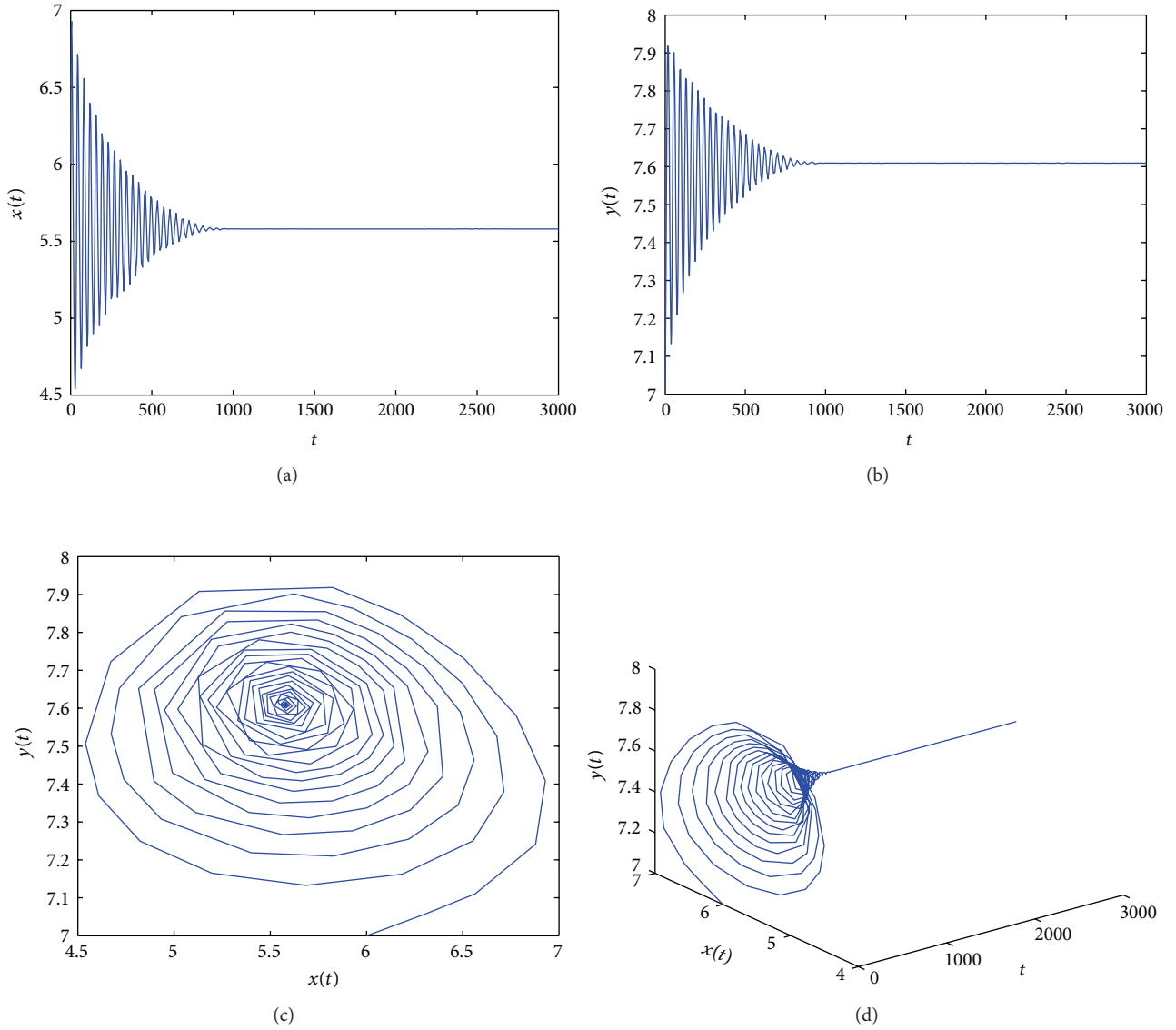
FIGURE 1: Trajectory portrait and phase portrait of system (57) with $\tau_1 = 0$, $\tau_2 = 2.2 < \tau_{2_0} \approx 2.3345$. The positive equilibrium $E(5.61, 7.02)$ is asymptotically stable. The initial value is $(5.6, 7.45)$.

we think that it is important to deal with the effect of time delay on the dynamics of system (1). To the best of the authors knowledge, there are very few works' which deal with this topic. In this paper, we will further investigate the stability and bifurcation of model (1) as a complementarity. It will be shown that the two different time delays τ_1 and τ_2 have different effects on the stability and Hopf bifurcation nature of system (1).

The remainder of the paper is organized as follows. In Section 2, we investigate the stability of the positive equilibrium and the occurrence of local Hopf bifurcations. In Section 3, numerical simulations are carried out to illustrate

the validity of the main results. Some main conclusions are drawn in Section 4.

2. Stability and Local Hopf Bifurcations

In this section, we shall study the stability of the positive equilibrium and the existence of local Hopf bifurcations.

From [27], we know that if the following conditions

(H1)

$$d < f < \frac{dc}{c - ma},$$
$$c > ma \quad \text{or} \quad f > d, c \leq ma$$

(2)

(a)

(b)

(c)

(d)

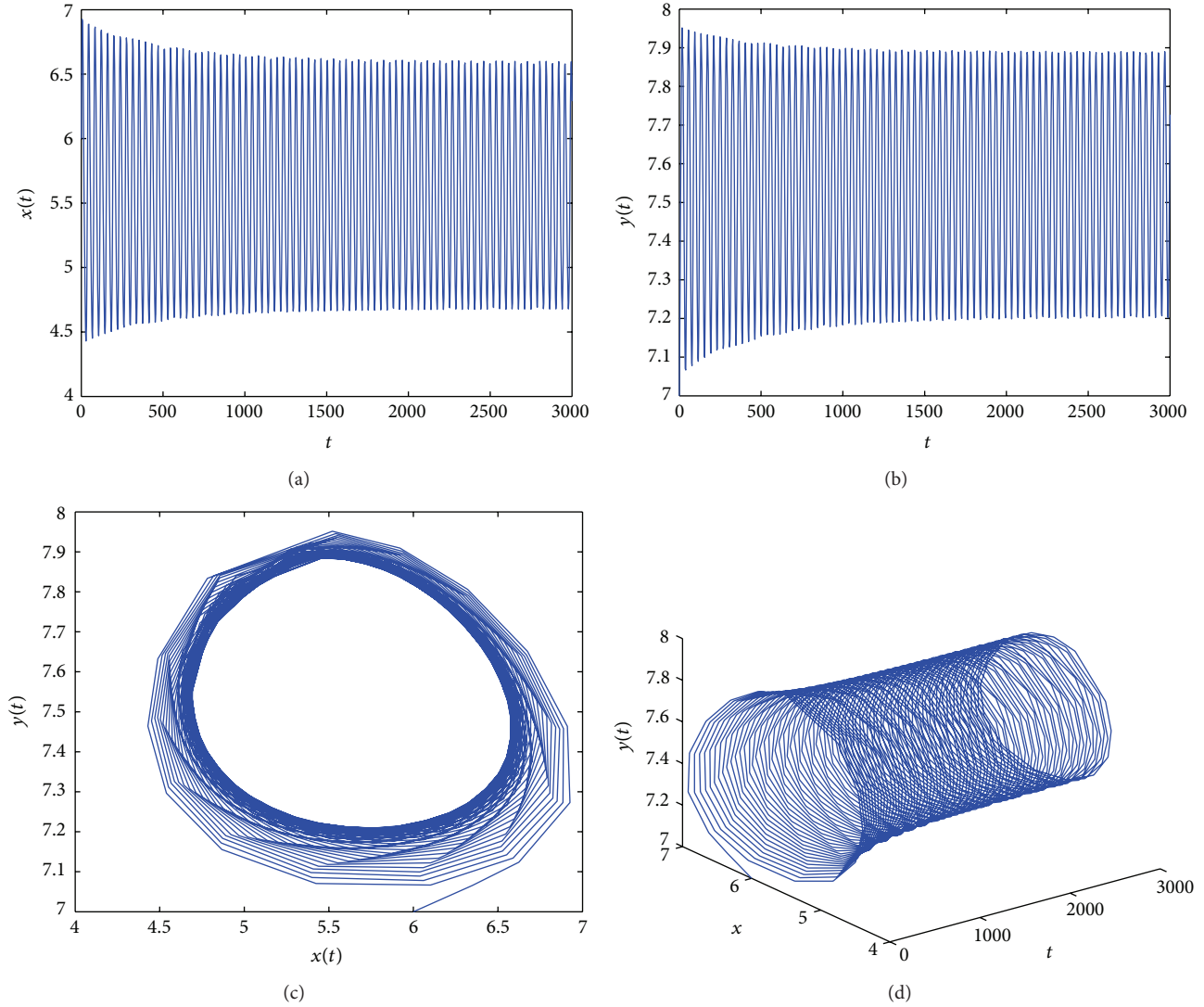FIGURE 2: Trajectory portrait and phase portrait of system (57) with $\tau_1 = 0, \tau_2 = 2.5 > \tau_{2_0} \approx 2.3345$. Hopf bifurcation occurs from the positive equilibrium $E(5.61, 7.02)$. The initial value is $(5.6, 7.45)$.

hold, then system (1) has a unique equilibrium point $E(x^*, y^*)$, where

$$x^* = \frac{akl - b_1(k+l)}{akl}, \qquad y^* = \frac{b_1}{kl}. \tag{3}$$

Let $\overline{x}(t) = x(t) - x^*$ and $\overline{y}(t) = y(t) - y^*$ and still denote $\overline{x}(t)$, $\overline{y}(t)$ by $x(t)$, $y(t)$, respectively. Then (1) reads as

$$\dot{x}(t) = p_1 x + p_2 y + p_3 x(t - \tau_1),$$
$$\dot{y}(t) = p_4 y + p_5 x(t - \tau_2) + p_6 y(t - \tau_2), \tag{4}$$

where

$$p_1 = \frac{c(f^2 - d^2) - 2maf^2}{mf^2},$$

$$p_2 = -\frac{d^2 c}{f^2},$$

$$p_3 = a,$$

$$p_4 = -d,$$

$$p_5 = \frac{(f-d)^2}{mf},$$

$$p_6 = \frac{d^2}{f}. \tag{5}$$

The characteristic equation of (4) is given by

$$\det \begin{pmatrix} \lambda - p_1 - p_3 e^{-\lambda \tau_1} & -p_2 \\ -p_5 e^{-\lambda \tau_2} & \lambda - p_4 - p_6 e^{-\lambda \tau_2} \end{pmatrix} = 0. \tag{6}$$
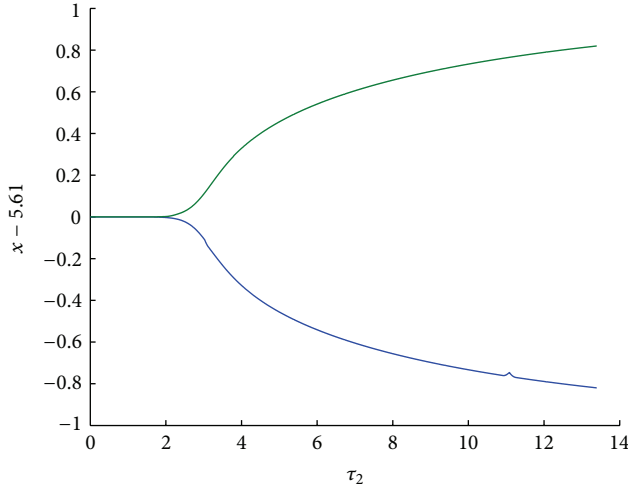
FIGURE 3: Bifurcation diagram with respect to the time delay τ_2 for system (57) with $\tau_1 = 0$.

That is,

$$\lambda^2 - (p_1 + p_4)\lambda - p_3\lambda e^{-\lambda\tau_1} - p_6\lambda e^{-\lambda\tau_2} + p_3 p_4 e^{-\lambda\tau_1}$$
$$+ (p_1 p_6 - p_2 p_5)e^{-\lambda\tau_2} + p_3 p_6 e^{-\lambda(\tau_1+\tau_2)} + p_1 p_4 = 0. \tag{7}$$

The following lemma is important for us to analyze the distribution of roots of the transcendental equation (7).

Lemma 1 (see [28]). *For the transcendental equation*

$$P\left(\lambda, e^{-\lambda\tau_1}, \ldots, e^{-\lambda\tau_m}\right)$$
$$= \lambda^n + p_1^{(0)}\lambda^{n-1} + \cdots + p_{n-1}^{(0)}\lambda + p_n^{(0)}$$
$$+ \left[p_1^{(1)}\lambda^{n-1} + \cdots + p_{n-1}^{(1)}\lambda + p_n^{(1)}\right]e^{-\lambda\tau_1}$$
$$+ \cdots + \left[p_1^{(m)}\lambda^{n-1} + \cdots + p_{n-1}^{(m)}\lambda + p_n^{(m)}\right]e^{-\lambda\tau_m} = 0, \tag{8}$$

as $(\tau_1, \tau_2, \tau_3, \ldots, \tau_m)$ vary, the sum of orders of the zeros of $P(\lambda, e^{-\lambda\tau_1}, \ldots, e^{-\lambda\tau_m})$ in the open right half plane can change, and only a zero appears on or crosses the imaginary axis.

In the sequel, we consider five cases.

Case 1. $\tau_1 = \tau_2 = 0$. Equation (7) becomes

$$\lambda^2 - (p_1 + p_3 + p_4 + p_6)\lambda^2$$
$$+ (p_3 p_4 + p_1 p_4 + p_1 p_6 - p_2 p_5 + p_3 p_6) = 0. \tag{9}$$

All roots of (9) have a negative real part if the following condition holds:

(H2)

$$p_1 + p_3 + p_4 + p_6 < 0,$$
$$p_3 p_4 + p_1 p_4 + p_1 p_6 - p_2 p_5 + p_3 p_6 > 0. \tag{10}$$

Then the equilibrium point $E(x^*, y^*)$ is locally asymptotically stable when the conditions (H1) and (H2) are satisfied.

Case 2. $\tau_1 = 0, \tau_2 > 0$. Equation (7) becomes

$$\lambda^2 - (p_1 + p_4)\lambda + p_3 p_4 + p_1 p_4$$
$$- (p_6\lambda + p_2 p_5 - p_1 p_6 - p_3 p_6)e^{-\lambda\tau_2} = 0. \tag{11}$$

For $\omega > 0$, let $i\omega$ be a root of (11). Then it follows that

$$(p_1 p_6 - p_2 p_5 + p_3 p_6)\cos\omega\tau_2 - p_6\omega\sin\omega\tau_2$$
$$= \omega^2 - p_3 p_4 - p_1 p_4,$$
$$p_6\omega\cos\omega\tau_2 + (p_1 p_6 - p_2 p_5 + p_3 p_6)\sin\omega\tau_2$$
$$= -(p_1 + p_3 + p_4)\omega \tag{12}$$

which is equivalent to

$$\omega^4 + r_1\omega^2 + r_2 = 0, \tag{13}$$

where

$$r_1 = (p_1 + p_3 + p_4)^2 - 2(p_3 p_4 - p_1 p_4) - p_6^2,$$
$$r_2 = (p_3 p_4 - p_1 p_4)^2 - (p_1 p_6 - p_2 p_5 + p_3 p_6)^2. \tag{14}$$

Define $\Delta_1 = r_1^2 - 4r_2$. Following Cao and Xiao [29] and Theorem 2.1 in Ge and Yan [30], we have the following result.

Lemma 2. *If (H1) holds, then*

(i) *if $r_1 < 0$ and $\Delta_1 = 0$, then (11) with $\tau_2 = \tau_{2_n}^+$ has a pair of pure imaginary roots $\pm i\omega_+$;*

(ii) *if $r_1 < 0$ and $\Delta_1 > 0$, then (11) with $\tau_2 = \tau_{2_n}^+$ has two pairs of pure imaginary roots $\pm i\omega_+$ and $\pm i\omega_-$, where*

$$\tau_{2_n}^{\pm} = \frac{1}{\omega_{\pm}}\arccos\left[\frac{(\omega_{\pm}^2 - p_3 p_4 - p_1 p_4)(p_1 p_6 - p_2 p_5 + p_3 p_6) - p_6(p_1 + p_3 + p_4)\omega_{\pm}^2}{(p_1 p_6 - p_2 p_5 + p_3 p_6)^2 + (p_6\omega_{\pm})^2}\right] + \frac{2n\pi}{\omega_{\pm}} \tag{15}$$

(a)

(b)

(c)

(d)

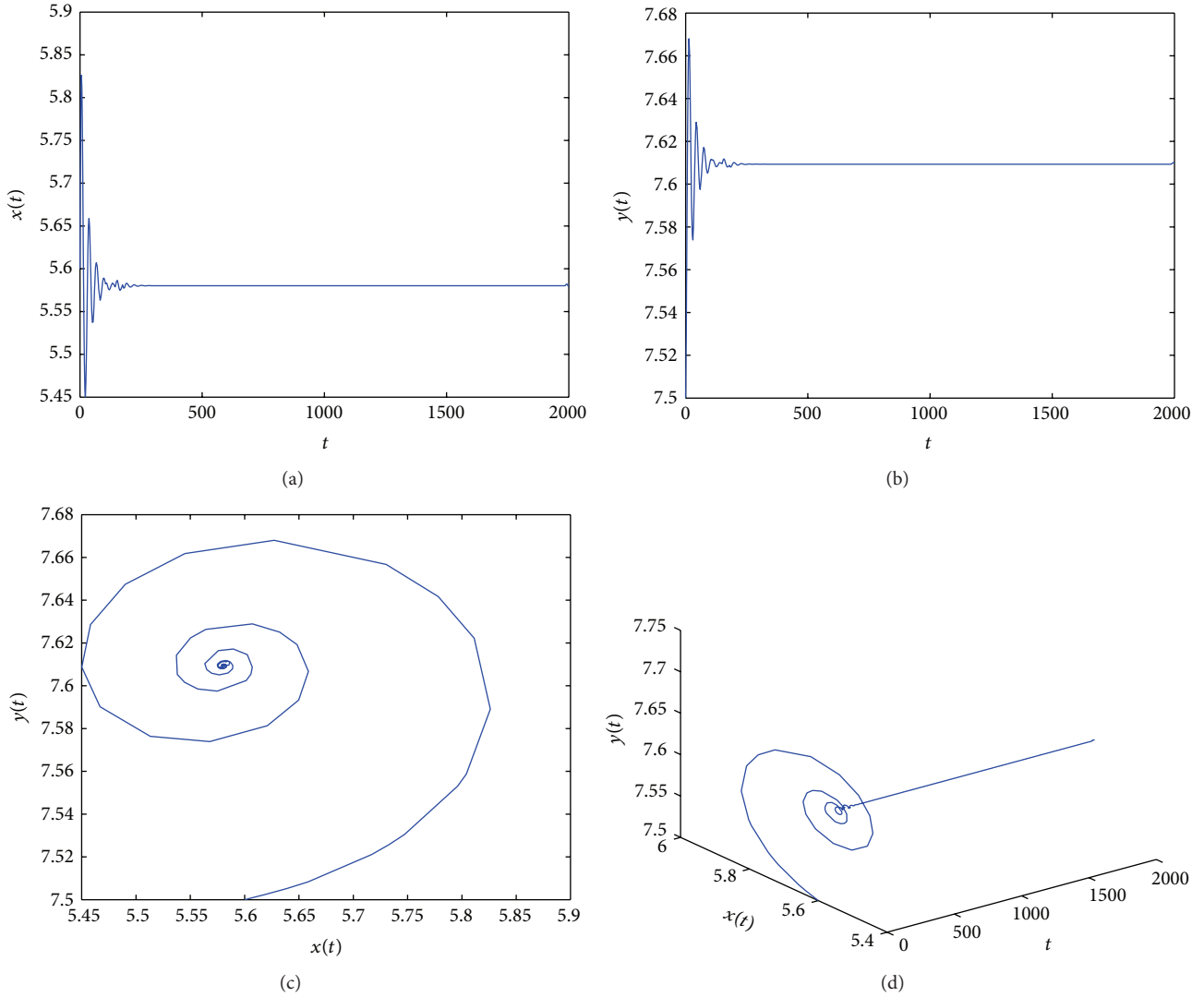FIGURE 4: Trajectory portrait and phase portrait of system (57) with $\tau_2 = 1.8$, $\tau_1 = 0.75 < \tau_{1_0} \approx 0.8013$. The positive equilibrium $E(5.61, 7.02)$ is asymptotically stable. The initial value is $(5.6, 7.45)$.

and $\omega_\pm$ satisfies

$$\omega_+^2 = \frac{-r_1 + \sqrt{\Delta_1}}{2}, \qquad \omega_-^2 = \frac{-r_1 - \sqrt{\Delta_1}}{2}; \qquad (16)$$

(iii) if $r_1 > 0$ or $\Delta_1 < 0$, then all the roots of (11) have negative real parts for $\tau_2 \geq 0$. From Lemma 2, one has the following result.

Theorem 3. Let $\tau_{2_n}^\pm$ be defined by (15). Under the condition (H1),

(i) if $r_1 < 0$ and $\Delta_1 = 0$, then the trivial solution $(x_*, y_*)^T$ of (11) is asymptotically stable for all $\tau_2 \in [0, \tau_{2_0})$ and unstable for $\tau_2 > \tau_{2_0}$. That is, Hopf bifurcation occurs when $\tau_2 = \tau_{2_0}$;

(ii) if $r_1 > 0$ or $\Delta_1 < 0$, then there are Hopf bifurcations near the trivial solution $(x_*, y_*)^T$ of (11) when $\tau_2 = \tau_{2_n}^+$ and $\tau_2 = \tau_{2_n}^-$.

Case 3. $\tau_1 > 0$, $\tau_2 = 0$. Equation (7) takes the form

$$\lambda^2 - (p_1 + p_3 + p_4 + p_6)\lambda + p_1 p_6 - p_2 p_5 + p_1 p_4$$
$$- (p_3 \lambda - p_3 p_4 - p_3 p_6) e^{-\lambda \tau_1} = 0. \qquad (17)$$

For $\eta > 0$, let $i\eta$ be a root of (17). Then it follows that

$$(p_3 p_4 + p_3 p_6) \cos \eta \tau_1 - p_3 \eta \sin \eta \tau_1$$
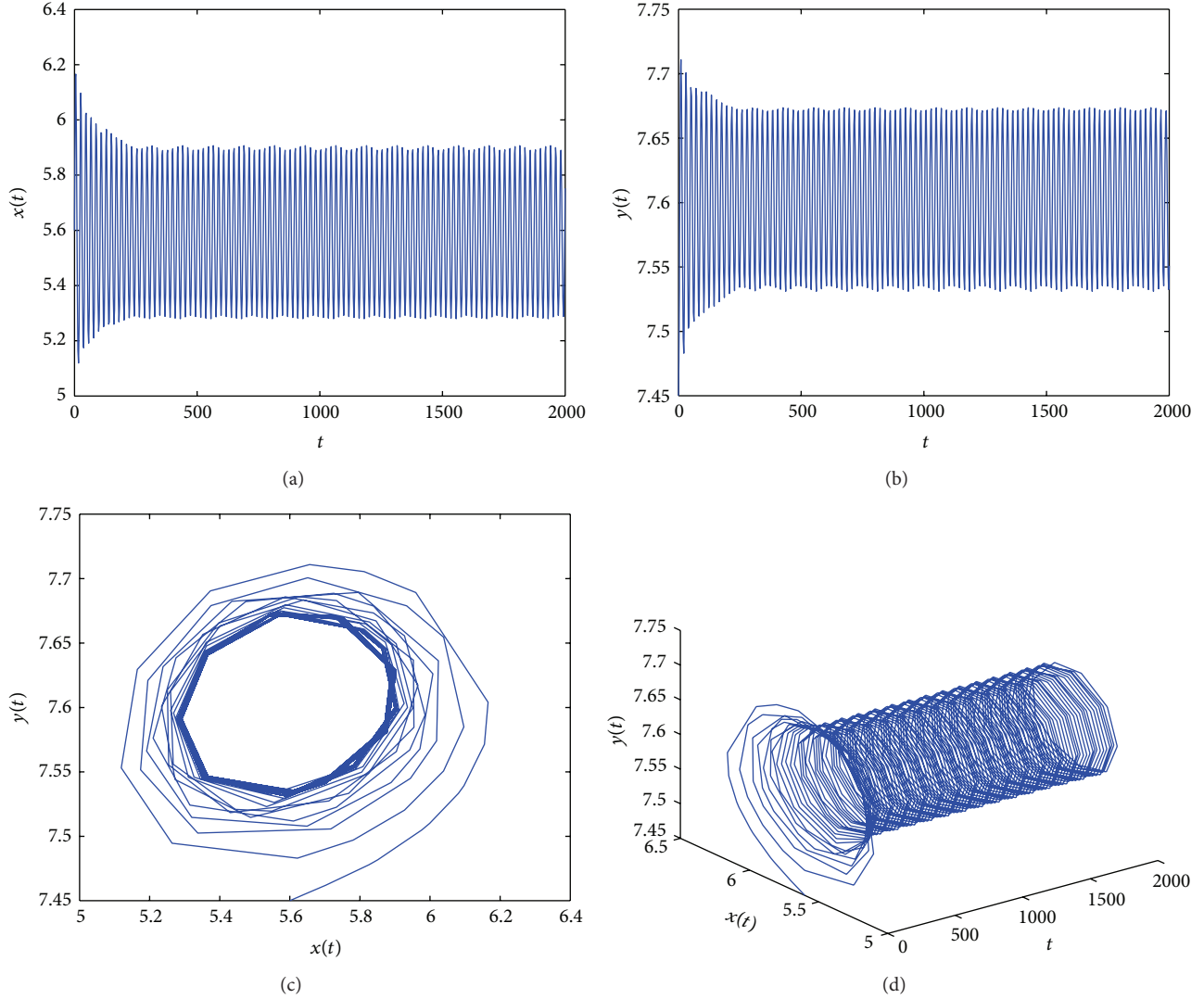$$= \eta^2 + p_2 p_5 - p_1 p_6 - p_1 p_4,$$

(a)

(b)

(c)

(d)

Figure 5: Trajectory portrait and phase portrait of system (57) with $\tau_2 = 1.8$, $\tau_1 = 0.98 > \tau_{1_0} \approx 0.8013$. Hopf bifurcation occurs from the positive equilibrium $E(5.61, 7.02)$. The initial value is $(5.6, 7.45)$.

$$p_3\eta\cos\eta\tau_1 + (p_3p_4 + p_3p_6)\sin\eta\tau_1$$

$$= -(p_1 + p_4 + p_6) \tag{18}$$

which is equivalent to

$$\eta^4 + s_1\eta^2 + s_2 = 0, \tag{19}$$

where

$$s_1 = (p_1 + p_4 + p_6)^2 + 2(p_2p_5 - p_1p_6 - p_1p_4) - p_3^2,$$

$$s_2 = (p_2p_5 - p_1p_6 - p_1p_4)^2 - (p_3p_4 + p_3p_6)^2. \tag{20}$$

Define $\Delta_2 = s_1^2 - 4s_2$. Following the Theorem 2.1 in Ge and Yan [30], we have the following result.

Lemma 4. *If (H1) holds, then*

(i) *if $s_1 < 0$ and $\Delta_2 = 0$, then (17) with $\tau_1 = \tau_{1_n}^+$ has a pair of pure imaginary roots $\pm i\eta_+$;*

(ii) *if $s_1 < 0$ and $\Delta_2 > 0$, then (17) with $\tau_1 = \tau_{1_n}^+$ has two pairs of pure imaginary roots $\pm i\eta_+$ and $\pm i\eta_-$, where*

$$\tau_{1_n}^\pm = \frac{1}{\eta_\pm}\arccos\left[\frac{(\eta_\pm^2 + p_2p_5 - p_1p_6 - p_1p_4)(p_3p_4 + p_3p_6) - p_3(p_1 + p_4 + p_6)\eta_\pm}{(p_3p_4 + p_3p_6)^2 + (p_3\eta_\pm)^2}\right] + \frac{2n\pi}{\eta_\pm} \tag{21}$$
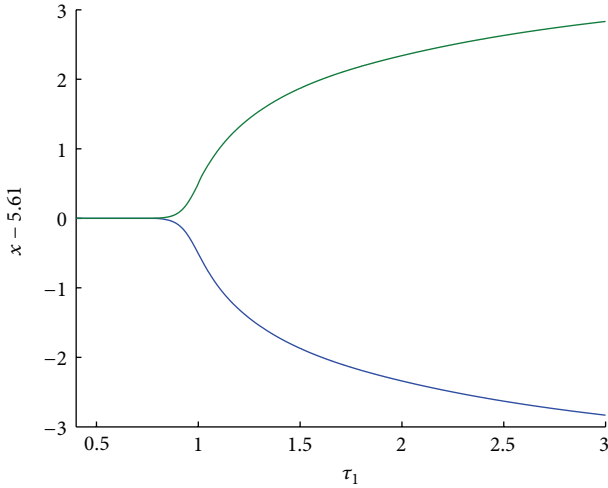
FIGURE 6: Bifurcation diagram with respect to the time delay τ_1 for system (57) with $\tau_2 = 1.8$.

and $\eta_{\pm}$ satisfies

$$\eta_+^2 = \frac{-s_1 + \sqrt{\Delta_2}}{2}, \qquad \eta_-^2 = \frac{-s_1 - \sqrt{\Delta_2}}{2}; \tag{22}$$

(iii) *if* $s_1 > 0$ *or* $\Delta_2 < 0$, *then all the roots of* (17) *have negative real parts for* $\tau_1 \geq 0$.

From Lemma 4, we have the following result.

Theorem 5. *Let* $\tau_{1_n}^{\pm}$ *be defined by* (21). *Under the condition* (H1),

(i) *if* $s_1 < 0$ *and* $\Delta_2 = 0$, *then the trivial solution* $(x_*, y_*)^T$ *of* (17) *is asymptotically stable for all* $\tau_1 \in [0, \tau_{1_0})$ *and unstable for* $\tau_1 > \tau_{1_0}$. *That is, Hopf bifurcation occurs when* $\tau_1 = \tau_{1_0}$;

(ii) *if* $s_1 > 0$ *or* $\Delta_2 < 0$, *then there are Hopf bifurcations near the trivial solution* $(x_*, y_*)^T$ *of* (17) *when* $\tau_1 = \tau_{1_n}^+$ *and* $\tau_1 = \tau_{1_n}^-$.

Case 4. $\tau_1 > 0$, $\tau_2 > 0$. We consider (7) with τ_2 in its stable interval, by regarding τ_1 as a parameter. Without loss of generality, we consider system (1) under the assumptions (H1) and (H2). Let $i\eta^*(\eta^* > 0)$ be a root of (7). Then we can obtain

$$\eta^{*4} + k_1 \eta^{*3} + k_2 \eta^{*2} + k_3 \eta^* + k_4 = 0, \tag{23}$$

where

$$k_1 = 2p_6 \sin \eta^* \tau_2,$$

$$k_2 = p_6^2 \sin^2 \eta^* \tau_2 - 2(p_1 p_6 - p_2 p_5) \cos \eta^* \tau_2$$

$$+ (p - 1 + p_4 + p_6 \cos \eta^* \tau_2)^2,$$

$$k_3 = -2p_6(p_1 p_6 - p_2 p_5) \sin \eta^* \tau_2 \cos \eta^* \tau_2,$$

$$k_4 = p_6(p_1 p_6 - p_2 p_5)^2 + (p_1 p_6 - p_2 p_5)^2 \sin^2 \eta^* \tau_2$$

$$+ 2(p_1 + p_4 + p_6 \cos \eta^* \tau_2)(p_1 p_6 - p_2 p_5) \sin \eta^* \tau_2$$

$$- (p_3 + p_4 + p_3 p_6 \cos \eta^* \tau_2)^2 - (p_3 \eta + p_3 p_6 \sin \eta^* \tau_2)^2. \tag{24}$$

Denote

$$H(\eta^*) = \eta^{*4} + k_1 \eta^{*3} + k_2 \eta^{*2} + k_3 \eta^* + k_4. \tag{25}$$

Assume that

(H3)

$$k_4 < 0. \tag{26}$$

It is easy to check that $H(0) < 0$ if (H5) holds and $\lim_{\eta^* \to +\infty} H(\eta^*) = +\infty$. We can obtain that (25) has finite positive roots $\eta_1^*, \eta_2^*, \ldots, \eta_n^*$. For every fixed η_i^*, $i = 1, 2, 3, \ldots, k$, there exists a sequence $\{\tau_{1_i}^j \mid j = 1, 2, 3, \ldots\}$, such that (25) holds. Let

$$\tau_{1_0} = \min \left\{ \tau_{1_i}^j \mid i = 1, 2, \ldots, k; j = 1, 2, \ldots \right\}. \tag{27}$$

When $\tau_1 = \tau_{1_0}$, (7) has a pair of purely imaginary roots $\pm i\tilde{\eta}^*$ for $\tau_2 \in [0, \tau_{2_0})$.

In the following, we assume that

(H4)

$$\left[\frac{d(\operatorname{Re}\lambda)}{d\tau_1} \right]_{\lambda = i\tilde{\eta}^*} \neq 0. \tag{28}$$

Thus, by the general Hopf bifurcation theorem for FDEs in Hale [31], we have the following result on the stability and Hopf bifurcation in system (1).

Theorem 6. *For system* (1), *suppose that* (H1), (H2), (H3), *and* (H4) *are satisfied and* $\tau_2 \in [0, \tau_{2_0})$. *Then the positive equilibrium* $E(x^*, y^*)$ *is asymptotically stable when* $\tau_1 \in [0, \tau_{1_0})$, *and system* (1) *undergoes a Hopf bifurcation at the positive equilibrium* $E(x^*, y^*)$ *when* $\tau_1 = \tau_{1_0}$.

Case 5. $\tau_1 > 0$, $\tau_2 > 0$. We consider (7) with τ_1 in its stable interval, by regarding τ_2 as a parameter. Without loss of generality, we consider system (1) under the assumptions (H1) and (H2). Let $i\omega^*(\omega^* > 0)$ be a root of (7). Then we can obtain

$$\omega^{*4} + t_1 \omega^{*3} + t_2 \omega^{*2} + t_3 \omega^* + t_4 = 0, \tag{29}$$

where

$$t_1 = 2p_3 \sin \omega^* \tau_1,$$

$$t_2 = p_3^2 \sin^2 \omega^* \tau_1 - 2p_3 p_4 \cos \omega^* \tau_1$$

$$- (p_1 + p_4 + p_3 \cos \omega^* \tau_1)^2,$$

$$t_3 = 2p_3 p_4 (p_1 + p_4 + p_3 \sin \omega^* \tau_1) \sin \omega^* \tau_1$$
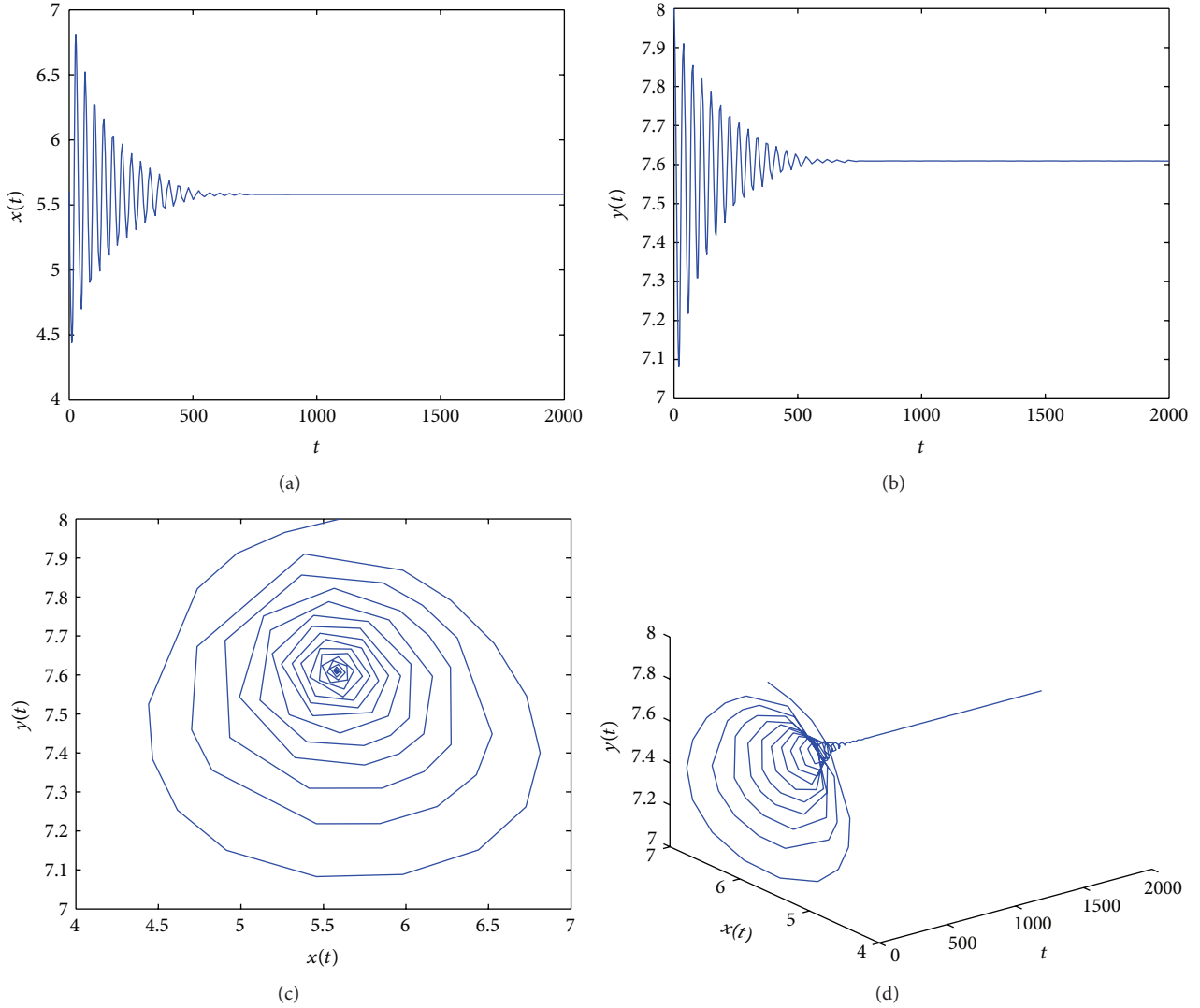
(a)

(b)

(c)

(d)

FIGURE 7: Trajectory portrait and phase portrait of system (57) with $\tau_2 = 0$, $\tau_1 = 0.90 < \tau_{1_0} \approx 0.9122$. The positive equilibrium $E(5.61, 7.02)$ is asymptotically stable. The initial value is $(5.6, 8)$.

$$- 2p_3^2 p_4 \sin \omega^* \tau_1 \cos \omega^* \tau_1,$$

$$t_4 = (p_3 p_4)^2 \sin^2 \omega^* \tau_2 - (p_1 p_6 - p_2 p_5 + p_3 p_6 \cos \omega^* \tau_1)^2$$

$$- (p_6 \omega^* + p_3 p_6 \sin \omega^* \tau_1)^2.$$

$$(30)$$

Denote

$$H_* (\omega^*) = \omega^{*4} + t_1 \omega^{*3} + t_2 \omega^{*2} + t_3 \omega^* + t_4. \qquad (31)$$

Obviously, $H(0) < 0$ if (H5) holds and $\lim_{\omega^* \to +\infty} H_*(\omega^*) = +\infty$. We can obtain that (31) has finite positive roots $\omega_1^*, \omega_2^*, \ldots, \omega_n^*$. For every fixed ω_i^*, $i = 1, 2, 3, \ldots, k$, there exists a sequence $\{\tau_{2_i}^j \mid j = 1, 2, 3, \ldots\}$, such that (31) holds. Let

$$\tau_{2_0} = \min \left\{ \tau_{2_i}^j \mid i = 1, 2, \ldots, k; j = 1, 2, \ldots \right\}. \qquad (32)$$

When $\tau_2 = \tau_{2_0}$, (7) has a pair of purely imaginary roots $\pm i \overline{\omega^*}$ for $\tau_1 \in [0, \tau_{1_0})$.

In the following, we assume that

(H5)

$$\left[\frac{d(\operatorname{Re} \lambda)}{d \tau_2} \right]_{\lambda = i \overline{\omega^*}} \neq 0. \qquad (33)$$

In view of the general Hopf bifurcation theorem for FDEs in Hale [31], we have the following result on the stability and Hopf bifurcation in system (1).

Theorem 7. *For system* (1)*, assume that* (H1)*,* (H2)*,* (H3)*, and* (H5) *are satisfied and* $\tau_1 \in [0, \tau_{1_0})$*. Then the positive equilibrium* $E(x^*, y^*)$ *is asymptotically stable when* $\tau_2 \in [0, \tau_{2_0})$*, and system* (1) *undergoes a Hopf bifurcation at the positive equilibrium* $E(x^*, y^*)$ *when* $\tau_2 = \tau_{2_0}$*.*
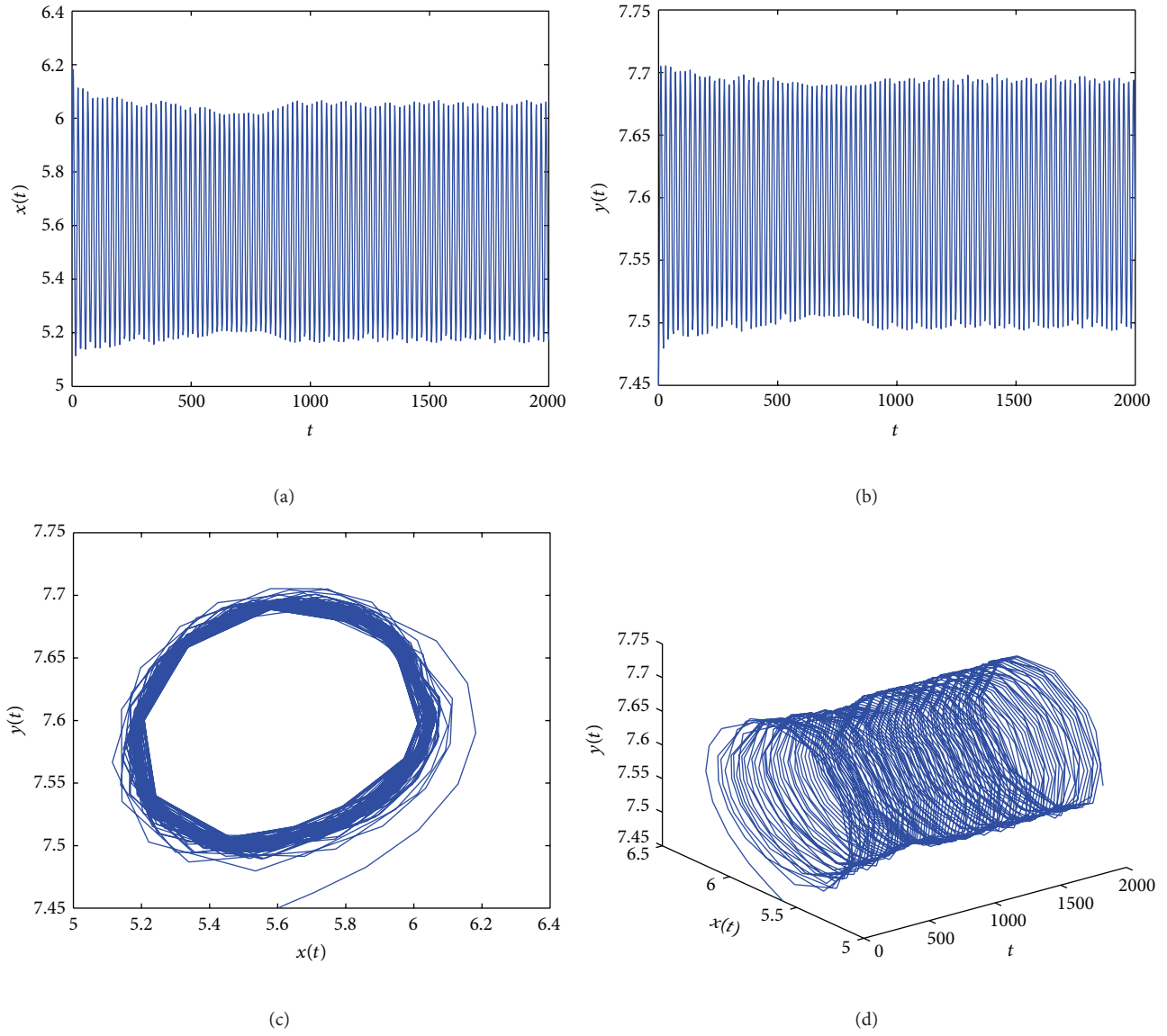
(a)

(b)

(c)

(d)

FIGURE 8: Trajectory portrait and phase portrait of system (57) with $\tau_2 = 0$, $\tau_1 = 1.0 > \tau_{1_0} \approx 0.9122$. Hopf bifurcation occurs from the positive equilibrium $E_0(5.61, 7.02)$. The initial value is $(5.6, 7.45)$.

Case 6. $\tau_1 = \tau_2 = \tau$. Equation (7) becomes

$$\lambda^2 - (p_1 + p_4)\lambda + p_1 p_4$$
$$- \left[(p_2 + p_6)\lambda - (p - 3p_4 + p_1 p_6 - p_2 p_5)\right] e^{-\lambda\tau} \quad (34)$$
$$+ p_3 p_6 \lambda e^{-2\lambda\tau} = 0,$$

which is equivalent to

$$\left[\lambda^2 - (p_1 + p_4)\lambda + p_1 p_4\right] e^{\lambda\tau}$$

$$- \left[(p_2 + p_6)\lambda - (p_3 p_4 + p_1 p_6 - p_2 p_5)\right]$$
$$+ p_3 p_6 \lambda e^{-\lambda\tau} = 0. \quad (35)$$

When $\tau = 0$, (35) becomes

$$\lambda^2 - (p_1 + p_3 + p_4 + p_6)\lambda^2$$
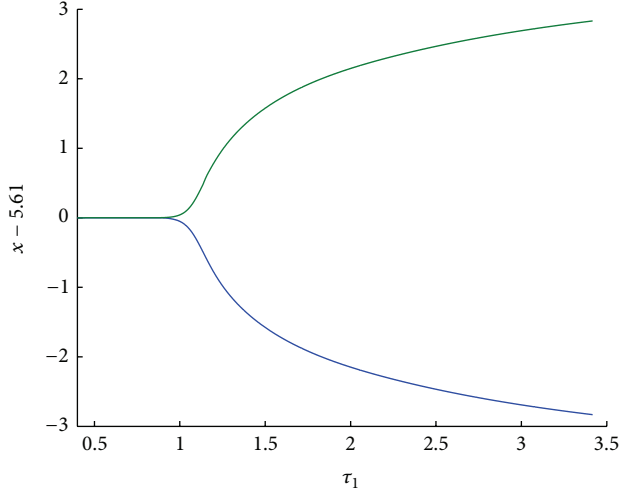$$+ (p_1 p_4 + p_3 p_4 + p_1 p_6 - p_2 p_5 + p_3 p_6) = 0. \quad (36)$$

FIGURE 9: Bifurcation diagram with respect to the time delay τ_1 for system (57) with $\tau_2 = 0$.

It is easy to see that if the condition (H2) holds, then all roots of (36) have a negative real part. Then the equilibrium point $E(x^*, y^*)$ is locally asymptotically stable when the conditions (H1) and (H2) are satisfied.

For $\theta > 0$, let $i\theta$ be a root of (35). Then it follows that

$$
\begin{aligned}
&\left(p_1 p_4 - \theta^2 + p_3 p_6\right) \cos \theta \tau_2 + \left(p_3 p_6 - p_1 p_4 + \theta^2\right) \sin \theta \tau \\
&\qquad = p_2 p_5 - p_3 p_4 - p_1 p_6, \\
&\left(p_1 + p_4\right) \theta \cos \theta \tau + \left(p_3 p_6 - p_1 p_4 + \theta^2\right) \sin \theta \tau \\
&\qquad = -\left(p_3 + p_6\right) \theta
\end{aligned}
\tag{37}
$$

which is equivalent to

$$
\sin \theta \tau = \frac{\left(p_2 p_5 - p_3 p_4 - p_1 p_6\right)\left(p_1 + p_4\right) \theta + \left(p_3 + p_6\right) \theta \left(p_1 p_4 - \theta^2 + p_3 p_6\right)}{\left[\left(p_1 + p_4\right) \theta\right]^2 - \left(p_3 p_6\right)^2 + \left(p_1 p_4 - \theta^2\right)^2},
\tag{38}
$$

$$
\cos \theta \tau = \frac{\left(p_3 p_4 + p_1 p_6 - p_2 p_5\right)\left(p_3 p_6 - p_1 p_4 + \theta^2\right) - \left(p_3 + p_6\right) \theta \left(p_1 + p_4\right) \theta}{\left[\left(p_1 + p_4\right) \theta\right]^2 - \left(p - 3 p_6\right)^2 + \left(p_1 p_4 - \theta^2\right)^2}.
\tag{39}
$$

It follows from $\sin^2 \theta \tau + \cos^2 \theta \tau = 1$ that

$$
\begin{aligned}
&\left[\left(p_2 p_5 - p_3 p_4 - p_1 p_6\right)\left(p_1 + p_4\right) \theta\right. \\
&\left.\quad + \left(p_3 + p_6\right) \theta \left(p_1 p_4 - \theta^2 + p_3 p_6\right)\right]^2 \\
&\quad + \left[\left(p_3 p_4 + p_1 p_6 - p_2 p_5\right)\left(p_3 p_6 - p_1 p_4 + \theta^2\right)\right. \\
&\left.\quad - \left(p_3 + p_6\right) \theta \left(p_1 + p_4\right) \theta\right]^2 \\
&\quad = \left\{\left[\left(p_1 + p_4\right) \theta\right]^2 - \left(p_3 p_6\right)^2 + \left(p_1 p_4 - \theta^2\right)^2\right\}^2.
\end{aligned}
\tag{40}
$$

Then we have

$$
\theta^8 + u_3 \theta^6 + u_2 \theta^4 + u_1 \theta^2 + u_0 = 0,
\tag{41}
$$

where

$$
\begin{aligned}
u_0 &= \left[\left(p_1 p_4\right)^2 - \left(p_3 p_6\right)^2\right]^2 \\
&\quad - \left[\left(p_2 p_5 - p_3 p_4 - p_1 p_6\right)\left(p_3 p_6 - p_1 p_4\right)\right]^2, \\
u_1 &= 2\left[\left(p_1 p_4\right)^2 - \left(p_3 p_6\right)^2\right]\left[\left(p_1 + p_2\right)^2 - 2 p_1 p_4\right]^2 \\
&\quad - \left[\left(p_2 p_5 - p_3 p_4 - p_1 p_6\right)\left(p_1 + p_4\right)\right.
\end{aligned}
$$

$$
\begin{aligned}
&\qquad + \left(p_3 + p_6\right)\left(p_1 p_4 + p_3 p_6\right)\big] \\
&\quad - 2\left(p_2 p_5 - p_3 p_4 - p_1 p_6\right)\left(p_1 + p_4\right) \\
&\quad \times \left[\left(p_2 p_5 - p_3 p_4 - p_1 p_6\right)\right. \\
&\left.\qquad + \left(p_3 + p_6\right)\left(p_1 + p_4\right)\right], \\
u_2 &= 2\left[\left(p_1 p_4\right)^2 - \left(p_3 p_6\right)^2\right] + \left[\left(p_1 + p_2\right)^2 - 2 p_1 p_4\right]^2 \\
&\quad + 2\left[\left(p_2 p_5 - p_3 p_4 - p_1 p_6\right)\left(p_1 + p_4\right)\right. \\
&\left.\qquad + \left(p_3 + p_6\right)\left(p_1 p_4 + p_3 p_6\right)\right]\left(p_3 + p_6\right) \\
&\quad - \left[\left(p - 2 p_5 - p_3 p_4 - p_1 p_6\right) + \left(p_3 + p_6\right)\left(p_1 + p_4\right)\right]^2, \\
u_3 &= 2\left[\left(p_1 + p_2\right)^2 - 2 p_1 p_4\right] - \left(p_3 + p_6\right)^2.
\end{aligned}
\tag{42}
$$

Let $z = \theta^2$. Then (41) becomes

$$
z^4 + u_3 z^3 + u_2 z^2 + u_1 z + u_0 = 0.
\tag{43}
$$

Denote

$$
h(z) = z^4 + u_3 z^3 + u_2 z^2 + u_1 z + u_0.
\tag{44}
$$

Then

$$
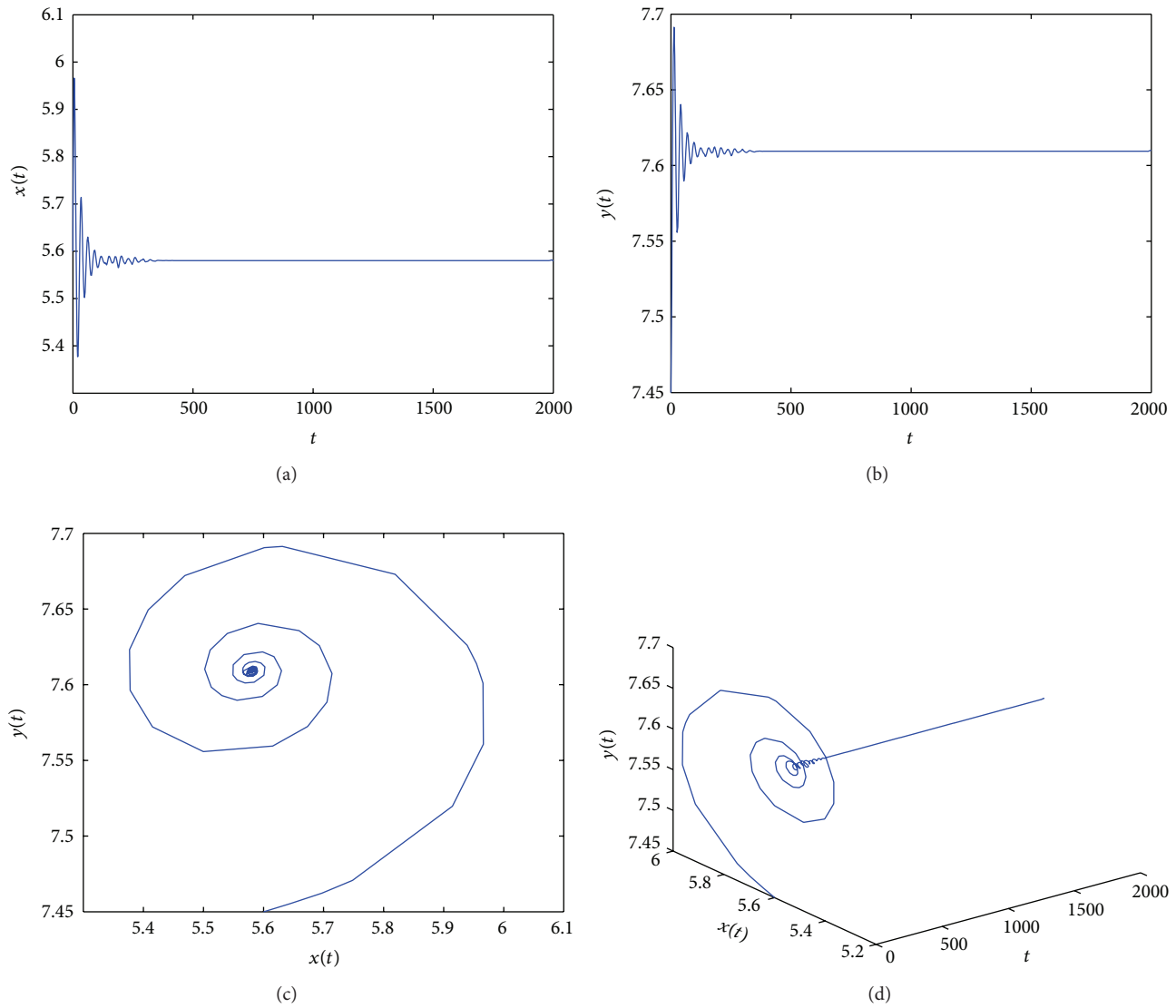h'(z) = 4 z^3 + 3 u_3 z^2 + 2 u_2 z + u_1.
\tag{45}
$$

(a)

(b)

(c)

(d)

FIGURE 10: Trajectory portrait and phase portrait of system (57) with $\tau_1 = 0.5$, $\tau_2 = 0.5 < \tau_{2_0} \approx 0.7723$. The positive equilibrium $E(5.61, 7.02)$ is asymptotically stable. The initial value is (5.6, 7.45).

Set

$$4z^3 + 3u_3 z^2 + 2u_2 z + u_1 = 0. \tag{46}$$

Let $y = z + u_3/4$. Then (46) becomes

$$y^3 + p_1 y + q_1 = 0, \tag{47}$$

where

$$p_1 = \frac{u_2}{2} - \frac{3}{16} u_3^2, \qquad q_1 = \frac{u_3^3}{32} - \frac{u_3 u_2}{8} + \frac{u_1}{4}. \tag{48}$$

Define

$$D = \left(\frac{q_1}{2}\right)^2 + \left(\frac{p_1}{3}\right)^3, \qquad \sigma = \frac{-1 + \sqrt{3}}{2},$$

$$y_1 = \sqrt[3]{-\frac{q_1}{2} + \sqrt{D}} + \sqrt[3]{-\frac{q_1}{2} - \sqrt{D}},$$

$$y_2 = \sqrt[3]{-\frac{q_1}{2} + \sqrt{D}\sigma} + \sqrt[3]{-\frac{q_1}{2} - \sqrt{D}\sigma^2},$$

$$y_3 = \sqrt[3]{-\frac{q_1}{2} + \sqrt{D}\sigma^2} + \sqrt[3]{-\frac{q_1}{2} - \sqrt{D}\sigma},$$

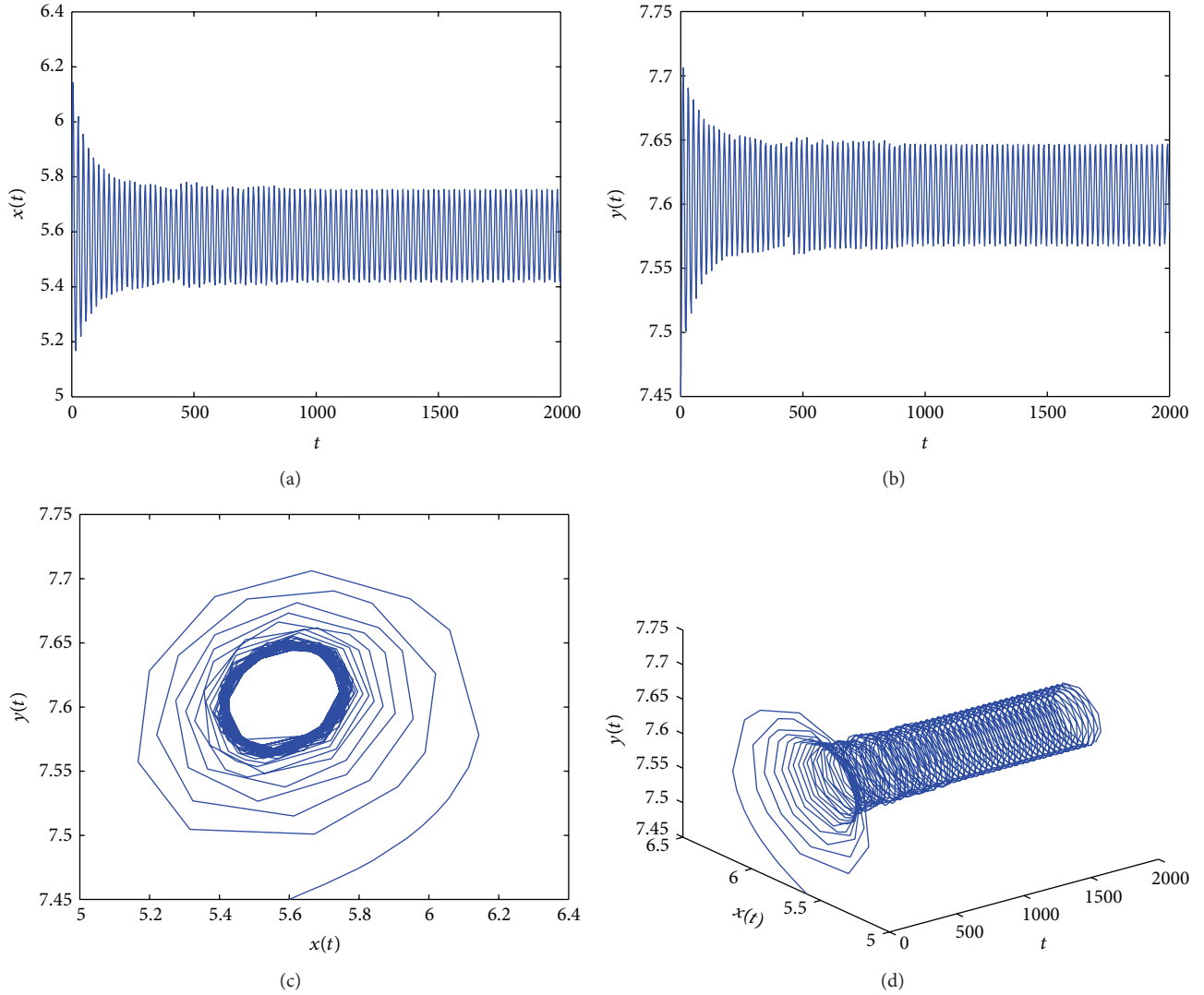$$z_i = y_i - \frac{p_1}{4}, \quad i = 1, 2, 3. \tag{49}$$

(a)

(b)

(c)

(d)

FIGURE 11: Trajectory portrait and phase portrait of system (57) with $\tau_1 = 0.5$, $\tau_2 = 1.6 > \tau_{2_0} \approx 0.7723$. Hopf bifurcation occurs from the positive equilibrium $E(5.61, 7.02)$. The initial value is $(5.6, 7.45)$.

From [32, 33], we have the following result.

Lemma 8. *If $u_0 < 0$, then (43) has at least one positive root.*

Lemma 9. *Assume that $u_0 \geq 0$. Then one has the following:*

(i) *if $D \geq 0$, then (43) has positive roots if and only if $z_1 > 0$ and $h'(z_1) < 0$;*

(ii) *if $D < 0$, then (43) has positive roots if and only if there exists at least one $z^* \in \{z_1, z_2, z_3\}$ such that $z^* > 0$ and $h'(z^*) \leq 0$.*

Without loss of generality, we assume that (43) has four positive roots, defined by z_1, z_2, z_3, z_4, respectively. Then (41) has four positive roots

$$\theta_1 = \sqrt{z_1}, \qquad \theta_2 = \sqrt{z_2},$$
$$\theta_3 = \sqrt{z_3}, \qquad \theta_4 = \sqrt{z_4}. \tag{50}$$

By (39), if we denote

$$\tau_k^{(j)} = \frac{1}{\theta_k} \left\{ \arccos \left[\frac{(p_3 p_4 + p_1 p_6 - p_2 p_5)(p_3 p_6 - p_1 p_4 + \theta^2) - (p_3 + p_6)\theta(p_1 + p_4)\theta}{[(p_1 + p_4)\theta]^2 - (p - 3p_6)^2 + (p_1 p_4 - \theta^2)^2} \right] + 2j\pi \right\}, \tag{51}$$
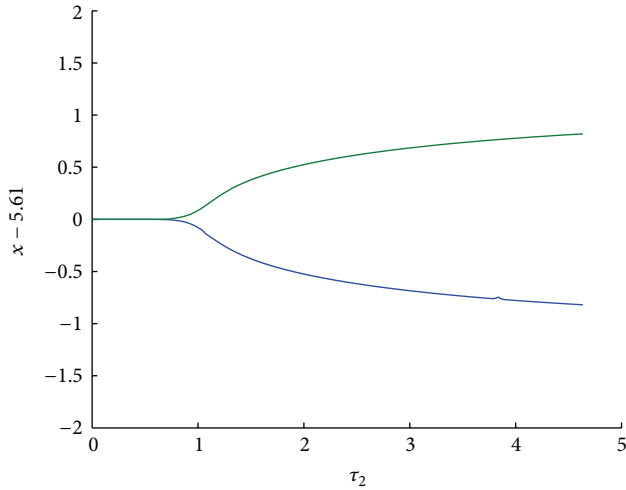
FIGURE 12: Bifurcation diagram with respect to the time delay τ_2 for system (57) with $\tau_1 = 0.5$.

where $k = 1, 2, 3, 4$; $j = 0, 1, \ldots$, then $\pm i\theta_k$ are a pair of purely imaginary roots of (35) with $\tau_k^{(j)}$. Define

$$\tau_0 = \tau_{k_0}^{(0)} = \min_{k \in \{1,2,3,4\}} \left\{ \tau_k^{(0)} \right\}, \quad \theta_0 = \theta_{k_0}. \quad (52)$$

Based on above analysis, we have the following result.

Lemma 10. *For $\tau_1 = \tau_2 = \tau$, if (H1) and (H2) hold, then all roots of (1) have a negative real part when $\tau \in [0, \tau_0)$, and (1) admits a pair of purely imaginary roots $\pm \theta_k i$ when $\tau = \tau_k^{(j)} (k = 1, 2, 3, 4, j = 0, 1, 2, \ldots)$.*

Let $\lambda(\tau) = \alpha(\tau) + i\theta(\tau)$ be a root of (35) near $\tau = \tau_k^{(j)}$, and let $\alpha(\tau_k^{(j)}) = 0$ and $\theta(\tau_k^{(j)}) = \theta_k$. Due to functional differential equation theory, for every $\tau_k^{(j)}$, $k = 1, 2, 3, 4$, $j = 0, 1, 2, \ldots$, there exists $\varepsilon > 0$ such that $\lambda(\tau)$ is continuously differentiable in τ for $|\tau - \tau_k^{(j)}| < \varepsilon$. Substituting $\lambda(\tau)$ into the left-hand side of (35) and taking derivative with respect to τ, we have

$$\left[\frac{d\lambda}{d\tau} \right]^{-1}$$

$$= -\frac{[2\lambda - (p_1 + p_4)] \lambda e^{\lambda\tau} - (p_2 + p_6) + p_3 p_6 e^{-\lambda\tau}}{\lambda [\lambda^2 - (p_1 + p_4)\lambda + p_1 p_4] e^{\lambda\tau} + p_3 p_6 \lambda^2 e^{-\lambda\tau}} \quad (53)$$

$$- \frac{\tau}{\lambda}.$$

We can easily obtain

$$\left[\frac{d(\operatorname{Re}\lambda(\tau))}{d\tau} \right]_{\tau = \tau_k^{(j)}}^{-1}$$

$$= -\operatorname{Re} \left\{ \frac{[2\lambda - (p_1 + p_4)] \lambda e^{\lambda\tau} - (p_2 + p_6) + p_3 p_6 e^{-\lambda\tau}}{\lambda [\lambda^2 - (p_1 + p_4)\lambda + p_1 p_4] e^{\lambda\tau} + p_3 p_6 \lambda^2 e^{-\lambda\tau}} \right\}_{\tau = \tau_k^{(j)}}$$

$$= -\operatorname{Re} \left\{ \frac{A_1 - A_2 i}{B_1 + B_2 i} \right\} = -\frac{A_1 B_1 - A_2 B_2}{B_1^2 + B_2^2}, \quad (54)$$

where

$$A_1 = \left(p_3 p_6 - 2\theta_k^2 \right) \cos \theta_k \tau_k^{(j)}$$

$$\quad + (p_1 + p_4)\theta_k \sin \theta_k \tau_k^{(j)} - (p_2 + p_6),$$

$$A_2 = (p_1 + p_4)\theta_k \tau_k^{(j)} \cos \theta_k \tau_k^{(j)} + \left(2\theta_k^2 + p_3 p_6 \right) \sin \theta_k \tau_k^{(j)},$$

$$B_1 = (p_1 + p_4)\theta_k^2 \cos \theta_k \tau_k^{(j)}$$

$$\quad - \left(p_1 p_4 - \theta_k^2 \right) \theta_k \sin \theta_k \tau_k^{(j)} - p_3 p_6 \theta_k^2,$$

$$B_2 = (p_1 + p_4)\theta_k^2 \sin \theta_k \tau_k^{(j)} + \left(p_1 p_4 - \theta_k^2 \right) \theta_k \cos \theta_k \tau_k^{(j)}$$

$$\quad + p_3 p_6 \theta_k^2 \sin \theta_k \tau_k^{(j)}. \quad (55)$$

Now we assume that

(H6)

$$A_1 B_1 \neq A_2 B_2. \quad (56)$$

In view of the above analysis and the results of Kuang [33] and Hale [31], we have the following.

Theorem 11. *For $\tau = 0$, if (H1) and (H2) hold, then the positive equilibrium $E(x^*, y^*)$ of system (1) is asymptotically stable for $\tau \in [0, \tau_0)$. In addition to the conditions (H1) and (H2), one further assumes that (H6) holds. Then system (1) undergoes a Hopf bifurcation at the positive equilibrium $E(x^*, y^*)$ when $\tau = \tau_k^{(j)}$, $k = 1, 2, 3, 4$, $j = 0, 1, 2, \ldots$.*

3. Computer Simulations

In this section, we present some numerical results of system (1) to verify the analytical predictions obtained in the previous section. Let us consider the following system:

$$\dot{x}(t) = 2x(t - \tau_1) - 0.12x^2 - \frac{2xy}{0.77y + x},$$

$$\dot{y}(t) = \frac{0.205x(t - \tau_2)y(t - \tau_2)}{0.77y(t - \tau_2) + x(t - \tau_2)} - 0.1y, \quad (57)$$

which has a positive equilibrium $E(5.61, 7.02)$. We can easily obtain that (H1)–(H6) hold true. When $\tau_1 = 0$, applying MATLAB 7.0, we can get $\omega_0 \approx 0.5524, \tau_{2_0} \approx 2.3345$. The positive equilibrium $E(5.61, 7.02)$ is asymptotically stable for $\tau_2 < \tau_{2_0} \approx 2.3345$ and unstable for $\tau_2 > \tau_{2_0} \approx 2.3345$ which is shown in **Figure 1**. When $\tau_2 = \tau_{2_0} \approx 2.3345$, (57) undergoes a Hopf bifurcation around the positive equilibrium

(a)

(b)

(c)

(d)
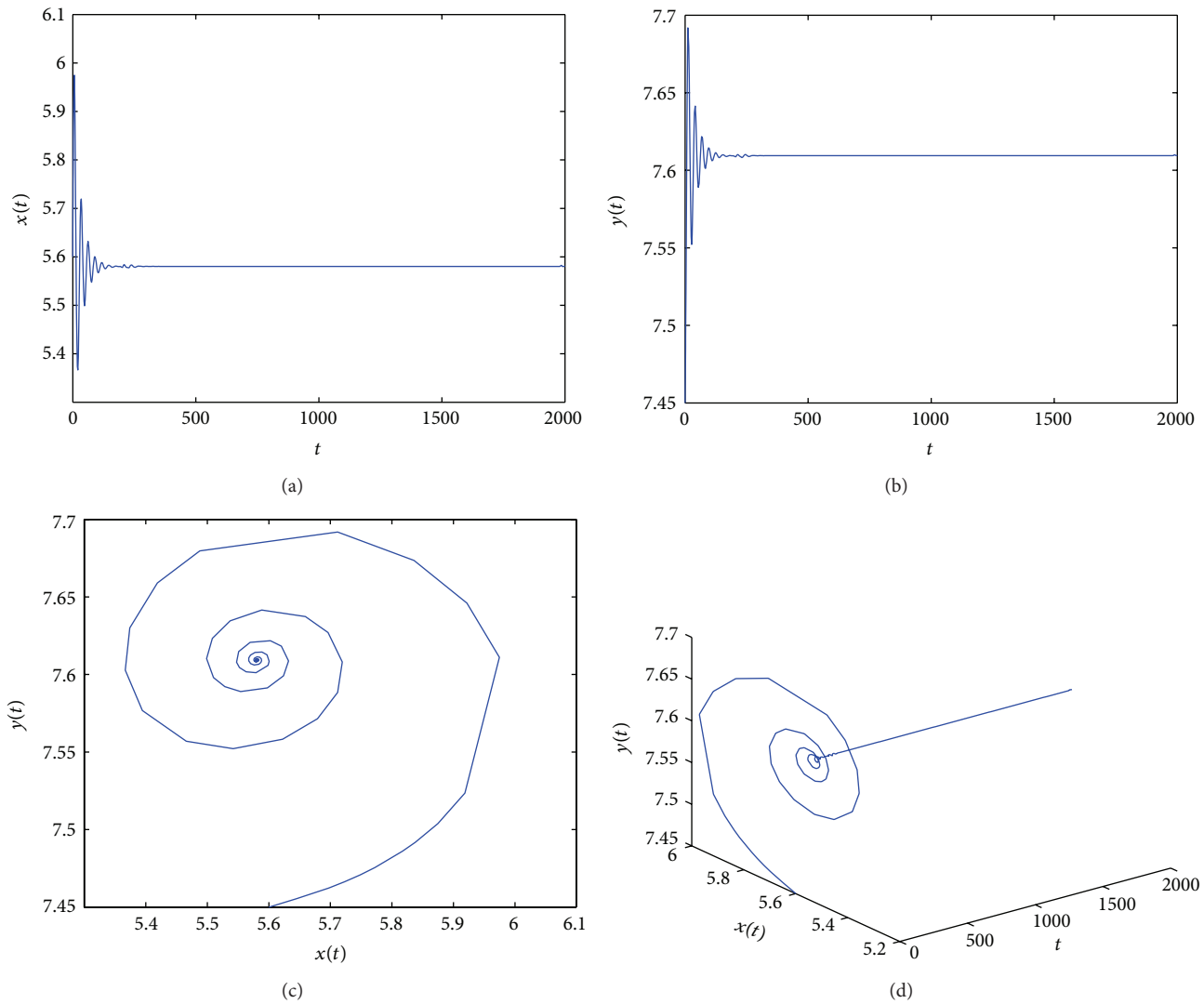
FIGURE 13: Trajectory portrait and phase portrait of system (57) with $\tau_1 = 0\tau_2 = \tau = 1.0 < \tau_0 \approx 1.2032$. The positive equilibrium $E(5.61, 7.02)$ is asymptotically stable. The initial value is $(5.6, 7.45)$.

$E(5.61, 7.02)$. That is, a small amplitude periodic solution occurs near $E(5.61, 7.02)$ when $\tau_1 = 0$ and τ_2 is close to $\tau_{2_0} = 2.3345$ which is shown in Figure 2. The bifurcation diagram of the case is shown in Figure 3.

Let $\tau_2 = 0.25 \in (0, 1.8)$, and choose τ_1 as a parameter. We have $\tau_{1_0} \approx 0.8013$. Then the positive equilibrium is asymptotically stable when $\tau_1 \in [0, \tau_{1_0})$. The Hopf bifurcation value of (57) is $\tau_{1_0} \approx 0.8013$ (see Figures 4 and 5) The bifurcation diagram of the case is shown in Figure 6.

When $\tau_2 = 0$, using MATLAB 7.0, we obtain $\eta_0 \approx 0.9056$, $\tau_{1_0} \approx 0.9122$. The positive equilibrium $E(5.61, 7.02)$ is asymptotically stable for $\tau_1 < \tau_{1_0} \approx 0.9122$ and unstable for $\tau_1 > \tau_{1_0} \approx 0.9122$ which is shown in Figure 7. When $\tau_1 = \tau_{1_0} \approx 0.9122$, (57) undergoes a Hopf bifurcation at the positive equilibrium $E_0(5.61, 7.02)$. That is, a small amplitude periodic solution occurs around $E(5.61, 7.02)$ when $\tau_2 = 0$ and τ_1 is

close to $\tau_{1_0} = 0.9122$ which is illustrated in Figure 8. The bifurcation diagram of the case is shown in Figure 9.

Let $\tau_1 = 0.5 \in (0, 0.9122)$, and choose τ_2 as a parameter. We have $\tau_{2_0} \approx 0.7723$. Then the positive equilibrium is asymptotically stable when $\tau_2 \in [0, \tau_{2_0})$. The Hopf bifurcation value of (57) is $\tau_{2_0} \approx 0.7723$ (see Figures 10 and 11). The bifurcation diagram of the case is shown in Figure 12.

When $\tau_1 = \tau_2 = \tau$, using MATLAB 7.0, we obtain $\theta_0 \approx 0.8725$, $\tau_0 \approx 1.2032$. The positive equilibrium $E(5.61, 7.02)$ is asymptotically stable for $\tau < \tau_0 \approx 1.2032$ and unstable for $\tau > \tau_0 \approx 1.2032$ which is shown in Figure 13. When $\tau = \tau_0 \approx 1.2032$, (57) undergoes a Hopf bifurcation at the positive equilibrium $E_0(5.61, 7.02)$. That is, a small amplitude periodic solution occurs around $E(5.61, 7.02)$ when τ is close to $\tau_0 \approx 1.2032$ which is illustrated in Figure 14. The bifurcation diagram of the case is shown in Figure 15.
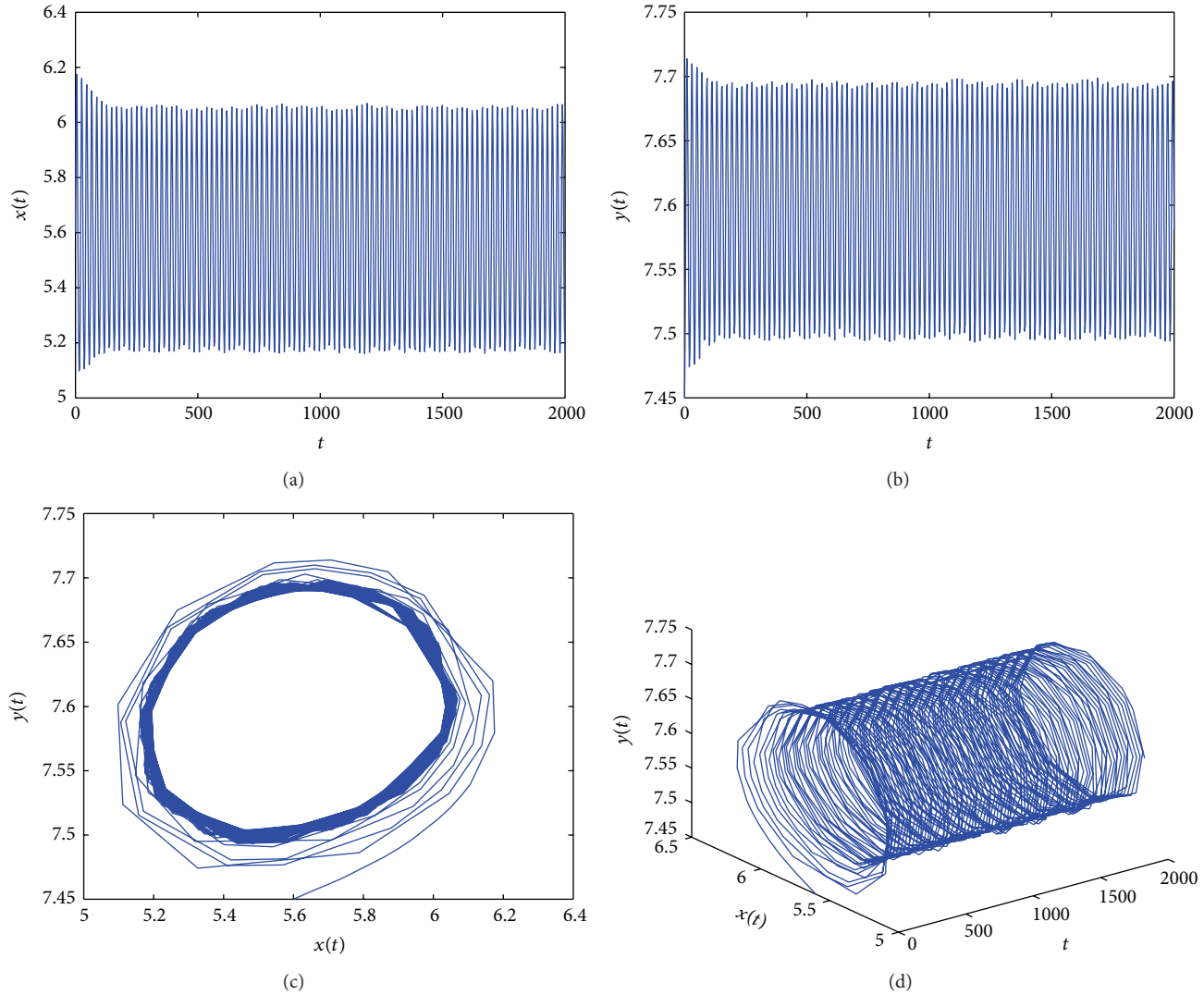
FIGURE 14: Trajectory portrait and phase portrait of system (57) with $\tau_1 = \tau_2 = \tau = 1.25 > \tau_0 \approx 1.2032$. Hopf bifurcation occurs from the positive equilibrium $E(5.61, 7.02)$. The initial value is $(5.6, 7.45)$.

4. Conclusions

In this paper, we have investigated local stability of the positive equilibrium $E(x^*, y^*)$ and local Hopf bifurcation of a ratio-dependent predator-prey model with two delays. It is shown that if some conditions hold true and $\tau_2 \in [0, \tau_{2_0})$, then the positive equilibrium $E(x^*, y^*)$ is asymptotically stable when $\tau_1 \in (0, \tau_{1_0})$. When the delay τ_1 increases, the positive equilibrium $E(x^*, y^*)$ loses its stability and a sequence of Hopf bifurcations occur at the positive equilibrium $E(x^*, y^*)$. That is, a family of periodic orbits bifurcates from the the positive equilibrium $E(x^*, y^*)$. We also showed that if a certain condition is satisfied and $\tau_1 \in [0, \tau_{1_0})$, then the positive equilibrium $E(x^*, y^*)$ is asymptotically stable when $\tau_2 \in (0, \tau_{2_0})$, when the delay τ_2 increases, the positive equilibrium $E(x^*, y^*)$ loses its stability and a sequence of Hopf bifurcations occur at the positive equilibrium $E(x^*, y^*)$.

In case $\tau_1 = \tau_2 = \tau$, we have shown that if some conditions are satisfied, and $\tau \in [0, \tau_0)$, then the positive equilibrium $E(x^*, y^*)$ is asymptotically stable when $\tau \in (0, \tau_0)$. When the delay τ increases, the positive equilibrium $E(x^*, y^*)$ loses its stability and a sequence of Hopf bifurcations occur at the positive equilibrium $E(x^*, y^*)$ which means a family of periodic orbits bifurcates from the the positive equilibrium $E(x^*, y^*)$. Some numerical simulations verifying our theoretical results are carried out. In addition, we must point out that although Ko and Ryu [6] have also investigated the the existence of Hopf bifurcation for system (1) with respect to positive equilibrium $E(x^*, y^*)$, it is assumed that $\tau_1 + \tau_2$ in a certain range and choose the delay τ_2 as bifurcation parameter to consider the Hopf bifurcation nature. But what effect different time delays have on the dynamics of system (1)? Ko and Ryu [6] did not deal with this issue. It is important for us to consider what effect the two different time delays has
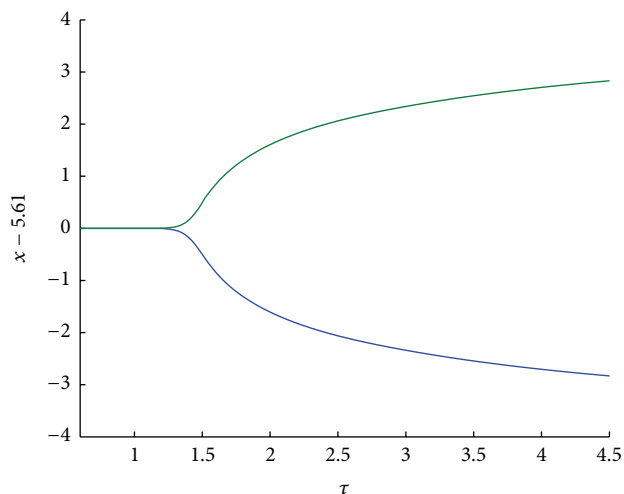
FIGURE 15: Bifurcation diagram with respect to the time delay τ for system (57) with $\tau_1 = \tau_2 = \tau$.

on the dynamical behavior of system (1). Thus we think that our work generalizes the known results of Ko and Ryu [6]. In addition, we can study the Hopf bifurcation nature of system (1) by regarding the delay τ_2 as bifurcation parameter. We will further focus on the topic elsewhere in the near future.

Acknowledgments

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